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An Exact Non-cooperative Support for the Sequential Raiffa Solution

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Abstract

This article provides an exact non-cooperative foundation of the sequential Raiffa solution for two person bargaining games. Based on an approximate foundation due to Myerson (1997) for any two-person bargaining game (S, d) an extensive form game $G^{S,d}$ is defined that has an infinity of weakly subgame perfect equilibria whose payoff vectors coincide with that of the sequential Raiffa solution of (S, d). Moreover all those equilibria share the same equilibrium path consisting of proposing the Raiffa solution and accepting it in the first stage of the game.

By a modification of $G^{S,d}$ the analogous result is provided for subgame perfect equilibria. Finally, it is indicated how these results can be extended to implementation of a sequential Raiffa (solution based) social choice rule in subgame perfect equilibrium.

Key words: Raiffa solution, non-cooperative foundation Nash program, implementation, solution based social choice rule

JEL Classification: C71; C72; C78

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1 Introduction

In two versions of a paper on arbitration schemes Raiffa (1951, 1953) almost simultaneously with Nash (1950, 1953) proposed and analyzed four different bargaining solutions, among them the sequential Raiffa solution. Despite attractive features this solution concept, mentioned also in Luce and Raiffa (1957) and Shubik (1985), remained for a long time unexplored. The first contribution to an analysis of this solution to my knowledge is a non-cooperative foundation in Myerson (1991, pp. 393, 394). There, the Raiffa solution is approximated by the payoff vector sequence of unique subgame perfect equilibria of extensive games with the number of rounds going to infinity.

A recent different approximate support for the discrete Raiffa solution can be found in Tanimura and Thoron (2009). Axiomatic characterizations have only recently been found (cf. Anbarci and Sun (2009), Trockel (2009)).

The present paper provides an exact non-cooperative foundation based on Myerson's approximate one. Moreover, it is indicated how this support result in the sense of the Nash program can be extended to mechanism theoretic implementation in subgame perfect Nash equilibrium.

2 Basic Concepts and notation

We shall use in the following two different types of game, namely 2-person cooperative bargaining games and 2-person non-cooperative games in extensive form, briefly extensive games. The definition of the latter ones is quite intricate though their illustration via game trees is very intuitive. I shall use this notion as treated in Myerson (1991, chapter 2) or in Mas-Colell at al. (1995, chapter 7).

As to bargaining games I shall use the following

Definition:

A two *person bargaining game* is a pair (S, d), where $d \in S \subset \mathbb{R}^2_+$ and $S \neq \emptyset$ is compact, convex and comprehensive (i.e. $d \leq y \leq x \in S \Longrightarrow y \in S$). The set of two-person bargaining games is denoted $\mathcal{B}$.

Definition:

A *bargaining solution* is a mapping

$L : \mathcal{B} \longrightarrow \mathbb{R}^2_+ : (S, d) \mapsto L(S, d) \in S.$

If we can associate with any $(S, d) \in \mathcal{B}$ some extensive game $G^{S,d}$ whose subgame perfect equilibrium payoff vectors coincide with $L(S, d)$ we say that the game $G^{S,d}$ *supports the solution* $L(S, d)$ of (S, d) by *subgame perfect equilibrium.* Such a support provides a *non-cooperative foundation* for the solution L in the sense of the *Nash program* (cf. Serrano (2005)).

In the present paper I want to provide such a non-cooperative foundation for the *sequential Raiffa solution.*

The relevant notion of a *subgame perfect Nash equilibrium* due to Selten (1965) is defined as a Nash equilibrium on an extensive game that on any subgame ($\backsim$ subtree) induces a Nash equilibrium.

3 The Myerson Game

Although Myerson (1991) is actually working with a slightly different class of bargaining games his construction is equally valid for the set $\mathcal{B}$ of bargaining games I am going to consider.

Consider for any $(S, d) \in \mathcal{B}$ the mapping $f^S : S \longrightarrow \mathbb{R}^2$ defined by

$f^S := (x_1, x_2) := (f_1^S(x_2), f_2^S(x_1))$ with

$f_1^S(x_2) := max\{x_1 \mid (x_1, x_2) \in S\}$ and

$f_2^S(x_1) := max\{x_2 \mid (x_1, x_2) \in S\}$.

Next define inductively the following sequence $(\pi^k)_{k\in\mathbb{N}}$ of payoff vectors:

·) $\pi^1 = (\pi_1^1, \pi_2^1) := d$

·) $\forall k \in \mathbb{N} : \pi^{k+1} := 1/2[(f_1^S(\pi_2^k), \pi_2^k) + (\pi_1^k, f_2^S(\pi_1^k)]$.

This process due to Myerson describes the procedure by which Raiffa (1951, 1953) had defined the sequential Raiffa solution (see Figure 1). It converges to $R(S, d)$, the sequential Raiffa solution of (S, d). This is illustrated in Figure 1.

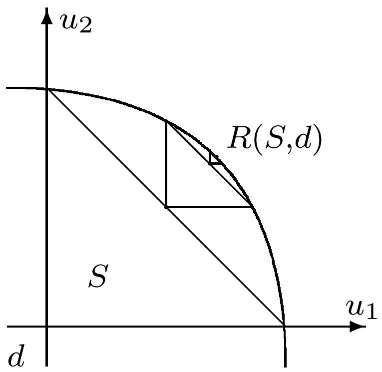

Figure 1:

Now I will present Myerson's (p. 393) extensive games with k rounds, for $k \in \mathbb{N}$ but with slightly different notation.

"At each round a player is selected at random to make an offer, which can be any point in S. After getting this offer, the other player can either accept or reject it. The first offer that is accepted is the payoff allocation that the players get in the game. If no offer is accepted at any of the k rounds, then the players get the disagreement payoff allocated. At each round, each player has probability 1/2 of being the one to make an offer, independently of the past history of the game."

According to Myerson (p. 393):

"This game has a unique subgame-perfect equilibrium in which the first round offer is accepted, and the expected payoff to each player i is π_i^{k-1}. In this equilibrium, at the ℓ-th round from the end, for any $\ell \in \{1, ..., k\}$, each player i would accept any offer that gives him at least π_i^ℓ because that would be his expected payoff in the subgame consisting of the last $\ell - 1$ rounds if no offer were accepted before; and so player 1 would make the offer $f_1^S(\pi_2^\ell)$, player 2 would make the offer $f_2^S(\pi_1^\ell)$. Thus, Raiffa's sequential bargaining solution is the limit of the expected outcomes of these games in equilibrium, as the number of possible offers goes to infinity."

This beautiful game is reminiscent of the Rubinstein game (cf. Binmore et al. (see References)) but differs in two aspects.

First, there is no "shrinking cake" or discounting in Myerson's games. In contrast to Rubinstein's game Myerson considers a sequence of games with increasing but finite lengths.

Secondly, in Myerson's games in each round the proposer is chosen again by random rather than having alternative offers after a beginner has been determined randomly. By making a crucial element of the model, namely discounting, more and more negligible the unique subgame perfect equilibrium of the Rubinstein game approximates the Nash solution.

The sequence of Myerson games by its construction exhibits convergence of its subgame perfect equilibria to the Raiffa solution. Both approaches provide only approximate non-cooperative foundations of the respective bargaining solutions.

4 An exact non-cooperative foundation

Denote the extensive form game with k rounds of Myerson described in the previous section by G^k and its subgame perfect equilibrium payoff vector by $\hat{\pi}^k, k \in \mathbb{N}$.

From Myerson's analysis we know that

$lim_{k \longrightarrow \infty} \hat{\pi}^k = \pi^* := R(S, d)$.

On this basis we define now an extensive form G as follows:

At round 0, one of the two players of the bargaining game (S, d) is selected randomly (again with probability 1/2) to make a proposal $x \in S$. After getting this offer, the other player can choose an element $k \in \mathbb{N}_\circ := \mathbb{N} \cup \{0\}$.

If he chooses 0 the proposal is accepted and the payoff will be realized. If he chooses $k \in \mathbb{N}$ the proposal is rejected and the game G^k will be played.

Like the Myerson games $G^k, k \in \mathbb{N}$, the game G has an infinity of Nash equilibria, among them the one where every player chooses always proposal $R(S, d)$ and always accepts this proposal and rejects any other one. Another one determines the Nash (or any other) bargaining solution in an analogous way. This is essentially the situation we have in Nash's simple demand game.

In our context salvation comes from the use of subgame perfect equilibrium. But in contrast to $G^k, k \in \mathbb{N}$, the game G does not have any subgame perfect equilibrium. But fortunately a weaker notion of equilibrium will do the job as well, namely a weak subgame perfect equilibrium.

Definition:

A Nash equilibrium of an extensive game is called *weakly subgame perfect* when it induces some Nash equilibrium in every subgame in which a Nash equilibrium exists.

I shall briefly discuss this concept in a separate section following the present one. Here I will use this concept in order to state and prove the first non-cooperative support result.

Proposition 1:

For any bargaining game $(S, d) \in \mathcal{B}$ the extensive game $G(= G^{S,d})$ as defined above has an infinity of weakly subgame perfect equilibria with identical equilibrium path and equilibrium payoff vector $R(S, d)$.

Proof:

Notice first that after proposal $x \in S, x \neq R(S, d)$ of some player a one-person subgame of the other player starts that has no optimal strategy for him, hence no Nash equilibrium. Therefore no Nash equilibrium of G can be subgame perfect. Notice further that as in any $G^k, k \in \mathbb{N}$, any efficient proposal and its acceptance defines Nash equilibria.

Consider the following strategy profile:

Every player, whenever chosen by random as a proposer chooses $R(S, d)$. And this will always be accepted by the other player. Any proposal $x \in S, x \neq R(S, d)$ of player $3 - i$ will be rejected by player i via some $k \in \mathbb{N}$ such that $\hat{\pi}_i^k > x_i (i \in \{1, 2\})$. In any $G^k, k \in \mathbb{N}$ $\hat{\pi}^k$ is played.

First we have to show that this defines a Nash equlibrium Deviation of $3 - i$ from $R(S, d)$ to some $x \in S, x \neq R(S, d)$ is either worse for himself or for both players or for i.

If x is worse only for i then i chooses some $k \in \mathbb{N}$ hence de facto some $\hat{\pi}^k$ close enough to $R(S, d)$ such that $\hat{\pi}_i^k > x_i$ and $\hat{\pi}_{3-i}^k < R_{3-i}(S, d)$.

Thus unilateral deviation of a proposer will not yield an improvement. If, however, $R(S, d)$ is proposed by $3 - i$ player i cannot to his advantage deviate from acceptance. If he rejects the best that can happen to him is a new accepted proposal $R(S, d)$, otherwise some $\hat{\pi}^k$ that is worse for him.

So the proposed strategy profile defines a Nash equilibrium. It remains to show that it is weakly subgame perfect.

So what are the subgames that do have Nash equilibria ?

The game G itself and the games $G^k, k \in \mathbb{N}$. In all these games Nash equilibria are induced. So the considered Nash equilibria are weakly subgame perfect. They differ only by the various sufficiently large $k \in \mathbb{N}$ chosen in the one-player games originating after a proposal $x \in S, x \neq R(S, d)$. So the equilibrium path is for all of them the same: propose $R(S, d)$ and accept. So is the equilibrium payoff vector $R(S, d)$

5 Remarks on weakly subgame equilibria

In contrast to the games $G^k, k \in \mathbb{N}$, the game G has an infinity of weakly subgame perfect equilibria. How bad is this ? There is no coordination problem involved as long as both players stay on the equilibrium path and the equilibrium payoff is uniquely determined. So the multiplicity of those equilibria appears to be harmless, in particular as subgame perfect equilibria also may have multiple ways of behavior off the equilibrium path.

So the criticism could better be based on the lack of credible threats to reject, because there is no optimal way of rejecting! But from a decision theoretical point of view this criticism is dubious. Almost everywhere in game theory we take participation in the game at hand as given. If there is a choice between money amounts $\{-50,\ 1,\ 2, ..., 10\}$, we take it for granted that -50 is rejected (via accepting 10). If the choice is among $\{-50\} \cup \mathbb{N}$, do we think -50 will be accepted because there is no best better alternative ? In real life we avoid worst cases even if we are unable to do that in an optimal way.

But obviously this is a controversial point. I will therefore provide a modified non-cooperative support via subgame perfect equilibria in the next section.

6 Subgame perfect direct support

When attempting to base a direct subgame perfect equilibrium foundation for the sequential Raiffa solution based on Myerson's games $G^k, k \in \mathbb{N}$, the dilemma is the appearance of those one-player subgames without Nash equilibrium. There are two potential ways out:

1. Add a best alternative to the set $\mathbb{N}_\circ$. This would be adding the possibility of repeating G! We would end up in an infinity of subgame perfect equilibria, all of which at some stage have proposal of $R(S, d)$ followed by acceptance. But the equilibrium paths are different and may have unbounded lengths. I do not find this an appealing model.

2. Stop those "one player subgames" from being subgames. In order to do so we modify our original game G in the following way.

   In the beginning the proposer is chosen randomly with probability 1/2. But the two players do not observe that random choice. So each player has probability 1/2 that he is the chosen proposer and 1/2 that he has to react to his opponent's proposal. Accordingly, both players' strategies have to contain full descriptions of what they would propose and how they would reject to any possible proposal $x \in S$.

For both players the strategy sets are the same, namely $S \times \mathbb{N}_\circ^S$, and both players when they have to make their moves do not know which part of their strategy x or f is relevant for the continuation and the final outcome. So if they choose strategies $(x^1, f^1), (x^2, f^2) \in S \times \mathbb{N}_\circ^S$, depending on the random move choosing 1 or 2 either $f^2(x^1)$ or $f^1(x^2)$ dictates the continuation of the play.

This modification of our original game G is illustrated in the equivalent Figures 2 and 3.

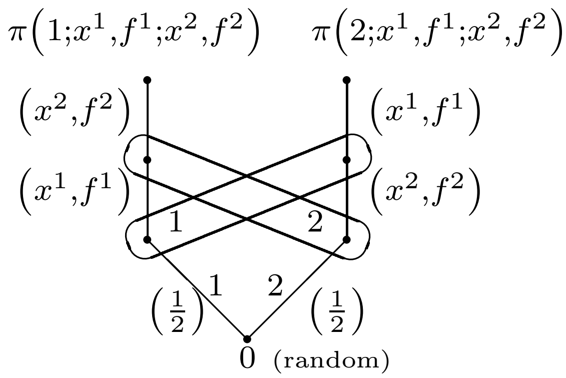

Figure 2:

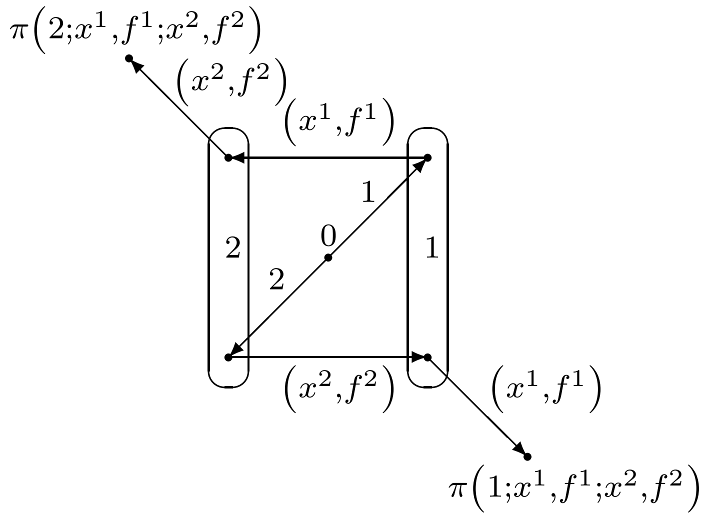

Figure 3:

Notice that these are only schematic illustrations rather than extensive game trees because the infinite sets of possible moves for players 1 and 2 are represented in these figures only by one typical move, namely $(x^i, f^i), i = 1,\ 2$. Also they only represent the reduced game where any root of a subgame $G^k, k \in \mathbb{N}$, is replaced by the subgame perfect equilibrium payoff vector of G^k.

The $\pi(i, x^1, f^1, x^2, f^2), i = 1, 2$, is the equilibrium payoff vector on the equilibrium path, the payoff vector of subgame perfect equilibria of the games $G^k, k \in \mathbb{N}$, otherwise.

The construction in Figure 2 is similar to the way Sudhölter et al. (2000, p.302) define the canonical extensive form for the Battle of Sexes game. It has precisely the intended effect in the present context. The one-player subgames without optimal moves have vanished now.

The only remaining subgames of the modified game $\tilde{G}$ are $\tilde{G}$ itself and the Myerson games $G^k, k \in \mathbb{N}$.

In $\tilde{G}$ any subgame perfect equilibrium is a pair $(x^1,\ f^1_1; x^2, f^2_{k^2})$ with $x^1 = x^2 = R(S, d)$ and $f^i_{k^i} : S \longrightarrow \mathbb{N}_\circ : x \mapsto f^i_{k^i}(x)$, such that

$$f^i_{k^i}(x) = \begin{cases} 0 & \text{for } x_i \geq R_i(S, d) \\ k^i & \text{for } x_i < R_i(S, d) \end{cases}$$

and $k^i \in \{k' \in \mathbb{N} \mid \hat{\pi}_i{}^{k'} > x_i\}, i = 1,\ 2$

After the first moves of both players the game ends in its equilibrium with payoff vector $R(S, d)$ or in some game $G^k, k \in \mathbb{N}$, where the unique subgame equilibrium is induced.

The multiplicity of subgame perfect equilibria comes in via the various k that can be possibly chosen for f. But the equilibrium path is unique.

I can formulate the modified version of the support result now as

Proposition 2:

For any bargaining game $(S, d) \in B$ the extensive game $\tilde{G}(= \tilde{G}^S)$ has an infinity of subgame perfect equilibria with a unique equilibrium and equilibrium payoff vector in the first round, namely $R(S, d)$

Proof:

Having Proposition 1 there is nothing to show as the only change resulting from substituting $\tilde{G}$ for G is the reduction of subgames to those having Nash equilibria.

Remark:

It is obvious that on the non-trivial information sets of players 1 and 2 we may introduce beliefs and deduce an infinity of sequential equilibria that result in the Raiffa solution.

7 Conclusion

We have provided an exact non-cooperative foundation of the sequential Raiffa solution R of a given bargainig game (S, d) in the sense of the Nash program by supporting $R(S, d)$ as a (weakly) subgame perfect equilibrium of an extensive game $\tilde{G}(G)$. There is an infinity of such equilibria, but all of them share the same unique equilibrium path that in the first round of the game leads to the equilibrium payoff vector $R(S, d)$.

As far as the implementation of R in the sense of mechanism theory is concerned one may derive weak implementation in subgame perfect Nash equilibrium of an R-induced social choice rule. Applying an embedding principle of Trockel (2002) this can be done in an similar way as for the Nash solution in Trockel (2000, section 5).

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