Four Logics for Minimal Belief Revision

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Abstract

It is natural to think of belief revision as the interaction of belief and information over time. Thus branching-time temporal logic seems a natural setting for a theory of belief revision. We propose a logic based on three modal operators: a belief operator, an information operator and a next-time operator. Four logics of increasing strength are proposed. The first is a logic that captures the most basic notion of minimal belief revision. The second characterizes the qualitative content of Bayes' rule. The third provides an axiomatization of the AGM theory of belief revision and the fourth provides a characterization of the notion of plausibility ordering of the set of possible worlds.

1 Introduction

The concepts of belief and belief revision are of central importance in many disciplines. A variety of approaches and formal tools have been employed in the study of these topics.

In their seminal contribution Alchourrón *et al* [1] model beliefs as sets of formulas in a given syntactic language and belief revision is construed as an operation that associates with every belief set K (thought of as the initial beliefs) and formula ϕ (thought of as new information) a new belief set K^*_ϕ representing the revised beliefs. Several requirements are imposed on this operation in order to capture the notion of "rational" belief change. Their approach has become known as the AGM theory of belief revision and has stimulated a large literature

The notion of static belief, on the other hand, starting with Hintikka's [13] seminal contribution, has been developed mainly within the context of modal

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logic. On the syntactic side a belief operator B is introduced, with the intended interpretation of $B\phi$ as "the individual believes that ϕ". Various properties of beliefs are then expressed by means of axioms, such as the positive introspection axiom $B\phi \rightarrow BB\phi$, which says that if the individual believes ϕ then she believes that she believes ϕ. On the semantic side Kripke structures (Kripke [15]) are used, consisting of a set of states (or possible worlds) Ω together with a binary relation $\mathcal{B}$ on Ω, with the interpretation of $\alpha\mathcal{B}\beta$ as "at state α the individual considers state β possible". The connection between syntax and semantics is then obtained by means of a valuation V which associates with every atomic sentence p the set of states where p is true. The pair $\langle \Omega, \mathcal{B} \rangle$ is called a frame and the addition of a valuation V to a frame yields a model. Rules are given for determining the truth of an arbitrary formula at every state of a model; in particular, the formula $B\phi$ is true at state α if and only if ϕ is true at every β such that $\alpha\mathcal{B}\beta$, that is, if ϕ is true at every state that the individual considers possible at α. A property of the accessibility relation $\mathcal{B}$ is said to correspond to an axiom if every instance of the axiom is true at every state of every model based on a frame that satisfies the property and *vice versa*. For example, the positive introspection axiom $B\phi \rightarrow BB\phi$ corresponds to transitivity of the relation $\mathcal{B}$.

This paper attempts to bridge the gap between these two strands of the literature, by proposing a framework that can accommodate both static beliefs and belief revision. It is natural to think of belief revision as the interaction of belief and information over time. Thus temporal logic is a natural starting point. On the syntactic side, besides the "next-time" operator $\bigcirc$, our language contains a belief operator B and an information operator I. The information operator is not a normal operator and is formally similar to the "only knowing" operator introduced by Levesque [16]. On the semantic side we use branching-time frames to represent different possible evolutions of beliefs. For every date t, beliefs and information are represented by binary relations $\mathcal{B}_t$ and $\mathcal{I}_t$ on a set of states Ω. As in the static setting, the link between syntax and semantics is provided by the notion of valuation and model. The truth of a formula in a model is defined at a state-instant pair (ω, t).

We propose a sequence of logics of increasing strength, beginning with a minimal logic for belief revision that captures the requirement that the agent not change his beliefs if he is informed of something which he already believes. The next logic, which we call $\mathbb{L}_{QBR}$, provides an axiomatization of the qualitative content of Bayes' rule, which is central to the modeling of belief revision in economics and game theory. A strengthening of $\mathbb{L}_{QBR}$ provides an axiomatization of the AGM theory of belief revision. We call this logic $\mathbb{L}_{AGM}$. The AGM theory has been shown to be related to the notion of plausibility ordering. Depending on the context, the plausibility ordering is defined either on the set of formulas (see [10]) or on the set of possible worlds (see [3], [12], [19]). Although we provide a semantic characterization of logic $\mathbb{L}_{AGM}$ which does not invoke the notion of plausibility ordering, it is natural to ask whether logic $\mathbb{L}_{AGM}$ implies the existence of a plausibility ordering on the set of states Ω that rationalizes belief revision. We show that the answer is negative. A strengthening of $\mathbb{L}_{AGM}$, which we call $\mathbb{L}_{PLS}$, is shown to provide the desired correspondence between a set of axioms for belief revision and the notion of plausibility ordering on the set of states.

The results that we provide are frame characterization results, that is, we show the correspondence between a set of axioms and a property of belief revision frames. The issue of completeness is explored in Bonanno [6].

2 Temporal belief revision frames

On the semantic side we consider branching-time structures with the addition of a belief relation and an information relation for every instant t.

Definition 1 *A* next-time branching frame *is a pair $\langle T, \rightarrowtail\rangle$ where T is a countable set of instants or dates and $\rightarrowtail$ is a binary relation on T satisfying the following properties: $\forall t_1, t_2, t_3 \in T$,*

(1)	backward uniqueness	if $t_1 \rightarrowtail t_3$ and $t_2 \rightarrowtail t_3$ then $t_1 = t_2$
(2)	acyclicity	if $\langle t_1, ..., t_n\rangle$ is a sequence with $t_i \rightarrowtail t_{i+1}$ for every $i = 1, ..., n-1$, then $t_n \neq t_1$.

The interpretation of $t_1 \rightarrowtail t_2$ is that t_2 is an *immediate successor* of t_1 or t_1 is the *immediate predecessor* of t_2: every instant has at most a unique immediate predecessor but can have several immediate successors.

Definition 2 *A* temporal belief revision frame *is a tuple $\langle T, \rightarrowtail, \Omega, \{\mathcal{B}_t, \mathcal{I}_t\}_{t\in T}\rangle$ where $\langle T, \rightarrowtail\rangle$ is a next-time branching frame, Ω is a set of states (or possible worlds) and, for every $t \in T$, $\mathcal{B}_t$ and $\mathcal{I}_t$ are binary relations on Ω.*

The interpretation of $\omega\mathcal{B}_t\omega'$ is that at state ω and time t the individual considers state ω' possible (an alternative expression is "ω' is a doxastic alternative to ω at time t"). On the other hand, the interpretation of $\omega\mathcal{I}_t\omega'$ is that at state ω and time t, according to the information received, it is possible that the true state is ω'. We shall use the following notation:

$$\mathcal{B}_t(\omega) = \{\omega' \in \Omega : \omega\mathcal{B}_t\omega'\} \text{ and, similarly, } \mathcal{I}_t(\omega) = \{\omega' \in \Omega : \omega\mathcal{I}_t\omega'\}.$$

Temporal belief frames can be used to describe either a situation where the objective facts describing the world do not change − so that only the beliefs of the agent change over time − or a situation where both the facts and the doxastic state of the agent change. In the computer science literature the first situation is called belief revision, while the latter is called belief update (see [14]). We shall focus on belief revision.

On the syntactic side we consider a propositional language with five modal operators: the next-time operator $\bigcirc$ and it inverse $\bigcirc^{-1}$, the belief operator B, the information operator I and the "all state" operator A. The intended interpretation is as follows:

$\bigcirc\phi$:	"at every next instant it will be the case that ϕ"
$\bigcirc^{-1}\phi$:	"at every previous instant it was the case that ϕ"
$B\phi$:	"the agent believes that ϕ"
$I\phi$:	"the agent is informed that ϕ"
$A\phi$:	"it is true at every state that ϕ".

The "all state" operator A is needed in order to capture the non-normality of the information operator I (see below). For a thorough discussion of the "all state" operator see Goranko and Passy [11].

Note that, while the other operators apply to arbitrary formulas, *we restrict the information operator to apply to Boolean formulas only*, that is, to formulas that do not contain any modal operators.

Definition 3 *Boolean formulas are defined recursively as follows: (1) every atomic proposition is a Boolean formula, and (2) if ϕ and ψ are Boolean formulas then so are $\neg\phi$ and $(\phi \vee \psi)$.*

Boolean formulas represent facts and *we restrict information to be about facts.*

Given a temporal belief revision frame $\langle T, \rightarrowtail, \Omega, \{\mathcal{B}_t, \mathcal{I}_t\}_{t\in T}\rangle$ one obtains a *model based on it* by adding a function $V : S \to 2^{\Omega}$ (where S is the set of atomic propositions and 2^{Ω} denotes the set of subsets of Ω) that associates with every atomic proposition q the set of states at which q is true. Note that defining a valuation this way is what frames the problem as one of belief revision, since the truth value of an atomic proposition q depends only on the state and not on the time.[1] Given a model, a state ω, an instant t and a formula ϕ, we write $(\omega, t) \models \phi$ to denote that ϕ is true at state ω and time t. Let $\|\phi\|$ denote the truth set of ϕ, that is, $\|\phi\| = \{(\omega, t) \in \Omega \times T : (\omega, t) \models \phi\}$ and let $\lceil\phi\rceil_t \subseteq \Omega$ denote the set of states at which ϕ is true at time t, that is, $\lceil\phi\rceil_t = \{\omega \in \Omega : (\omega, t) \models \phi\}$. Truth of an arbitrary formula at a pair (ω, t) is defined recursively as follows:

if $q \in S$,	$(\omega, t) \models q$ if and only if $\omega \in V(q)$;
$(\omega, t) \models \neg\phi$	if and only if $(\omega, t) \not\models \phi$;
$(\omega, t) \models \phi \vee \psi$	if and only if either $(\omega, t) \models \phi$ or $(\omega, t) \models \psi$ (or both);
$(\omega, t) \models \bigcirc\phi$	if and only if $(\omega, t') \models \phi$ for every t' such that $t \rightarrowtail t'$;
$(\omega, t) \models \bigcirc^{-1}\phi$	if and only if $(\omega, t'') \models \phi$ for every t'' such that $t'' \rightarrowtail t$;
$(\omega, t) \models B\phi$	if and only if $\mathcal{B}_t(\omega) \subseteq \lceil\phi\rceil_t$, that is, if $(\omega', t) \models \phi$ for all $\omega' \in \mathcal{B}_t(\omega)$;
$(\omega, t) \models I\phi$	if and only if $\mathcal{I}_t(\omega) = \lceil\phi\rceil_t$, that is, if (1) $(\omega', t) \models \phi$ for all $\omega' \in \mathcal{I}_t(\omega)$, and (2) if $(\omega', t) \models \phi$ then $\omega' \in \mathcal{I}_t(\omega)$;
$(\omega, t) \models A\phi$	if and only if $\|\phi\|_t = \Omega$, that is, if $(\omega', t) \models \phi$ for all $\omega' \in \Omega$.

Note that, while the truth condition for the operator B is the standard one, the truth condition for the operator I is non-standard: instead of simply

[1] Belief update would require a valuation to be defined as a function $V : S \to 2^{\Omega\times T}$.

requiring that $\mathcal{I}_t(\omega) \subseteq \lceil\phi\rceil_t$ we require equality: $\mathcal{I}_t(\omega) = \lceil\phi\rceil_t$. Thus our information operator is formally similar to the "only knowing" operator introduced by Levesque (see [16]), although the interpretation is different.

The following proposition (proved in [5], p. 148) says that the truth value of a Boolean formula does not change over time: it is only a function of the state. We denote by Φ^B the set of Boolean formulas.

Proposition 4 *Let $\phi \in \Phi^B$. Fix an arbitrary model and suppose that $(\omega, t) \models \phi$. Then, for every $t' \in T$, $(\omega, t') \models \phi$.*

A formula ϕ is *valid in a model* if $\|\phi\| = \Omega \times T$, that is, if ϕ is true at every state-instant pair (ω, t). A formula ϕ is *valid in a frame* if it is valid in every model based on it.

3 The basic logic

The formal language is built in the usual way (see [2]) from a countable set of atomic propositions, the connectives $\neg$ and $\vee$ (from which the connectives $\wedge$, $\rightarrow$ and $\leftrightarrow$ are defined as usual) and the modal operators $\bigcirc$, $\bigcirc^{-1}$, B, I and A, with the restriction that $I\phi$ is a well-formed formula if and only if ϕ is a Boolean formula. Let $\Diamond\phi \stackrel{def}{=} \neg\bigcirc\neg\phi$, and $\Diamond^{-1}\phi \stackrel{def}{=} \neg\bigcirc^{-1}\neg\phi$. Thus the interpretation of $\Diamond\phi$ is "at *some* next instant it will be the case that ϕ" while the interpretation of $\Diamond^{-1}\phi$ is "at some immediately preceding instant it was the case that ϕ".

We denote by $\mathbb{L}_0$ the basic logic defined by the following axioms and rules of inference.

AXIOMS:

1. All propositional tautologies.
2. Axiom K for $\bigcirc$, $\bigcirc^{-1}$, B and A: for $\Box \in \{\bigcirc, \bigcirc^{-1}, B, A\}$

$$(\Box\phi \wedge \Box(\phi \rightarrow \psi)) \rightarrow \Box\psi \qquad \text{(K)}$$

3. Temporal axioms relating $\bigcirc$ and $\bigcirc^{-1}$:

$$\begin{array}{ll} \phi \rightarrow \bigcirc\Diamond^{-1}\phi & (\mathrm{O}_1) \\ \phi \rightarrow \bigcirc^{-1}\Diamond\phi & (\mathrm{O}_2) \end{array}$$

4. Backward Uniqueness axiom:

$$\Diamond^{-1}\phi \rightarrow \bigcirc^{-1}\phi \qquad \text{(BU)}$$

5. S5 axioms for A:

$$\begin{array}{ll} A\phi \rightarrow \phi & (\mathrm{T}_A) \\ \neg A\phi \rightarrow A\neg A\phi & (5_A) \end{array}$$

6. Inclusion axiom for B (note the absence of an analogous axiom for I):

$$A\phi \to B\phi \quad (Incl_B)$$

7. Axioms to capture the non-standard semantics for I: for ϕ and ψ Boolean,

$$\begin{array}{rl} (I\phi \wedge I\psi) \to A(\phi \leftrightarrow \psi) & (\mathrm{I}_1) \\ A(\phi \leftrightarrow \psi) \to (I\phi \leftrightarrow I\psi) & (\mathrm{I}_2) \end{array}$$

RULES OF INFERENCE:

1. Modus Ponens: $\frac{\phi, \quad \phi \to \psi}{\psi}$ (MP)

2. Necessitation for A, $\bigcirc$ and $\bigcirc^{-1}$: for every $\Box \in \{A, \bigcirc, \bigcirc^{-1}\}$, $\frac{\phi}{\Box\phi}$ (Nec).

Note that from MP, axiom $Incl_B$ and Necessitation for A one can derive necessitation for B. On the other hand, necessitation for I is *not* a rule of inference of this logic (indeed it is not validity preserving).

Remark 5 *By MP, axiom K and Necessitation, the following is a derived rule of inference for the operators $\bigcirc$, $\bigcirc^{-1}$, B and A: $\frac{\phi \to \psi}{\Box\phi \to \Box\psi}$ for $\Box \in \{\bigcirc, \bigcirc^{-1}, B, A\}$. We call this rule RK. On the other hand, rule RK is not a valid rule of inference for the operator I (despite the fact that axiom K for I can be shown to be a theorem of $\mathbb{L}_0$).*

4 The weakest logic of belief revision

Our purpose is to model how the *factual* beliefs of an individual change over time in response to *factual* information. Thus *the axioms we introduce are restricted to Boolean formulas*, which are formulas that do not contain any modal operators (cf. Definition 3).

We shall consider axioms of increasing strength that capture the notion of minimal change of beliefs.

The first axiom says that if ϕ and ψ are facts (Boolean formulas) and - currently - the agent believes that ϕ and also believes that ψ and his belief that ϕ is non-trivial (in the sense that he considers ϕ possible) then - at every next instant - if he is informed that ϕ it will still be the case that he believes that ψ. That is, if at a next instant he is informed of some fact that he currently believes, then he *cannot drop* any of his current factual beliefs ('W' stands for

'Weak' and 'ND' for 'No Drop'):[2] if ϕ and ψ are Boolean,

$$(B\phi \wedge \neg B\neg\phi \wedge B\psi) \rightarrow \bigcirc(I\phi \rightarrow B\psi). \qquad (WND)$$

The second axiom says that if ϕ and ψ are facts (Boolean formulas) and - currently - the agent believes that ϕ and does not believe that ψ, then - at every next instant - if he is informed that ϕ it will still be the case that he does not believe that ψ. That is, at any next instant at which he is informed of some fact that he currently believes he *cannot add* a factual belief that he does not currently have ('W' stands for 'Weak' and 'NA' stands for 'No Add'):[3] if ϕ and ψ are Boolean,

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[2] The following axiom, which says that if the individual is informed of some fact that he believed non-trivially at a previous instant then he must continue to believe every fact that he believed at that time, is equivalent to (WND): if ϕ and ψ are Boolean,

$$\Diamond^{-1}(B\phi \wedge B\psi \wedge \neg B\neg\phi) \wedge I\phi \rightarrow B\psi$$

(this, in turn, is propositionally equivalent to $\Diamond^{-1}(B\phi \wedge B\psi \wedge \neg B\neg\phi) \rightarrow (I\phi \rightarrow B\psi)$.
We prove equivalence by deriving each from the other. Derivation of (WND) from the above axiom ('PL' stands for 'Propositional Logic'):

1.	$\Diamond^{-1}(B\phi \wedge B\psi \wedge \neg B\neg\phi) \rightarrow (I\phi \rightarrow B\psi)$	above axiom
2.	$\bigcirc\Diamond^{-1}(B\phi \wedge B\psi \wedge \neg B\neg\phi) \rightarrow \bigcirc(I\phi \rightarrow B\psi)$	1, rule RK for $\bigcirc$
3.	$(B\phi \wedge B\psi \wedge \neg B\neg\phi) \rightarrow \bigcirc\Diamond^{-1}(B\phi \wedge B\psi \wedge \neg B\neg\phi)$	Temporal axiom O_1
4.	$(B\phi \wedge B\psi \wedge \neg B\neg\phi) \rightarrow \bigcirc(I\phi \rightarrow B\psi)$	2,3, PL.

Derivation of the above axiom from (WND):

1.	$(B\phi \wedge B\psi \wedge \neg B\neg\phi) \rightarrow \bigcirc(I\phi \rightarrow B\psi)$	Axiom WND
2.	$\neg\bigcirc(I\phi \rightarrow B\psi) \rightarrow \neg(B\phi \wedge B\psi \wedge \neg B\neg\phi)$	1, PL
3.	$\bigcirc^{-1}\neg\bigcirc(I\phi \rightarrow B\psi) \rightarrow \bigcirc^{-1}\neg(B\phi \wedge B\psi \wedge \neg B\neg\phi)$	2, rule RK for $\bigcirc^{-1}$
4.	$\Diamond^{-1}(B\phi \wedge B\psi \wedge \neg B\neg\phi) \rightarrow \Diamond^{-1}\bigcirc(I\phi \rightarrow B\psi)$	3, PL, definition of $\Diamond^{-1}$
5.	$\neg(I\phi \rightarrow B\psi) \rightarrow \bigcirc^{-1}\Diamond\neg(I\phi \rightarrow B\psi)$	Temporal axiom O_2
6.	$\Diamond^{-1}\bigcirc(I\phi \rightarrow B\psi) \rightarrow (I\phi \rightarrow B\psi)$	5, PL, definition of $\Diamond^{-1}$ and $\Diamond$
7.	$\Diamond^{-1}(B\phi \wedge B\psi \wedge \neg B\neg\phi) \rightarrow (I\phi \rightarrow B\psi)$	4, 6, PL.

[3] The following is an equivalent formulation of (WNA): if ϕ and ψ are Boolean,

$$\Diamond^{-1}(B\phi \wedge \neg B\psi) \wedge I\phi \rightarrow \neg B\psi.$$

We prove equivalence by deriving each from the other. Derivation of (WNA) from the above axiom:

1.	$\Diamond^{-1}(B\phi \wedge \neg B\psi) \wedge I\phi \rightarrow \neg B\psi$	above axiom
2.	$\Diamond^{-1}(B\phi \wedge \neg B\psi) \rightarrow (I\phi \rightarrow \neg B\psi)$	1, PL
3.	$\bigcirc\Diamond^{-1}(B\phi \wedge \neg B\psi) \rightarrow \bigcirc(I\phi \rightarrow \neg B\psi)$	2, rule RK for $\bigcirc$
4.	$(B\phi \wedge \neg B\psi) \rightarrow \bigcirc\Diamond^{-1}(B\phi \wedge \neg B\psi)$	Temporal axiom O_1
5.	$(B\phi \wedge \neg B\psi) \rightarrow \bigcirc(I\phi \rightarrow \neg B\psi)$	3, 4, PL.

Now derivation of the above axiom from (WNA):

1.	$(B\phi \wedge \neg B\psi) \rightarrow \bigcirc(I\phi \rightarrow \neg B\psi)$	Axiom WNA
2.	$\neg\bigcirc(I\phi \rightarrow \neg B\psi) \rightarrow \neg(B\phi \wedge \neg B\psi)$	1, PL
3.	$\bigcirc^{-1}\neg\bigcirc(I\phi \rightarrow \neg B\psi) \rightarrow \bigcirc^{-1}\neg(B\phi \wedge \neg B\psi)$	2, rule RK for $\bigcirc^{-1}$
4.	$\Diamond^{-1}(B\phi \wedge \neg B\psi) \rightarrow \Diamond^{-1}\bigcirc(I\phi \rightarrow \neg B\psi)$	3, PL and definition of $\Diamond^{-1}$
5.	$\neg(I\phi \rightarrow \neg B\psi) \rightarrow \bigcirc^{-1}\Diamond\neg(I\phi \rightarrow \neg B\psi)$	Temporal axiom O_2
6.	$\Diamond^{-1}\bigcirc(I\phi \rightarrow \neg B\psi) \rightarrow (I\phi \rightarrow \neg B\psi)$	5, PL
7.	$\Diamond^{-1}(B\phi \wedge \neg B\psi) \rightarrow (I\phi \rightarrow \neg B\psi)$	4, 6, PL
8.	$\Diamond^{-1}(B\phi \wedge \neg B\psi) \wedge I\phi \rightarrow \neg B\psi$	7, PL.

$$(B\phi \wedge \neg B\psi) \to \bigcirc(I\phi \to \neg B\psi). \qquad (WNA)$$

Thus, by WND, no belief can be dropped and, by WNA, no belief can be added, at any next instant at which the individual is informed of a fact that he currently believes.

Definition 6 *An axiom is characterized by (or characterizes) a property of frames if it is valid in a frame if and only if the frame satisfies that property.*

Proposition 7 *Axiom WND is characterized by the following property:*

$$\text{if } t_1 \rightarrowtail t_2,\ \mathcal{B}_{t_1}(\omega) \neq \varnothing \text{ and } \mathcal{B}_{t_1}(\omega) \subseteq \mathcal{I}_{t_2}(\omega) \text{ then } \mathcal{B}_{t_2}(\omega) \subseteq \mathcal{B}_{t_1}(\omega). \qquad (P_{WND})$$

Proof. Fix a frame that satisfies P_{WND}, an arbitrary model based on it and arbitrary $\alpha \in \Omega$, $t_1 \in T$ and Boolean formulas ϕ and ψ and suppose that $(\alpha, t_1) \models (B\phi \wedge B\psi \wedge \neg B\neg\phi)$. Since $(\alpha, t_1) \models \neg B\neg\phi$, there exists an $\omega \in \mathcal{B}_{t_1}(\alpha)$ such that $(\omega, t_1) \models \phi$. Thus $\mathcal{B}_{t_1}(\alpha) \neq \varnothing$. Fix an arbitrary $t_2 \in T$ such that $t_1 \to t_2$ and suppose that $(\alpha, t_2) \models I\phi$. Then $\mathcal{I}_{t_2}(\alpha) = \lceil\phi\rceil_{t_2}$. Fix an arbitrary $\beta \in \mathcal{B}_{t_1}(\alpha)$. Since $(\alpha, t_1) \models B\phi$, $(\beta, t_1) \models \phi$. Since ϕ is Boolean, by Proposition 4 $(\beta, t_2) \models \phi$, that is, $\beta \in \lceil\phi\rceil_{t_2}$ Hence $\beta \in \mathcal{I}_{t_2}(\alpha)$. Thus $\mathcal{B}_{t_1}(\alpha) \subseteq \mathcal{I}_{t_2}(\alpha)$. Hence, by P_{WND} $\mathcal{B}_{t_2}(\alpha) \subseteq \mathcal{B}_{t_1}(\alpha)$. Fix an arbitrary $\omega \in \mathcal{B}_{t_2}(\alpha)$. Then $\omega \in \mathcal{B}_{t_1}(\alpha)$ and, since $(\alpha, t_1) \models B\psi$, $(\omega, t_1) \models \psi$. Since ψ is Boolean, by Proposition 4 $(\omega, t_2) \models \psi$. Thus $(\alpha, t_2) \models B\psi$.

Conversely, suppose that P_{WND} is violated. Then there exist $\alpha \in \Omega$ and $t_1, t_2 \in T$ such that $t_1 \rightarrowtail t_2$, $\mathcal{B}_{t_1}(\alpha) \neq \varnothing$, $\mathcal{B}_{t_1}(\alpha) \subseteq \mathcal{I}_{t_2}(\alpha)$ and $\mathcal{B}_{t_2}(\alpha) \nsubseteq \mathcal{B}_{t_1}(\alpha)$. Let p and q be atomic propositions and construct a model where $||p|| = \mathcal{I}_{t_2}(\alpha) \times T$ and $||q|| = \mathcal{B}_{t_1}(\alpha) \times T$. Then $(\alpha, t_1) \models (Bp \wedge Bq \wedge \neg B\neg q)$. By hypothesis, there exists a $\beta \in \mathcal{B}_{t_2}(\alpha)$ such that $\beta \notin \mathcal{B}_{t_1}(\alpha)$, so that $(\beta, t_2) \nvDash q$. Hence $(\alpha, t_2) \nvDash Bq$ while $(\alpha, t_2) \models Ip$, so that $(\alpha, t_2) \nvDash Ip \to Bq$. Thus, since $t_1 \rightarrowtail t_2$, WND is falsified at (α, t_1). ■

Proposition 8 *Axiom WNA is characterized by the following property:*

$$\text{if } t_1 \rightarrowtail t_2 \text{ and } \mathcal{B}_{t_1}(\omega) \subseteq \mathcal{I}_{t_2}(\omega) \text{ then } \mathcal{B}_{t_1}(\omega) \subseteq \mathcal{B}_{t_2}(\omega). \qquad (P_{WNA})$$

Proof. Fix a frame that satisfies P_{WNA}, an arbitrary model based on it and arbitrary $\alpha \in \Omega$, $t_1 \in T$ and Boolean formulas ϕ and ψ and suppose that $(\alpha, t_1) \models B\phi \wedge \neg B\psi$. Then there exists a $\beta \in \mathcal{B}_{t_1}(\alpha)$ such that $(\beta, t_1) \models \neg\psi$. Fix an arbitrary $t_2 \in T$ such that $t_1 \rightarrowtail t_2$ and suppose that $(\alpha, t_2) \models I\phi$. Then $\mathcal{I}_{t_2}(\alpha) = \lceil\phi\rceil_{t_2}$. Fix an arbitrary $\omega \in \mathcal{B}_{t_1}(\alpha)$. Since $(\alpha, t_1) \models B\phi$, $(\omega, t_1) \models \phi$. Since ϕ is Boolean, by Proposition 4 $(\omega, t_2) \models \phi$ and therefore $\omega \in \mathcal{I}_{t_2}(\alpha)$. Thus $\mathcal{B}_{t_1}(\alpha) \subseteq \mathcal{I}_{t_2}(\alpha)$ and, by P_{WNA}, $\mathcal{B}_{t_1}(\alpha) \subseteq \mathcal{B}_{t_2}(\alpha)$. Since $(\beta, t_1) \models \neg\psi$ and $\neg\psi$ is Boolean (because ψ is), by Proposition 4 $(\beta, t_2) \models \neg\psi$. Since $\beta \in \mathcal{B}_{t_1}(\alpha)$ and $\mathcal{B}_{t_1}(\alpha) \subseteq \mathcal{B}_{t_2}(\alpha)$, $\beta \in \mathcal{B}_{t_2}(\omega)$ and therefore $(\alpha, t_2) \models \neg B\psi$.

Conversely, suppose that P_{WNA} is violated. Then there exist $\alpha \in \Omega$ and $t_1, t_2 \in T$ such that $t_1 \rightarrowtail t_2$ and $\mathcal{B}_{t_1}(\alpha) \subseteq \mathcal{I}_{t_2}(\alpha)$ and $\mathcal{B}_{t_1}(\alpha) \nsubseteq \mathcal{B}_{t_2}(\alpha)$. Let p

and q be atomic propositions and construct a model where $||p|| = \mathcal{I}_{t_2}(\alpha) \times T$ and $||q|| = \mathcal{B}_{t_2}(\alpha) \times T$. Then $(\alpha, t_1) \models Bp \wedge \neg Bq$ and $(\alpha, t_2) \models Ip \wedge Bq$, so that $(\alpha, t_1) \models (Bp \wedge \neg Bq) \wedge \neg \bigcirc (Ip \to \neg Bq)$. ■

Let $\mathbb{L}_W$ (where 'W' stands for 'Weak') be the logic obtained by adding WND and WNA to $\mathbb{L}_0$. We denote this by writing $\mathbb{L}_W = \mathbb{L}_0 + WNA + WND$. The following is a corollary of Propositions 7 and 8.

Corollary 9 *Logic $\mathbb{L}_W$ is characterized by the class of temporal belief revision frames that satisfy the following property:*

$$\text{if } t_1 \rightarrowtail t_2,\ \mathcal{B}_{t_1}(\omega) \neq \varnothing \text{ and } \mathcal{B}_{t_1}(\omega) \subseteq \mathcal{I}_{t_2}(\omega) \text{ then } \mathcal{B}_{t_1}(\omega) = \mathcal{B}_{t_2}(\omega).$$

Logic $\mathbb{L}_W$ captures a weak notion of minimal change of beliefs in that it requires the agent not to change his beliefs if he is informed of some fact that he already believes. This requirement is stated explicitly in the following axiom ('WNC' stand for 'Weak No Change'): if ϕ and ψ are Boolean formulas,

$$(I\phi \wedge \Diamond^{-1}(B\phi \wedge \neg B\neg\phi)) \to (B\psi \leftrightarrow \Diamond^{-1}B\psi). \qquad (WNC)$$

WNC says that if the agent is informed of a fact that he believed non-trivially in the immediately preceding past, then he now believes a fact if and only if he believed it then.

Proposition 10 *WNC is a theorem of $\mathbb{L}_W$.*

Proof. First of all, note that, since $\bigcirc^{-1}$ is a normal operator, the following is a theorem of $\mathbb{L}_0$ (hence of $\mathbb{L}_W$):

$$\Diamond^{-1}\chi \wedge \bigcirc^{-1}\xi \to \Diamond^{-1}(\chi \wedge \xi). \qquad (*)$$

It follows from (*) and axiom BU that the following is a theorem of $\mathbb{L}_0$:

$$\Diamond^{-1}\chi \wedge \Diamond^{-1}\xi \to \Diamond^{-1}(\chi \wedge \xi). \qquad (**)$$

The following is a syntactic derivation of WNC:

1.	$\Diamond^{-1}(B\phi \wedge \neg B\neg\phi) \wedge \Diamond^{-1}B\psi \to \Diamond^{-1}(B\phi \wedge \neg B\neg\phi \wedge B\psi)$	Theorem of L_0 (see ** above)
2.	$\Diamond^{-1}(B\phi \wedge \neg B\neg\phi \wedge B\psi) \wedge I\phi \to B\psi$	equivalent to axiom WND (see Footnote 2)
3.	$\Diamond^{-1}(B\phi \wedge \neg B\neg\phi) \wedge \Diamond^{-1}B\psi \wedge I\phi \to B\psi$	1, 2, PL
4.	$I\phi \wedge \Diamond^{-1}(B\phi \wedge \neg B\neg\phi) \to (\Diamond^{-1}B\psi \to B\psi)$	3, PL
5.	$\Diamond^{-1}(B\phi \wedge \neg B\neg\phi) \wedge \bigcirc^{-1}\neg B\psi \to \Diamond^{-1}(B\phi \wedge \neg B\neg\phi \wedge \neg B\psi)$	Theorem of L_0 (see * above)
6.	$\neg(B\phi \wedge \neg B\psi) \to \neg(B\phi \wedge \neg B\neg\phi \wedge \neg B\psi)$	Tautology
7.	$\bigcirc^{-1}\neg(B\phi \wedge \neg B\psi) \to \bigcirc^{-1}\neg(B\phi \wedge \neg B\neg\phi \wedge \neg B\psi)$	6, rule RK for $\bigcirc^{-1}$
8.	$\Diamond^{-1}(B\phi \wedge \neg B\neg\phi \wedge \neg B\psi) \to \Diamond^{-1}(B\phi \wedge \neg B\psi)$	7, PL, definition of $\Diamond^{-1}$
9.	$\Diamond^{-1}(B\phi \wedge \neg B\neg\phi) \wedge \bigcirc^{-1}\neg B\psi \to \Diamond^{-1}(B\phi \wedge \neg B\psi)$	5, 8, PL
10.	$\Diamond^{-1}(B\phi \wedge \neg B\psi) \wedge I\phi \to \neg B\psi$	equivalent to axiom WNA (see Footnote 3)
11.	$\Diamond^{-1}(B\phi \wedge \neg B\neg\phi) \wedge \bigcirc^{-1}\neg B\psi \wedge I\phi \to \neg B\psi$	9, 10, PL
12.	$I\phi \wedge \Diamond^{-1}(B\phi \wedge \neg B\neg\phi) \to (\bigcirc^{-1}\neg B\psi \to \neg B\psi)$	11, PL
13.	$(\bigcirc^{-1}\neg B\psi \to \neg B\psi) \to (B\psi \to \Diamond^{-1}B\psi)$	tautology and definition of $\Diamond^{-1}$
14.	$I\phi \wedge \Diamond^{-1}(B\phi \wedge \neg B\neg\phi) \to (B\psi \to \Diamond^{-1}B\psi)$	12, 13, PL
15.	$I\phi \wedge \Diamond^{-1}(B\phi \wedge \neg B\neg\phi) \to (B\psi \leftrightarrow \Diamond^{-1}B\psi)$	4, 14, PL.

■

5 The logic of the Qualitative Bayes Rule

Logic $\mathbb{L}_W$ imposes no restrictions on belief revision whenever the individual is informed of some fact that he did not previously believe. We now consider a stronger logic than $\mathbb{L}_W$. The following axiom strengthens (WND) by requiring the individual not to drop any of his current factual beliefs at any next instant at which he is informed of some fact that he currently *considers possible* (without necessarily believing it: the condition $B\phi$ in the antecedent of WND is dropped): if ϕ and ψ are Boolean,

$$(\neg B\neg\phi \wedge B\psi) \to \bigcirc(I\phi \to B\psi). \tag{ND}$$

The corresponding strengthening of (WNA) requires that if the individual considers it possible that $(\phi\wedge\neg\psi)$ then at any next instant where he is informed that ϕ he does not believe that ψ:[4] if ϕ and ψ are Boolean,

[4] Axiom NA can alternatively be written as $\neg B(\phi \to \psi) \to \bigcirc(I\phi \to \neg B\psi)$, which says that if the individual does not believe that whenever ϕ is the case then ψ is the case, then - at any next instant - if he is informed that ϕ then he cannot believe that ψ. Another, propositionally equivalent, formulation of NA is the following: $\Diamond(I\phi \wedge B\psi) \to B(\phi \to \psi)$, which says that if there is a next instant at which the individual is informed that ϕ and believes that ψ, then

$$\neg B \neg(\phi \wedge \neg\psi) \rightarrow \bigcirc(I\phi \rightarrow \neg B\psi). \qquad (NA)$$

Proposition 11 *Axiom ND is characterized by the following property:*

$$\text{if } t_1 \rightarrowtail t_2 \text{ and } \mathcal{B}_{t_1}(\omega) \cap \mathcal{I}_{t_2}(\omega) \neq \varnothing \text{ then } \mathcal{B}_{t_2}(\omega) \subseteq \mathcal{B}_{t_1}(\omega). \qquad (P_{ND})$$

Proof. Fix a frame that satisfies P_{ND}, an arbitrary model based on it and arbitrary $\alpha \in \Omega$, $t_1 \in T$ and Boolean formulas ϕ and ψ and suppose that $(\alpha, t_1) \models \neg B\neg\phi \wedge B\psi$. Fix an arbitrary $t_2 \in T$ such that $t_1 \rightarrowtail t_2$ and $(\alpha, t_2) \models I\phi$. Then $\mathcal{I}_{t_2}(\alpha) = \lceil\phi\rceil_{t_2}$. Since $(\alpha, t_1) \models \neg B\neg\phi$, there exists a $\beta \in \mathcal{B}_{t_1}(\alpha)$ such that $(\beta, t_1) \models \phi$. Since ϕ is Boolean, by Proposition 4 $(\beta, t_2) \models \phi$ and, therefore, $\beta \in \mathcal{I}_{t_2}(\alpha)$. Thus $\mathcal{B}_{t_1}(\alpha) \cap \mathcal{I}_{t_2}(\alpha) \neq \varnothing$ and, by P_{ND}, $\mathcal{B}_{t_2}(\alpha) \subseteq \mathcal{B}_{t_1}(\alpha)$. Fix an arbitrary $\omega \in \mathcal{B}_{t_2}(\alpha)$. Then $\omega \in \mathcal{B}_{t_1}(\alpha)$ and, since $(\alpha, t_1) \models B\psi$, $(\omega, t_1) \models \psi$. Since ψ is Boolean, by Proposition 4, $(\omega, t_2) \models \psi$. Hence $(\alpha, t_2) \models B\psi$.

Conversely, fix a frame that does not satisfy P_{ND}. Then there exist $\alpha \in \Omega$ and $t_1, t_2 \in T$ such that $t_1 \rightarrowtail t_2$, $\mathcal{B}_{t_1}(\alpha) \cap \mathcal{I}_{t_2}(\alpha) \neq \varnothing$ and $\mathcal{B}_{t_2}(\alpha) \nsubseteq \mathcal{B}_{t_1}(\alpha)$. Let p and q be atomic propositions and construct a model where $\|p\| = \mathcal{B}_{t_1}(\alpha) \times T$ and $\|q\| = \mathcal{I}_{t_2}(\alpha) \times T$. Then $(\alpha, t_1) \models \neg B\neg q \wedge Bp$ and $(\alpha, t_2) \models Iq$. By hypothesis there exists a $\beta \in \mathcal{B}_{t_2}(\alpha)$ such that $\beta \notin \mathcal{B}_{t_1}(\alpha)$. Thus $(\beta, t_2) \nvDash p$ and therefore $(\alpha, t_2) \models \neg Bp$. Hence $(\alpha, t_1) \models \neg B\neg q \wedge Bp \wedge \neg \bigcirc (Iq \rightarrow Bp)$. ■

Proposition 12 *Axiom NA is characterized by the following property:*

$$\text{if } t_1 \rightarrowtail t_2 \text{ then } \mathcal{B}_{t_1}(\omega) \cap \mathcal{I}_{t_2}(\omega) \subseteq \mathcal{B}_{t_2}(\omega). \qquad (P_{NA})$$

Proof. Fix a frame that satisfies P_{NA}, an arbitrary model based on it and arbitrary $\alpha \in \Omega$, $t_1 \in T$ and Boolean formulas ϕ and ψ and suppose that $(\alpha, t_1) \models \neg B\neg(\phi \wedge \neg\psi)$. Fix an arbitrary $t_2 \in T$ such that $t_1 \rightarrowtail t_2$ and suppose that $(\alpha, t_2) \models I\phi$. Then $\mathcal{I}_{t_2}(\alpha) = \lceil\phi\rceil_{t_2}$. Since $(\alpha, t_1) \models \neg B\neg(\phi \wedge \neg\psi)$, there exists a $\beta \in \mathcal{B}_{t_1}(\alpha)$ such that $(\beta, t_1) \models \phi \wedge \neg\psi$. Since ϕ and ψ are Boolean, by Proposition 4 $(\beta, t_2) \models \phi \wedge \neg\psi$. Thus $\beta \in \mathcal{I}_{t_2}(\alpha)$ and, by P_{NA}, $\beta \in \mathcal{B}_{t_2}(\alpha)$. Thus, since $(\beta, t_2) \models \neg\psi$, $(\alpha, t_2) \models \neg B\psi$.

Conversely, fix a frame that does not satisfy P_{NA}. Then there exist $\alpha \in \Omega$ and $t_1, t_2 \in T$ such that $t_1 \rightarrowtail t_2$ and $\mathcal{B}_{t_1}(\alpha) \cap \mathcal{I}_{t_2}(\alpha) \nsubseteq \mathcal{B}_{t_2}(\alpha)$. Let p and q be atomic propositions and construct a model where $\|p\| = \mathcal{I}_{t_2}(\alpha) \times T$ and $\|q\| = \mathcal{B}_{t_2}(\alpha) \times T$. Then $(\alpha, t_2) \models Ip \wedge Bq$ and, therefore, $(\alpha, t_1) \models \neg \bigcirc (Ip \rightarrow \neg Bq)$. Since $\mathcal{B}_{t_1}(\alpha) \cap \mathcal{I}_{t_2}(\alpha) \nsubseteq \mathcal{B}_{t_2}(\alpha)$ there exists a $\beta \in \mathcal{B}_{t_1}(\alpha) \cap \mathcal{I}_{t_2}(\alpha)$ such that $\beta \notin \mathcal{B}_{t_2}(\alpha)$. Thus $(\beta, t_1) \models p \wedge \neg q$. Hence $(\alpha, t_1) \models \neg B\neg(p \wedge \neg q)$, so that axiom NA is falsified at (α, t_1). ■

One of the axioms of the AGM theory of belief revision (see [10]) is that information is believed. Such axiom is often referred to as "Success" or "Acceptance". The following axiom is a weaker form of it: information is believed when it is not surprising. If the agent considers a fact ϕ possible, then he will

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he must now believe that whenever ϕ is the case then ψ is the case.

believe ϕ at any next date at which he is informed that ϕ. We call this axiom "*Qualified Acceptance" (QA)*: if ϕ is Boolean,

$$\neg B\neg\phi \to \bigcirc(I\phi \to B\phi). \tag{QA}$$

Proposition 13 *Axiom (QA) is characterized by the following property:*

$$\text{if } t_1 \rightarrowtail t_2 \text{ and } \mathcal{B}_{t_1}(\omega) \cap \mathcal{I}_{t_2}(\omega) \neq \varnothing \text{ then } \mathcal{B}_{t_2}(\omega) \subseteq \mathcal{T}_{t_2}(\omega). \tag{P_{QA}}$$

Proof. Fix a frame that satisfies P_{QA}, an arbitrary model based on it and arbitrary $\alpha \in \Omega$, $t_1 \in T$ and Boolean formula ϕ and suppose that $(\alpha, t_1) \models \neg B\neg\phi$. Then there exists a $\beta \in \mathcal{B}_{t_1}(\alpha)$ such that $(\beta, t_1) \models \phi$. Fix an arbitrary t_2 such that $t_1 \rightarrowtail t_2$ and suppose that $(\alpha, t_2) \models I\phi$. Then $\mathcal{I}_{t_2}(\alpha) = \lceil\phi\rceil_{t_2}$. Since ϕ is Boolean and $(\beta, t_1) \models \phi$, by Proposition 4 $(\beta, t_2) \models \phi$. Thus $\beta \in \mathcal{I}_{t_2}(\alpha)$ and, therefore, $\mathcal{B}_{t_1}(\alpha) \cap \mathcal{I}_{t_2}(\alpha) \neq \varnothing$. By P_{QA}, $\mathcal{B}_{t_2}(\alpha) \subseteq \mathcal{T}_{t_2}(\alpha)$. Thus $(\alpha, t_2) \models B\phi$. Hence $(\alpha, t_1) \models \bigcirc(I\phi \to B\phi)$.

Conversely, suppose that P_{QA} is violated. Then there exist $\alpha \in \Omega$ and $t_1, t_2 \in T$ such that $t_1 \rightarrowtail t_2$, $\mathcal{B}_{t_1}(\alpha) \cap \mathcal{I}_{t_2}(\alpha) \neq \varnothing$ and $\mathcal{B}_{t_2}(\alpha) \nsubseteq \mathcal{T}_{t_2}(\alpha)$. Let p be an atomic proposition and construct a model where $\|p\| = \mathcal{I}_{t_2}(\alpha) \times T$. Then $(\alpha, t_1) \models \neg B\neg p$ and $(\alpha, t_2) \models Ip$. By hypothesis, there exists a $\beta \in \mathcal{B}_{t_2}(\alpha)$ such that $\beta \notin \mathcal{T}_{t_2}(\alpha)$. Thus $(\beta, t_2) \nvDash p$ and therefore $(\alpha, t_2) \models \neg Bp$. Hence $(\alpha, t_1) \nvDash \bigcirc(Ip \to Bp)$. ■

We call the following property of temporal belief revision frames "Qualitative Bayes Rule" (QBR): $\forall t_1, t_2 \in T, \forall \omega \in \Omega$,

$$\text{if } t_1 \rightarrowtail t_2 \text{ and } \mathcal{B}_{t_1}(\omega) \cap \mathcal{I}_{t_2}(\omega) \neq \varnothing \text{ then } \mathcal{B}_{t_2}(\omega) = \mathcal{B}_{t_1}(\omega) \cap \mathcal{I}_{t_2}(\omega). \tag{QBR}$$

The expression "Qualitative Bayes Rule" is motivated by the following observation (see [4]). In a probabilistic setting, let P_{ω,t_1} be the probability measure over a set of states Ω representing the individual's beliefs at state ω and time t_1; let $F \subseteq \Omega$ be an event representing the information received by the individual at a later date t_2 and let P_{ω,t_2} be the posterior probability measure representing the revised beliefs at state ω and date t_2. Bayes' rule requires that, if $P_{\omega,t_1}(F) > 0$, then, for every event $E \subseteq \Omega$, $P_{\omega,t_2}(E) = \frac{P_{\omega,t_1}(E \cap F)}{P_{\omega,t_1}(F)}$. Bayes' rule thus implies the following (where $supp(P)$ denotes the support of the probability measure P):

$$\text{if } \; supp(P_{\omega,t_1}) \cap F \neq \varnothing, \text{ then } \; supp(P_{\omega,t_2}) = supp(P_{\omega,t_1}) \cap F.$$

If we set $\mathcal{B}_{t_1}(\omega) = supp(P_{\omega,t_1})$, $F = \mathcal{I}_{t_2}(\omega)$ (with $t_1 \rightarrowtail t_2$) and $\mathcal{B}_{t_2}(\omega) = supp(P_{\omega,t_2})$ then we get the Qualitative Bayes Rule as stated above. Thus in a probabilistic setting the proposition "at date t the individual believes ϕ" would be interpreted as "the individual assigns probability 1 to the event $\lceil\phi\rceil_t \subseteq \Omega$".

Proposition 14 *The conjunction of axioms ND, NA and QA characterizes the Qualitative Bayes Rule.*

Proof. It is a corollary of Propositions 11, 12 and 13. ■

Let $\mathbb{L}_{QBR} = \mathbb{L}_0 + ND + NA + QA$.

Remark 15 *Logic $\mathbb{L}_{QBR}$ contains (is a strengthening of) $\mathbb{L}_W$. In fact, WND is a theorem of logic $\mathbb{L}_0 + ND$, since $(B\phi \wedge \neg B\neg\phi \wedge B\psi) \to (\neg B\neg\phi \wedge B\psi)$ is a tautology, and WNA is a theorem of logic $\mathbb{L}_0 + NA$ as the following derivation shows:*

1.	$\neg B(\phi \to \psi) \to \bigcirc(I\phi \to \neg B\psi)$	*Axiom NA*[5]
2.	$B(\phi \to \psi) \to (B\phi \to B\psi)$	*Axiom K for B*
3.	$(B\phi \wedge \neg B\psi) \to \neg B(\phi \to \psi)$	*2, PL*
4.	$(B\phi \wedge \neg B\psi) \to \bigcirc(I\phi \to \neg B\psi)$	*1, 3, PL.*

6 The logic of AGM

We now strengthen logic $\mathbb{L}_{QBR}$ by adding four more axioms.

The first axiom is the Acceptance axiom, which is a strengthening of Qualified Acceptance:

$$I\phi \to B\phi. \tag{A}$$

Remark 16 *It is straightforward to show that axiom A is characterized by the following property: $\forall \omega \in \Omega, \forall t \in T,\ \mathcal{B}_t(\omega) \subseteq \mathcal{I}_t(\omega)$.*

The second axiom says that if there is a next instant where the individual is informed that $\phi \wedge \psi$ and believes that χ, then at every next instant it must be the case that if the individual is informed that ϕ then he must believe that $(\phi \wedge \psi) \to \chi$ (we call this axiom $K7$ because it corresponds to AGM postulate (K*7)): if ϕ, ψ and χ are Boolean formulas,

$$\Diamond(I(\phi \wedge \psi) \wedge B\chi) \to \bigcirc(I\phi \to B\left((\phi \wedge \psi) \to \chi\right)). \tag{K7}$$

Proposition 17 *Axiom $(K7)$ is characterized by the following property:*

$$\begin{array}{l}\textit{if } t_1, t_2, t_3 \textit{ and } \alpha \textit{ are such that } t_1 \rightarrowtail t_2,\ t_1 \rightarrowtail t_3 \textit{ and} \\ \mathcal{I}_{t_3}(\alpha) \subseteq \mathcal{I}_{t_2}(\alpha) \textit{ then } \mathcal{I}_{t_3}(\alpha) \cap \mathcal{B}_{t_2}(\alpha) \subseteq \mathcal{B}_{t_3}(\alpha).\end{array} \tag{P_{K7}}$$

[5] Note that $\neg(\phi \wedge \neg\psi)$ is tautologically equivalent to $(\phi \to \psi)$, so that $\neg B\neg(\phi \wedge \neg\psi)$ is equivalent to $\neg B(\phi \to \psi)$.

Proof. Fix a frame that satisfies property P_{K7}. Let α and t_1 be such that $(a,t_1) \models \Diamond(I(\phi \wedge \psi) \wedge B\chi)$, where ϕ, ψ and χ are Boolean formulas. Then there exists a t_3 such that $t_1 \rightarrowtail t_3$ and $(\alpha,t_3) \models I(\phi \wedge \psi) \wedge B\chi$. It follows that $\mathcal{I}_{t_3}(\alpha) = \lceil \phi \wedge \psi \rceil_{t_3}$. Fix an arbitrary t_2 such that $t_1 \rightarrowtail t_2$ and suppose that $(\alpha,t_2) \models I\phi$. Then $\mathcal{I}_{t_2}(\alpha) = \lceil \phi \rceil_{t_2}$. Since ϕ and ψ are Boolean, by Proposition 4 $\lceil \phi \wedge \psi \rceil_{t_3} = \lceil \phi \wedge \psi \rceil_{t_2}$. Thus, since $\lceil \phi \wedge \psi \rceil_{t_2} \subseteq \lceil \phi \rceil_{t_2}$, $\mathcal{I}_{t_3}(\alpha) \subseteq \mathcal{I}_{t_2}(\alpha)$. Hence by P_{K7}, $\mathcal{I}_{t_3}(\alpha) \cap \mathcal{B}_{t_2}(\alpha) \subseteq \mathcal{B}_{t_3}(\alpha)$. Fix an arbitrary $\beta \in \mathcal{B}_{t_2}(\alpha)$. If $(\beta,t_2) \models \neg(\phi \wedge \psi)$ then $(\beta,t_2) \models (\phi \wedge \psi) \to \chi$. If $(\beta,t_2) \models \phi \wedge \psi$, then, by Proposition 4, $(\beta,t_3) \models \phi \wedge \psi$ and, therefore, $\beta \in \mathcal{I}_{t_3}(\alpha)$. Hence $\beta \in \mathcal{B}_{t_3}(\alpha)$. Since $(\alpha,t_3) \models B\chi$, $(\beta,t_3) \models \chi$ and, therefore, $(\beta,t_3) \models (\phi \wedge \psi) \to \chi$. Since $(\phi \wedge \psi) \to \chi$ is Boolean (because ϕ, ψ and χ are), by Proposition 4, $(\beta,t_2) \models (\phi \wedge \psi) \to \chi$. Thus, since $\beta \in \mathcal{B}_{t_2}(\alpha)$ was chosen arbitrarily, $(\alpha,t_2) \models B((\phi \wedge \psi) \to \chi)$.

Conversely, suppose that P_{K7} is violated. Then there exist t_1, t_2, t_3 and α such that $t_1 \rightarrowtail t_2$, $t_1 \rightarrowtail t_3$, $\mathcal{I}_{t_3}(\alpha) \subseteq \mathcal{I}_{t_2}(\alpha)$ and $\mathcal{I}_{t_3}(\alpha) \cap \mathcal{B}_{t_2}(\alpha) \nsubseteq \mathcal{B}_{t_3}(\alpha)$. Let p, q and r be atomic propositions and construct a model where $||p|| = \mathcal{I}_{t_2}(\alpha) \times T$, $||q|| = \mathcal{I}_{t_3}(\alpha) \times T$ and $||r|| = \mathcal{B}_{t_3}(\alpha) \times T$. Then, $(\alpha,t_3) \models Br$ and, since $\mathcal{I}_{t_3}(\alpha) \subseteq \mathcal{I}_{t_2}(\alpha)$, $\mathcal{I}_{t_3}(\alpha) = \lceil p \wedge q \rceil_{t_3}$ so that $(\alpha,t_3) \models I(p \wedge q)$. Thus, since $t_1 \to t_3$, $(\alpha,t_1) \models \Diamond(I(p \wedge q) \wedge Br)$. By construction, $(\alpha,t_2) \models Ip$. Since $\mathcal{I}_{t_3}(\alpha) \cap \mathcal{B}_{t_2}(\alpha) \nsubseteq \mathcal{B}_{t_3}(\alpha)$, there exists a $\beta \in \mathcal{I}_{t_3}(\alpha) \cap \mathcal{B}_{t_2}(\alpha)$ such that $\beta \notin \mathcal{B}_{t_3}(\alpha)$. Thus $(\beta,t_2) \nvDash r$; furthermore, since $\beta \in \mathcal{I}_{t_3}(\alpha)$, $(\beta,t_3) \models p \wedge q$ and, by Proposition 4, $(\beta,t_2) \models p \wedge q$. Thus, $(\beta,t_2) \nvDash (p \wedge q) \to r$. Since $\beta \in \mathcal{B}_{t_2}(\alpha)$ it follows that $(\alpha,t_2) \nvDash B((p \wedge q) \to r)$. Hence, since $t_1 \rightarrowtail t_2$, $(\alpha,t_1) \nvDash \bigcirc(Ip \to B((p \wedge q) \to r)$ so that axiom $K7$ is falsified at (α,t_1). ■

The third axiom says that if there is a next instant where the individual is informed that ϕ, considers $\phi \wedge \psi$ possible and believes that $\psi \to \chi$, then at every next instant it must be the case that if the individual is informed that $\phi \wedge \psi$ then he believes that χ (we call this axiom $K8$ because it corresponds to AGM postulate (K*8)): if ϕ, ψ and χ are Boolean formulas,

$$\Diamond(I\phi \wedge \neg B\neg(\phi \wedge \psi) \wedge B(\psi \to \chi)) \to \bigcirc(I(\phi \wedge \psi) \to B\chi). \qquad (K8)$$

Proposition 18 *Axiom $(K8)$ is characterized by the following property:*

if t_1, t_2, t_3 and α are such that $t_1 \rightarrowtail t_2$, $t_1 \rightarrowtail t_3$, $\mathcal{I}_{t_3}(\alpha) \subseteq \mathcal{I}_{t_2}(\alpha)$ and $\mathcal{I}_{t_3}(\alpha) \cap \mathcal{B}_{t_2}(\alpha) \neq \varnothing$ then $\mathcal{B}_{t_3}(\alpha) \subseteq \mathcal{I}_{t_3}(\alpha) \cap \mathcal{B}_{t_2}(\alpha)$. $\qquad (P_{K8})$

Proof. Fix a frame that satisfies property P_{K8}. Let ϕ, ψ and χ be Boolean formulas and let α and t_1 be such that $(\alpha,t_1) \models \Diamond(I\phi \wedge \neg B\neg(\phi \wedge \psi) \wedge B(\psi \to \chi))$. Then there exists a t_2 such that $t_1 \rightarrowtail t_2$ and $(\alpha,t_2) \models I\phi \wedge \neg B\neg(\phi \wedge \psi) \wedge B(\psi \to \chi)$. Thus $\mathcal{I}_{t_2}(\alpha) = \lceil \phi \rceil_{t_2}$ and there exists a $\beta \in \mathcal{B}_{t_2}(\alpha)$ such that $(\beta,t_2) \models \phi \wedge \psi$. Fix an arbitrary t_3 such that $t_1 \rightarrowtail t_3$ and suppose that

$(\alpha, t_3) \models I(\phi \wedge \psi)$. Then $\mathcal{I}_{t_3}(\alpha) = \lceil \phi \wedge \psi \rceil_{t_3}$. Since $\phi \wedge \psi$ is a Boolean formula and $(\beta, t_2) \models \phi \wedge \psi$, by Proposition 4 $(\beta, t_3) \models \phi \wedge \psi$ and therefore $\beta \in \mathcal{I}_{t_3}(\alpha)$. Hence $\mathcal{I}_{t_3}(\alpha) \cap \mathcal{B}_{t_2}(\alpha) \neq \varnothing$. Furthermore, since ϕ is Boolean, by Proposition 4 $\lceil \phi \rceil_{t_3} = \lceil \phi \rceil_{t_2}$. Thus, since $\lceil \phi \wedge \psi \rceil_{t_3} \subseteq \lceil \phi \rceil_{t_3}$ it follows that $\mathcal{I}_{t_3}(\alpha) \subseteq \mathcal{I}_{t_2}(\alpha)$. Hence, by property P_{K8}, $\mathcal{B}_{t_3}(\alpha) \subseteq \mathcal{I}_{t_3}(\alpha) \cap \mathcal{B}_{t_2}(\alpha)$. Fix an arbitrary $\gamma \in \mathcal{B}_{t_3}(\alpha)$. Then $\gamma \in \mathcal{I}_{t_3}(\alpha) \cap \mathcal{B}_{t_2}(\alpha)$ and, since $(\alpha, t_2) \models B(\psi \to \chi)$, $(\gamma, t_2) \models \psi \to \chi$. Since $\psi \to \chi$ is a Boolean formula, by Proposition 4 $(\gamma, t_3) \models \psi \to \chi$. Since $\gamma \in \mathcal{I}_{t_3}(\alpha)$ and $\mathcal{I}_{t_3}(\alpha) = \lceil \phi \wedge \psi \rceil_{t_3}$, $(\gamma, t_3) \models \psi$. Thus $(\gamma, t_3) \models \chi$. Hence $(\alpha, t_3) \models B\chi$.

Conversely, fix a frame that does not satisfy property P_{K8}. Then there exist t_1, t_2, t_3 and α such that $t_1 \rightarrowtail t_2$, $t_1 \rightarrowtail t_3$, $\mathcal{I}_{t_3}(\alpha) \cap \mathcal{B}_{t_2}(\alpha) \neq \varnothing$, $\mathcal{I}_{t_3}(\alpha) \subseteq \mathcal{I}_{t_2}(\alpha)$ and $\mathcal{B}_{t_3}(\alpha) \nsubseteq \mathcal{I}_{t_3}(\alpha) \cap \mathcal{B}_{t_2}(\alpha)$. Let p, q and r be atomic propositions and construct a model where $||p|| = \mathcal{I}_{t_2}(\alpha) \times T$, $||q|| = \mathcal{I}_{t_3}(\alpha) \times T$ and $||r|| = (\mathcal{I}_{t_3}(\alpha) \cap \mathcal{B}_{t_2}(\alpha)) \times T$. Then $(\alpha, t_2) \models Ip$ and, since $\mathcal{I}_{t_3}(\alpha) \subseteq \mathcal{I}_{t_2}(\alpha)$, if $\omega \in \mathcal{I}_{t_3}(\alpha)$ then $(\omega, t) \models p \wedge q$ for every $t \in T$. Thus, since $\mathcal{I}_{t_3}(\alpha) \cap \mathcal{B}_{t_2}(\alpha) \neq \varnothing$, $(\alpha, t_2) \models \neg B \neg (p \wedge q)$. Fix an arbitrary $\omega \in \mathcal{B}_{t_2}(\alpha)$; if $\omega \in \mathcal{I}_{t_3}(\alpha)$ then $(\omega, t_2) \models r$; if $\omega \notin \mathcal{I}_{t_3}(\alpha)$ then $(\omega, t_2) \models \neg q$; in either case $(\omega, t_2) \models q \to r$. Thus $(\alpha, t_2) \models B(q \to r)$. Hence $(\alpha, t_2) \models Ip \wedge \neg B \neg (p \wedge q) \wedge B(q \to r)$ and thus $(\alpha, t_1) \models \Diamond (Ip \wedge \neg B \neg (p \wedge q) \wedge B(q \to r))$. Since $\mathcal{I}_{t_3}(\alpha) = \lceil q \rceil_{t_3}$ and $\mathcal{I}_{t_2}(\alpha) = \lceil p \rceil_{t_2}$ and, by Proposition 4 , $\lceil p \rceil_{t_2} = \lceil p \rceil_{t_3}$ and $\mathcal{I}_{t_3}(\alpha) \subseteq \mathcal{I}_{t_2}(\alpha)$, if follows that $\mathcal{I}_{t_3}(\alpha) = \lceil p \wedge q \rceil_{t_3}$, so that $(\alpha, t_3) \models I(p \wedge q)$. Since $\mathcal{B}_{t_3}(\alpha) \nsubseteq \mathcal{I}_{t_3}(\alpha) \cap \mathcal{B}_{t_2}(\alpha)$, there exists a $\beta \in \mathcal{B}_{t_3}(\alpha)$ such that $\beta \notin \mathcal{I}_{t_3}(\alpha) \cap \mathcal{B}_{t_2}(\alpha)$. Then $(\beta, t_3) \nvDash r$ and therefore $(\alpha, t_3) \nvDash Br$. Thus $(\alpha, t_3) \nvDash I(p \wedge q) \to Br$ and hence, $(\alpha, t_1) \nvDash \bigcirc (I(p \wedge q) \to Br)$, so that axiom $K8$ is falsified at (a, t_1). ■

The fourth axiom says that if the individual receives consistent information then his beliefs are consistent, in the sense that he does not simultaneously believe a formula and its negation ('WC' stands for 'Weak Consistency'): if ϕ is a Boolean formula,

$$(I\phi \wedge \neg A \neg \phi) \to (B\psi \to \neg B \neg \psi). \qquad (WC)$$

Proposition 19 *Axiom WC is characterized by the following property:* $\forall \omega \in \Omega$, $\forall t \in T$, *if* $\mathcal{I}_t(\omega) \neq \varnothing$ *then* $\mathcal{B}_t(\omega) \neq \varnothing$.

Proof. Let ϕ be a Boolean formula, $\alpha \in \Omega$, $t \in T$ and suppose that $(\alpha, t) \models I\phi \wedge \neg A \neg \phi$. Then $\mathcal{I}_t(\alpha) = \lceil \phi \rceil_t$ and there exist $\beta \in \Omega$ that $(\beta, t) \models \phi$. Thus $\mathcal{I}_t(\alpha) \neq \varnothing$ and, by the above property, $\mathcal{B}_t(\alpha) \neq \varnothing$. Fix an arbitrary formula ψ and suppose that $(\alpha, t) \models B\psi$. Then, $\forall \omega \in \mathcal{B}_t(\alpha)$, $(\omega, t) \models \psi$. Since $\mathcal{B}_t(\alpha) \neq \varnothing$, there exists a $\gamma \in \mathcal{B}_t(\alpha)$. Thus $(\gamma, t) \models \psi$ and hence $(\alpha, t) \models \neg B \neg \psi$.

Conversely, fix a frame that does not satisfy the above property. Then there exist $\alpha \in \Omega$ and $t \in T$ such that $\mathcal{I}_t(\alpha) \neq \varnothing$ while $\mathcal{B}_t(\alpha) = \varnothing$. Let p be an atomic proposition and construct a model where $\|p\| = \mathcal{I}_t(\alpha) \times T$. Then $(\alpha, t) \models Ip$. Furthermore, since $\mathcal{I}_t(\alpha) \neq \varnothing$, there exists a $\beta \in \mathcal{I}_t(\alpha)$. Thus $(\beta, t) \models p$ and hence $(\alpha, t) \models \neg A \neg p$. Since $\mathcal{B}_t(\alpha) = \varnothing$, $(\alpha, t) \models B\psi$ for every formula ψ, so that $(\alpha, t) \models Bp \wedge B\neg p$. Thus WC is falsified at (α, t). ■

Let $\mathbb{L}_{AGM} = \mathbb{L}_0 + A + ND + NA + K7 + K8 + WC$. Since QA can be derived from A, logic $\mathbb{L}_{AGM}$ contains (is a strengthening of) logic $\mathbb{L}_{QBR}$.

It is shown in [5] that $\mathbb{L}_{AGM}$ can be viewed as an axiomatization of the theory of belief revision due to Alchourrón *et al* [1], known as the AGM theory. For a precise statement of this result and a proof the reader is referred to [5].

Definition 20 *An* $\mathbb{L}_{AGM}$-frame *is a temporal belief revision frame that satisfies the following properties:*

(1) the Qualitative Bayes Rule,

(2) $\forall\omega \in \Omega$, $\forall t \in T$, $\mathcal{B}_t(\omega) \subseteq \mathcal{I}_t(\omega)$,

(3) $\forall\omega \in \Omega$, $\forall t \in T$, *if* $\mathcal{I}_t(\omega) \neq \varnothing$ *then* $\mathcal{B}_t(\omega) \neq \varnothing$.

(4) $\forall\omega \in \Omega$, $\forall t_1, t_2, t_3 \in T$,
if $t_1 \rightarrowtail t_2$, $t_1 \rightarrowtail t_3$, $\mathcal{I}_{t_3}(\alpha) \subseteq \mathcal{I}_{t_2}(\alpha)$ *and* $\mathcal{I}_{t_3}(\alpha) \cap \mathcal{B}_{t_2}(\alpha) \neq \varnothing$
then $\mathcal{B}_{t_3}(\alpha) = \mathcal{I}_{t_3}(\alpha) \cap \mathcal{B}_{t_2}(\alpha)$.

An $\mathbb{L}_{AGM}$-model *is a model based on an* $\mathbb{L}_{AGM}$*-frame.*

Remark 21 *It follows from Remark 16 and Propositions 11, 12, 17, 18 and 19 that logic* $\mathbb{L}_{AGM}$ *is characterized by the class of* $\mathbb{L}_{AGM}$*-frames.*

7 The logic of plausibility orderings

As is well-known the AGM axioms can be associated with the existence of a plausibility ordering: depending on the context, the plausibility ordering is defined either on the set of formulas (see [10]) or on the set of possible worlds (see [3], [12], [19]). Although our previous results establish a semantic characterization of logic $\mathbb{L}_{AGM}$ which does not invoke the notion of plausibility ordering, it is natural to ask whether logic $\mathbb{L}_{AGM}$ implies the existence of a plausibility ordering on the set of states Ω that rationalizes belief revision.

Definition 22 *A* plausibility well-ordering *of* Ω *is a binary relation* $\precsim$ *on* Ω *that satisfies the following properties:*

(1) completeness (or connectedness): $\forall\omega, \omega' \in \Omega$, *either* $\omega \precsim \omega'$ *or* $\omega' \precsim \omega$;

(2) transitivity: if $\omega \precsim \omega'$ *and* $\omega' \precsim \omega''$ *then* $\omega \precsim \omega''$;

(3) well-foundedness: there is no infinite sequence $\langle \omega_1, \omega_2, ..., \omega_n, ... \rangle$ *such that* $...\omega_n \prec \omega_{n-1} \prec ... \prec \omega_1$ *(where* $\omega \prec \omega'$ *if and only if* $\omega \precsim \omega'$ *and* $\omega' \not\precsim \omega$*).*

Thus a plausibility ordering is a total order of Ω satisfying the property that every non-empty subset of Ω has minimal elements. The interpretation of $\omega \precsim \omega'$ is that state ω is at least as plausible as state ω' ($\omega \prec \omega'$ means that ω is more plausible than ω' and $\omega \sim \omega'$ means that ω is as plausible as ω', where $\omega \sim \omega'$ if and only if $\omega \precsim \omega'$ and $\omega' \precsim \omega$). If $\Sigma \subseteq \Omega$ we define $\min_{\precsim} \Sigma = \{\omega \in \Sigma : \omega \precsim \omega', \ \forall \omega' \in \Sigma\}$. Well-foundedness ensures that if $\Sigma \neq \varnothing$ then $\min_{\precsim} \Sigma \neq \varnothing$.

We want to investigate the conditions under which belief revision is guided by a plausibility ordering, in the sense that the states that the individual considers possible (doxastically accessible) are precisely those that are the most plausible among the ones that are consistent with the information received.

Definition 23 *A* plausibility frame *is a belief revision frame that satisfies the following property: for every $\omega \in \Omega$ and $t \in T$, there exists a plausibility ordering $\precsim_{\omega,t}$ of Ω such that*

(1) $\mathcal{B}_t(\omega) = \min_{\precsim_{\omega,t}} \mathcal{I}_t(\omega)$,

(2) if $\omega' \in \mathcal{B}_t(\omega)$ and $\omega'' \in \Omega \backslash \mathcal{B}_t(\omega)$, then $\omega' \prec_{\omega,t} \omega''$,

(3) for every $t' \in T$, if $t \rightarrowtail t'$ then $\mathcal{B}_{t'}(\omega) = \min_{\precsim_{\omega,t}} \mathcal{I}_{t'}(\omega)$.

Thus in a plausibility frame the set of states that the individual considers possible at state ω and time t', where t' is an immediate successor of t, is the set of most plausible states among the ones that are compatible with the information received at state ω and time t', using the plausibility relation associated with (ω, t). Furthermore, the states that the individual considers possible at (ω, t) are more plausible than all the other states. This requirement is necessary in order to ensure that a plausibility frame is an $\mathbb{L}_{AGM}$ frame. To see this, consider the following frame: $T = \{t_1, t_2\}$, $\rightarrowtail\ = \{(t_1, t_2)\}$, $\Omega = \{\alpha, \beta\}$, $\mathcal{B}_{t_1} = \mathcal{I}_{t_1} = \{(\alpha, \alpha), (\beta, \beta)\}$, $\mathcal{B}_{t_2} = \mathcal{I}_{t_2} = \{(\alpha, \alpha), (\alpha, \beta), (\beta, \alpha), (\beta, \beta)\}$. The frame is illustrated in Figure 1 where the relations $\mathcal{I}_t$ are represented by rectangles and the relations $\mathcal{B}_t$ are represented by ovals.

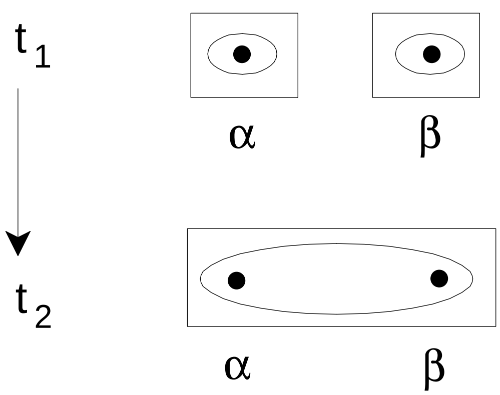

Figure 1

Consider the following plausibility relation: $\precsim_{\alpha,t_1} = \{(\alpha, \alpha), (\alpha, \beta), (\beta, \alpha), (\beta, \beta)\}$. Then the frame satisfies conditions (1) and (3) of Definition 23 but not condition (2). Indeed this frame is not an $\mathbb{L}_{AGM}$ frame since it does not satisfy the Qualitative Bayes Rule: $\mathcal{B}_{t_1}(\alpha) \cap \mathcal{I}_{t_2}(\alpha) = \{\alpha\} \neq \varnothing$ and yet $\mathcal{B}_{t_2}(\alpha) = \{\alpha, \beta\} \neq \mathcal{B}_{t_1}(\alpha) \cap \mathcal{I}_{t_2}(\alpha)$.

The following lemma will be used in the proof of the next proposition.

Lemma 24 *Let $\precsim$ be a complete and transitive binary relation on Ω and $X \subseteq Y \subseteq \Omega$. If $\left(\min_{\precsim} Y\right) \cap X \neq \varnothing$ then $\min_{\precsim} X = \left(\min_{\precsim} Y\right) \cap X$.*

Proof. First we show that $\left(\min_{\precsim} Y\right) \cap X \subseteq \min_{\precsim} X$. If $\left(\min_{\precsim} Y\right) \cap X = \varnothing$ there is nothing to prove. Therefore let $\beta \in \left(\min_{\precsim} Y\right) \cap X$. Then $\beta \in X$ and $\beta \precsim \gamma$ for all $\gamma \in Y$. Since $X \subseteq Y$, it follows that $\beta \in \min_{\precsim} X$. Next we show that if $\left(\min_{\precsim} Y\right) \cap X \neq \varnothing$ then $\min_{\precsim} X \subseteq \left(\min_{\precsim} Y\right) \cap X$. Let $\beta \in \left(\min_{\precsim} Y\right) \cap X$. Fix an arbitrary $\gamma \in \min_{\precsim} X$. Then $\gamma \in X$ and $\gamma \precsim \beta$. Suppose that $\gamma \notin \min_{\precsim} Y$. Then there exists a $\delta \in Y$ such that $\delta \prec \gamma$ (that is, $\delta \precsim \gamma$ and $\gamma \not\precsim \delta$). By transitivity (since $\gamma \precsim \beta$), $\delta \prec \beta$, contradicting the fact that $\beta \in \min_{\precsim} Y$. ■

Proposition 25 *Every plausibility frame is an $\mathbb{L}_{AGM}$ frame, but the converse is not true.*

Proof. Fix an arbitrary plausibility frame and arbitrary $\omega \in \Omega$ and $t \in T$. By (1) of Definition 23, there exists a plausibility ordering $\precsim_{\omega,t}$ of Ω such that $\mathcal{B}_t(\omega) = \min_{\precsim_{\omega,t}} \mathcal{I}_t(\omega)$. Thus $\mathcal{B}_t(\omega) \subseteq \mathcal{I}_t(\omega)$ and property (2) of definition 20 is satisfied. Furthermore, if $\mathcal{I}_t(\omega) \neq \varnothing$, by (3) of definition 22, $\min_{\precsim_{\omega,t}} \mathcal{I}_t(\omega) \neq \varnothing$. Hence property (3) of Definition 20 is satisfied. Next we show that the Qualitative Bayes Rule is satisfied. Let t_1 be such that $t \rightarrowtail t_1$ and $\mathcal{B}_t(\omega) \cap \mathcal{I}_{t_1}(\omega) \neq \varnothing$. By (3) of definition 23, $\mathcal{B}_{t_1}(\omega) = \min_{\precsim_{\omega,t}} \mathcal{I}_{t_1}(\omega)$. First we show that if $\alpha \in \mathcal{B}_t(\omega) \cap \mathcal{I}_{t_1}(\omega)$ then $\alpha \in \mathcal{B}_{t_1}(\omega)$. Suppose not. Then there exists a $\beta \in \mathcal{I}_{t_1}(\omega)$ such that $\beta \prec_{\omega,t} \alpha$. If $\beta \in \mathcal{B}_t(\omega)$, then since, $\mathcal{B}_t(\omega) = \min_{\precsim_{\omega,t}} \mathcal{I}_t(\omega)$, $\beta \sim_{\omega,t} \alpha$, yielding a contradiction. On the other hand, if $\beta \notin \mathcal{B}_t(\omega)$, then, by property (2) of Definition 23, since $\alpha \in \mathcal{B}_t(\omega)$, $\alpha \prec_{\omega,t} \beta$, again yielding a contradiction. Next we show that if $\alpha \in \mathcal{B}_{t_1}(\omega)$ and $\mathcal{B}_t(\omega) \cap \mathcal{I}_{t_1}(\omega) \neq \varnothing$ then $\alpha \in \mathcal{B}_t(\omega) \cap \mathcal{I}_{t_1}(\omega)$. Suppose that $\alpha \notin \mathcal{B}_t(\omega) \cap \mathcal{I}_{t_1}(\omega)$. Since $\mathcal{B}_{t_1}(\omega) \subseteq \mathcal{I}_{t_1}(\omega)$ (proved above), $\alpha \in \mathcal{I}_{t_1}(\omega)$. Thus it must be that $\alpha \notin \mathcal{B}_t(\omega)$. Let $\beta \in \mathcal{B}_t(\omega) \cap \mathcal{I}_{t_1}(\omega)$. Then by property (2) of Definition 23, $\beta \prec_{\omega,t} \alpha$. Since $\beta \in \mathcal{I}_{t_1}(\omega)$ this contradicts the fact that $\mathcal{B}_{t_1}(\omega) = \min_{\precsim_{\omega,t}} \mathcal{I}_{t_1}(\omega)$. Finally we need to show that property (4) of Definition 20 is satisfied. Let t_2 and t_3 be such that $t \rightarrowtail t_2$, $t \rightarrowtail t_3$, $\mathcal{I}_{t_3}(\omega) \subseteq \mathcal{I}_{t_2}(\omega)$ and $\mathcal{I}_{t_3}(\omega) \cap \mathcal{B}_{t_2}(\omega) \neq \varnothing$. We need to show that $\mathcal{B}_{t_3}(\omega) = \mathcal{I}_{t_3}(\omega) \cap \mathcal{B}_{t_2}(\omega)$. By (3) of Definition 23, $\mathcal{B}_{t_2}(\omega) = \min_{\precsim_{\omega,t}} \mathcal{I}_{t_2}(\omega)$ and $\mathcal{B}_{t_3}(\omega) = \min_{\precsim_{\omega,t}} \mathcal{I}_{t_3}(\omega)$. The desired property then follows from Lemma 24, letting $X = \mathcal{I}_{t_3}(\omega)$ and $Y = \mathcal{I}_{t_2}(\omega)$.

To complete the proof we give an example of an $\mathbb{L}_{AGM}$ frame which is not a plausibility frame. Let $T = \{t_1, t_2, t_3\}$, $\rightarrowtail \; = \{(t_1, t_2), (t_1, t_3)\}$; $\Omega = \{\alpha, \beta, \gamma, \delta, \varepsilon\}$; for every $\omega \in \Omega$, $\mathcal{I}_{t_1}(\omega) = \Omega$ and $\mathcal{B}_{t_1}(\omega) = \{\varepsilon\}$; $\mathcal{I}_{t_2}(\alpha) = \mathcal{I}_{t_2}(\beta) = \mathcal{I}_{t_2}(\gamma) = \{\alpha, \beta, \gamma\}$, $\mathcal{I}_{t_2}(\delta) = \mathcal{I}_{t_2}(\varepsilon) = \{\delta, \varepsilon\}$, $\mathcal{B}_{t_2}(\alpha) = \mathcal{B}_{t_2}(\beta) = \mathcal{B}_{t_2}(\gamma) = \{\alpha, \beta\}$, $\mathcal{B}_{t_2}(\delta) = \mathcal{B}_{t_2}(\varepsilon) = \{\varepsilon\}$; $\mathcal{I}_{t_3}(\alpha) = \mathcal{I}_{t_3}(\gamma) = \mathcal{I}_{t_3}(\delta) = \{\alpha, \gamma, \delta\}$, $\mathcal{I}_{t_3}(\beta) = \mathcal{B}_{t_3}(\beta) = \{\beta\}$, $\mathcal{I}_{t_3}(\varepsilon) = \mathcal{B}_{t_3}(\varepsilon) = \{\varepsilon\}$, $\mathcal{B}_{t_3}(\alpha) = \mathcal{B}_{t_3}(\gamma) = \mathcal{B}_{t_3}(\delta) = \{\alpha, \gamma\}$. This frame is illustrated in Figure 2. It is straightforward to check that it is an $\mathbb{L}_{AGM}$ frame (note, in particular, that property (4) of Definition 20 is satisfied at ε and vacuously satisfied at every other state). However, it is not a plausibility frame. To see this, suppose that $\precsim_{\alpha,t_1}$ is a plausibility relation on Ω that satisfies the properties of Definition 23. Then, since $\gamma \in \mathcal{I}_{t_2}(\alpha)$ and $\gamma \notin \mathcal{B}_{t_2}(\alpha) = \{\alpha, \beta\}$ it must be that $\gamma \not\precsim_{\alpha,t_1} \alpha$. On the other hand, since $\gamma \in \mathcal{B}_{t_3}(\alpha) \cap \mathcal{I}_{t_3}(\alpha)$ and $\mathcal{B}_{t_3}(\alpha) = \{\alpha, \gamma\}$, it must be that $\gamma \precsim_{\alpha,t_1} \alpha$, yielding a contradiction. ■

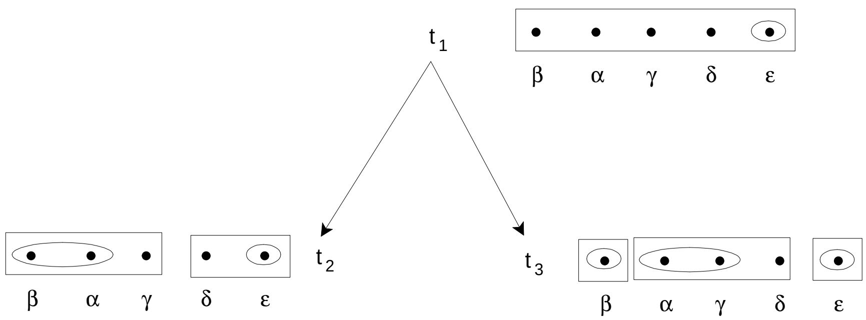

Figure 2

In order to capture the semantic notion of plausibility frame we need a stronger logic than $\mathbb{L}_{AGM}$.

The following axiom says that if there is a next instant where the agent is informed that ϕ and believes that χ, then at every next instant it must be the case that he believes that $\phi \to \chi$: if ϕ and χ are Boolean formulas

$$\Diamond(I\phi \wedge B\chi) \to \bigcirc B\,(\phi \to \chi)\,. \qquad (K7s)$$

Proposition 26 *Axiom $(K7s)$ is characterized by the following property:*

$$\begin{array}{l} \text{if } t_1, t_2, t_3 \text{ and } \alpha \text{ are such that } t_1 \rightarrowtail t_2,\ t_1 \rightarrowtail t_3 \text{ and} \\ \mathcal{I}_{t_3}(\alpha) \cap \mathcal{B}_{t_2}(\alpha) \neq \varnothing \text{ then } \mathcal{I}_{t_3}(\alpha) \cap \mathcal{B}_{t_2}(\alpha) \subseteq \mathcal{B}_{t_3}(\alpha). \end{array} \qquad (P_{K7s})$$

Proof. Fix a frame that satisfies property P_{K7s}. Let α and t_1 be such that $(a, t_1) \models \Diamond(I(\phi \wedge B\chi)$, where ϕ and χ are Boolean formulas. Then there exists a t_3 such that $t_1 \rightarrowtail t_3$ and $(\alpha, t_3) \models I\phi \wedge B\chi$. It follows that $\mathcal{I}_{t_3}(\alpha) = \lceil\phi\rceil_{t_3}$. Fix an arbitrary t_2 such that $t_1 \rightarrowtail t_2$ and an arbitrary $\beta \in \mathcal{B}_{t_2}(\alpha)$. If $\beta \notin \mathcal{I}_{t_3}(\alpha)$ then $(\beta, t_3) \nvDash \phi$ and, since ϕ is Boolean, by Proposition 4, $(\beta, t_2) \nvDash \phi$, so that $(\beta, t_2) \models (\phi \to \chi)$. If $\beta \in \mathcal{I}_{t_3}(\alpha)$, then $(\beta, t_3) \models \phi$; furthermore, by property P_{K7s}, $\beta \in \mathcal{B}_{t_3}(\alpha)$ and, therefore, since $(\alpha, t_3) \models B\chi$, $(\beta, t_3) \models \chi$. Thus $(\beta, t_3) \models (\phi \to \chi)$. Since $(\phi \to \chi)$ is Boolean (because ϕ and χ are), by Proposition 4, $(\beta, t_2) \models (\phi \to \chi)$. Thus $(\alpha, t_2) \models B(\phi \to \chi)$.

Conversely, suppose that P_{K7s} is violated. Then there exist t_1, t_2, t_3 and α, β such that $t_1 \rightarrowtail t_2$, $t_1 \rightarrowtail t_3$, $\beta \in \mathcal{I}_{t_3}(\alpha) \cap \mathcal{B}_{t_2}(\alpha)$ and $\beta \notin \mathcal{B}_{t_3}(\alpha)$. Let p and q be atomic propositions and construct a model where $||p|| = \mathcal{I}_{t_3}(\alpha) \times T$ and $||q|| = \mathcal{B}_{t_3}(\alpha) \times T$. Then, $(\alpha, t_3) \models Ip \wedge Bq$ so that $(\alpha, t_1) \models \Diamond(Ip \wedge Bq)$. Since $\beta \notin \mathcal{B}_{t_3}(\alpha)$, $(\beta, t_2) \nvDash q$ and since $\beta \in \mathcal{I}_{t_3}(\alpha)$, $(\beta, t_2) \models p$. Thus $(\beta, t_2) \models \neg(p \to q)$. Hence, since $\beta \in \mathcal{B}_{t_2}(\alpha)$, $(\alpha, t_2) \nvDash B(p \to q)$ and therefore $(\alpha, t_1) \nvDash \bigcirc B(p \to q)$. ■

Let $\mathbb{L}_{PLS} = \mathbb{L}_0 + A + ND + NA + K7s + K8 + WC$ ('PLS' stands for 'plausibility') be the logic obtained from $\mathbb{L}_{AGM}$ by replacing axiom $K7$ with

$K7s$. Then logic $\mathbb{L}_{PLS}$ contains (is an extension of) logic $\mathbb{L}_{AGM}$. In fact, $K7$ can be derived from $K7s$.[6]

Definition 27 *An* $\mathbb{L}_{PLS}$-frame *is a temporal belief revision frame that satisfies the following properties:*

(1) the Qualitative Bayes Rule,

(2) $\forall \omega \in \Omega$, $\forall t \in T$, $\mathcal{B}_t(\omega) \subseteq \mathcal{I}_t(\omega)$,

(3) $\forall \omega \in \Omega$, $\forall t \in T$, *if* $\mathcal{I}_t(\omega) \neq \varnothing$ *then* $\mathcal{B}_t(\omega) \neq \varnothing$,

(4) $\forall \omega \in \Omega$, $\forall t_1, t_2, t_3 \in T$, *if* $t_1 \rightarrowtail t_2$, $t_1 \rightarrowtail t_3$, $\mathcal{I}_{t_3}(\omega) \subseteq \mathcal{I}_{t_2}(\omega)$ *and* $\mathcal{I}_{t_3}(\omega) \cap \mathcal{B}_{t_2}(\omega) \neq \varnothing$ *then* $\mathcal{B}_{t_3}(\omega) \subseteq \mathcal{I}_{t_3}(\omega) \cap \mathcal{B}_{t_2}(\omega)$,

(5) $\forall \omega \in \Omega$, $\forall t_1, t_2, t_3 \in T$, *if* $t_1 \rightarrowtail t_2$, $t_1 \rightarrowtail t_3$ *and* $\mathcal{I}_{t_3}(\omega) \cap \mathcal{B}_{t_2}(\omega) \neq \varnothing$ *then* $\mathcal{I}_{t_3}(\omega) \cap \mathcal{B}_{t_2}(\omega) \subseteq \mathcal{B}_{t_3}(\omega)$.

Remark 28 *It follows from Remark 16 and Propositions 11, 12, 18, 19 and 26 that logic* $\mathbb{L}_{PLS}$ *is characterized by the class of* $\mathbb{L}_{PLS}$*-frames.*

The next proposition shows that every $\mathbb{L}_{PLS}$-frame is a plausibility frame. *Thus logic* $\mathbb{L}_{PLS}$ *captures the notion of belief revision based on a plausibility ordering of the set of states.*

Note, however, that not every plausibility frame is an $\mathbb{L}_{PLS}$-frame, that is, the set of $\mathbb{L}_{PLS}$-frames is a proper subset of the set of plausibility frames, as the following example shows: $T = \{t_1, t_2, t_3\}$, $\rightarrowtail\ = \{(t_1, t_2), (t_1, t_3)\}$; $\Omega = \{\alpha, \beta, \gamma, \delta\}$; for every $\omega \in \Omega$, $\mathcal{I}_{t_1}(\omega) = \Omega$ and $\mathcal{B}_{t_1}(\omega) = \{\delta\}$; $\mathcal{I}_{t_2}(\alpha) = \mathcal{I}_{t_2}(\beta) = \{\alpha, \beta\}$, $\mathcal{I}_{t_2}(\gamma) = \mathcal{I}_{t_2}(\delta) = \{\gamma, \delta\}$, $\mathcal{B}_{t_2}(\alpha) = \mathcal{B}_{t_2}(\beta) = \{\beta\}$, $\mathcal{B}_{t_2}(\gamma) = \mathcal{B}_{t_2}(\delta) = \{\delta\}$; $\mathcal{I}_{t_3}(\alpha) = \{\alpha\}$, $\mathcal{I}_{t_3}(\beta) = \mathcal{I}_{t_3}(\gamma) = \{\beta, \gamma\}$; $\mathcal{I}_{t_3}(\delta) = \{\delta\}$, $\mathcal{B}_{t_3}(\alpha) = \{\alpha\}$, $\mathcal{B}_{t_3}(\beta) = \mathcal{B}_{t_3}(\gamma) = \{\gamma\}$, $\mathcal{B}_{t_3}(\delta) = \{\delta\}$. This frame is illustrated in Figure 3. It is a plausibility frame, based on the following plausibility ordering: for every $\omega \in \Omega$, $\delta \prec_{\omega,t_1} \gamma \prec_{\omega,t_1} \beta \prec_{\omega,t_1} \alpha$. However, it is not an $\mathbb{L}_{PLS}$-frame since it fails to satisfy property (5) of Definition 27: $\mathcal{I}_{t_3}(\beta) \cap \mathcal{B}_{t_2}(\beta) = \{\beta\} \nsubseteq \mathcal{B}_{t_3}(\beta) = \{\gamma\}$.

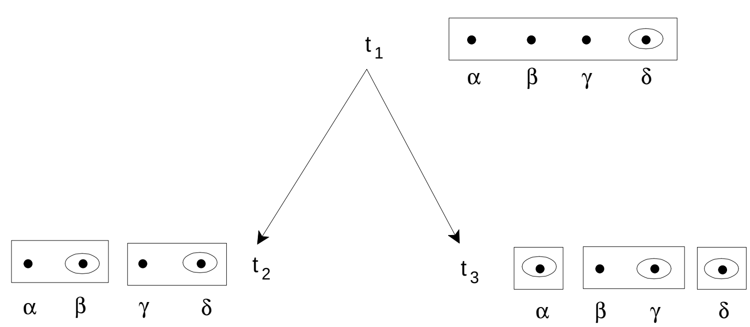

Figure 3

[6] *Proof.*

1.	$\Diamond(I(\phi \wedge \psi) \wedge B\chi) \to \bigcirc(B((\phi \wedge \psi) \to \chi))$.	instance of $K7s$
2.	$B((\phi \wedge \psi) \to \chi) \to (I\phi \to B((\phi \wedge \psi) \to \chi))$	tautology
3.	$\bigcirc B((\phi \wedge \psi) \to \chi) \to \bigcirc(I\phi \to B((\phi \wedge \psi) \to \chi))$	2, rule RK for $\bigcirc$
4.	$\Diamond(I(\phi \wedge \psi) \wedge B\chi) \to \bigcirc(I\phi \to B((\phi \wedge \psi) \to \chi))$	1, 3, PL.

Proposition 29 *Every $\mathbb{L}_{PLS}$-frame is a plausibility frame.*

Proof. Fix an arbitrary $\mathbb{L}_{PLS}$-frame and arbitrary $\alpha_0 \in \Omega$ and $t_0 \in T$. We will construct a plausibility ordering $\precsim_{\alpha_0,t_0}$ that satisfies the three properties of Definition 23. Let $T_0 = \{t \in T : t_0 \rightarrowtail t\}$. Since T is countable, so is T_0. Fix a numbering of the elements of T_0: $T_0 = \{t_{01}, t_{02}, ..., t_{0n}, ...\}$. For $n \in \mathbb{N}$ (where $\mathbb{N}$ is the set of natural numbers), define $f_n : \mathcal{B}_{t_{0n}}(\alpha_0) \to \mathbb{N}$ (with the convention that $t_{00} = t_0$) recursively as follows :

- $f_0(\omega) = 0$, for every $\omega \in \mathcal{B}_{t_0}(\alpha_0)$
- for $n > 0$, let $A_n = \mathcal{B}_{t_{0n}}(\alpha_0) \cap \left(\bigcup_{i=0}^{n-1} \mathcal{B}_{t_{0i}}(\alpha_0)\right)$. If $A_n \neq \varnothing$, for every $\omega \in A_n$ let $m_{n,\omega} = \min\{f_i(\omega) : i \in \{0,...,n-1\} \text{ and } \omega \in \mathcal{B}_{t_{0i}}(\alpha_0)\}$ and $m_n = \min\{m_{n,\omega} : \omega \in A_n\}$. For every $\omega \in \mathcal{B}_{t_{0n}}(\alpha_0)$, define $f_n(\omega) = \begin{cases} n & \text{if } A_n = \varnothing \\ m_n & \text{otherwise} \end{cases}$.[7]

These functions can be defined for every frame, not necessarily plausibility frames. For example, in the frame of Figure 2, letting $t_0 = t_1$, $t_{01} = t_2$, $t_{02} = t_3$ and $\alpha_0 = \beta$ we have $f_0(\varepsilon) = 0$, $f_1(\alpha) = f_1(\beta) = 1$ and $f_2(\alpha) = f_2(\gamma) = 1$. In the frame of Figure 3 we have $f_0(\delta) = 0$, $f_1(\beta) = 1$ and $f_2(\gamma) = 2$.

Finally let $g : \Omega \setminus (\bigcup_{i=0}^{\infty} \mathcal{B}_{t_{0i}}(\alpha_0)) \to \{\infty\}$ and, identifying functions with sets of ordered pairs, define

$$f = \left(\bigcup_{n=0}^{\infty} f_n\right) \cup g.$$

For example, in the frame of Figure 2 we have $f(\alpha) = f(\beta) = f(\gamma) = 1$, $f(\delta) = \infty$ and $f(\varepsilon) = 0$, while in the frame of Figure 3 we have $f(\alpha) = \infty$, $f(\beta) = 1$, $f(\gamma) = 2$ and $f(\delta) = 0$.

Now define the following relation $\precsim_{\alpha_0,t_0}$ on Ω (with the convention that, for every $n \in \mathbb{N}$, $n < \infty$):

$\omega \precsim_{\alpha_0,t_0} \omega'$ if and only if $f(\omega) \leq f(\omega')$.

This relation is clearly a plausibility relation (complete, transitive and well-founded). We want to show that if the frame we started with is an $\mathbb{L}_{PLS}$-frame then

(1) $\mathcal{B}_{t_0}(\alpha_0) = \min_{\precsim_{\alpha_0,t_0}} \mathcal{I}_{t_0}(\alpha_0)$,

(2) if $\omega' \in \mathcal{B}_{t_0}(\alpha_0)$ and $\omega'' \in \Omega \backslash \mathcal{B}_{t_0}(\alpha_0)$, then $\omega' \prec_{\alpha_0,t_0} \omega''$, that is, $f(\omega') < f(\omega'')$,

(3) for every $n > 0$, $\mathcal{B}_{t_{0n}}(\alpha_0) = \min_{\precsim_{\alpha_0,t_0}} \mathcal{I}_{t_{0n}}(\omega)$.

Properties (1) and (2) are satisfied since, by (2) of Definition 27, $\mathcal{B}_{t_0}(\alpha_0) \subseteq \mathcal{I}_{t_0}(\alpha_0)$ and by construction, for every $\omega \in \Omega$, $f(\omega) = 0$ if $\omega \in \mathcal{B}_{t_0}(\alpha_0)$ and $f(\omega) > 0$ if $\omega \notin \mathcal{B}_{t_0}(\alpha_0)$. Fix an arbitrary $n > 0$. By (2) of Definition 27,

[7] Note that, for every $n \in \mathbb{N}$, if $A_n \neq \varnothing$, $m_n \leq n$. We prove this by induction. First of all, for $n = 1$ we have that $A_1 = \mathcal{B}_{t_{01}}(\alpha_0) \cap \mathcal{B}_{t_0}(\alpha_0)$ and if $A_1 \neq \varnothing$ then $m_1 = 0 < 1$. Assume that the statement is true for every $p \leq n$, that is, if $p \leq n$ and $A_p \neq \varnothing$ then $m_p \leq p$. We want to show that it is true for $n + 1$. Suppose that $A_{n+1} \neq \varnothing$. Fix an arbitrary $\omega \in A_{n+1}$. Then for every $i \in \{0, ..., n\}$, $f_i(\omega)$ is either equal to i or to m_i and by the induction hypothesis $m_i \leq i$. Thus $m_{n+1,\omega} \leq n$, so that $m_{n+1} \leq n < n + 1$.

$\mathcal{B}_{t_{0n}}(\alpha_0) \subseteq \mathcal{I}_{t_{0n}}(\alpha_0)$. By construction, $\forall \omega, \omega' \in \mathcal{B}_{t_{0n}}(\alpha_0)$, $f(\omega) = f(\omega')$. Thus we only need to show that if $\omega \in \mathcal{B}_{t_{0n}}(\alpha_0)$ and $\omega' \in \mathcal{I}_{t_{0n}}(\alpha_0) \setminus \mathcal{B}_{t_{0n}}(\alpha_0)$ then $f(\omega) < f(\omega')$. Since $f(\omega)$ is either equal to n or to m_n and, by Footnote 7, $m_n \leq n$, $f(\omega) \leq n$. Suppose that $f(\omega') < f(\omega)$. Hence $f(\omega') < n$. Then there exists a $p < n$ such that $\omega' \in \mathcal{B}_{t_{0p}}(\alpha_0)$. Thus $\omega' \in \mathcal{B}_{t_{0p}}(\alpha_0) \cap \mathcal{I}_{t_{0n}}(\alpha_0)$. It follows from property (4) of Definition 27 that $\mathcal{B}_{t_{0p}}(\alpha_0) \cap \mathcal{I}_{t_{0n}}(\alpha_0) \subseteq \mathcal{B}_{t_{0n}}(\alpha_0)$, contradicting the hypothesis that $\omega' \in \mathcal{I}_{t_{0n}}(\alpha_0) \setminus \mathcal{B}_{t_{0n}}(\alpha_0)$. ■

8 Conclusion

We proposed a temporal logic where information and beliefs are modeled by means of two modal operators I and B, respectively. A third modal operator, the next-time operator $\bigcirc$, enables one to express the dynamic interaction of information and beliefs over time. The proposed logic can be viewed as a temporal generalization of the theory of static belief pioneered by Hintikka [13] and has the advantage of achieving a uniform treatment of static belief and of belief revision by providing a modal logic translation of the AGM theory of belief revision pioneered by Alchourrón *et al* [1].

The combined syntactic-semantic approach of modal logic allows one to state properties of beliefs in a clear and transparent way by means of axioms and to show the correspondence between axioms and semantic properties, such as the qualitative version of Bayes' rule. The proposed framework can accommodate not only the AGM theory of belief revision but also *iterated revision*, a topic that has received considerable attention in recent years. In this literature a belief state is represented by a set of beliefs together with a plausibility ordering. After revising beliefs in response to a particular piece of information, the plausibility ordering may change and there is much discussion in the literature about the criteria that should guide such a change. (see, for example, [8], [17] and [18]).

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