Department of Economics
Working Paper Series

# Sylvester Matrix and Common Factors in Polynomial Matrices

Leon Wegge
University of California, Davis

Leon Wegge
University of California, Davis

March 13, 2007

Paper # 07-4

With the coefficient matrices of the polynomial matrices replacing the scalar coefficients in the standard Sylvester matrix, common factors exist if and only if this (generalized) Sylvester matrix is singular and the coefficient matrices commute. If the coefficient matrices do not commute, a necessary and sufficient condition for a common factor to exist is that a submatrix of the ratio (transfer) coefficient matrices is of less than full row rank. Whether coefficient matrices commute or not, a nonsingular (generalized) Sylvester matrix is always a sufficient condition for no common factors to exist. These conditions hold whether common factors are unimodular or not unimodular. These results follow from requiring that in the potential alternative pair of polynomial matrices with the same matrix ratio, i.e. with the same transfer function, all coefficient matrices beyond the given integers and are null matrices. Algebraically these requirements take the form of linear equations in the coefficient matrices of the inverse of the potential common factor. Lower block triangular Toeplitz matrices appear in these equations and the sequential inverse of these matrices generates sequentially the coefficient matrices of the inverse of the common factor. The conclusions follow from the properties of infinite dimensional diagonally dominant matrices.

UCDAVIS

Department of Economics
One Shields Avenue
Davis, CA 95616
(530)752-0741

http://www.econ.ucdavis.edu/working_search.cfm

Sylvester Matrix and Common Factors in Polynomial Matrices

Leon L. Wegge, Economics, University of California Davis
One Shields Avenue, Davis, CA 95616
llwegge@ucdavis.edu

ABSTRACT: With the coefficient matrices of the polynomial matrices replacing the scalar coefficients in the standard Sylvester matrix, common factors exist if and only if this (generalized) Sylvester matrix is singular and the coefficient matrices commute. If the coefficient matrices do not commute, a necessary and sufficient condition for a common factor to exist is that a submatrix of the ratio (transfer) coefficient matrices is of less than full row rank. Whether coefficient matrices commute or not, a nonsingular (generalized) Sylvester matrix is always a sufficient condition for no common factors to exist. These conditions hold whether common factors are unimodular or not unimodular.

These results follow from requiring that in the potential alternative pair of polynomial matrices with the same matrix ratio, i.e. with the same transfer function, all coefficient matrices beyond the given integers $p$ and $q$ are null matrices. Algebraically these requirements take the form of linear equations in the coefficient matrices of the inverse of the potential common factor. Lower block triangular Toeplitz matrices appear in these equations and the sequential inverse of these matrices generates sequentially the coefficient matrices of the inverse of the common factor. The conclusions follow from the properties of infinite dimensional diagonally dominant matrices.

Abbreviated Title: Sylvester Matrix

Subject Classification: 15A09, 15A06, 15A23, 15A27.

Keywords. Sylvester matrix, common factor, polynomial matrix, transfer function, commutativity, Toeplitz matrix.

# Sylvester Matrix and Common Factors in Polynomial Matrices

Leon L. Wegge, Economics, University of California Davis
One Shields Avenue, Davis, CA 95616
llwegge@ucdavis.edu

With the coefficient matrices of the polynomial matrices replacing the scalar coefficients in the standard Sylvester matrix, common factors exist if and only if this generalized Sylvester matrix is singular and the coefficient matrices commute. If the coefficient matrices do not commute, a necessary and sufficient condition for a common factor to exist is that a submatrix of the ratio (transfer) coefficient matrices is of less than full row rank. Whether coefficient matrices commute or not, a nonsingular Sylvester matrix is always a sufficient condition for no common factors to exist. These conditions hold whether common factors are unimodular or not unimodular.

The paper is divided in ten short sections. The first three sections contain the definitions of polynomial matrix, common factor, truncated transfer functions and the Toeplitz matrices that generate the latter. Section 4 states the assumptions and the method of analysis. Theorem 1 is the main result. It is stated and proved in the longest Section 5. The Sylvester matrix for polynomial matrices is defined in Section 6 and its relevance in characterizing the common factor existence conditions is stated under Corollary 1 of Section 7. Alternative necessary and sufficient conditions for the existence of a common factor with finite degree inverse or of finite degree are obtained in Section 8. Examples with a singular Sylvester matrix but without a common factor are constructed in Section 9. A short comparison of our method of analysis with that existing in the linear system literature and a statement of the relevance of the polynomial matrix results for uniqueness conditions in time series Arma(p,q) models conclude the paper.

## 1. *Polynomial Matrices*.

With $p$ and $q$ finite positive integers, consider the $m \times m$ polynomial matrices

(1) $$\Pi(z,p) = \sum_{i=0}^{p} \Pi(i) z^i, \quad \Psi(z,q) = \sum_{k=0}^{q} \Psi(k) z^k, \quad z \in C, \; \Pi(0) = I_m,$$

of degrees $p$ and $q$ respectively, where the coefficient matrices $\Pi(i) \in \Re^{m\times m}$, $\Psi(k) \in \Re^{m\times m}$, $\Pi(p)$ and $\Psi(q)$ not null. If originally $\Pi(0)$ is nonsingular, the polynomial matrix $\Pi(z,p)$ is obtainable in the form (1) after post-multiplication by $\Pi(0)$ inverse. If originally $\Pi(p)$ is nonsingular and $\Pi(0)$ singular, obtain (1) after renumbering the coefficient matrices in the inverse order.

## 2. *Common Factor.*

The $m \times m$ polynomial matrix $\mathrm{X}(z,\infty) = \sum_{j=0}^{\infty} \mathrm{X}(j) z^j$, $\mathrm{X}(j) \in \Re^{m\times m}$, $\mathrm{X}(z,\infty) \neq I_m$, is a common factor if and only if there exist $m \times m$ polynomial matrices $(\Pi^1(z,p), \Psi^1(z,q))$ such that

(2) $$(\Pi(z,p), \Psi(z,q)) = \mathrm{X}(z,\infty)\,(\Pi^1(z,p), \Psi^1(z,q))$$

with

(3) $$\Pi^1(z,p) = \sum_{i=0}^{p} \Pi^1(i) z^i, \quad \Psi^1(z,q) = \sum_{k=0}^{q} \Psi^1(k) z^k .$$

A common factor is defined here as a common left divisor with $(\Pi^1(z,p),\Psi^1(z,q))$ having degrees not exceeding $p$ and $q$, respectively. The concern is to find conditions under which more than one pair of polynomial matrices, each pair satisfying the degree requirements $(p,q)$, has the same transfer function. It is understood that $\mathrm{X}(z,\infty)=I_m$ is not commonly called a common factor.

3. *Truncated Transfer Function.*

$\mathrm{T}(z,\infty)=(\Pi(z,p)^{-1}\Psi(z,q)$ is the transfer function. The truncated transfer function is the polynomial matrix

(4) $$\mathrm{T}(z,n)=I_m+\mathrm{T}(1)z+...+\mathrm{T}(n)z^n.$$

Under (2) the pairs $(\Pi(z,p),\Psi(z,q))$ and $(\Pi^1(z,p),\Psi^1(z,q))$ share the same transfer and truncated transfer function.

In what follows we represent the system of equations to be satisfied by the coefficient matrices in $(\Pi^1(z,p),\Psi^1(z,q))$, their relationship to the polynomial matrices in $(\Pi(z,p),\Psi(z,q))$ and to the truncated transfer function through lower block triangular Toeplitz matrices.

Let $(Z)_K^L$ be the matrix of elements of a block matrix $Z$ in its row blocks indexed in $K$ and its column blocks indexed in $L$. Define the index set $I(n,\ell)=\{1,...,n-\ell\}$ and its complement $II(n,\ell)=\{n-\ell+1,...,n\}$. With these symbols also define the following block Toeplitz matrices and their partitions

(5) $$P_n=\begin{pmatrix} P_{n-p} & 0 \\ (P_n)_{II(n,p)}^{I(n,p)} & P_p \end{pmatrix}=\begin{pmatrix} I_m & .. & 0 & & & \\ .. & .. & .. & & 0 & \\ \Pi(n-p-1)z^{n-p-1} & .. & I_m & & & \\ \Pi(n-p)z^{n-p} & .. & \Pi(1)z & I_m & .. & 0 \\ .. & .. & .. & .. & .. & .. \\ \Pi(n-1)z^{n-1} & .. & \Pi(p)z^p & \Pi(p-1)z^{p-1} & .. & I_m \end{pmatrix},$$

where $\Pi(i)=0$ for integer $i\neq[0,p]$,

(6) $$Q_n=\begin{pmatrix} (Q_n)_{I(n,p)}^{I(n,q)} & (Q_n)_{I(n,p)}^{II(n,q)} \\ (Q_n)_{II(n,p)}^{I(n,q)} & (Q_n)_{II(n,p)}^{II(n,q)} \end{pmatrix}$$

$$=\begin{pmatrix} \Psi(0) & .. & 0 & 0 & .. & 0 \\ .. & .. & .. & .. & .. & .. \\ \Psi(n-p-1)z^{n-p-1} & .. & \Psi(q-p)z^{q-p} & \Psi(q-p-1)z^{q-p-1} & .. & \Psi(-p)z^{-p} \\ \Psi(n-p)z^{n-p} & .. & \Psi(q-p+1)z^{q-p+1} & \Psi(q-p)z^{q-p} & .. & \Psi(1-p)z^{1-p} \\ .. & .. & .. & .. & .. & .. \\ \Psi(n-1)z^{n-1} & .. & \Psi(q)z^q & \Psi(q-1)z^{q-1} & .. & \Psi(0) \end{pmatrix}$$

where $\Psi(i)=0$, for integer $i \neq [0,q]$,

$$(7) \qquad \mathrm{T}_n = P_n^{-1}Q_n = \begin{pmatrix} \mathrm{T}(0) & 0 & 0 & .. & 0 \\ \mathrm{T}(1)z & \mathrm{T}(0) & 0 & .. & 0 \\ \mathrm{T}(2)z^2 & \mathrm{T}(1)z & \mathrm{T}(0) & .. & 0 \\ .. & .. & .. & .. & .. \\ \mathrm{T}(n-1)z^{n-1} & \mathrm{T}(n-2)z^{n-2} & \mathrm{T}(n-3)z^{n-3} & .. & \mathrm{T}(0) \end{pmatrix}.$$

$\mathrm{T}_n$ is the matrix generating the coefficient matrices of the truncated transfer function. In writing (7) we used the property that the blocks in the first column block determine all the block matrices in lower triangular block Toeplitz matrices.

4. *The Assumption and the Analysis.*

a) The transfer function exists as a sum of matrices $\mathrm{T}(j)z^j, \mathrm{T}(j) \in \Re^{m\times m}$, that converges in a neighborhood of $z=0$.

b) The common factor $\mathrm{X}(z,\infty)=\sum_{j=0}^{\infty}\mathrm{X}(j)z^j$, with $\mathrm{X}(0)=I_m$, has an inverse $\sum_{j=0}^{\infty}\mathrm{H}(j)z^j$, with $\mathrm{H}(0)=I_m$, in a neighborhood of $z=0$.

c) From (2), the coefficient of $z^j$ in $\Pi^1(z,p)=(\sum_{j=0}^{n}\mathrm{H}(j)z^j)\Pi(z,p),\ p<j$, and in $\Psi^1(z,q)=(\sum_{j=0}^{n}\mathrm{H}(j)z^j)\Psi(z,q),\ q<j$, must be null matrices. This is the equation system

$$(8) \qquad \left(\Pi^1(n),...,\Pi^1(p+1),\Psi^1(n),...,\Psi^1(q+1)\right)=0$$

that must hold for any finite $n$, with $Max(p,q)<n$ and for $n\to\infty$.

The assumptions and the algebraic analysis are based on properties of diagonally dominant matrices when the number of equations tends to infinity, e.g. Theorem 5, p. 127 of [1]. With its emphasis on coprime polynomial matrices and unimodular common factors, this body of theory is not included in [2] as a potential source of algebraic manipulation. Given the interest here in the existence of common factors, whether unimodular or not, the diagonally dominant matrix theory and the operations through lower block triangular Toeplitz matrices are compelling.

5. *Theorem 1.*

*The polynomial matrix* $\mathrm{X}(z,\infty)=\sum_{j=0}^{\infty}\mathrm{X}(j)z^{j}$, $\mathrm{X}(0)=I_m$, *having inverse* $\sum_{j=0}^{\infty}\mathrm{H}(j)z^{j}$, *with* $\mathrm{H}(0)=I_m$, *is a common factor of the pair of polynomial matrices* $(\Pi(z,p),\Psi(z,q))$ *if and only if*

$$(9)\qquad \big(\mathrm{H}(p),...,\mathrm{H}(1)\big)\begin{pmatrix} I_m & .. & 0\\ .. & .. & ..\\ \Pi(p-1) & .. & I_m\end{pmatrix}\Im(n_q)=\big(\Pi^1(p)-\Pi(p),...,\Pi^1(1)-\Pi(1)\big)\Im(n_q)=0,$$

*where* $pm\times pm^2$ $\Im(n_q)=\begin{pmatrix}\mathrm{T}(n_q-p) & .. & \mathrm{T}(q-p+1)\\ .. & .. & ..\\ \mathrm{T}(n_q-1) & .. & \mathrm{T}(q)\end{pmatrix}$, *with* $n_q=pm+q$, $\mathrm{T}(i)=0$ *for* $i<0$.

*When the matrices* $\Pi(i)$, $i=1,...,p$, $\Psi(j)$, $j=1,...,q$, *commute,* $\mathrm{X}(z,\infty)$ *is a common factor if and only if*

$$(10)\qquad \big(\mathrm{H}(p),...,\mathrm{H}(1)\big)\begin{pmatrix} I_m & .. & 0\\ .. & .. & ..\\ \Pi(p-1) & .. & I_m\end{pmatrix}\Im(p+q)=\big(\Pi^1(p)-\Pi(p),...,\Pi^1(1)-\Pi(1)\big)\Im(p+q)=0,$$

*where* $pm\times pm$ $\Im(p+q)=\begin{pmatrix}\mathrm{T}(q) & .. & \mathrm{T}(q-p+1)\\ .. & .. & ..\\ \mathrm{T}(p+q-1) & .. & \mathrm{T}(q)\end{pmatrix}$.

Proof.

From (8) the matrices $\mathrm{H}(j)z^j$ must satisfy the linear equations

$$(11)\qquad 0=\big(\mathrm{H}(n)z^n,...,\mathrm{H}(p+1)z^{p+1},\quad \mathrm{H}(p)z^p,...,\mathrm{H}(1)z\big)\begin{pmatrix} P_{n-p} & (Q_n)^{I(n,q)}_{I(n,p)}\\ (P_n)^{I(n,p)}_{II(n,p)} & (Q_n)^{I(n,q)}_{II(n,p)}\end{pmatrix}.$$

Eliminating the top matrices, (11) is equivalent with the two subsets of equations

$$(12)\qquad \big(\mathrm{H}(p)z^p,...,\mathrm{H}(1)z\big)\big((Q_n)^{I(n,q)}_{II(n,p)}-(P_n)^{I(n,p)}_{II(n,p)}P^{-1}_{n-p}(Q_n)^{I(n,q)}_{I(n,p)}\big)$$
$$=\big(\mathrm{H}(p)z^p,...,\mathrm{H}(1)z\big)P_p\,G(z,n),\ \text{ with } G(z,n)=\begin{pmatrix}\mathrm{T}(n-p)z^{n-p} & .. & \mathrm{T}(q-p+1)z^{q-p+1}\\ .. & .. & ..\\ \mathrm{T}(n-1)z^{n-1} & .. & \mathrm{T}(q)z^q\end{pmatrix},$$
$$=\big((\Pi^1(p)-\Pi(p))z^p,...,(\Pi^1(p)-\Pi(p))z\big)G(z,n)=0,$$

$$(13)\qquad \big(\mathrm{H}(n)z^n,...,\mathrm{H}(p+1)z^{p+1}\big)P_{n-p}=-\big(\mathrm{H}(p)z^p,...,\mathrm{H}(1)z\big)(P_n)^{I(n,p)}_{II(n,p)}.$$

The first equality in (12) follows from the equality between the partitions

$$\begin{pmatrix}(\mathrm{T}_n)^{I(n,q)}_{I(n,p)}\\ (\mathrm{T}_n)^{I(n,q)}_{II(n,p)}\end{pmatrix}=\begin{pmatrix}P_{n-p} & 0\\ (P_n)^{I(n,p)}_{II(n,p)} & P_p\end{pmatrix}^{-1}\begin{pmatrix}(Q_n)^{I(n,q)}_{I(n,p)}\\ (Q_n)^{I(n,q)}_{II(n,p)}\end{pmatrix}=\begin{pmatrix}P^{-1}_{n-p}(Q_n)^{I(n,q)}_{I(n,p)}\\ P_p^{-1}((Q_n)^{I(n,q)}_{II(n,p)}-(P_n)^{I(n,p)}_{II(n,p)}P^{-1}_{n-p}(Q_n)^{I(n,q)}_{I(n,p)}\end{pmatrix}$$

and the second equality follows from (2). The matrix $(\mathrm{H}(p)z^p,...,\mathrm{H}(1)z)$ is determined by the equations (12) only.

The matrix $(\mathrm{H}(n)z^n,...,\mathrm{H}(p+1)z^{p+1})$ is determined in (13) and is different from a null matrix only if $(\mathrm{H}(p)z^p,...,\mathrm{H}(1)z)$ is not a null matrix. A solution exists for all finite $n$. Since $P_{n-p}$ is a block diagonally dominant matrix in a neighborhood of $z=0$, the limit of this solution converges to the solution when $n\to\infty$ [1]. Under this assumption (13) can always be satisfied.

To end the proof of the first part, observe the relation between the column blocks

$$(14)\qquad \begin{pmatrix} \mathrm{T}(q+t-p+1) \\ .. \\ \mathrm{T}(q+t-1) \\ \mathrm{T}(q+t) \end{pmatrix} = J_p \begin{pmatrix} \mathrm{T}(q+t-p) \\ .. \\ \mathrm{T}(q+t-2) \\ \mathrm{T}(q+t-1) \end{pmatrix},\ t\geq 1, \text{ with } J_p = \begin{pmatrix} 0 & I_m & .. & 0 \\ .. & .. & .. & .. \\ 0 & 0 & .. & I_m \\ -\Pi(p) & -\Pi(p-1) & .. & -\Pi(1) \end{pmatrix}.$$

To see this, pre-multiply the first column block of $\mathrm{T}_{q+t+1} = P_{q+t+1}^{-1}Q_{q+t+1}$ by the bottom block row of $P_{q+t+1}$, the matrix $(\Pi(q+t)z^{q+t},...,\Pi(1)z,I_m)$, and verify the equation

$$(15)\qquad \Pi(p)z^p\mathrm{T}(q+t-p)z^{q+t-p} + ... + \Pi(1)z\mathrm{T}(q+t-1)z^{q+t-1} + \mathrm{T}(q+t)z^{q+t} = \Psi(q+t)z^{q+t}.$$

For a finite $n$, evaluate (12) at $z=1$ and let $\Im(n)=G(n,1)$. By the Caley-Hamilton theorem, all $n-q$ block equalities in (12) hold if the first $pm$ block equalities are satisfied. As a consequence, with $n_q = mp+q$, the matrix $\Im(n)$ can be replaced by $\Im(n_q)$ and the number of block equations is thereby reduced to the finite number $n_q - q = mp$, independently of $n\to\infty$.

The second part of the Theorem follows from observing that when the coefficient matrices commute we have the relation

$$(16)\qquad \begin{pmatrix} \mathrm{T}(t-p+1) \\ .. \\ \mathrm{T}(t) \end{pmatrix} = \begin{pmatrix} \mathrm{T}(t-p) & .. & \mathrm{T}(t-2p+1) \\ .. & .. & .. \\ \mathrm{T}(t-1) & .. & \mathrm{T}(t-p) \end{pmatrix} \begin{pmatrix} \Pi(1) \\ .. \\ \Pi(p) \end{pmatrix},\ p+q\leq t,$$

and the matrix $\Im(n_q)$ in (9) can be replaced by its submatrix $\Im(p+q)$.

6. *Sylvester Matrix.*

The generalized Sylvester Matrix is the $m(p+q)\times m(p+q)$ matrix

$$(17)\qquad S(p,q) = \begin{pmatrix} I_m & .. & 0 & \Psi(0) & .. & 0 \\ .. & .. & .. & .. & .. & .. \\ \Pi(q-1) & .. & I_m & \Psi(q-1) & .. & \Psi(q-p) \\ \Pi(q) & .. & \Pi(1) & \Psi(q) & .. & \Psi(q-p+1) \\ .. & .. & .. & .. & .. & .. \\ \Pi(p+q-1) & .. & \Pi(p) & \Psi(q+p-1) & .. & \Psi(q) \end{pmatrix},$$

$\Pi(i)=0$ for $i\notin[0,p]$, $\Psi(j)=0$ for $j\notin[0,q]$.

A different generalized Sylvester matrix with the number of row blocks increasing with $m$ is introduced in [3]. It is shown there that the two polynomial matrices are coprime if the Sylvester matrix, so generalized, is of full rank. Still another generalized Sylvester matrix of expanding dimensions is considered in [4], [5]. They show that the null space of their matrix must contain the coefficients of the coprime polynomial matrices $(A(z,\infty), B(z,\infty))$ that satisfy $A(z,\infty)\Pi(z,p)+B(z,\infty)\Psi(z,q)=0$. These last equations imply that $(\Pi(z,p), \Psi(z,q))$ are coprime. In both publications the emphasis is on analyzing the connection between coprime polynomials and their chosen generalized Sylvester matrix.

This paper's definition (17) is the most natural generalization of the Sylvester matrix and with it one gets a simple interpretation of the common factor existence conditions of Theorem 1.

## 7. *Corollary 1.*

*The polynomial matrix* $\mathrm{X}(z,\infty)=\sum_{j=0}^{\infty}\mathrm{X}(j)z^j$, $\mathrm{X}(0)=I_m$, *having inverse* $\sum_{j=0}^{\infty}\mathrm{H}(j)z^j$, $\mathrm{H}(0)=I_m$, *is a common factor of the pair of polynomial matrices* $(\Pi(z,p),\Psi(z,q))$ *only if*

(18) $$0=\big(\mathrm{H}(p+q),...,\mathrm{H}(p+1),\quad \mathrm{H}(p),...,\mathrm{H}(1)\big)S(p,q),$$

*If the coefficient matrices commute, (18) is also sufficient.*

Proof.

The requirements (11) for $n=p+q$ and $z=1$ are the equations (18). Hence they are necessary. The conditions (18) are equivalent to the two subsystems

(19a) $$0=\big(\mathrm{H}(p+q),...,\mathrm{H}(p+1)\big)\begin{pmatrix} I_m & .. & 0\\ .. & .. & ..\\ \Pi(q-1) & .. & I_m\end{pmatrix}=-\big(\mathrm{H}(p),...,\mathrm{H}(1)\big)\begin{pmatrix} \Pi(q) & .. & \Pi(1)\\ .. & .. & ..\\ \Pi(p+q-1) & .. & \Pi(p)\end{pmatrix}.$$

(19b) $$0=\big(\mathrm{H}(p),...,\mathrm{H}(1)\big)\Bigg(\begin{pmatrix} \Psi(q) & .. & \Psi(q-p+1)\\ .. & .. & ..\\ \Psi(p+q-1) & .. & \Psi(q)\end{pmatrix}$$

$$-\begin{pmatrix} \Pi(q) & .. & \Pi(1)\\ .. & .. & ..\\ \Pi(p+q-1) & .. & \Pi(p)\end{pmatrix}\begin{pmatrix} I_m & .. & 0\\ .. & .. & ..\\ \Pi(q-1) & .. & I_m\end{pmatrix}^{-1}\begin{pmatrix} \Psi(0) & .. & 0\\ .. & .. & ..\\ \Psi(q-1) & .. & \Psi(q-p)\end{pmatrix}\Bigg)$$

$$=\big(\mathrm{H}(p),...,\mathrm{H}(1)\big)\begin{pmatrix} I_m & .. & 0\\ .. & .. & ..\\ \Pi(p-1) & .. & I_m\end{pmatrix}\begin{pmatrix} \mathrm{T}(q) & .. & \mathrm{T}(q-p+1)\\ .. & .. & ..\\ \mathrm{T}(p+q-1) & .. & \mathrm{T}(q)\end{pmatrix},$$

since by definition

$$\begin{pmatrix} (\mathrm{T}_{p+q})^{I(p+q,q)}_{I(p+q,p)}\\ (\mathrm{T}_{p+q})^{I(p+q,q)}_{II(p+q,p)}\end{pmatrix}=\begin{pmatrix} P_q & 0\\ (P_{p+q})^{I(p+q,p)}_{II(p+q,p)} & P_p\end{pmatrix}^{-1}\begin{pmatrix} (Q_{p+q})^{I(p+q,q)}_{I(p+q,p)}\\ (Q_{p+q})^{I(p+q,q)}_{II(p+q,p)}\end{pmatrix}.$$

(19a) is always satisfied and the condition (19b) is exactly the same as the condition (10) for the case when coefficient matrices commute.

The result may be summarized as follows.

*The generalized Sylvester matrix* $S(p,q)$ *being nonsingular is a sufficient condition for two polynomial matrices not to have a common factor different from the unit matrix. If* $S(p,q)$ *is singular a common factor different from the unit matrix always exists if the coefficient matrices commute.*

By requiring that the coefficient matrices beyond $p$ and $q$ be null matrices, the analysis above proves Sylvester's result for the scalar case and the extension to polynomial matrices.

## 8. *Dual classes of common factors.*

Theorem 1 characterized the existence conditions of a common factor of any degree. In this section alternative necessary and sufficient conditions are stated for the existence of special common factors. When the inverse of the common factor is of finite degree, or when the common factor itself is of finite degree, special conditions are derived in Theorems 2 and 3.

*Theorem 2.*

*The polynomial matrix* $\mathrm{X}(z,\infty)=(I_m+H(1)z+...+H(s)z^s)^{-1}$, $H(s)\neq 0$, $s$ *finite, is a common factor of the pair of polynomial matrices* $(\Pi(z,p),\Psi(z,q))$ *if and only if*

(20) $$(H(s),...,H(2),H(1))W(s,p,q)=0,$$

*with* $ms\times 2ms$

$$W(s,p,q)=\begin{pmatrix}\Pi(p) & .. & \Pi(-s+p+2) & \Pi(-s+p+1) & \Psi(q) & .. & \Psi(-s+q+2) & \Psi(-s+q+1)\\ .. & .. & .. & .. & .. & .. & .. & ..\\ 0 & .. & \Pi(p) & \Pi(p-1) & 0 & .. & \Psi(q) & \Psi(q-1)\\ 0 & .. & 0 & \Pi(p) & 0 & .. & 0 & \Psi(q)\end{pmatrix}.$$

Proof. If $\mathrm{X}(z,\infty)$ has an inverse of degree $s$, from (2) the alternative coefficient matrices $\Pi^1(i)$, $p+s<i$, $\Psi^1(j)$, $q+s<j$, are null matrices and it is a common factor because the remaining alternative matrix coefficients satisfy

$$(\Pi^1(p+s),...,\Pi^1(p+1),\Psi^1(q+s),...,\Psi^1(q+1))=(H(s),...,H(2),H(1))W(s,p,q)=0.$$

*Remarks.*

I. For this class of common factors (20) implies (9). To see this, if $s\leq p$ the latter conditions require that for $k=0,...,mp-1$ we have

$$\sum_{j=1}^{s}H(j)\sum_{i=j}^{p}\Pi(i-j)T(q+k+1-i)$$
$$=\sum_{j=1}^{s}H(j)(\Psi(q+k+1-j)-\sum_{i=p-j+1}^{p}\Pi(i)T(q+k+1-j-i))$$
$$=\sum_{j=k+1}^{s}H(j)\Psi(q+k+1-j)-\sum_{\ell=1}^{s}(\sum_{j=\ell}^{s}H(j)\Pi(p-j+\ell))T(q+k+1-p-\ell)=0.$$

The first equality follows from (15) and the first term in the last equality is a null matrix for integer $k \notin [0,...,s-1]$. Under (20) the first term and the coefficient of $T(q+k+1-p-\ell)$ are null matrices for all $k$ and $\ell$, showing that (9) is satisfied. In the same way, (9) follows from the subset of the conditions (20) that involve $(H(p),...,H(1))$ when $p<s$.

II. From (20) if the polynomial matrix $(I_m + H(1)z + ... + H(s)z^s)^{-1}$, $H(s) \neq 0$, is a common factor then $(I_m + H(2)z + ... + H(s)z^{s-1})^{-1}, \ldots, (I_m + H(s)z)^{-1}$ are common factors. The last polynomial is a common factor if and only if $H(s)(\Pi(p), \Psi(q)) = 0$.

III. Common factors of coprime polynomial matrices are common factors with finite degree inverse and the condition on the end coefficient matrices $(\Pi(p), \Psi(q))$ is well known. Here it is shown that $\rho(\Pi(p), \Psi(q)) < m$ is the necessary and sufficient condition for a common factor of this class to exist, unimodular or not unimodular, Under this condition on the end coefficient matrices the Sylvester matrix is singular since these are the only non-null matrices in the last row block.

IV. If $(I_m + H(1)z + ... + H(s)z^s)^{-1}$ is a common factor, $(I_m + G(1)z + ... + G(s)z^s)^{-1}$ is a common factor, where $G(i) = FH(i)$ with $m \times m$ $F$, $FH(s) \neq 0$. This follows from (20) being a system of linear homogeneous equations.

V. With scalar polynomials this class of common factors is empty, since the condition requires that the end coefficients are zeros and therefore the degrees are less than $p$ and $q$.

*Theorem 3.*

*The polynomial matrix* $X(z,s) = I_m + X(1)z + ... + X(s)z^s$, $X(s) \neq 0$, *of finite degree* $s$ *and with inverse* $\sum_{i=0}^{\infty} H(i)z^i$, $H(0) = I_m$, *is a common factor of the pair of polynomial matrices* $(\Pi(z,p), \Psi(z,q))$ *if and only if*

(21) $\quad (X(s),...,X(2),X(1))W^1(s,p,q) = 0$,

*with* $ms \times 2ms$

$$W^1(s,p,q) = \begin{pmatrix} \Pi^1(p) & .. & \Pi^1(-s+p+1) & \Psi^1(q) & .. & \Psi^1(-s+q+1) \\ .. & .. & .. & .. & .. & .. \\ 0 & .. & \Pi^1(p) & 0 & .. & \Psi^1(q) \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{k=0}^{p} H(k)\Pi(p-k) & .. & \sum_{k=0}^{p-s+1} H(k)\Pi(-s+p+1-k) & \sum_{k=0}^{q} H(k)\Psi(q-k) & .. & \sum_{k=0}^{q-s+1} H(k)\Psi(-s+q+1-k) \\ .. & .. & .. & .. & .. & .. \\ 0 & .. & \sum_{k=0}^{p} H(k)\Pi(p-k) & 0 & .. & \sum_{k=0}^{q} H(k)\Psi(q-k) \end{pmatrix}.$$

*Remarks.*

I. This class of finite degree common factors is dual to the class of common factors with finite degree inverse. The assumed and the alternative coefficient matrices are reversed. Condition (21) imposes on the alternative coefficient matrices the requirement that the given assumed matrices satisfy $(\Pi(p+s),...,\Pi(p+1),\Psi(q+s),...,\Psi(q+1))=0$.

II. When $s=1$, the equality $X(1)(\Pi^1(p),\Psi^1(q))=0$, with $\Pi^1(p)=\sum_{k=0}^{p}(-\mathrm{X}(1))^k\Pi(p-k)$ and $\Psi^1(q)=\sum_{k=0}^{q}(-\mathrm{X}(1))^k\Psi(q-k)$ is the required condition. These matrices could be constructed in the Sylvester matrix $S(p,q)$ by adding together all its row blocks after pre-multiplication of the $i-th$ row block by $H(p+q-i)=(-X(1))^{p+q-i}$, $i=1,...,p+q$. The product of $X(1)$ with the sum row block so constructed is $X(1)\left((-\mathrm{X}(1))^{q-1}\Pi^1(p),\,...,\,\Pi^1(p),(-\mathrm{X}(1))^{p-1}\Psi^1(q),\,...,\,\Psi^1(q)\right)$. This is a null matrix under (21) and the Sylvester matrix is singular regardless of the rank of the assumed end coefficient matrices $(\Pi(p),\Psi(q))$.

III. For general $s$, the coefficient matrix $W^1(s,p,q)$ could be constructed in the Sylvester matrix $S(p,q)$ by replacing

1) the bottom row block by the sum total of the $i-th$ row block after pre-multiplication by $H(p+q-i)$, $i=1,...,p+q$,

2) the second to last row block by the sum total of the $i-th$ row block after pre-multiplication by $H(p+q-1-i)$, $i=1,...,p+q-1,\ldots$,

s) the $s-th$ to last row block by the sum total of the $i-th$ row block after pre-multiplication by $H(p+q-s+1-i)$, $i=1,...,p+q-s+1$.

The bottom $s$ row blocks so constructed are

$$\Sigma(s)=\begin{pmatrix}\Pi^1(p+q-s),...,\Pi^1(p-s+1),\Psi^1(q+p-s),...,\Psi^1(q-s+1)\\ .... \quad ..., \quad ..., \quad ..., \quad ..., \quad ...\\ \Pi^1(p+q-1),..., \quad \Pi^1(p), \quad \Psi^1(q+p-1),..., \quad \Psi^1(q)\end{pmatrix},$$

since, with $H(k)=0$ for $k$ negative, we have constructed

$$\sum_{k=0}^{p}H(k)\Pi(i-k)=\sum_{k=0}^{p}H(k+i-p)\Pi(p-k)=\Pi^1(i)$$

and

$$\sum_{k=0}^{q}H(k)\Psi(j-k)=\sum_{k=0}^{q}H(k+j-q)\Psi(q-k)=\Psi^1(j).$$

Also observe that the coefficient matrices of the inverse of the potential common factor satisfy $H(k+1)=-\sum_{\ell=1}^{s}X(\ell)H(k-\ell+1)$. From this, for $i=1,...,q$, conditions (21) imply sequentially

$$0=\sum_{\ell=1}^{s}X(\ell)\Pi^1(p-\ell+i)=\sum_{k=0}^{p}(\sum_{\ell=1}^{s}X(\ell)H(k-\ell+i))\Pi(p-k)=-\Pi^1(p+i),$$

Repeating the same argument for the last $p$ columns of $\Sigma(s)$ find that under conditions (21) $\Psi^1(q+j)=0$, $j=1,...,p$. Therefore, $\Sigma(s)$ has less than full row rank and the Sylvester matrix is singular regardless of the rank of the end coefficient matrices $(\Pi(p),\Psi(q))$.

As shown above, a common factor with finite degree inverse exists only if the Sylvester matrix is singular because its bottom block row has less than full row rank. Here it is shown that a common factor of finite degree exists only if the Sylvester matrix is singular because linear combinations of all its row blocks have less than full row rank.

*Illustration*. Consider the matrix polynomials

$$(\Pi(z,2),\, \Psi(z,2)) = (I_m + (X(1)+\Pi^1(1))z + (X(1)\Pi^1(1)+\Pi^1(2))z^2,$$
$$I_m + (X(1)+\Psi^1(1))z + (X(1)\Psi^1(1)+\Psi^1(2))z^2),\ X(1)\neq 0,$$

satisfying $X(1)(\Pi^1(2),\Psi^1(2))=0$, with common factor $I_m + X(1)z$ and Sylvester matrix

$$S(2,2)=\begin{pmatrix} I_m & 0 & I_m & 0 \\ X(1)+\Pi^1(1) & I_m & X(1)+\Psi^1(1) & I_m \\ X(1)\Pi^1(1)+\Pi^1(2) & X(1)+\Pi^1(1) & X(1)\Psi^1(1)+\Psi^1(2) & X(1)+\Psi^1(1) \\ 0 & X(1)\Pi^1(1)+\Pi^1(2) & 0 & X(1)\Psi^1(1)+\Psi^1(2) \end{pmatrix}.$$

Verify that $(X(1)^4,-X(1)^3,X(1)^2,-X(1))S(2,2)=0$, where the matrix of powers of $X(1)$ are the second to fifth terms in the expansion of $(I_m + X(1)z)^{-1}$. The common factor $I_m + X(1)z$ is unique if the null space of $S(2,2)$ contains only one set of $m$ vectors with the $i-th$ vector consisting of the $i-th$ rows of the four powers of $X(1)$, $i=1,\ldots,m$. In the scalar case $\Pi^1(2)=\Psi^1(2)=0$ and the alternative is a pair $(\Pi^1(z,1),\,\Psi^1(z,1))$ both of degree one. When $m>1$ both the assumed and the alternative pair can be of the same degrees.

This example illustrates the necessary condition (18) of Corollary 1 and the fact that Condition (21) is a system of nonlinear equations in the coefficient matrices of the inverse of the common factor. Unlike the common factors with finite degree inverses, $I_m + X(1)z$ a common factor does not imply that $I_m + FX(1)z$ with $m\times m$ $F$, is a common factor. The illustration also constructs the system of matrix equations to be solved in finding the common factor of finite degree of this example.

Theorem 1 defines a general solution technique if the truncated transfer function $T(z,n)$ is given. Theorems 2 and 3 define their solution techniques for given coefficient matrices of the polynomial matrices $(\Pi(z,p),\Psi(z,q))$. Whereas linear equations are involved in Theorem 2, in general the solution techniques take the form of nonlinear matrix equations.

9. *A singular Sylvester matrix without a common factor*.

Simple examples with a singular Sylvester matrix, yet without a common factor can be constructed from a pair of polynomial matrices of degrees $(p,q)$ with $p=1$. From Theorem 1 the matrix

$$\Im(m+q)=\left((-\Pi(1))^{m-1}\mathrm{T}(q),\ldots,-\Pi(1)\mathrm{T}(q),\mathrm{T}(q)\right),\ \mathrm{T}(q)=\sum_{i=0}^{q}(-\Pi(1))^i\,\Psi(q-i)$$

is to have rank $m$, whereas the determinant of the Sylvester matrix $|S(1,q)|=|\mathrm{T}(q)|$ is to be zero. In general the elements of $\Psi(q)$ can be chosen in many ways, given the other matrices, to satisfy both $\rho(\Im(m+q))=m$ and $|\mathrm{T}(q)|=0$, as long as in those choices $\Pi(1)\mathrm{T}(q)\neq 0$, $(\Pi(1),\,\mathrm{T}(q))$ do not commute and $\rho(\Pi(1),\Psi(q))=m$.

Observe that if $(\Pi(1), \mathrm{T}(q))$ commute, $\Im(m+q) = \mathrm{T}(q)((-\Pi(1))^{m-1}, ..., -\Pi(1), \quad I_m)$, so that its rank cannot exceed that of $\mathrm{T}(q)$ and a common factor exists, confirming Corollary 1. Also verify that if there exists $H(1) \neq 0$ such that $H(1)(\Pi(1), \Psi(q)) = 0$, then $H(1)\Im(m+q) = 0$ and $(I_m + H(1)z)^{-1}$ is a common factor, confirming Theorem 2.

## 10. *Conclusion*.

In the linear systems and time-series literature the problem of a common factor in the two polynomial matrices that after division generate a given transfer function has been analyzed under the restriction that potential common factors are unimodular. Polynomial matrix algebraic results such as developed in [6] have been utilized in this analysis. In this paper we followed the Sylvester tradition [7], in which the existence condition does not depend on whether the common factor is unimodular or not. Furthermore, Sylvester's result does not depend on whether common factors are small or large.

From the algebraic point of view, the proof of the generalization of Sylvester's result is based on defining the inverse of a polynomial matrix $(I_m + X(1)z + ...)^{-1}$ in a neighborhood of zero, as another infinite series $I_m + \mathrm{H}(1)z + ...$, obtainable sequentially through diagonally dominant Toeplitz matrices [1] and not on the notion that an inverse is a matrix of ratios of determinants. The latter notion drives the concept of unimodular common factors, whereas the former permits the collection of terms with the same power in the vector product multiplications, regardless if the determinant of the common factor is a constant or not.

The generalization of Sylvester's criterion to polynomial matrices has not been studied in the literature and the results, some expected and some non-expected alike, are pure algebra results. From the applied perspective, they add to our understanding of problems of non-uniqueness of representations in linear systems. The time series Arma(p,q) model literature considers the uniqueness problem for models in coprime form [8]. If it is desirable to transform a model not in coprime form to one in coprime form, the Sylvester criterion might help in discovering the common factors that are not unimodular. Interestingly, under the generalized Sylvester approach of this paper models need not be in coprime form.

## *References*.


[1] Davis, H.T. (1936), *The Theory of Linear Operators*, The Principia Press, Bloomington, Indiana.

[2] Barnett S. (1990), Algebraic Manipulation of Polynomial Matrices: A Survey, *IMA Journal of Mathematical Control & Information*, 7, 249-256

[3] Rowe, A. (1972), The Generalized Resultant Matrix, *J. Inst. Maths Applics*, 9, 390-396.

[4] Kung, S., Kailath T. & Morf, M. (1976), A Generalized Resultant Matrix, *Proc. 1976 IEEE Conference on Decision & Control*, 892-895, Florida.

[5] Kailath, T. (1980), *Linear Systems*, Prentice-Hall, Inc., N.J.

[6] MacDuffee, C.C. (1933), *The Theory of Matrices*, Springer, Berlin.

[7] Sylvester, J. (1840), A Method of determining by mere Inspection the Derivatives from two Equations of any Degree, *The London and Edinburgh Philosophical Magazine and Journal of Science*, January-June, Vol. XVI, 132-135.

[8] Hannan, E.J. & Deistler, M. (1988), *The Statistical Theory of Linear Systems*, John Wiley & Sons, New York.