A characterization of sequential equilibrium in terms of AGM belief revision

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Abstract

In [G. Bonanno, Rational choice and AGM belief revision, *Artificial Intelligence*, 2009] a semantics for one-stage AGM belief revision was proposed based on choice frames, borrowed from the rational choice literature. In this paper we extend the semantics of choice frames to deal with iterated belief revision and use the corresponding structures to analyze extensive-form games. Choice frames can be used to represent a player's initial beliefs and disposition to change those beliefs when informed that it is her turn to move. If the frame satisfies AGM-consistency and a natural postulate for iterated belief revision, then it is rationalizable by a total pre-order on the set of histories. We show that three properties of this total pre-order, together with the hypothesis of agreement among players, provide a characterization of the notion of consistent assessment, which is the central component of the notion of sequential equilibrium proposed by Kreps and Wilson [*Econometrica*, 1982].

Keywords: Choice function, AGM belief revision, extensive-form game, sequential equilibrium, iterated belief revision, backward induction.

1 Introduction

In [7] the notion of choice frame, borrowed from the rational choice literature, was proposed as a semantics for the theory of one-stage belief revision put forward by Alchourrón, Gärdenfors and Makinson [1]. In this paper we extend the semantics of choice frames to deal with iterated belief revision and use the corresponding structures to analyze extensive-form games. With every extensive

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form one can associate a (possibly iterated) choice frame for every player, representing the player's initial beliefs and disposition to change those beliefs when informed that it is her turn to move. If the structure satisfies AGM-consistency and a natural postulate for iterated belief revision, then it is rationalizable by a total pre-order on the set of histories. We show that three properties of this total pre-order, together with the hypothesis of agreement among players, provide a characterization of the notion of consistent assessment, which is the central component of the notion of sequential equilibrium introduced by Kreps and Wilson [15]. Consistent assessments were proposed by Kreps and Wilson as an attempt to capture the concept of minimal belief revision. A number of authors have tried to shed light on the technical notion of consistent assessment by relating it to more intuitive concepts, such as "structural consistency" ([16]), "generally reasonable extended assessment" ([11]), "stochastic independence" ([2], [14]).[1] Our result provides a characterization of consistent assessments in terms of the AGM theory of belief revision, through the notion of AGM-consistent choice frame.

Consistent assessments were proposed as an embodiment of the notion of "minimal" belief revision. Our characterization makes this precise, by relating the technical notion of consistency (as the limit of a sequence of completely mixed strategies and corresponding Bayesian beliefs) to the belief revision postulates of the AGM theory.

The paper is organized as follows. In Section 2 we review the AGM postulates for belief revision and the semantics based on choice frames. In Section 3 we review the definition of extensive-form game, show how to associate with an extensive form a choice frame for every player and prove the main characterization result. In Section 4 we discuss the issue of iterated belief revision that arises in extensive forms where some players move more than once along some play. Section 5 deals with the other component of sequential equilibrium, namely sequential rationality and provides a characterization of pure sequential equilibria. A characterization of backward induction in extensive-form games with perfect information is also provided. Section 6 concludes.

2 Choice frames and AGM-consistent beliefs

In this section we briefly review the AGM theory of belief revision ([1], [12]) and its semantics based on choice frames ([7]).

Let Φ be the set of formulas of a propositional language based on a countable set S of atoms. Given a subset $K \subseteq \Phi$, its deductive closure, denoted by $[K]$, is defined as follows: $\psi \in [K]$ if and only if there exist $\phi_1, ..., \phi_n \in K$ (with $n \geq 0$) such that $(\phi_1 \wedge ... \wedge \phi_n) \to \psi$ is a tautology. A set $K \subseteq \Phi$ is *deductively closed* if $K = [K]$ and it is *consistent* if $[K] \neq \Phi$. Let K be a consistent and deductively closed set representing the agent's initial beliefs and let $\Psi \subseteq \Phi$ be a set of formulas representing possible items of information. A *belief revision function based on* K is a function $B_K : \Psi \to 2^\Phi$ (where 2^Φ denotes the set

[1] Perea et al [23] offer an algebraic characterization of consistent assessments

of subsets of Φ) that associates with every formula $\phi \in \Psi$ (thought of as new information) a set $B_K(\phi) \subseteq \Phi$ (thought of as the revised beliefs). If $\Psi \neq \Phi$ then B_K is called a *partial* belief revision function, while if $\Psi = \Phi$ then B_K is called a *full* belief revision function.

Let $B_K : \Psi \to 2^\Phi$ be a (partial) belief revision function and $B_K^* : \Phi \to 2^\Phi$ a full belief revision function. We say that B_K^* is an *extension* of B_K if, for every $\phi \in \Psi$, $B_K^*(\phi) = B_K(\phi)$.

A *full* belief revision function is called an *AGM function* if it satisfies the following properties, known as the AGM postulates: $\forall \phi, \psi \in \Phi$,

(AGM1)	$B_K(\phi) = [B_K(\phi)]$.
(AGM2)	$\phi \in B_K(\phi)$.
(AGM3)	$B_K(\phi) \subseteq [K \cup \{\phi\}]$.
(AGM4)	if $\neg\phi \notin K$, then $[K \cup \{\phi\}] \subseteq B_K(\phi)$.
(AGM5)	$B_K(\phi) = \Phi$ if and only if ϕ is a contradiction.
(AGM6)	if $\phi \leftrightarrow \psi$ is a tautology then $B_K(\phi) = B_K(\psi)$.
(AGM7)	$B_K(\phi \wedge \psi) \subseteq [B_K(\phi) \cup \{\psi\}]$.
(AGM8)	if $\neg\psi \notin B_K(\phi)$, then $[B_K(\phi) \cup \{\psi\}] \subseteq B_K(\phi \wedge \psi)$.

AGM1 requires the revised belief set to be deductively closed. AGM2 requires that the information be believed. AGM3 says that beliefs should be revised minimally, in the sense that no new formula should be added unless it can be deduced from the information received and the initial beliefs. AGM4 says that if the information received is compatible with the initial beliefs, then any formula that can be deduced from the information and the initial beliefs should be part of the revised beliefs. AGM5 requires the revised beliefs to be consistent, unless the information ϕ is a contradiction (that is, $\neg\phi$ is a tautology). AGM6 requires that if ϕ is propositionally equivalent to ψ then the result of revising by ϕ be identical to the result of revising by ψ. AGM7 and AGM8 are a generalization of AGM3 and AGM4 that requires $B_K(\phi \wedge \psi)$ to be the same as the expansion of $B_K(\phi)$ by ψ, as long as ψ is compatible with $B_K(\phi)$.

Choice frames provide a set-theoretic semantics for belief revision functions.

Definition 1 *A* choice frame *is a triple* $\langle \Omega, \mathcal{E}, f \rangle$ *where*

Ω *is a non-empty set of* states*; subsets of* Ω *are called* events.

$\mathcal{E} \subseteq 2^\Omega$ *is a collection of events such that* $\varnothing \notin \mathcal{E}$ *and* $\Omega \in \mathcal{E}$.

$f : \mathcal{E} \to 2^\Omega$ *is a function that associates with every event* $E \in \mathcal{E}$ *an event* $f(E)$ *satisfying the following properties: (1)* $f(E) \subseteq E$ *and (2)* $f(E) \neq \varnothing$.

In rational choice theory a set $E \in \mathcal{E}$ is interpreted as a set of available alternatives and $f(E)$ is interpreted as the subset of E which consists of the chosen alternatives (see, for example, [25] and [27]). In our case, we think of the elements of $\mathcal{E}$ as possible items of information and the interpretation of $f(E)$ is that, if informed that event E has occurred, the agent considers as possible all and only the states in $f(E)$. The set $f(\Omega)$ is interpreted as the states that are initially considered possible.

Note that in the literature (see, for example [25]) it is common to impose some structure on the collection of events $\mathcal{E}$ (for example, that it be closed under finite unions). On the contrarry, we allow $\mathcal{E}$ to be an arbitrary subset of 2^Ω and typically think of $\mathcal{E}$ as containing only a small number of events. This is typically the case in extensive-form games, as shown in the following section.

In order to interpret a choice frame $\langle \Omega, \mathcal{E}, f \rangle$ in terms of belief revision we need to add a *valuation* $V : S \to 2^\Omega$ that associates with every atomic formula $p \in S$ the set of states at which p is true. The quadruple $\langle \Omega, \mathcal{E}, f, V \rangle$ is called a *model* (or an *interpretation*) *of* $\langle \Omega, \mathcal{E}, f \rangle$. Given a model $\mathcal{M} = \langle \Omega, \mathcal{E}, f, V \rangle$, truth of an arbitrary formula at a state is defined recursively as follows ($\omega \models_{\mathcal{M}} \phi$ means that formula ϕ is true at state ω in model $\mathcal{M}$):
(1) for $p \in S$, $\omega \models_{\mathcal{M}} p$ if and only if $\omega \in V(p)$, (2) $\omega \models_{\mathcal{M}} \neg\phi$ if and only if $\omega \not\models_{\mathcal{M}} \phi$ and (3) $\omega \models_{\mathcal{M}} (\phi \vee \psi)$ if and only if either $\omega \models_{\mathcal{M}} \phi$ or $\omega \models_{\mathcal{M}} \psi$ (or both). The truth set of formula ϕ in model $\mathcal{M}$ is denoted by $\|\phi\|_{\mathcal{M}}$, that is, $\|\phi\|_{\mathcal{M}} = \{\omega \in \Omega : \omega \models_{\mathcal{M}} \phi\}$.

Given a model $\mathcal{M} = \langle \Omega, \mathcal{E}, f, V \rangle$ we say that

- the agent *initially believes that* ψ if and only if $f(\Omega) \subseteq \|\psi\|_{\mathcal{M}}$,
- the agent *believes that* ψ *upon learning that* ϕ if and only if (1) $\|\phi\|_{\mathcal{M}} \in \mathcal{E}$ and (2) $f(\|\phi\|_{\mathcal{M}}) \subseteq \|\psi\|_{\mathcal{M}}$.

Accordingly, we can associate with every model $\mathcal{M}$ a (partial) belief revision function as follows. Let

$$\begin{aligned} &K_{\mathcal{M}} = \{\phi \in \Phi : f(\Omega) \subseteq \|\phi\|_{\mathcal{M}}\}, \\ &\Psi_{\mathcal{M}} = \{\phi \in \Phi : \|\phi\|_{\mathcal{M}} \in \mathcal{E}\}, \\ &B_{K_{\mathcal{M}}} : \Psi_{\mathcal{M}} \to 2^\Phi \text{ given by } B_{K_{\mathcal{M}}}(\phi) = \{\psi \in \Phi : f(\|\phi\|_{\mathcal{M}}) \subseteq \|\psi\|_{\mathcal{M}}\}. \end{aligned} \tag{1}$$

What properties must a choice frame satisfy in order for it to be the case that the (typically partial) belief revision function associated with an *arbitrary* interpretation of it can be extended to a full AGM belief revision function? This question motivates the following definition.

Definition 2 *A choice frame* $\langle \Omega, \mathcal{E}, f \rangle$ *is* AGM-consistent *if, for every model* $\mathcal{M} = \langle \Omega, \mathcal{E}, f, V \rangle$ *based on it, the (partial) belief revision function* $B_{K_{\mathcal{M}}}$ *associated with* $\mathcal{M}$ *(see (1)) can be extended to a full belief revision function that satisfies the AGM postulates.*

Recall that a binary relation $\precsim$ on Ω is a *total pre-order* if it is complete ($\forall \omega, \omega' \in \Omega$, either $\omega \precsim \omega'$ or $\omega' \precsim \omega$) and transitive ($\forall \omega, \omega', \omega'' \in \Omega$, if $\omega \precsim \omega'$ and $\omega' \precsim \omega''$ then $\omega \precsim \omega''$).

Definition 3 *A choice frame* $\langle \Omega, \mathcal{E}, f \rangle$ *is* rationalizable *if there exists a total pre-order* $\precsim$ *on* Ω *such that, for every* $E \in \mathcal{E}$, $f(E) = \{\omega \in E : \omega \precsim \omega', \forall \omega' \in E\}$.

The interpretation of $\omega \precsim \omega'$ is that state ω is at least as plausible as state ω' (or ω' is more implausible than or as implausible as ω). Thus in a rationalizable choice frame $\langle \Omega, \mathcal{E}, f \rangle$, for every $E \in \mathcal{E}$, $f(E)$ is the set of most plausible states in E. The following proposition is proved in [7]:

Proposition 4 *Let $\langle \Omega, \mathcal{E}, f \rangle$ be a choice frame where Ω is finite. Then $\langle \Omega, \mathcal{E}, f \rangle$ is AGM-consistent if and only if it is rationalizable*

On the basis of Proposition 4, rationalizable choice frames can be viewed as providing a semantics for one-stage partial belief revision functions that obey the AGM postulates.[2] In the next section we use choice frames to analyze extensive-form games.

3 Choice frames in extensive-form games

We shall use the history-based definition of extensive-form game (see, for example, [19]).[3] If A is a set, we denote by A^* the set of finite sequences in A. If $h = \langle a_1, ..., a_k \rangle \in A^*$ and $1 \leq j \leq k$, the sequence $h' = \langle a_1, ..., a_j \rangle$ is called a *prefix* of h. If $h = \langle a_1, ..., a_k \rangle \in A^*$ and $a \in A$, we denote the sequence $\langle a_1, ..., a_k, a \rangle \in A^*$ by ha.

A *finite extensive form* is a tuple $\left\langle A, H, N, P, \{\approx_i\}_{i \in N} \right\rangle$:[4]

- A finite set of actions A with a distinguished element called the null action and denoted by $\emptyset$.
- A finite set of histories $H \subseteq A^*$ which is closed under prefixes (that is, if $h \in H$ and $h' \in A^*$ is a prefix of h, then $h' \in H$) and is such that the null action is a prefix of every history (that is, $\forall h \in H$, $\langle \emptyset \rangle$ is a prefix of h). A history $h \in H$ such that, for every $a \in A$, $ha \notin H$, is called a *terminal history*. The set of terminal histories is denoted by Z. Let $D = H \setminus Z$ denote the set of non-terminal or *decision* histories. For every history $h \in H$, we denote by $A(h)$ the set of actions available at h, that is, $A(h) = \{a \in A : ha \in H\}$. Thus $A(h) \neq \varnothing$ if and only if $h \in D$. We shall assume throughout that $A = \bigcup_{h \in D} A(h)$ (that is, we restrict attention to actions that are available at some decision history).
- A finite set $N = \{1, ...n\}$ of players. An additional player, called *chance*, might also be added.

[2] For an investigation of the relationship between choice frames and non-monotonic inference see Lehmann [17].

[3] Similar structures were introduced in the computer science literature by Parikh and Ramanujam ([21], [22]; see also [20]). These sructures are more general than extensive-form games in that they specify a player's information at *every* node, that is, not only at nodes where the player himself has to move. Hoewever, as shown in [3] and [4], it is possible to extend the definition of extensive-form game by specifying, for every node, the information that every player has at that node.

[4] Given an extensive form, one obtains an *extensive game* by adding, for every player $i \in N$, a *utility* or *payoff function* $U_i : Z \to \mathbb{R}$ (where $\mathbb{R}$ denotes the set of real numbers and Z the set of terminal histories).

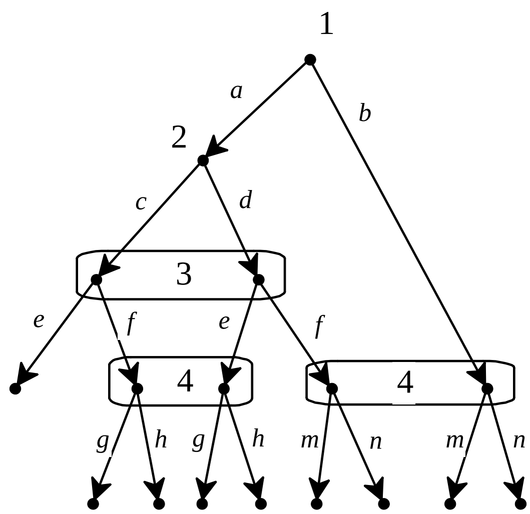
- A function $P : D \to N \cup \{chance\}$ that assigns a player to each non-terminal history. Thus $P(h)$ is the player who moves at history h. A game is said to be *without chance moves* if $P(h) \in N$ for every $h \in D$. For every $i \in N \cup \{chance\}$, let $D_i = P^{-1}(i)$ be the histories assigned to player i. Thus $\{D_{chance}, D_1, ..., D_n\}$ is a partition of D. We follow Kreps and Wilson [15] in assuming that chance moves occur at most at the beginning of the game, that is, either $D_{chance} = \varnothing$ or $D_{chance} = \{\langle\emptyset\rangle\}$ (recall that $\emptyset$ is the null action). If $\langle\emptyset\rangle$ is assigned to chance, then a probability distribution over $A(\langle\emptyset\rangle)$ is given.

- For each player $i \in N$, an equivalence relation $\approx_i$ on D_i. The interpretation of $h \approx_i h'$ is that, when choosing an action at history $h \in D_i$, player i does not know whether she is moving at h or at h'. The equivalence class of $h \in D_i$ is denoted by $I_i(h)$ and is called an *information set of player i*; thus $I_i(h) = \{h' \in D_i : h \approx_i h'\}$. The following restriction applies: if $h' \in I_i(h)$ then $A(h') = A(h)$, that is, the set of actions available to a player is the same at any two histories that belong to the same information set of that player.

- The following property, known as *perfect recall*, is satisfied: for every player $i \in N$, if $h_1, h_2 \in D_i$, $a \in A(h_1)$ and $h_1 a$ is a prefix of h_2 then for every $h' \in I_i(h_2)$ there exists an $h \in I_i(h_1)$ such that ha is a prefix of h'. Intuitively, perfect recall requires a player to remember what she knew in the past and what actions she took previously.[5]

Figure 1 shows an extensive form without chance moves where $A = \{a, b, c, d, e, f, g, h, m, n\}$, $H = D \cup Z$ with (to simplify the notation we write a instead of $\langle\emptyset, a\rangle$, ac instead of $\langle\emptyset, a, c\rangle$, etc.) $D = \{\emptyset, a, b, ac, ad, acf, ade, adf\}$, $Z = \{ace, acfg, acfh, adeg, adeh, adfm, adfn, bm, bn\}$, $A(\emptyset) = \{a, b\}$, $A(a) = \{c, d\}$, $A(ac) = A(ad) = \{e, f\}$, $A(acf) = A(ade) = \{g, h\}$, $A(adf) = A(b) = \{m, n\}$, $N = \{1, 2, 3, 4\}$, $P(\emptyset) = 1$, $P(a) = 2$, $P(ac) = P(ad) = 3$, $P(acf) = P(ade) = P(adf) = P(b) = 4$, $\approx_1 \ = \{(\emptyset, \emptyset)\}$, $\approx_2 \ = \{(a, a)\}$, $\approx_3 \ = \{(ac, ac), (ac, ad), (ad, ac), (ad, ad)\}$ and $\approx_4 \ = \{(acf, acf), (acf, ade), (ade, acf), (ade, ade), (adf, adf), (adf, b), (b, adf), (b, b)\}$. The information sets containing more than one history are shown as rounded rectangles. Thus, for example, $I_4(b) = \{adf, b\}$. The root of the tree represents the null action $\emptyset$.

[5] For an investigation of the conceptual content of the property of perfect recall see [6].



An extensive form without chance moves
Figure 1

Notation 5 *We shall write $h \in I(h')$ as a short-hand for "$h \in I_i(h')$ for some $i \in N$".*

Remark 6 *In order to simplify the notation in the proofs, we shall assume that the same action cannot be available at different information sets: $\forall h, h' \in H, \forall a \in A, \quad$ if $a \in A(h) \cap A(h')$ then $h \in I(h')$.*[6] *The extensive form of Figure 1 satisfies this assumption.*

We begin by restricting attention to the class of extensive forms where no player moves more than once along any history. That is, for every history $h = \langle \emptyset, a_1, ..., a_m \rangle$, if h_1 and h_2 are prefixes of h with $P(h_1) = P(h_2)$ then $h_1 = h_2$ (recall that $P(h)$ is the player who moves at h). The extensive form represented in Figure 1 satisfies this property.

Choice frames can be used to represent, for every player, her initial beliefs and her disposition to change those beliefs when it is her turn to move. Given an extensive form, we can associate with every $i \in N$ a choice frame $\langle \Omega, \mathcal{E}_i, f_i \rangle$ as follows: $\Omega = H$ (the set of histories), $E \in \mathcal{E}_i$ if and only if either $E = H$ or E consists of an information set of player i together with all the continuation histories. Recall that, if $h \in D_i$, player i's information set that contains h is denoted by $I_i(h)$; that is, $I_i(h) = \{h' \in H : h' \approx_i h\}$. We shall denote by $\vec{I_i}(h)$ the set $I_i(h)$ together with the continuation histories: for $h \in D_i$,

$$\vec{I_i}(h) = \{x \in H : \exists h' \in I_i(h) \text{ such that } h' \text{ is a prefix of } x\}. \tag{2}$$

[6] See Footnote 17 in the Appendix for an explanation of how the proofs would have to be written without this convention.

Thus

$$\mathcal{E}_i = \{H\} \cup \{\vec{I}_i(h) : h \in D_i\}. \tag{3}$$

We call $\vec{I}_i(h)$ the *augmented* information set of player i at decision history $h \in D_i$.[7]

Finally, the function f_i provides conditional beliefs about past and future moves.[8]

Note that in the class of extensive forms we are restricting attention to in this section (namely extensive forms where no player moves more than once along any history), for every player $i \in N$ and for every $h, h' \in D_i$, either $\vec{I}_i(h) = \vec{I}_i(h')$ or $\vec{I}_i(h) \cap \vec{I}_i(h') = \varnothing$. That is, any two different augmented information sets are disjoint.

If we assume that the choice frame of player i is AGM consistent, then, by Proposition 4, there exists a total pre-order $\precsim_i$ on H that rationalizes f_i (that is, for every $E \in \mathcal{E}_i$, $f_i(E) = \{h \in E : h \precsim_i h', \forall h' \in E\}$).

What are natural properties to impose on these total pre-orders, that is, on the associated beliefs? We shall discuss four properties and show that they characterize the notion of consistent assessment, which is the central component of the notion of sequential equilibrium introduced in [15].

The first property expresses the notion of *agreement of beliefs*, in the sense that the players share the same initial beliefs and the same disposition to change those beliefs in response to the same information:[9]

$$\exists \precsim \; \subseteq H \times H : \quad \forall i \in N, \quad \precsim_i \; = \; \precsim . \tag{$P1$}$$

Note that $P1$ is consistent with the players holding different beliefs during any particular play of the game, since they will typically receive different information.

The remaining properties will be stated in terms of the common pre-order $\precsim$ given by $P1$. Recall that the interpretation of $h \precsim h'$ is that history h is at least as plausible as history h' (or h' is more implausible than or as implausible as h).

[7] For example, in the extensive form of Figure 1, $\mathcal{E}_4 = \{H, E_1, E_2\}$, where $E_1 = \{acf, ade, acfg, acfh, adeg, adeh\}$ and $E_2 = \{adf, b, adfm, adfn, bm, bn\}$.

[8] For example, in the extensive form of Figure 1 one possibility for Player 4 is: $f_4(H) = \{a, ac, ace\}$, $f_4(E_1) = \{acf, acfh\}$ and $f(E_2) = \{b, bm\}$, where E_1 and E_2 are as given in the previous footnote. The interpretation of this is that Player 4 initially believes that Player 1 will play a, Player 2 will follow with c and Player 3 with e (so that Player 4 does not expect to be asked to make any choices). If informed that she is at her information set on the left then whe would continue to believe that Player 1 played a and Player 2 followed with c, but she would now believe that Player 3 chose f and she herself plans to choose h. If informed that she is at her information set on the right then whe would believe that Player 1 played b and she herself plans to choose m.

[9] This property can be viewed as an expression of the notion of a "common prior" (see, for example, [5]), which is pervasive in game theory.

The second property says that adding an action to a history h cannot yield a more plausible history than h itself:

$$\forall h \in D, \quad \forall a \in A(h), \quad h \precsim ha. \tag{$P2$}$$

Remark 7 *It follows from Property $P2$ that, for every $h, h' \in H$, if h' is a prefix of h then $h' \precsim h$.*[10]

The third property says that at every decision history h there is some action a such that adding a to h yields a history which is at least as plausible as h; furthermore, any such action a performs the same role with any other history that belongs to the same information set:

$$\begin{array}{c} \forall h \in D, \\ (1)\ \exists a \in A(h):\ ha \precsim h \text{ and} \\ (2)\ \forall a \in A(h),\ \forall h' \in I(h),\ \text{if } ha \precsim h \text{ then } h'a \precsim h'. \end{array} \tag{$P3$}$$

Notation 8 *We write $h \sim h'$ (with the interpretation that h is as plausible as h') as a short-hand for "$h \precsim h'$ and $h' \precsim h$" and we write $h \prec h'$ (with the interpretation that h is more plausible than h') as a short-hand for "$h \precsim h'$ and $h' \not\precsim h$".*

Remark 9 *It follows from Properties $P2$ and $P3$ that, for every decision history h, there is at least one action a at h such that, for every h' in the same information set as h, $h'a$ is just as plausible as h': $\forall h \in D, \exists a \in A(h) : h \sim ha$ and if $h' \in I(h)$, then $h' \sim h'a$. We call such actions* plausibility preserving.

A function $F : H \to \mathbb{N}$ (where $\mathbb{N}$ denotes the set of natural numbers) is an *integer-valued representation* of $\precsim$ if $F(\langle\emptyset\rangle) = 0$ and, $\forall h, h' \in H$, $h \precsim h'$ if and only if $F(h) \leq F(h')$. Let $\mathcal{R}$ be the set of integer-valued representations of $\precsim$. Since H is finite, $\mathcal{R} \neq \varnothing$.[11] We call an integer-valued representation F of $\precsim$ *action-based* if,

$$\begin{array}{l} \forall h, h' \in D, \forall a \in A(h), \text{if } h' \in I(h) \text{ then} \\ F(ha) - F(h) = F(h'a) - F(h'). \end{array} \tag{4}$$

[10] In fact, if h' is a prefix of h then $h = h'a_1...a_m$ for some (possibly none) $a_1, ..., a_m \in A$, so that, by Property $P2$, $h' \precsim h'a_1 \precsim h'a_1a_2 \precsim ... \precsim h'a_1...a_m = h$ and thus, by transitivity of $\precsim$, $h' \precsim h$.

[11] In fact, a natural integer-valued representation is the following. Define $H_0 = \{h \in H : h \precsim x, \ \forall x \in H\}$, $H_1 = \{h \in H \setminus H_0 : h \precsim x, \ \forall x \in H \setminus H_0\}$ and, in general for every integer $k \geq 1$, $H_k = \{h \in H \setminus H_0 \cup ... \cup H_{k-1} : h \precsim x, \ \forall x \in H \setminus H_0 \cup ... \cup H_{k-1}\}$. Since H is finite, there is an $m \in \mathbb{N}$ such that $\{H_0, ..., H_m\}$ is a partition of H and, for every $j, k \in \mathbb{N}$, with $j < k \leq m$, and for every $h, h' \in H$, if $h \in H_j$ and $h' \in H_k$ then $h \prec h'$. Define $F : H \to \mathbb{N}$ as follows: $F(h) = k$ if and only if $h \in H_k$. The function F so defined is an integer-valued representation of $\precsim$.

For example, consider the extensive form represented in Figure 2 and the following total pre-order $\precsim$: $\emptyset \sim a \prec b \sim be \prec bf \prec c \sim ce \prec d \prec cf$. The first column in Table 1 reproduces this total pre-order with the convention that if x and y are on the same line, then $x \sim y$ and if x is above y then $x \prec y$; the second and third columns give two integer-valued representations of $\precsim$, F_1 and F_2. F_1 is the representation described in Footnote 11 and is not action-based, since $c \in I(b)$ and $F_1(bf) - F_1(b) = 2 - 1 = 1$ while $F_1(cf) - F_1(c) = 5 - 3 = 2$. On the other hand, F_2 is action-based.

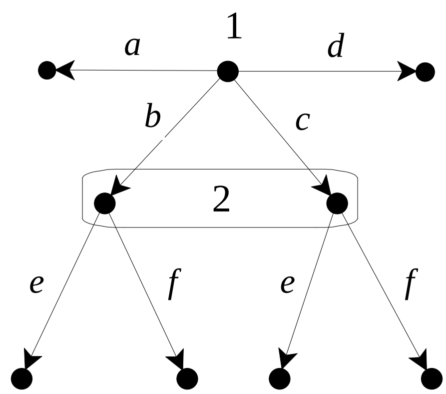


Figure 2

$\precsim$	F_1	F_2
$\emptyset,\ a$	0	0
$b,\ be$	1	1
bf	2	3
$c,\ ce$	3	4
d	4	5
cf	5	6

Two representations of a total pre-order for the extensive form of Figure 2.

Table 1

The fourth (and last) property says that among the integer-valued representations of $\precsim$ there is at least one which is action-based:

$$\text{There exists an } F \in \mathcal{R} \text{ which is action-based.} \qquad (P4)$$

Not every total pre-order satisfies property $P4$. To see this, consider the extensive form of Figure 3 and the following total pre-order, which is illustrated in Table 2:

$$\emptyset \sim a \prec b \sim bg \sim d \sim dr \prec bh \sim ds \prec c \sim cg \sim e \sim er \prec f \sim es \prec ch.$$

This total pre-order does not have an action-based integer-valued representation. In fact, fix an arbitrary integer-valued representation F (we know that there exists at least one). Then, since $b \sim d$ and $bh \sim ds$, it must be that $F(b) = F(d)$ and $F(bh) = F(ds)$. Thus $F(bh) - F(b) = F(ds) - F(d)$. Furthermore, since $c \sim e$ and $es \prec ch$, it must be that $F(c) = F(e)$ and $F(es) < F(ch)$.Thus $F(es) - F(e) < F(ch) - F(c)$, so that if $F(ds) - F(d) = F(es) - F(e)$ then $F(bh) - F(b) < F(ch) - F(c)$. Hence action basedness is violated, since it requires $F(ds) - F(d) = F(es) - F(e)$ and $F(bh) - F(b) = F(ch) - F(c)$ (because $c \in I(b)$ and $e \in I(d)$).

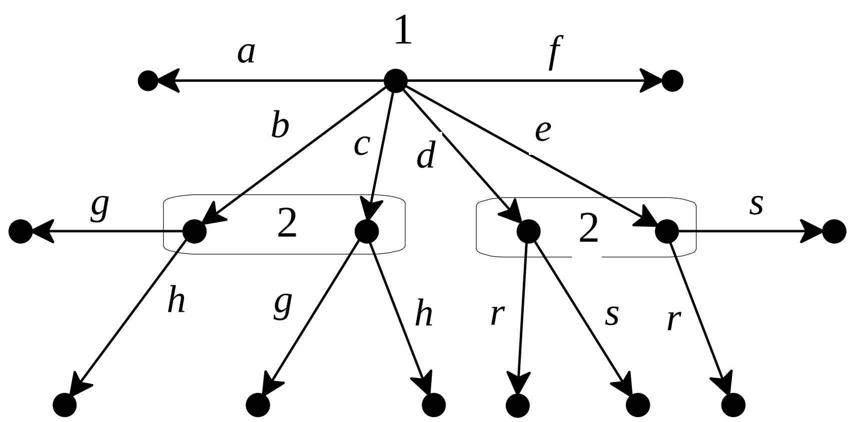


Figure 3

$\emptyset,\ a$
$b,\ bg,\ d,\ dr$
$bh,\ ds$
$c,\ cg,\ e,\ er$
$f,\ es$
ch

A total pre-order for the extensive form of Figure 3
which does not have an action-based representation.
Table 2

Note that Properties $P2$-$P4$ are independent of each other. For example, the total pre-order of Table 2 (for the extensive form illustrated in Figure 3) satisfies properties $P2$ and $P3$ (the plausibility preserving actions are a, g and r) but violates $P4$. Similar examples can be constructed that satisfy two of the properties but not the remaining one.[12]

If the beliefs of player i are rationalized by a total pre-order $\precsim$ on H, then the following holds: if the play of the game reaches history $h \in D_i$ then player i receives information $\vec{I}_i(h)$ and revises her previous beliefs to $f_i(\vec{I}_i(h)) = \{h' \in$

[12] Consider the extensive form of Figure 2 and the following total pre-order: $\emptyset \prec a \prec b \sim be \prec bf \prec c \sim ce \prec d \prec cf$, which violates $P3$ since there is no plausibility preserving action at the root. However, it satisfies $P2$ and $P4$. In fact, the following is an action-based representation:

$\emptyset$	a	$b,\ be$	bf	$c,\ ce$	d	cf
0	1	2	4	5	6	7

.

Now consider the following total pre-order: $\emptyset \sim a \prec bf \prec b \sim be \prec cf \prec d \prec c \sim ce$, which violates $P2$ since $bf \prec b$. However, it satisfies $P3$ (the plausibility preserving actions are a and e) and $P4$. In fact, the following is an action-based representation:

$\emptyset$	a	bf	$b,\ be$	cf	d	$c,\ ce$
0	1	2	4	5	6	7

.

$\vec{I}_i(h) : h' \precsim x,\ \forall x \in \vec{I}_i(h)\}$, that is, the histories that are most plausible given her information constitute her revised beliefs.

Before we proceed to our main result, we recall the notion of sequential equilibrium ([15]), which is the most widely used solution concept in economics and applied game theory. Given an extensive form, a *pure strategy* of player $i \in N$ is a function that associates with every information set of player i a choice at that information set, that is, a function $s_i : D_i \to A$ such that (1) $s_i(h) \in A(h)$ and (2) if $h' \in I_i(h)$ then $s_i(h') = s_i(h)$. For example, one of the pure strategies of Player 4 in the extensive form illustrated in Figure 1 is $s_4(acf) = s_4(ade) = g$, $s_4(adf) = s_4(b) = m$. A *behavior strategy* of player i is a collection of probability distributions, one for each information set, over the actions available at that information set; that is, a function $\sigma_i : D_i \to \Delta(A)$ (where $\Delta(A)$ denotes the set of probability distributions over A) such that (1) $\sigma_i(h)$ is a probability distribution over $A(h)$ and (2) if $h' \in I_i(h)$ then $\sigma_i(h') = \sigma_i(h)$. We denote by $\sigma_i(h)(a)$ the probability assigned to $a \in A(h)$ by $\sigma_i(h)$. Note that a pure strategy is a special case of a behavior strategy where each probability distribution is degenerate. A behavior strategy σ_i of player i is *completely mixed* if, for every $h \in D_i$ and for every $a \in A(h)$, $\sigma_i(h)(a) > 0$. A behavior strategy *profile* is an n-tuple $\sigma = (\sigma_1, ..., \sigma_n)$ where, for every $i \in N$, σ_i is a behavior strategy of player i.

A *system of beliefs*, is a collection of probability distributions, one for every information set, over the elements of that information set, that is, a function $\mu : D \to \Delta(H)$ such that (1) if $h \in D_i$ then $\mu(h)$ is a probability distribution over $I_i(h)$ and (2) if $h \in D_i$ and $h' \in I_i(h)$ then $\mu(h) = \mu(h')$. Note that a completely mixed behavior strategy profile yields, using Bayes' rule, a unique system of beliefs.

An *assessment* is a pair (σ, μ) where σ is a behavior strategy profile and μ is a system of beliefs. An assessment (σ, μ) is *consistent* if there is an infinite sequence $\langle \sigma^1, ..., \sigma^m, ... \rangle$ of completely mixed strategy profiles such that, letting μ^m be the unique system of beliefs associated - using Bayes' rule - to σ^m, $\lim_{m \to \infty}(\sigma^m, \mu^m) = (\sigma, \mu)$.

Kreps and Wilson ([15]) proposed the notion of consistent assessment as an attempt to capture the concept of minimal belief revision. As noted in the Introduction, a number of authors have tried to shed light on the notion of consistent assessment by relating it to more intuitive notions. The following proposition, which is proved in the Appendix, provides a characterization of consistent assessments in terms of the AGM theory of belief revision [1], through the notion of AGM-consistency of choice frames.[13]

Given an extensive form, we say that a profile $\{\langle \Omega, \mathcal{E}_i, f_i \rangle\}_{i \in N}$ of AGM-consistent choice frames (where $\Omega = H$ and $\mathcal{E}_i$ is given by (3)) satisfies properties $P1$-$P4$ if the collection of total pre-orders $\{\precsim_i\}_{i \in N}$ that rationalize $\{\langle \Omega, \mathcal{E}_i, f_i \rangle\}_{i \in N}$ (whose existence is guaranteed by Proposition 4) satisfies properties $P1$-$P4$ (that

[13] Sequential rationality is discussed in Section 5.

is, there exists a common total pre-order $\precsim$ on H that rationalizes those choice frames and satisfies properties $P2 - P4$).

Proposition 10 *Fix an extensive form where no player moves more than once along any history. Then*

(a) If the players' initial beliefs and disposition to revise those beliefs are represented by a profile of AGM-consistent choice frames that satisfies properties P1-P4 then there exists a consistent assessment (σ, μ) *such that (letting* $\precsim$ *be a total pre-order on* H *that rationalizes those choice frames), for all* $i \in N$*,* $h \in D_i$ *and* $a \in A(h)$*, (i)* $\sigma_i(h)(a) > 0$ *if and only if* $h \sim ha$ *and (ii)* $\mu(h) > 0$ *if and only if* $h \precsim h'$ *for every* $h' \in \vec{I}_i(h)$*;*[14]

(b) if (σ, μ) *is a consistent assessment then there exists a profile of AGM-consistent choice frames that satisfies properties P1-P4 such that (letting* $\precsim$ *be a total pre-order on* H *that rationalizes those choice frames), for every* $i \in N$*,* $h \in D_i$ *and* $a \in A(h)$*, (i)* $h \sim ha$ *if and only if* $\sigma_i(h)(a) > 0$ *and (ii) for every* $h' \in \vec{I}_i(h)$*,* $h \precsim h'$ *if and only if* $\mu(h) > 0$.

4 Iterated belief revision

In an arbitrary extensive form there may be players who move more than once along some histories. If i is such a player, then the set $\mathcal{E}_i$ defined above (see (3)) will contain two sets E and F such that $F \subseteq E$ and there is a history along which player i receives first information E and then, at a later moment, information F. Indeed, it is a consequence of the property of perfect recall that, for every player $i \in N$ and for every $h, h' \in D_i$, if h is a prefix of h' then $\vec{I}_i(h') \subseteq \vec{I}_i(h)$. Because of the possibility of sequential informational inputs, we are outside the scope of one-shot belief revision and it is no longer sufficient to appeal to AGM consistency in order to guarantee the existence of a total pre-order that rationalizes the beliefs of a player. In this section we argue that - within the context of extensive forms - rationalizability by a total pre-order is a natural property even in situations involving sequential informational inputs.

Iterated belief revision, that is, the evolution of beliefs over time in response to sequences of informational inputs, has been investigated extensively in the literature (see, for example, [8], [9], [10], [13], [18], [26]). AGM belief revision functions map a belief set $K \subseteq \Phi$ and an informational input $\phi \in \Phi$ into a new belief set $B_K(\phi) \subseteq \Phi$. While such functions are sufficient for modeling one-stage belief revision, it has been argued (see, for example, [18] and [24]) that, in the context of iterated belief revision, one should model the evolutions of *belief states* or *epistemic states*, rather than simply of belief sets. A belief state is a pair (K, B_K), consisting of a belief set together with a disposition

[14] Recall - see (2) - that $\vec{I}_i(h)$ is the information set that contains h together with the continuation histories and that if $\langle \Omega, \mathcal{E}_i, f_i \rangle$ is the choice frame of player i then $\Omega = H$ and $\mathcal{E}_i = \{H\} \cup \{\vec{I}_i(h) : h \in D_i\}$ (see (3)).

to revise one's beliefs, as captured by the belief revision function B_K. Thus iterated belief revision should be construed as a function that maps a belief state (K, B_K) and an informational input ϕ into a new belief state (K', B'_K). In particular, one should allow for the possibility that, after learning that ϕ, one changes the disposition to revise one's beliefs; in other words, in general it is possible that $B'_K \neq B_K$.

Fix the set of states Ω. A choice frame $\langle \Omega, \mathcal{E}, f \rangle$ incorporates both the initial beliefs and the disposition to revise those beliefs; thus it can be regarded as representing a belief state. In accordance with the view expressed above, iterated belief revision can be captured semantically by a pair $\langle \mathbb{C}, \mathbb{B} \rangle$ where $\mathbb{C}$ is a set of choice frames and $\mathbb{B}$ is a function that maps a pair consisting of a choice frame $\langle \Omega, \mathcal{E}, f \rangle \in \mathbb{C}$ and an information input $E \in \mathcal{E}$ into a new choice frame in $\mathbb{C}$. We call the pair $\langle \mathbb{C}, \mathbb{B} \rangle$ an *iterated choice structure*. We require that, for all $\langle \Omega, \mathcal{E}, f \rangle \in \mathbb{C}$, $\mathbb{B}(\langle \Omega, \mathcal{E}, f \rangle, \Omega) = \langle \Omega, \mathcal{E}, f \rangle$, that is, the trivial informational input Ω does not change the belief state. Furthermore, if $\langle \Omega, \mathcal{E}', f' \rangle = \mathbb{B}(\langle \Omega, \mathcal{E}, f \rangle, E)$, consistency requires that

$$f'(\Omega) = f(E). \tag{5}$$

In fact, if the agent's initial beliefs are $f(\Omega)$ and she learns that E, then - according to her initial belief state - her revised beliefs are $f(E)$ and these constitute the initial beliefs in the new belief state $\langle \Omega, \mathcal{E}', f' \rangle$, which are given by $f'(\Omega)$.

Definition 11 *An iterated choice structure $\langle \mathbb{C}, \mathbb{B} \rangle$ is* AGM-consistent *if every choice frame $\langle \Omega, \mathcal{E}, f \rangle \in \mathbb{C}$ is AGM-consistent (see Definition 2).*

From now on we shall restrict attention to AGM-consistent iterated choice structures.

Iterated choice structures can be represented by means of rooted trees. Let t_0 be the root of the tree. Associate with it the initial belief state $\langle \Omega, \mathcal{E}, f \rangle$. For every $E \in \mathcal{E}$ draw an arrow out of t_0 leading to a new node t and associate with t the choice frame $\langle \Omega, \mathcal{E}', f' \rangle = \mathbb{B}(\langle \Omega, \mathcal{E}, f \rangle, E)$ and proceed similarly for every $E' \in \mathcal{E}'$. Figure 4 provides an illustration where $\Omega = \{\alpha, \beta, \gamma, \delta\}$ and the initial belief state $\langle \Omega, \mathcal{E}, f \rangle$ is given by $\mathcal{E} = \{\Omega, \{\beta, \gamma, \delta\}, \{\gamma, \delta\}\}$, $f(\Omega) = \{\alpha\}$, $f(\{\beta, \gamma, \delta\}) = \{\beta\}$ and $f(\{\gamma, \delta\}) = \{\gamma\}$. We represent the elements of $\mathcal{E}$ as rectangles and the function f as ovals inside the rectangles. Letting $\langle \Omega, \mathcal{E}', f' \rangle = \mathbb{B}(\langle \Omega, \mathcal{E}, f \rangle, \{\beta, \gamma, \delta\})$ we have that $\mathcal{E}' = \{\Omega, \{\alpha, \beta, \delta\}, \{\gamma, \delta\}\}$ and $f'(\Omega) = \{\beta\}$, $f'(\{\alpha, \beta, \delta\}) = \{\beta\}$ and $f'(\{\gamma, \delta\}) = \{\delta\}$. Note that, as required by (5), $f'(\Omega) = f(\{\beta, \gamma, \delta\}) = \{\beta\}$. Note also that the two choice frames associated with nodes t_0 and t_3 in Figure 4 are AGM-consistent, since they are rationalizable (although not by the same total pre-order).

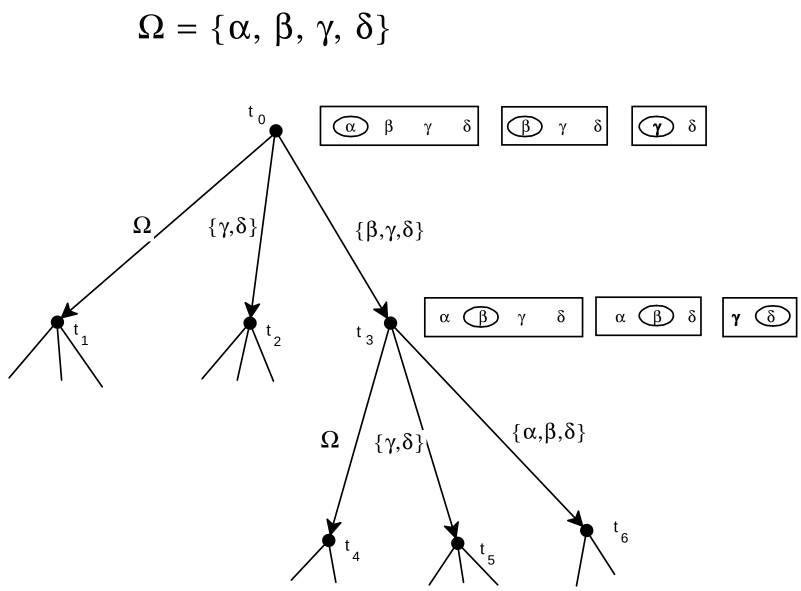


Figure 4

Representing an iterated choice structure by means of a tree.

If $\langle \Omega, \mathcal{E}', f' \rangle = \mathbb{B}\left(\langle \Omega, \mathcal{E}, f \rangle, E\right)$ (with $E \in \mathcal{E}$) with abuse of notation we shall denote f' by $\mathbb{B}_{f,E}$ (thus $\mathbb{B}_{f,E} : \mathcal{E}' \to 2^{\Omega}$).

The proof of the following lemma is given in the Appendix.

Lemma 12 *Let $\langle \Omega, \mathcal{E}, f \rangle$ be a choice frame and $E, F \in \mathcal{E}$ be such that $F \subseteq E$ and $f(E) \cap F \neq \varnothing$. Let $\langle \Omega, \mathcal{E}', f' \rangle = \mathbb{B}\left(\langle \Omega, \mathcal{E}, f \rangle, E\right)$ and suppose that $F \in \mathcal{E}'$. Then if both $\langle \Omega, \mathcal{E}, f \rangle$ and $\langle \Omega, \mathcal{E}', f' \rangle$ are AGM-consistent, $f'(F) = f(F)$. More succinctly:*

$$\text{if } F \subseteq E \text{ and } f(E) \cap F \neq \varnothing \text{ then } \mathbb{B}_{f,E}(F) = f(F). \tag{6}$$

Lemma 12 says that the following is an implication of AGM-consistency: when $F \subseteq E$, if the agent is first informed that E and, in her revised beliefs, she does not rule out F, then - if she is next informed that F - the propositions that she believes are the same as the ones that she would have believed had she been informed that F to start with.[15] Lemma 12 is trivially satisfied if the condition $f(E) \cap F \neq \varnothing$ does not hold. For example, in the structure illustrated in Figure 4, taking $E = \{\beta, \gamma, \delta\}$ and $F = \{\gamma, \delta\}$ we have that, since

[15] A stronger result than (6) can be proved, namely that if $E, F \in \mathcal{E}$ are such that $(E \cap F) \in \mathcal{E}$ and $f(E) \cap F \neq \varnothing$ then $\mathbb{B}_{f,E}(F) = f(E \cap F)$. However, for our purposes it is sufficient to focus on the weaker version (6).

$f(E) = \{\beta\}$, $f(E) \cap F = \varnothing$ and thus (6) is vacuously satisfied; yet the agent responds differently to the *initial* information that F ($f(F) = \{\gamma\}$) relative to the situation where the information that F is preceded by the less precise information that E ($f'(F) = \{\delta\}$). It seems that an introspective agent ought to be reluctant to manifest such a capricious disposition to change his beliefs. Thus we shall assume the following strengthening of (6) (obtained by dropping the clause $f(E) \cap F \neq \varnothing$):

$$\text{if } F \subseteq E \text{ then } \mathbb{B}_{f,E}(F) = f(F). \tag{7}$$

According to (7) the agent will hold the same beliefs no matter whether he is informed that F right away or whether he is first informed that E and then that F, whenever $F \subseteq E$. Note that this principle is implied by the best-known theories of iterated belief revision (see, for example, [8], [9], [10], [13], [18], [26]).

We shall now restrict attention to iterated choice structures that satisfy the following property, which we call *information refinement*:

$$\text{if } \langle \Omega, \mathcal{E}', f' \rangle = \mathbb{B}\left(\langle \Omega, \mathcal{E}, f \rangle, E\right) \text{ then, for every } S \in \mathcal{E}', \ \ S \subseteq E. \tag{8}$$

Information refinement says that if the agent is first informed that E and, later on, is informed that F, then $F \subseteq E$. Hence the agent never receives information that contradicts earlier information. Note, however, that (8) does not rule out the possibility that every new piece of information contradicts the agent's previous beliefs, as illustrated in Figure 5.

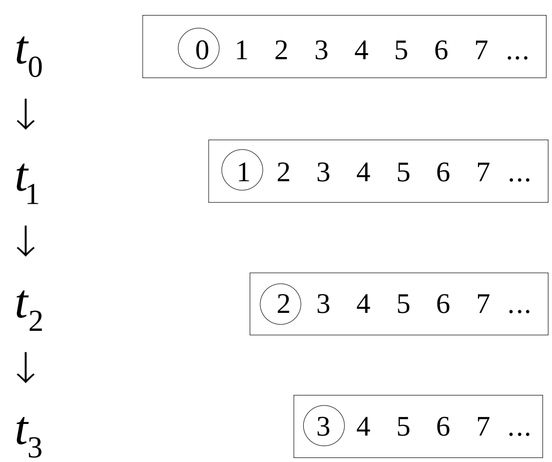


Figure 5
Information refinement does not rule out repeated surprises.

The property of information refinement is satisfied in extensive forms. In fact, as noted in the previous section, it follows from the property of perfect recall that, for every player $i \in N$ and for every $h, h' \in D_i$, if h is a prefix of h' then $\vec{I}_i(h') \subseteq \vec{I}_i(h)$.

Applying the iterated belief revision principle (7) to iterated choice structures that satisfy information refinement we obtain the following property. Let $\precsim$ be the total pre-order on Ω that rationalizes the choice frame $\langle \Omega, \mathcal{E}, f \rangle$ (see Proposition 4), that is, for every $S \in \mathcal{E}$, $f(S) = \{\omega \in S : \omega \precsim x, \forall x \in S\}$. Let $E \in \mathcal{E}$ and $\langle \Omega, \mathcal{E}', f' \rangle = \mathbb{B}\left(\langle \Omega, \mathcal{E}, f \rangle, E\right)$. Then

$$\forall T \in \mathcal{E} \cap \mathcal{E}', \quad f'(T) = f(T) = \{\omega \in T : \omega \precsim x, \forall x \in T\} \tag{9}$$

that is, the same total pre-order $\precsim$ rationalizes both $f(T)$ and $f'(T)$. The above considerations motivate the following definition.

Definition 13 *An iterated choice structure with information refinement* $\langle \mathbb{C}, \mathbb{B} \rangle$ *is* rationalizable *if there exists a total pre-order* $\precsim$ *of* Ω *such that, for every choice frame* $\langle \Omega, \mathcal{E}, f \rangle \in \mathbb{C}$ *and for every* $E \in \mathcal{E}$, $f(E) = \{\omega \in E : \omega \precsim x, \forall x \in E\}$.

Thus a rationalizable iterated choice structure with information refinement $\langle \mathbb{C}, \mathbb{B} \rangle$ *is equivalent to a one-stage choice frame* $\langle \Omega, \mathcal{E}, f \rangle$ where $\mathcal{E}$ is the union of the domains of the choice frames that belong to $\mathbb{C}$. For example, let $\langle \mathbb{C}, \mathbb{B} \rangle$ be the following iterated choice structure: $\mathbb{C} = \{\langle \Omega, \mathcal{E}_1, f_1 \rangle, \langle \Omega, \mathcal{E}_2, f_2 \rangle, \langle \Omega, \mathcal{E}_3, f_3 \rangle\}$ with $\mathcal{E}_1 = \{\Omega, E, F\}$, $\mathcal{E}_2 = \{\Omega, E_1, E_2\}$, $\mathcal{E}_3 = \{\Omega, F_1, F_2\}$ and, for $k = 1, 2$, $E_k \subseteq E$ and $F_k \subseteq F$; $\mathbb{B}\left(\langle \Omega, \mathcal{E}_1, f_1 \rangle, E\right) = \langle \Omega, \mathcal{E}_2, f_2 \rangle$ and $\mathbb{B}\left(\langle \Omega, \mathcal{E}_1, f_1 \rangle, F\right) = \langle \Omega, \mathcal{E}_3, f_3 \rangle$. Then $\langle \mathbb{C}, \mathbb{B} \rangle$ is equivalent to the choice frame $\langle \Omega, \mathcal{E}, f \rangle$ where $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3$ and f is defined by $f(S) = \{\omega \in S : \omega \precsim x, \forall x \in S\}$ (for every $S \in \mathcal{E}$),where $\precsim$ is a total pre-order that rationalizes $\langle \mathbb{C}, \mathbb{B} \rangle$.

The characterization of consistent assessments given in Proposition 10 requires, for every player, the existence of a total pre-order on the set of histories that rationalizes the beliefs of that player. For extensive forms where no player moves more than once along any play, the existence of a total pre-order is guaranteed by AGM-consistency of the choice frame representing a player's initial beliefs and disposition to revise those beliefs. For arbitrary extensive forms one needs to add to the hypothesis of AGM-consistency the natural iterated belief revision principle discussed above, which requires an agent's beliefs to be the same in the case where he learns that F as in the case where he first learns that E and then that F, whenever $F \subseteq E$. Such a principle is captured by the requirement that the iterated choice structure be rationalizable (Definition 13). We argued above that, *in the context of information refinement* (which is necessarily satisfied in extensive forms with perfect recall), this principle of iterated belief revision seems very plausible and indeed it is implied by the best-known theories of iterated belief revision (see [8], [9], [10], [13], [18], [26]). Thus Proposition 10 can be generalized as follows, without the need to make any adjustments to the proof.

Proposition 14 *Fix an arbitrary extensive form. Then*

(a) If the players' beliefs and belief revision policies are represented by a profile of (possibly iterated) rationalizable choice frames that satisfies properties $P1$-$P4$ then there exists a consistent assessment (σ, μ) such that (letting $\precsim$ be a total pre-order on H that rationalizes those choice frames), for all $i \in N$, $h \in D_i$ and $a \in A(h)$, (i) $\sigma_i(h)(a) > 0$ if and only if $h \sim ha$ and (ii) $\mu(h) > 0$ if and only if $h \precsim h'$ for every $h' \in \vec{I}_i(h)$;

(b) if (σ, μ) is a consistent assessment then there exists a profile of (possibly iterated) rationalizable choice frames that satisfies properties $P1$-$P4$ such that (letting $\precsim$ be a total pre-order on H that rationalizes those choice frames), for every $i \in N$, $h \in D_i$ and $a \in A(h)$, (i) $h \sim ha$ if and only if $\sigma_i(h)(a) > 0$ and (ii) for every $h' \in \vec{I}_i(h)$, $h \precsim h'$ if and only if $\mu(h) > 0$.

5 Sequential rationality, pure sequential equilibria and backward induction

A sequential equilibrium is an assessment (σ, μ) which is consistent and sequentially rational. Sequential rationality requires that - at each information set - the strategy of each player be optimal starting from there according to the player's beliefs over the nodes in the information set (as captured by the relevant part of μ) and the strategies of everyone else. Conceptually, little is gained by expressing sequential rationality in terms of the total pre-order underlying the consistent assessment (σ, μ). However, there is one case where sequential rationality can be expressed very simply and that is the case where the restriction of the total pre-order $\precsim$ to the set Z of terminal histories is antisymmetric:

$$\text{if } z, z' \in Z, \quad \text{and} \quad z \sim z' \quad \text{then} \quad z = z'. \qquad (P5)$$

The following lemma is proved in the Appendix.

Lemma 15 *Let $\precsim$ be a total pre-order on H that satisfies Properties $P2$, $P3$ and $P5$. Then, for every history $h \in H$, there is a unique terminal history z such that $h \sim z$. Call this terminal history $z(h)$ (if $h \in Z$ then $z(h) = h$). Furthermore, for every decision history $h \in D$, (a) there is a unique action $a \in A(h)$ such that $h \sim a$ and (b) for all $h' \in \vec{I}_i(h)$, if $h \sim h'$ then h is a prefix of h'.*

We will show that, under the hypotheses of Lemma 15, sequential rationality can be expressed as follows (recall - see Footnote 4 - that, for every player $i \in N$, $U_i : Z \to \mathbb{R}$ is i's payoff function):

$$\forall i \in N, \forall h \in D_i, \forall a \in A(h), \quad U_i(z(h)) \geq U_i(z(ha)). \qquad (P6)$$

Call a sequential equilibrium (σ, μ) *pure* if the strategy σ_i of each player $i \in N$ is a pure strategy and μ consists of degenerate probability distributions (that is, if $h \in D_i$ and $h' \in I_i(h)$ then either $\mu(h)(h') = 0$ or $\mu(h)(h') = 1$). The following proposition is proved in the Appendix.

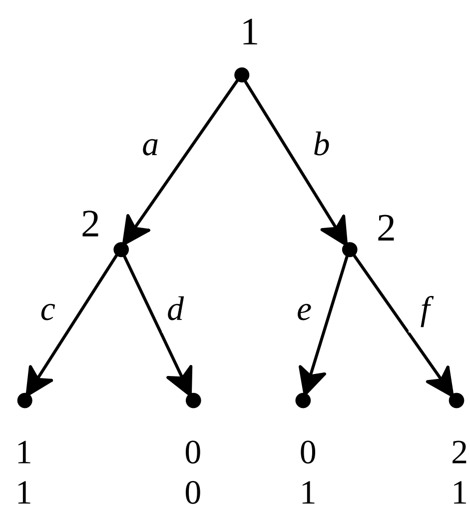
Proposition 16 *Fix an extensive-form game without chance moves. Then,*

(a) If the players' beliefs and belief revision policies are represented by a profile of (possibly iterated) rationalizable choice frames that satisfies properties $P1$*-*$P6$ *then the assessment* (σ, μ) *given by (letting* $\precsim$ *be a total pre-order on* H *that rationalizes those choice frames), for all* $i \in N$*,* $h \in D_i$ *and* $a \in A(h)$*, (i)* $\sigma_i(h)(a) > 0$ *if and only if* $h \sim ha$ *and (ii)* $\mu(h) > 0$ *if and only if* $h \precsim h'$ *for every* $h' \in \vec{I}_i(h)$*, is a pure sequential equilibrium.*

(b) if (σ, μ) *is a pure sequential equilibrium then there exists a profile of (possibly iterated) rationalizable choice frames that satisfies properties* $P1$*-*$P4$ *and* $P6$ *such that (letting* $\precsim$ *be a total pre-order on* H *that rationalizes those choice frames), for every* $i \in N$*,* $h \in D_i$ *and* $a \in A(h)$*, (i)* $h \sim ha$ *if and only if* $\sigma_i(h)(a) = 1$ *and (ii) for every* $h' \in \vec{I}_i(h)$*,* $h \precsim h'$ *if and only if* $\mu(h) = 1$*.*

Note that, since we ruled out chance moves, Proposition 16 does not require the payoff function U_i of player i to satisfy the von Neumann-Morgenstern axioms of expected utility; indeed, it could be an ordinal payoff function (that is, a numerical representation of a total pre-order over Z expressing player i's preferences over the elements of Z).

An extensive form has *perfect information* if every information set is a singleton. Figure 6 shows a perfect information game. Associated with each terminal history are two numbers: the top one is Player 1's utility (or payoff) and the bottom one is Player 2's utility.



An extensive-form game with perfect information
Figure 6

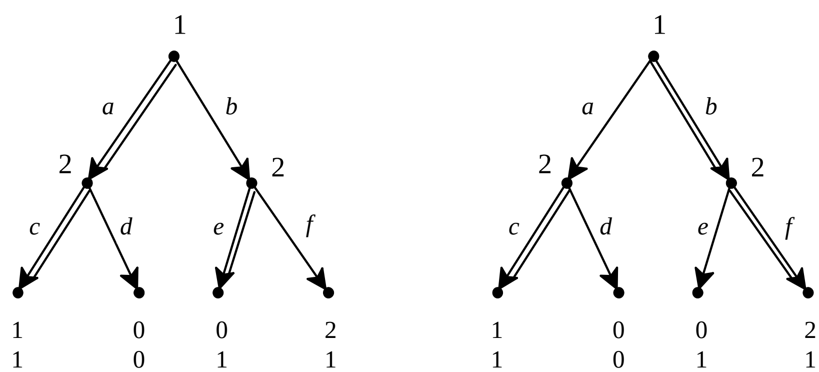
We shall restrict attention to perfect-information games without chance moves. The solution concept that is most commonly used for perfect-information games is that of *backward induction*, obtained using the following algorithm, which yields a function $\lambda : A \to \{0,1\}$ (recall that A is the set of actions). Start from a decision history $h \in D$ whose immediate successors (that is, histories of the form ha with $a \in A(h)$) are all terminal histories.[16] Let $a \in A(h)$ be an action at h that maximizes the payoff of the player assigned to h (that is, if $P(h) = i$, then $U_i(ha) \geq U_i(ha')$ for all $a' \in A(h)$). Set $\lambda(a) = 0$ and $\lambda(a') = 1$ for all $a' \in A(h)\backslash\{a\}$. Now let $h \in D$ be a decision history such that every immediate successor is either a terminal history or a history h' such that λ has been defined on $A(h')$. Select an action a at h that maximizes the payoff of the player assigned to h, call him player i, under the convention that the payoff of action $a' \in A(h)$ is $U_i(ha')$ if ha' is a terminal history or it is $U_i(ha'a_1...a_m)$ where $ha'a_1...a_m$ is the terminal history reached from ha' by following actions a_k with $\lambda(a_k) = 0$ $(k = 1,...,m)$. As before, set $\lambda(a) = 0$ and $\lambda(a') = 1$ for all $a' \in A(h)\backslash\{a\}$. Repeat the procedure until the function λ has been defined on the entire set A. We call $\{a \in A : \lambda(a) = 0\}$ a backward-induction solution.

When applied to the game illustrated in Figure 6, the backward induction algorithm yields two solutions, shown in Figure 7, where the actions assigned the value 0 are shown as double edges. Thus the function λ corresponding to the solution on the left is a follows:

action	a	b	c	d	e	f
λ	0	1	0	1	0	1

while the function λ corresponding to the solution on the right is given by:

action	a	b	c	d	e	f
λ	1	0	0	1	1	0



The two backward-induction solutions of the game of Figure 6
Figure 7

[16] Such a history h exists because of finiteness of the set of actions A and the assumption that no action is available at two different information sets (see Remark 6).

Note that the backward induction solution yields an actual play (that is, a terminal history: ac in the first solution shown in Figure 7 and bd in the second solution) as well as a strategy profile ((a, ce) in the first solution and (b, cf) in the second solution in Figure 7.

In perfect-information games Property $P4$ is trivially satisfied, since $h' \in I(h)$ implies that $h' = h$. We now show that, for every perfect-information game, there is a one-to one correspondence between the set of backward-induction solutions and the set of total pre-orders on H that satisfy properties $P2$, $P3$ and the following property, which is a generalization of $P6$. First of all some notation. Fix a total pre-order $\precsim$ on H. For every decision history $h \in D$, let $A_0(h) = \{a \in A(h) : h \sim ha\}$ and let $Z(h) = \{z \in Z : z \sim ha$ for some $a \in A_0(h)\}$. If $\precsim$ satisfies Properties $P2$ and $P3$ then $A_0(h) \neq \varnothing$ and $Z(h) \neq \varnothing$. We can now introduce the generalization of $P6$:

$$\begin{gathered}\forall i \in N,\ \forall h \in D_i,\ \forall z \in Z(h),\ \forall a \in A(h),\ \forall z' \in Z(ha), \\ U_i(z) \geq U_i(z').\end{gathered} \tag{$P7$}$$

Property $P7$ says that if h is a decision history of player i then the utility of any terminal history reached from h by following only plausibility preserving actions is not less than the utility of a terminal node reached by taking an arbitrary action a at h and then continuing from ha by following only plausibility preserving actions. The following proposition is proved in the Appendix.

Proposition 17 *Fix an arbitrary finite perfect-information game. There is a one-to-one correspondence between the set of backward-induction solutions and the set of total pre-orders on H that satisfy properties $P2$, $P3$ and $P7$.*

Proposition 17 thus provides a characterization of backward induction in terms of beliefs and belief revision policies that are represented by profiles $\{\langle \Omega, \mathcal{E}_i, f_i \rangle\}_{i \in N}$ of (possibly iterated) rationalizable choice frames (where $\Omega = H$ and $\mathcal{E}_i$ is given by (3)) that satisfy Properties $P1$-$P3$ and $P7$.

6 Conclusion

We applied the notion of AGM-consistent choice frame developed in [7] to extensive-form games and provided a characterization of the notion of consistent assessment, which is the central part of the notion of sequential equilibrium introduced by Kreps and Wilson [15]. AGM-consistency is adequate if the extensive form is such that no player moves more than once along any terminal history, otherwise the issue of iterated belief revision arises. In extensive forms with perfect recall the (suitably encoded) information that a player receives at a later moment is always a refinement of earlier information. This property of information refinement together with AGM-consistency and a simple postulate for iterated belief revision ensures that the beliefs of each player are rationalizable by a total pre-order on the set of histories. When the same total pre-order rationalizes the beliefs of all the players, four properties of it yield a characterization of consistent assessments.

A characterization of pure sequential equiilibria and of backward induction in perfect-information games was also provided.

In future work we plan to extend the analysis of this paper by exploring different solution concepts for extensive-form games.

A Appendix

In order to prove Proposition 10 we need some preliminary definitions and lemmas.

Remark 18 *Recall our assumption that* $A = \bigcup_{h \in D} A(h)$. *Thus, for every* $a \in A$ *there is an* $h \in D$ *such that* $a \in A(h)$. *Recall also the assumption that if* $h, h' \in H$ *are such that* $A(h) \cap A(h') \neq \varnothing$ *then* $h' \in I(h)$ *(and thus* $A(h) = A(h')$*).*

Notation 19 *For every* $h \in H$ *and* $a \in A$ *we write* $a \in h$ *if there exists an* $h' \in H$ *such that* $h'a$ *is a prefix of* h *(thus* $a \in A(h')$*; recall that the null action* $\emptyset$ *is a prefix of every history and thus we set* $\emptyset \in h$ *for every* $h \in H$*).*

Remark 20 *If* $\precsim$ *satisfies Property* $P2$ *then* $\emptyset \precsim h$ *for every* $h \in H$.

Definition 21 *An* A-weighting *is a function* $\lambda : A \to \mathbb{N}$ *such that (1)* $\lambda(\emptyset) = 0$ *and (2) for every* $h \in D$, *there is at least one* $a \in A(h)$ *with* $\lambda(a) = 0$. *Given a* A*-weighting* λ, *define* $\Lambda : H \to \mathbb{N}$ *as follows:* $\Lambda(h) = \sum_{a \in h} \lambda(a)$.

Lemma 22 *Let* $\precsim$ *be a total pre-order that satisfies properties* $P2$*-*$P4$. *Let* $F : H \to \mathbb{N}$ *be an action-based representation of* $\precsim$ *(it exists by* $P4$*). Define* $\lambda_{\precsim} : A \to \mathbb{N}$ *as follows:* $\lambda_{\precsim}(\emptyset) = 0$ *and, for* $a \neq \emptyset$, $\lambda_{\precsim}(a) = F(ha) - F(h)$ *for some* h *such that* $a \in A(h)$. *Then*

(i) $\lambda_{\precsim}$ *is an* A*-weighting;*

(ii) if $\Lambda_{\precsim} : H \to \mathbb{N}$ *is the associated function (given by* $\Lambda_{\precsim}(h) = \sum_{a \in h} \lambda_{\precsim}(a)$*) then, for every* $h \in H$, $\Lambda_{\precsim}(h) = F(h)$ *(so that, for every* $h, h' \in H$, $\Lambda_{\precsim}(h) \leq \Lambda_{\precsim}(h')$ *if and only if* $h \precsim h'$*).*

Proof. (i) First of all, $\lambda_{\precsim}$ is well defined since if h and h' are such that $a \in A(h) \cap A(h')$ then, by Remark 18, $h' \in I(h)$[17] and thus, by definition of action basedness, $F(ha) - F(h) = F(h'a) - F(h')$. By $P2$, $F(ha) \geq F(h)$ and thus, since F is integer-valued, $\lambda_{\precsim}(a) \in \mathbb{N}$. By $P3$, for every $h \in D$ there is an $a \in A(h)$ such that $ha \sim h$ and thus $F(ha) = F(h)$ so that $\lambda_{\precsim}(a) = 0$.

[17] Without this convention the domain of the weigthing function λ would have to be chosen as the set of action-history pairs (a, h) with $a \in A(h)$. This would make the notation more complicated, but the proofs would go through.

(ii) Fix an arbitrary $h = \langle \emptyset, a_1, ..., a_m \rangle \in H$. Then $\Lambda_{\precsim}(h) = \lambda_{\precsim}(\emptyset) + \underbrace{[F(\emptyset a_1) - F(\emptyset)]}_{\lambda_{\precsim}(a_1)} + \underbrace{[F(\emptyset a_1 a_2) - F(\emptyset a_1)]}_{\lambda_{\precsim}(a_2)} + ... + \underbrace{[F(h) - F(\emptyset a_1 a_2 ... a_{m-1})]}_{\lambda_{\precsim}(a_m)} = \lambda_{\precsim}(\emptyset) - F(\emptyset) + F(h) = F(h)$ (since $\lambda_{\precsim}(\emptyset) = F(\emptyset) = 0$). ■

Lemma 23 *Let $\precsim$ be a total pre-order that satisfies properties $P2$-$P4$. Then, for every $i \in N$ and $\hat{h} \in D_i$, $\varnothing \neq \{h : h \precsim h', \forall h' \in I_i(\hat{h})\} \subseteq \{h : h \precsim h', \forall h' \in \vec{I}_i(\hat{h})\}$ (where $\vec{I}_i(\hat{h})$ is defined in (2)).*

Proof. Fix arbitrary $i \in N$ and $\hat{h} \in D_i$. Then, since $\hat{h} \in I_i(\hat{h})$, $I_i(\hat{h}) \neq \varnothing$. It follows from this and the fact that $\precsim$ is a total pre-order that $\varnothing \neq \{h : h \precsim h', \forall h' \in I_i(\hat{h})\}$. Now let $E = \{h : h \precsim h', \forall h' \in I_i(\hat{h})\}$ and $F = \{h : h \precsim h', \forall h' \in \vec{I}_i(\hat{h})\}$. Let $h \in E$ and fix an arbitrary $h_1 \in \vec{I}_i(\hat{h})$. We want to show that $h \precsim h_1$, so that $h \in F$. By definition of $\vec{I}_i(\hat{h})$, there exists an $h_0 \in I_i(\hat{h})$ such that h_0 is a prefix of h_1. By $P2$ (see Remark 7), $h_0 \precsim h_1$. By hypothesis, since $h \in E$ and $h_0 \in I_i(\hat{h})$, $h \precsim h_0$. Thus, by transitivity of $\precsim$, $h \precsim h_1$. ■

Lemma 24 *Fix an extensive form and let $\lambda : A \to \mathbb{N}$ be an A-weighting with corresponding $\Lambda : H \to \mathbb{N}$ (given by $\Lambda(h) = \sum_{a \in h} \lambda(a)$). Define the following total pre-order $\precsim_\lambda$ on H: $h \precsim_\lambda h'$ if and only if $\Lambda(h) \leq \Lambda(h')$. Then, (i) $\precsim_\lambda$ satisfies properties $P2$-$P4$. Furthermore, for every $i \in N$, $h \in D_i$ and $a \in A(h)$ (ii) $h \sim_\lambda ha$ if and only if $\lambda(a) = 0$ and (iii) if h is such that $h \precsim_\lambda h'$, for every $h' \in I_i(h)$, then $h \precsim_\lambda h'$, for every $h' \in \vec{I}_i(h)$ (where $\vec{I}_i(h)$ is defined in (2)).*

Proof. (i) Fix $i \in N$, $h \in D_i$ and $a \in A(h)$. Property ($P2$) is satisfied because $\Lambda(h) \leq \Lambda(h) + \lambda(a) = \Lambda(ha)$ and thus, by definition of $\precsim_\lambda$, $h \precsim_\lambda ha$.

By definition of A-weighting, for every $h \in D$ there exists an $a \in A(h)$ such that $\lambda(a) = 0$; thus $\Lambda(ha) = \Lambda(h) + \lambda(a) = \Lambda(h)$ and therefore $ha \precsim_\lambda h$; furthermore, for every $a \in A(h)$, by definition of A-weighting, $\lambda(a) \geq 0$ and thus if $ha \precsim_\lambda h$ then $\Lambda(ha) = \Lambda(h) + \lambda(a) \leq \Lambda(h)$ which implies that $\lambda(a) = 0$ so that, for every $h' \in I(h)$, $\Lambda(h') + \lambda(a) = \Lambda(h')$ and hence $h'a \precsim_\lambda h'$. Thus Property $P3$ is satisfied.

By construction, Λ is an integer-valued representation of $\precsim_\lambda$. Fix arbitrary $h, h' \in D$, with $h' \in I(h)$, and $a \in A(h)$. Then $\Lambda(ha) - \Lambda(h) = \Lambda(h) + \lambda(a) - \Lambda(h) = \lambda(a)$. Similarly, $\Lambda(h'a) - \Lambda(h') = \lambda(a)$. Thus Property $P4$ is satisfied.

(ii) If $\lambda(a) = 0$ then $\Lambda(ha) = \Lambda(h) + \lambda(a) = \Lambda(h)$ and thus $h \sim ha$. Conversely, if $h \sim ha$ then $\Lambda(ha) = \Lambda(h) + \lambda(a)$ and thus $\lambda(a) = 0$.

(iii) Let $i \in N$ and $h \in H$ be such that, for every $h' \in I_i(h)$, $h \precsim h'$ (that is, $\Lambda(h) \leq \Lambda(h')$). We want to show that $\Lambda(h) \leq \Lambda(h')$ for every $h' \in \vec{I}_i(h)$. Fix an arbitrary $\hat{h} \in \vec{I}_i(h)$. By definition of $\vec{I}_i(h)$ (see (2)) there exists an $h' \in I_i(h)$ which is a prefix of $\hat{h}$. Thus $\Lambda(h') \leq \Lambda(\hat{h})$. By hypothesis, $\Lambda(h) \leq \Lambda(h')$; hence $\Lambda(h) \leq \Lambda(\hat{h})$ and thus $h \precsim_\lambda \hat{h}$. ■

The following result is proved in Kreps and Wilson ([15], Lemma A.1, p. 887; we have re-written the result in terms of the notation used in this paper and slightly reworded it).

Lemma 25 *Fix an arbitrary extensive form (with perfect recall). Then*

(a) if (σ,μ) is a consistent assessment there exists an A-weighting $\lambda : A \to \mathbb{N}$ such that, $\forall i \in N, \forall h \in D_i, \forall a \in A(h)$, (i) $\lambda(a) = 0$ if and only if $\sigma_i(h)(a) > 0$, and (ii) $\mu(h) > 0$ if and only if $\Lambda(h) \leq \Lambda(h')$ for all $h' \in I_i(h)$;

(b) if $\lambda : A \to N$ is an A-weighting, then there exists a consistent assessment (σ,μ) such that, $\forall i \in N, \forall h \in D_i, \forall a \in A(h)$, (i) $\sigma_i(h)(a) > 0$ if and only if $\lambda(a) = 0$ and (ii) $\mu(h) > 0$ if and only if $\Lambda(h) \leq \Lambda(h')$ for all $h' \in I_i(h)$.

Proof of Proposition 10. Fix an extensive form where no player moves more than once along any history. (a) Let $\{\langle \Omega, \mathcal{E}_i, f_i \rangle\}_{i \in N}$ be a profile of AGM-consistent choice frames (where $\Omega = H$ and $\mathcal{E}_i$ is given by (3)) that satisfies properties $P1$-$P4$. Let $\precsim$ be a total pre-order of H that rationalizes those frames (it exists by Proposition 4 and by the hypothesis that $P1$ is satisfied). Then, by hypothesis, $\precsim$ satisfies properties $P2$-$P4$. Let $\lambda_{\precsim}$ be the A-weighting defined in Lemma 22). Define the following assessment (σ,μ): $\forall i \in N, \forall h \in D_i, \forall a \in A(h)$, (i) $\sigma_i(h)(a) > 0$ if and only if $\lambda_{\precsim}(a) = 0$, and (ii) $\mu(h) > 0$ if and only if $\Lambda_{\precsim}(h) \leq \Lambda_{\precsim}(h')$ for all $h' \in I_i(h)$. Then, by Lemma 25, (σ,μ) is a consistent assessment. Furthermore, by definition of $\lambda_{\precsim}$, $\lambda_{\precsim}(a) = 0$ if and only if $h \sim ha$ and thus $\sigma_i(h)(a) > 0$ if and only if $h \sim ha$. by Lemma 23, $\mu(h) > 0$ if and only if $h \precsim h'$, for all $h' \in \vec{I}_i(h)$.

(b) Let (σ,μ) be a consistent assessment. By (a) of Lemma 25 there exists a A-weighting $\lambda : A \to \mathbb{N}$ such that, $\forall i \in N, \forall h \in D_i, \forall a \in A(h)$, (i) $\lambda(a) = 0$ if and only if $\sigma_i(h)(a) > 0$, and (ii) $\mu(h) > 0$ if and only if $\Lambda(h) \leq \Lambda(h')$ for all $h' \in I_i(h)$. Let $\precsim_\lambda$ be the total pre-order on H defined by: $h \precsim_\lambda h'$ if and only if $\Lambda(h) \leq \Lambda(h')$. For every player $i \in N$, let $\langle \Omega, \mathcal{E}_i, f_i \rangle$ be the following choice frame: $\Omega = H$, $\mathcal{E}_i$ is given by (3) and f_i is given $f_i(E) = \{h \in E : h \precsim_\lambda h', \forall h' \in E\}$. Then, by Lemma 24, the profile $\{\langle \Omega, \mathcal{E}_i, f_i \rangle\}_{i \in N}$ satisfies properties $P1$-$P4$. Furthermore, by Lemma 24, (i) $h \sim_\lambda ha$ if and only if $\sigma_i(h)(a) > 0$ and (ii) for every $h' \in \vec{I}_i(h)$, $h \precsim_\lambda h'$ if and only if $\mu(h) > 0$. ■

Proof of Lemma 12. It is shown in [7] that if $\langle \Omega, \mathcal{E}, f \rangle$ is an AGM-consistent choice frame then it satisfies the following property, known as Arrow's Axiom:

$$\forall S, T \in \mathcal{E}, \text{ if } T \subseteq S \text{ and } f(S) \cap T \neq \varnothing \text{ then } f(T) = f(S) \cap T. \qquad (10)$$

By Definition 1, $f(E) \subseteq E$ so that $f(E) \cap E = f(E)$; furthermore, $f(E) \neq \varnothing$. By (5) $f'(\Omega) = f(E)$. Hence, applying (10) to f' (with $S = \Omega$ and $T = E$) we get $f'(E) = f'(\Omega)$ and thus

$$f'(E) = f(E). \qquad (11)$$

By hypothesis, $F \subseteq E$ and $f(E) \cap F \neq \varnothing$. Thus, by (11), $f'(E) \cap F \neq \varnothing$. Hence, applying (10) to f' (with $S = E$ and $T = F$) we get that $f'(F) = f'(E) \cap F$. Using this and (11) we get

$$f'(F) = f(E) \cap F. \qquad (12)$$

Applying (10) to f (with $S = E$ and $T = F$) we get that $f(F) = f(E) \cap F$. It follows from this and (12) that $f'(F) = f(F)$. ■

Proof of Lemma 15. Let $\precsim$ be a total pre-order on H that satisfies Properties $P2$, $P3$ and $P5$. Fix an arbitrary $h \in H$. We want to show that there exists a unique $z \in Z$ such that $h \sim z$. If $h \in Z$ then it follows from Property $P5$ (and the fact that $\precsim$ is reflexive). Suppose, therefore, that $h \in D$. First we show that there is a $z \in Z$ such that $h \sim z$. By $P2$ and $P3$ there exists an $a_1 \in A(h)$ such that $h \sim ha_1$. If $ha_1 \in Z$ then we are done; otherwise, by $P2$ and $P3$ again, there exists an $a_2 \in A(ha_1)$ such that $ha_1 \sim ha_1a_2$. Repeating this argument a finite number of times we get that $h \sim ha_1 \sim ha_1a_2 \sim ... \sim ha_1a_2...a_m \in Z$.[18] Thus the desired result follows from transitivity of $\sim$. Now suppose that $h \sim z$ and $h \sim z'$ with $z, z' \in Z$. Then, by transitivity of $\sim$, $z \sim z'$ and thus, by Property $P5$, $z = z'$.

Now fix an arbitrary $h \in D$. (a) Let $a_1, a_2 \in A(h)$ be such that $h \sim ha_1$ and $h \sim ha_2$. Then $h \sim ha_1 \sim z(ha_1)$ and $h \sim ha_2 \sim z(ha_2)$, so that - as shown above - $z(ha_1) = z(ha_2)$ which implies that $a_1 = a_2$. (b) Suppose that $h' \in \vec{I}_i(h)$ is such that $h \sim h'$. By definition of $\vec{I}_i(h)$, there exists an $h_0 \in I(h)$ such that h_0 is a prefix of h' and thus, by Property $P2$, $h_0 \precsim h'$; hence, by transitivity of $\precsim$, $h \sim h_0$. Then, since $h \sim z(h)$ and $h_0 \sim z(h_0)$, $z(h) \sim z(h_0)$, so that, by $P5$, $z(h) = z(h_0)$ and thus $h = h_0$. ■

Proof of Proposition 16. Fix an extensive game without chance moves.

(a) Let $\{\langle \Omega, \mathcal{E}_i, f_i \rangle\}_{i \in N}$ be a profile of AGM-consistent choice frames that satisfies properties $P1$-$P6$. Let $\precsim$ be a total pre-order of H that rationalizes those frames. Then, by (a) of Proposition 14, the assessment (σ, μ) given by $\sigma_i(h)(a) > 0$ if and only if $h \sim ha$ and $\mu(h) > 0$ if and only if $h \precsim h'$, for all $h' \in \vec{I}_i(h)$ is a consistent assessment. By Lemma 15, for every $h \in D$, there is a unique $a \in A(h)$ such that $h \sim ha$. Hence it must be that $\sigma_i(h)(a) = 1$. By (b) of Lemma 15, there is no $h'' \in I(h)$ such that $h'' \neq h$ and $h'' \precsim h'$, for all $h' \in \vec{I}_i(h)$. Thus it must be that $\mu(h) = 1$. It follows that, for every $h \in D$, $z(h)$ (as defined in Lemma 15) is the unique terminal history reached from h by the strategy profile σ. Thus, by Property $P6$, (σ, μ) is sequentially rational and therefore it is a pure sequential equilibrium.

(b) Let (σ, μ) be a pure sequential equilibrium. Then (σ, μ) is a consistent assessment and by (b) of Proposition 14, there exists a profile of (possibly iterated) rationalizable choice frames that satisfies properties $P1$-$P4$ such that (letting $\precsim$ be a total pre-order on H that rationalizes those choice frames), for every $i \in N$, $h \in D_i$ and $a \in A(h)$, (i) $h \sim ha$ if and only if $\sigma_i(h)(a) = 1$ and (ii) for every $h' \in \vec{I}_i(h)$, $h \precsim h'$ if and only if $\mu(h) = 1$ (since (σ, μ) is a pure assessment). It follows that, for every $h \in D$ there exists a unique $a \in A(h)$ such that $h \sim ha$ and thus a unique $z \in Z$ such that h is a prefix of z and $h \sim z$. Property $P6$ is an immediate consequence of sequential rationality. ■

[18] Since the set of actions is finite and no action is available at more than one information set (see Remark 6), for every $h \in D$ there is a sequence of actions $\langle a_1, a_2, ..., a_m \rangle$ such that $ha_1a_2...a_m \in Z$.

Proof of Proposition 17. Fix an arbitrary perfect-information game without chance moves and an arbitrary backward-induction solution. Let $\lambda : A \to \{0,1\}$ be the associated function obtained in the construction of the solution. Define $\Lambda : H \to \mathbb{N}$ as follows: $\Lambda(\emptyset) = 0$ and if $h = \langle \emptyset, a_1, ..., a_m \rangle$ then $\Lambda(h) = \sum\limits_{k=1,...,m} \lambda(a_k)$. Let $\precsim$ be the total pre-order on H defined by $h \precsim h'$ if and only if $\Lambda(h) \leq \Lambda(h')$. Then for every $h \in D$ and $a \in A(h)$, $\Lambda(ha) = \Lambda(h) + \lambda(a) \leq \Lambda(h)$ so that $h \precsim ha$ and thus Property $P2$ is satisfied. Property $P3$ is also satisfied, because, by definition of λ, for every $h \in D$ there exists an $a \in A(h)$ such that $\lambda(a) = 0$ and therefore $\Lambda(ha) = \Lambda(h) + \lambda(a) = \Lambda(h)$ and thus $h \sim ha$. Finally Property $P7$ is satisfied by construction.

Conversely, let $\precsim$ be a total pre-order on H that satisfies Properties $P2$, $P3$ and $P7$. Define $\lambda : A \to \{0,1\}$ as follows: for every $h \in D$ choose an arbitrary $a \in A(h)$ such that $h \sim ha$ (its existence is guaranteed by Properties $P2$ and $P3$) and set $\lambda(a) = 0$ and $\lambda(a') = 1$ for every $a' \in A(h) \backslash \{a\}$. We want to show that $B = \{a \in A : \lambda(a) = 0\}$ is a backward induction solution. This requires that if i is the player assigned to $h \in D$ and $a \in A(h)$ is such that $\lambda(a) = 0$ then $U_i(haa_1...a_m) \geq U_i(ha'b_1...b_p)$ where both $haa_1...a_m$ and $ha'a_1...a_p$ are terminal histories, a' is an arbitrary action in $A(h)$ and, for every $j = 1, ..., m$ and $k = 1, ..., p$, $\lambda(a_j) = \lambda(b_j) = 0$. But this is precisely what Property $P7$ guarantees. ■

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