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Departamento de Estadística  
Universidad Carlos III de Madrid  
Calle Madrid, 126  
28903 Getafe (Spain)  
Fax (34-91) 6249849

## BOOTSTRAP PREDICTION MEAN SQUARED ERRORS OF UNOBSERVED STATES BASED ON THE KALMAN FILTER WITH ESTIMATED PARAMETERS\*

Alejandro Rodriguez<sup>1</sup> and Esther Ruiz<sup>2</sup>

### Abstract

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Prediction intervals in State Space models can be obtained by assuming Gaussian innovations and using the prediction equations of the Kalman filter, where the true parameters are substituted by consistent estimates. This approach has two limitations. First, it does not incorporate the uncertainty due to parameter estimation. Second, the Gaussianity assumption of future innovations may be inaccurate. To overcome these drawbacks, Wall and Stoffer (2002) propose to obtain prediction intervals by using a bootstrap procedure that requires the backward representation of the model. Obtaining this representation increases the complexity of the procedure and limits its implementation to models for which it exists. The bootstrap procedure proposed by Wall and Stoffer (2002) is further complicated by fact that the intervals are obtained for the prediction errors instead of for the observations. In this paper, we propose a bootstrap procedure for constructing prediction intervals in State Space models that does not need the backward representation of the model and is based on obtaining the intervals directly for the observations. Therefore, its application is much simpler, without losing the good behavior of bootstrap prediction intervals. We study its finite sample properties and compare them with those of the standard and the Wall and Stoffer (2002) procedures for the Local Level Model. Finally, we illustrate the results by implementing the new procedure to obtain prediction intervals for future values of a real time series.

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**Keywords:** NAIRU, output gap, Parameter uncertainty, Prediction Intervals, State Space Models.

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<sup>1</sup> Department of Statistics, Universidad Carlos III de Madrid, e-mail: [afrodrig@est-econ.uc3m.es](mailto:afrodrig@est-econ.uc3m.es).

<sup>2</sup> Department of Statistics, Universidad Carlos III de Madrid, e-mail: [ortega@est-econ.uc3m.es](mailto:ortega@est-econ.uc3m.es).

# Bootstrap Prediction Mean Squared Errors of Unobserved States based on the Kalman filter with estimated parameters \*

Alejandro F. Rodríguez and Esther Ruiz<sup>†</sup>

*Departamento de Estadística  
Universidad Carlos III de Madrid  
28903 Getafe (Madrid), Spain*

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## Abstract

In the context of linear state space models with Gaussian errors and known parameters, the Kalman filter generates best linear unbiased predictions of the underlying components together with their corresponding prediction mean squared errors (PMSE). However, in practice, the filter is run by substituting some parameters of the model by consistent estimates. In these circumstances, the PMSEs given by the Kalman filter do not take into account the parameter uncertainty and, consequently, they underestimate the true PMSEs. In this paper, we propose two new bootstrap procedures to obtain PMSE of the unobserved states based on obtaining replicates of the underlying states conditional on the information available at each moment of time. By conditioning on the available information, we simplify the procedure with respect to alternative bootstrap proposals previously available in the literature. Furthermore, we show that the new procedures proposed in this paper have better finite sample properties than the alternatives. We implement the proposed procedures for estimating the PMSE of several key unobservable US macroeconomic variables as the output gap, the non accelerating inflation rate of unemployment (NAIRU), the long-run investment rate and the core inflation. We show how taking into account the parameter uncertainty may change the prediction intervals constructed for those unobservable macroeconomic variables and, in particular, change the conclusions about the utility of the NAIRU as a macroeconomic indicator for expansions and recessions.

Keywords: NAIRU, output gap, Parameter uncertainty, Prediction Intervals, State Space Models.

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<sup>†</sup>Corresponding author. Tel.: +34-91624-9851; Fax: +34-91624-9849; E-mail address: ortega@est-econ.uc3m.es

# 1 Introduction

State space models and the Kalman filter are very popular tools for describing the dynamic evolution of a large range of economic and financial time series in which there are unobserved variables of interest. Some applications include the estimation of regression models with time varying parameters as in Cooley and Prescott (1973), models with unobserved expectations of agents as Hamilton (1985), models with unobserved factors like in Engle and Watson (1981) for metropolitan wage prices or models with measurement errors like in Howrey (1984). There are also many applications in which the unobserved states are components with a direct interpretation; see, for example, Orphanides and van Norden (2002) and Doménech and Gómez (2006) for estimating unobserved the output gap in several economies, Pedregal and Young (2006) for electricity load demand with unobserved modulated periodic components, Stock and Watson (2007) for a trend-cycle model with stochastic volatility fitted to US inflation or Rueda and Rodríguez (2009) to estimate fertility rates, just to cite a few recent empirical applications. One of the main attractiveness of state space models is that they allow the estimation of the underlying states which, as the previous applications illustrate, are often of interest in themselves.

Obviously, there is also an interest in obtaining measures of the uncertainty associated with the corresponding estimates of the states. In the context of linear state space models with Gaussian errors and known parameters, the Kalman filter generates best unbiased one-step-ahead predictions of the unobserved components together with their corresponding prediction mean squared errors (PMSE). However, in practice, the filter is run by substituting some parameters of the model by consistent estimates. In this case, the PMSEs resulting from the filter underestimate the true PMSEs because they do not take into account the additional uncertainty due to the parameter estimation. Counting for this uncertainty may be important in many practical situations; see, for example, Durbin and Koopman (2000). In this paper, we focus on analyzing how the parameter uncertainty can change the intervals and consequently,

the conclusions about the evolution of the output gap, the non accelerating inflation rate of unemployment (NAIRU), the long-run investment rate and the core inflation in the US which are obviously variables of interest in the context of macroeconomic policy; see Orphanides and van Norden (2002), Kang et al. (2009) and Sinclair (2009) for empirical applications of unobserved component models to estimate some of these variables. We build on previous work by Doménech and Gómez (2006) who propose a multivariate unobserved components model for the US economy with the four unobserved variables mentioned above. They obtain prediction intervals of the unobserved output gap, NAIRU, core inflation and structural investment rate that do not incorporate the parameter uncertainty. We show that taking into account the additional uncertainty associated with the estimation of the parameters, the conclusions about the utility of the NAIRU as a macroeconomic indicator for expansions and recessions can be changed.

There are several alternatives proposed in the literature to incorporate the estimation uncertainty into the PMSEs of the Kalman filter estimates of the unobserved components. They can be classified into three main groups. First, several proposals are based on the asymptotic distribution of the parameter estimator; see Ansley and Kohn (1986), Hamilton (1986) and Quenneville and Singh (2000). These procedures can be inadequate in small samples because, in this case, the asymptotic distribution could be a poor approximation of the finite sample distribution of the parameter estimator. The second group of alternatives is based on using a full Bayesian approach. However, in relatively large models, the computations can become very heavy and time consuming; see Harvey (2000). Finally, Pfeffermann and Tiller (2005) propose using bootstrap procedures which have the advantage of being computationally simple even in relatively complicated models. The bootstrap PMSEs proposed by Pfeffermann and Tiller (2005) are based on obtaining the unconditional PMSEs of the estimates of the underlying states. However, one should note that the Kalman filter is designed to generate PMSEs conditional on the available information set. Although this distinction is irrelevant in state space models with time-invariant system matrices, it could be important when the

system matrices in the state space model are observation dependent. Furthermore, by taking into account this distinction, it is possible to simplify the bootstrap procedures proposed by Pfeiffermann and Tiller (2005) improving at the same time their finite sample performance.

In this paper, we propose two new bootstrap procedures to obtain PMSEs of the Kalman filter estimates of the unobserved states in Gaussian state space models that incorporate the parameter uncertainty. By obtaining replicates of the underlying states conditional on the information available at each moment of time, we simplify the procedures with respect to alternative bootstrap PMSEs. The first bootstrap procedure proposed is parametric and it is based on resampling from the assumed distribution of the errors. The second procedure is based on resampling from the residuals of the estimated model and consequently, does not assume any particular error distribution. We carry out Monte Carlo experiments to analyze the finite sample performance of our procedures and show that, they reduce the biases of the asymptotic procedure of Hamilton (1986) and the bootstrap PMSEs proposed by Pfeiffermann and Tiller (2005). We also show that the proposed procedures can be implemented in the context of non-Gaussian models with good performance.

In the empirical application to estimate the macroeconomic unobservable variables describe above, we show that, given that the sample size is relatively large, the PMSEs and, consequently, the prediction intervals constructed for these variables, are rather similar when using the standard Kalman filter or the asymptotic approximation of Hamilton (1986). Doménech and Gómez (2006) use the standard Kalman filter PMSE to construct prediction intervals for the NAIRU and conclude that it is an adequate indicator for predicting expansions and recessions. However, the intervals are wider when using the bootstrap methods, specially, when using the procedures proposed in this paper that take into account the added uncertainty due to the parameter estimation. Because of this additional uncertainty, our intervals support the doubts of Staiger et al. (2001) on the ability of the difference between the NAIRU and the unemployment rate for predicting periods of expansion and recession.

The rest of the paper is organized as follows. Section 2 describes the Kalman filter and shows the biases incurred when estimating the PMSEs of the estimates of the underlying states by running the filter with estimated parameters. We also briefly describe the asymptotic procedure of Hamilton (1986) and the bootstrap procedures proposed by Pfeiffermann and Tiller (2005) to overcome these biases. In Section 3, we propose two new bootstrap procedures to obtain PMSEs of the one-step-ahead estimator of the unobserved states that take into account the parameter uncertainty in the context of conditionally Gaussian state space models. Their finite sample properties are analyzed and compared with those of the standard Kalman filter, the asymptotic and bootstrap PMSEs proposed by Hamilton (1986) and Pfeiffermann and Tiller (2005) respectively. All the results are illustrated with simulated data in the context of the random walk plus noise (RWN) model with homoscedastic and heteroscedastic errors. Section 4 analyzes the finite sample performance of the new bootstrap procedures in non-Gaussian models and compare them with those obtained using the alternative procedures. Section 5 contains the empirical application in which we estimate the uncertainty associated with the unobserved quarterly output gap, NAIRU, investment rate and core inflation in the US. Finally, Section 6 concludes the paper.

## 2 PMSE of Kalman filter estimators of unobserved components

Unobserved component models can be casted into the following state space model

$$Y_t = Z_t\alpha_t + d_t + R_{1t}\varepsilon_t, \quad (1a)$$

$$\alpha_t = T_t\alpha_{t-1} + c_t + R_{2t}\eta_t, \quad t = 1, \dots, T, \quad (1b)$$

where  $Y_t$  is an  $N \times 1$  vector time series observed at time  $t$ ,  $\alpha_t$  is the  $m \times 1$  vector of unobservable state variables,  $\varepsilon_t$  is a  $k \times 1$  vector of independent white noises with zero mean and covariance matrix  $H_t$  and  $\eta_t$  is a  $g \times 1$  vector of serially uncorrelated disturbances with zero mean and covariance matrix  $Q_t$ . The disturbances  $\varepsilon_t$  and  $\eta_t$  are uncorrelated with each other in all time

periods. Finally, the initial state vector,  $\alpha_1$ , has mean  $a_{1|0}$  and covariance matrix  $P_{1|0}$ . All the system matrices,  $Z_t$ ,  $d_t$ ,  $T_t$ ,  $c_t$ ,  $R_{1t}$ ,  $R_{2t}$ ,  $H_t$  and  $Q_t$ , are assumed to be known one-step-ahead<sup>1</sup>. The model in (1) is time-invariant when, with the exception of  $d_t$  and  $c_t$ , all the system matrices are time-invariant.

## 2.1 PMSE of the Kalman filter with estimated parameters

The Kalman filter provides estimates of the underlying states,  $\alpha_t$ , and their corresponding PMSE, which are denoted by  $a_{t|t-1}$  and  $P_{t|t-1}$  respectively, given the information available at time  $t - 1$ , i.e.  $\{Y_1, \dots, Y_{t-1}\}$ <sup>2</sup>. If the errors are further assumed to have a conditional joint Normal distribution, using well known results of this distribution,  $a_{t|t-1}$  and  $P_{t|t-1}$  are the conditional mean and its conditional PMSE respectively. In particular, it is possible to see that

$$\begin{matrix} \alpha_t \\ Y_t \end{matrix} \Big| Y_1, \dots, Y_{t-1} \sim \mathcal{N} \left[ \begin{pmatrix} a_{t|t-1} \\ Z_t a_{t|t-1} + d_t \end{pmatrix}, \begin{pmatrix} P_{t|t-1} & P_{t|t-1} Z_t' \\ Z_t P_{t|t-1} & F_t \end{pmatrix} \right] \quad (2)$$

where

$$a_{t|t-1} = T_t a_{t-1|t-2} + c_t + K_t F_{t-1}^{-1} V_{t-1} \quad (3a)$$

$$P_{t|t-1} = T_t P_{t-1|t-2} T_t' - K_t F_{t-1}^{-1} K_t' + R_{2t} Q_t R_{2t}', \quad (3b)$$

and  $K_t = T_t P_{t-1|t-2} Z_{t-1}'$ , is the filter gain,  $V_t = Y_t - d_t - Z_t a_{t|t-1}$  is the one-step-ahead vector of innovations and  $F_t$  is their covariance matrix given by  $F_t = Z_t P_{t|t-1} Z_t' + R_{1t} H_t R_{1t}'$ ; see, Harvey (1989) for details. When running the Kalman filter equations in (3), it is assumed that all the parameters involved in the system matrices and the initial conditions  $a_{1|0}$  and  $P_{1|0}$  are known. It is important to observe that in linear models in which the system matrices are independent of the observations, the PMSE,  $P_{t|t-1}$ , is also independent of the observations.

<sup>1</sup>When dealing with smoothing filters (based on observations up to time  $T$ ), the system matrices together with  $a_{1|0}$  and  $P_{1|0}$  are assumed to be known in all time periods.

<sup>2</sup>The Kalman filter also provides updated estimates of the underlying components at time  $t$  based on the information available up to time  $t$ . Furthermore, it is also possible to estimate  $\alpha_t$  based on all the information in the time series  $\{Y_1, \dots, Y_T\}$ . These latter estimates are known as smoothing estimates. Although in this paper we focus on one-step-ahead estimates of the underlying components, the results can be easily extended to updated and smoothing estimates.

Therefore, in this case,  $P_{t|t-1}$  is also the unconditional error covariance matrix associated with the conditional mean estimator of the underlying state.

Finally, it is also useful for the bootstrap procedures described later in this paper to express the state space model in (1) in what is known as the Innovation Form (IF) which depends on a unique disturbance vector instead of two. The IF is given by equation (3a) together with

$$Y_t = Z_t a_{t|t-1} + d_t + V_t. \quad (3c)$$

Note that the unique disturbance vector in the IF is the one-step-ahead vector of innovations,  $V_t$ .

Consider, as an illustration, the univariate RWN model. Although we consider this particular model for its simplicity, it has been successfully applied for explaining the dynamic evolution of many real time series; see, for example, Koopman and Bos (2004) and Stock and Watson (2007) who fit it for explaining monthly US inflation. The RWN model is defined by the following equations

$$y_t = \mu_t + \varepsilon_t \quad (4a)$$

$$\mu_t = \mu_{t-1} + \eta_t \quad (4b)$$

where  $y_t$  is the observation at time  $t$  of the series of interest and  $\varepsilon_t$  and  $\eta_t$  are mutually independent Gaussian white noises with variances  $\sigma_\varepsilon^2$  and  $\sigma_\eta^2 = \sigma_\varepsilon^2 q$  respectively, where  $q$  is known as the signal-to-noise ratio. We generate  $R = 1000$  replicates of  $\{y_t^{(j)}, \mu_t^{(j)}, j = 1, \dots, R\}$  by model (4) with  $\sigma_\varepsilon^2 = 1$  and  $q = 0.25$ , sample sizes  $T = 40, 100$  and  $500$ , and initial value equal to zero,  $\mu_0 = 0$ . For each replicate, we run the Kalman filter in (3) with known parameters to obtain one-step-ahead estimates of the underlying level,  $\mu_t^{(j)}$ , denoted by  $m_{t|t-1}^{(j)}$  and their PMSE, denoted by  $P_{t|t-1}^{(j)}$ . Furthermore, for each simulated series  $j$  and moment of time  $t$ , we also generate 10000 replicates of  $\mu_{t+1}^{(j)}$ , denoted by  $\mu_{t+1}^{(j,i)}$ ,  $i = 1, \dots, 10000$ , from the corresponding conditional distribution in (2). Finally, at each moment of time, we compute the empirical conditional PMSE of  $m_{t|t-1}^{(j)}$  given by  $PMSE_{t|t-1}^{(j)} = \frac{1}{10000} \sum_{i=1}^{10000} \left( \mu_{t+1}^{(j,i)} - m_{t|t-1}^{(j)} \right)^2$



and the relative bias<sup>3</sup>  $d_t^{(j)} = P_{t|t-1}^{(j)} / PMSE_{t|t-1}^{(j)} - 1$ . Figure 1 plots the averages over the Monte Carlo replicates of  $d_t^{(j)}$  denoted as KF1 which, as expected, evolve around zero through time regardless of the sample size considered. The average and standard deviations through time of the Monte Carlo averages plotted in Figure 1 have been reported in the left column of Table 1. Note that the average relative biases and their standard deviations, which as expected are very small and do not depend on the sample size, can be attributed to the simulation error.

We consider a second illustration with a slightly more complicate model in which the system of matrices are time-varying. In particular, we consider the RWN model in (4) but where the transitory noise,  $\varepsilon_t$ , is heteroscedastic and given by

$$\varepsilon_t = \varepsilon_t^\dagger \sigma_t, \quad (5)$$

where  $\sigma_t^2 = \alpha_0 + \alpha_1 v_{t-1}^2$ ,  $v_t$  is the innovation given by  $v_t = y_t - m_{t|t-1}$ , and  $\varepsilon_t^\dagger$  is a Gaussian white noise process with variance 1, distributed independently of  $\eta_t$ . Note that, given the specification of  $\sigma_t^2$  and assuming that the parameters are known, the model is still conditionally Gaussian since knowledge of past observations and past estimates of the state implies knowledge of the past innovations  $v_{t-1}$ . This model is related with the STARCH model described by Harvey et al. (1992) but they differ in that the STARCH model assumes that  $\sigma_t^2 = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2$  and, consequently, it is not conditionally Gaussian. Unobserved component models with heteroscedastic disturbances are becoming very popular to represent the dynamic evolution of macroeconomic variables as, for example, inflation or electricity prices; see, Broto and Ruiz (2009), Jungbacker et al. (2009) and Stock and Watson (2007) among many others. In this case, we also generate  $R = 1000$  series from the heteroscedastic RWN model with  $\varepsilon_t$  defined as in (5) with  $\alpha_0 = 0.6719$ ,  $\alpha_1 = 0.2$  and  $\sigma_\eta^2 = 0.25$ . The initial conditions are given by  $\mu_0 = 0$  and  $\sigma_1^2$  equal to the marginal variance of  $\varepsilon_t$ , which is one. As above, we run the Kalman filter for each simulated series and compute  $m_{t|t-1}^{(j)}$  and  $P_{t|t-1}^{(j)}$  and the corresponding

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<sup>3</sup>All programs for maximizing the log-likelihood and subsequent estimation of the unobserved components and PMSE were written by the first author in MATLAB.

biases  $d_t^{(j)}$ . Figure 2 plots the corresponding average biases denoted by KF2 which, similarly to those plotted in Figure 1 for the time-invariant model, are very close to zero for all sample sizes. The central column of Table 1 reports the averages and standard deviations through time of the averages of  $d_t^{(j)}$  plotted in Figure 1 which are identical to those reported in the time-invariant case.

Up to now, we have assumed that the parameters of the model are known when running the Kalman filter. However, in practice, some of these parameters are unknown and have to be substituted by consistent estimates. In this paper, we consider the Quasi-Maximum Likelihood (QML) estimator of the parameters; see, for example, Harvey (1989) and Durbin and Koopman (2001) for details. Denote by  $\hat{Z}_t, \hat{d}_t, \hat{H}_t, \hat{R}_{1t}, \hat{T}_t, \hat{c}_t, \hat{R}_{2t}$  and  $\hat{Q}_t$  the system of matrices where the unknown parameters have been substituted by their QML estimates. Furthermore, the initial conditions for the filter are also unknown. The usual practice is to assume that they are given by the unconditional distribution of the unobserved state in case of stationary states or by a diffuse prior distribution when they are non-stationary; see Harvey (1989). Then, equations (3a) and (3b) of the Kalman filter can be run with the system matrices substituted by their respective estimates providing  $\hat{a}_{t|t-1}$  and  $\hat{P}_{t|t-1}$ . Note that,  $\hat{a}_{t|t-1}$ , is an estimate of the conditional mean of the state,  $a_{t|t-1}$ . However,  $\hat{P}_{t|t-1}$  is not the PMSE of  $\hat{a}_{t|t-1}$  as it does not take into account the parameter uncertainty involved in its computation. Therefore,  $\hat{P}_{t|t-1}$  will underestimate the true conditional PMSE of  $\hat{a}_{t|t-1}$ .

To illustrate the biases incurred when  $\hat{P}_{t|t-1}$  is considered as the PMSE of  $\hat{a}_{t|t-1}$ , we consider again the same design of the Monte Carlo experiments carried out above with the homoscedastic and heteroscedastic RWN models. In this case, for each replicate, we estimate the parameters by QML using as starting values for the filter  $\hat{m}_{1|0} = 0$  and  $\hat{P}_{1|0} = \infty$  for both the homoscedastic and heteroscedastic models. Then, as before, the empirical PMSE of  $\hat{m}_{t|t-1}$  is calculated as  $PMSE_{t|t-1}^{(j)} = \frac{1}{10000} \sum_{i=1}^{10000} \left( \mu_t^{(j,i)} - \hat{m}_{t|t-1}^{(j)} \right)^2$  where  $\hat{m}_{t|t-1}$ , is the one-step-ahead estimate of  $\mu_t$  provided by the Kalman filter with estimated parameters. Figures 1 and

2 plot the averages through all replicates of the relative biases  $d_t^{(j)}$  for the homoscedastic and heteroscedastic models, respectively<sup>4</sup>. Note that in the homoscedastic model  $\widehat{P}_{t|t-1}$  underestimate the true PMSE of  $\widehat{m}_{t|t-1}$  denoted as KF2 in approximately 9% when  $T = 40$ . Obviously, because this bias is caused by using estimated parameters which are consistent, it disappears as the sample size increases. Something similar can be observed in the heteroscedastic model although in this case, the biases are clearly larger. For example, when  $T = 40$ , the bias is approximately 18%. The average biases reported in Table 1 show that when the Kalman filter is run with estimated parameters  $\widehat{P}_{t|t-1}$  is a negatively biased estimator the conditional true PMSE of  $\widehat{m}_{t|t-1}$ . These biases can be very important in small samples specially when the model is time-varying.

As we mentioned in the introduction, there have been several proposals in the literature to compute the PMSE of the estimator of the unobserved components that take into account the parameter uncertainty. Next, we describe how to obtain PMSE of  $\widehat{a}_{t|t-1}$  based on the asymptotic distribution of the parameter estimator as proposed by Hamilton (1986) and the bootstrap PMSE proposed by Pfeiffermann and Tiller (2005).

## 2.2 Asymptotic approximation

Hamilton (1986) proposes to estimate the PMSE of  $\widehat{a}_{t|t-1}$  by considering the following decomposition of  $PMSE_{t|t-1} = E_{t-1} \left[ (\widehat{a}_{t|t-1} - \alpha_t) (\widehat{a}_{t|t-1} - \alpha_t)' \right]$

$$PMSE_{t|t-1} = E_{t-1} \left[ (\widehat{a}_{t|t-1} - a_{t|t-1}) (\widehat{a}_{t|t-1} - a_{t|t-1})' \right] + E_{t-1} \left[ (a_{t|t-1} - \alpha_t) (a_{t|t-1} - \alpha_t)' \right] \quad (6)$$

where the  $t - 1$  under the expectation means that it is taken conditional on  $\{Y_1, \dots, Y_{t-1}\}$ .

Note that the cross-product  $E_{t-1} \left[ (\widehat{a}_{t|t-1} - a_{t|t-1}) (a_{t|t-1} - \alpha_t)' \right]$  is zero under the assumption of conditional Normality. The second term in (6) is denoted by Hamilton (1986) as filter

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<sup>4</sup>Note that when the Kalman filter is run the effect of the initial values on the estimates of the PMSE vanishes in approximately five iterations; see Ray (1989). Consequently, we remove  $\widehat{P}_{t|t-1}$ , for  $t = 1$  to 5, for calculating the corresponding biases  $d_t^{(j)}$ .

uncertainty. It represents how far would the state be from its estimate when the parameters are known. This uncertainty is due to the uncertainty in separating signal and noise and it is inherent to the Kalman filter. On the other hand, the first term in (6), denoted as parameter uncertainty, represents the discrepancy between the estimates of the unobserved states obtained with known and unknown parameters. In order to estimate the PMSE in (6), Hamilton (1986) considers the following relationship

$$PMSE_{t|t-1} = E_{\theta} \left\{ E_{t-1} \left[ (\hat{a}_{t|t-1} - a_{t|t-1}) (\hat{a}_{t|t-1} - a_{t|t-1})' | \theta \right] \right\} + E_{\theta} \left\{ E_{t-1} \left[ (a_{t|t-1} - \alpha_t) (a_{t|t-1} - \alpha_t)' | \theta \right] \right\}, \quad (7)$$

where  $\theta$  is the vector of model parameters

Once the parameters are estimated, a large number,  $M$ , of realizations of  $\hat{\theta}^{(i)}$  is generated from the asymptotic distribution of the estimator. Then, the Kalman filter is run using each of the realizations  $\hat{\theta}^{(i)}$  and the original observations,  $\{Y_1, \dots, Y_T\}$ , obtaining a series of estimates the state and their corresponding PMSE, denoted by  $\hat{a}_{t|t-1}^{(i)}$  and  $\hat{P}_{t|t-1}^{(i)}$ , respectively. In this way, an analogue of the expectations within squared brackets in (7) can be obtained by  $(\hat{a}_{t|t-1}^{(i)} - \hat{a}_{t|t-1}) (\hat{a}_{t|t-1}^{(i)} - \hat{a}_{t|t-1})'$  and  $P_{t|t-1}^{(i)}$  respectively. Then, the sample averages for all possible values of the parameters are obtained to estimate the expectation over all values of  $\theta$ . Finally, the estimate of PMSE in (7) is given by

$$\widehat{PMSE}_{t|t-1}^{Asy} = \frac{1}{M} \sum_{i=1}^M \hat{P}_{t|t-1}^{(i)} + \frac{1}{M} \sum_{i=1}^M (\hat{a}_{t|t-1}^{(i)} - \hat{a}_{t|t-1}) (\hat{a}_{t|t-1}^{(i)} - \hat{a}_{t|t-1})'. \quad (8)$$

Figures 1 and 2 plot the relative biases of the asymptotic estimator of the PMSE in (8) denoted as Asy for the RWN models considered above. We can observed that when  $T = 40$  the biases are even larger in absolute value than when the PMSE are computed with estimated parameters. Table 1, that reports the averages through time of theses averages, shows that regardless of whether the RWN model is homoscedastic or heteroscedastic, the relative bias is around 20%, while the relative biases of the PMSEs obtained from the Kalman filter with estimated parameters are -8% and -15% in the homoscedastic and heteroscedastic

models respectively. Obviously, as the ML estimator is consistent, the biases decrease with the sample size. However, it is important to note that even when  $T = 100$  the biases are rather larger when the model is heteroscedastic.

### 2.3 Bootstrap procedures

We have seen that in small sample sizes, the asymptotic approximation can be poor and, consequently, the asymptotic estimator of the PMSE in (8) can also have poor properties. Alternatively, Pfeiffermann and Tiller (2005) propose using bootstrap procedures to obtain PMSEs of  $\hat{a}_{t|t-1}$  that incorporate the parameter uncertainty. They propose parametric and non-parametric bootstrap procedures. Next, we describe only the parametric bootstrap that has the best performance according to our simulation results. They consider the decomposition of the PMSE in (6) but with the expectations taken over all possible point realizations of  $\{Y_1, \dots, Y_T\}$  and  $\{\alpha_1, \dots, \alpha_T\}$  instead of expectations conditional on the available data set. The parametric bootstrap analogue of (6) is obtained as follows:

**Step 1.** Given a realization of  $\{Y_1, \dots, Y_T\}$ , estimate the parameters,  $\hat{\theta}$ , and implement the Kalman filter to obtain the estimates of the underlying state,  $\hat{a}_{t|t-1}(\hat{\theta})$  and the corresponding PMSE,  $\hat{P}_{t|t-1}(\hat{\theta})$ .<sup>5</sup>

**Step 2.** Obtain a bootstrap replicate of the series  $\{Y_1^*, \dots, Y_T^*\}$ , and of the underlying state  $\{\alpha_1^*, \dots, \alpha_T^*\}$ , by extracting realizations,  $\varepsilon_t^*$  and  $\eta_t^*$ ,  $t = 1, \dots, T$ , from the joint Gaussian distribution of  $\varepsilon_t$  and  $\eta_t$ , using them and the estimated parameters,  $\hat{\theta}$ , substituted in model (1). Then, estimate the bootstrap parameters,  $\hat{\theta}^*$ .

**Step 3.** Implement again the Kalman filter with the bootstrap estimates,  $\hat{\theta}^*$ , and the bootstrap replicates  $\{Y_1^*, \dots, Y_T^*\}$ , to obtain bootstrap estimates of the state,  $\hat{a}_{t|t-1}^*(\hat{\theta}^*)$ , and their corresponding PMSE,  $\hat{P}_{t|t-1}^*(\hat{\theta}^*)$ .

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<sup>5</sup>We add explicitly the dependence of the estimates of the unobserved states and their corresponding PMSE on the estimated parameters to clarify the procedure.

**Step 4.** Using the bootstrap series  $\{Y_1^*, \dots, Y_T^*\}$  and the parameters estimated in step 1,  $\hat{\theta}$ , run the Kalman filter to obtain the estimates of the state denoted by  $\hat{a}_{t|t-1}^*(\hat{\theta})$ .

Repeat  $B$  times steps 2 to 4. Finally, the bootstrap analogue of the PMSE of  $\hat{a}_{t|t-1}$  in (6) is estimated by<sup>6</sup>

$$\begin{aligned} \widehat{PMSE}_t^{PT} &= \frac{1}{B} \sum_{j=1}^B \left( \hat{a}_{t|t-1}^{*(j)}(\hat{\theta}^*) - \hat{a}_{t|t-1}^*(\hat{\theta}) \right) \left( \hat{a}_{t|t-1}^{*(j)}(\hat{\theta}^*) - \hat{a}_{t|t-1}^*(\hat{\theta}) \right)' \\ &\quad + 2\hat{P}_{t|t-1} - \frac{1}{B} \sum_{j=1}^B \hat{P}_{t|t-1}^{*(j)}(\hat{\theta}^*). \end{aligned} \quad (9)$$

In order to illustrate the performance of the bootstrap procedure proposed by Pfeiffermann and Tiller (2005), we consider again the Monte Carlo design carried out in previous section for the RWN model. Figures 1 and 2 plot the Monte Carlo averages of the relative biases denoted by PT for the homoscedastic and heteroscedastic models respectively. We can observe that in the small sample size,  $T = 40$ , the relative biases of  $PMSE_t^{PT}$  are smaller than those of the Hamilton (1986) approximation but only slightly smaller than the biases obtained when the Kalman filter is run with estimated parameters. Table 1, that reports the averages through time of the relative biases, shows that in the homoscedastic model the bias is -7.62% compared with -8.02% in the Kalman filter. The reduction in the biases is a little bit larger in the heteroscedastic model. In the moderate sample size,  $T = 100$ , the procedures proposed by Hamilton (1986) and Pfeiffermann and Tiller (2005) have similar relative biases in the homoscedastic model while in the heteroscedastic model, there is a larger reduction when the bootstrap procedure is implemented. Finally, in large sample size, both procedures are approximately unbiased.

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<sup>6</sup>Pfeiffermann and Tiller (2005) also propose a non-parametric bootstrap for estimating the PMSE which is based on obtaining the bootstrap replicates of  $\{Y_1^*, \dots, Y_T^*\}$  by using the IF of the model in (3a) and (3c) and random extractions,  $\{V_1^*, \dots, V_T^*\}$ , from the empirical distribution of the standardized innovations,  $\hat{V}_i/\sqrt{\hat{F}_i}$ ; see, Stoffer and Wall (1991) and Rodriguez and Ruiz (2009) for its practical implementation. This non-parametric bootstrap does not assume any particular distribution of the errors. In our comparisons, we do not consider this non-parametric bootstrap because the results are always worse than for the parametric bootstrap.

As we noted above, the PMSE in (9) is computed by taking expectations over all bootstrap realizations of the original series. However, the Kalman filter is designed to obtain conditional estimates of the underlying state and their corresponding PMSE. Therefore, it could be possible to simplify computationally the bootstrap procedure and, simultaneously improve its performance by computing the PMSEs conditional on the available data set. This is the proposal of this paper that we develop in the following section.

### 3 A new bootstrap procedure

In this section, we propose new bootstrap procedures to estimate the conditional PMSE of the one-step-ahead estimator of the unobserved components obtained by the Kalman filter run with estimated parameters. Our proposed procedures are similar to that proposed by Hamilton (1986) in the sense that we compute PMSE conditional on the available information set. However, instead of dealing with the parameter uncertainty by simulating the parameters from the asymptotic distribution of the corresponding estimator, we simulate them from a bootstrap distribution. In this way we obtain PMSEs with better properties in small samples than those of Hamilton (1986). On the other hand, dealing with conditional PMSE allows us to simplify computationally the procedure with respect to the bootstrap procedures proposed by Pfeiffermann and Tiller (2005) and, at the same time, we improve their performance in small samples. Furthermore, from an analytical point of view, the distinction between conditional and unconditional PMSEs can be important when dealing with models in which the system matrices are time-variant.

The first procedure proposed in this paper is a parametric bootstrap procedure based on resampling from the assumed joint Gaussian distribution of the noises. Alternatively, we also propose a non-parametric procedure, based on resampling from the empirical distribution of the standardized one-step-ahead innovations,  $\widehat{V}_t^s = \widehat{V}_t / \sqrt{\widehat{F}_t}$ , which does not assume any particular distribution of the errors.

First, we describe the proposed parametric bootstrap algorithm.

**Step 1.** Given the realization  $\{Y_1, \dots, Y_T\}$ , estimate the parameters,  $\hat{\theta}$ , and implement the Kalman filter to obtain the estimates of the underlying state,  $\hat{a}_{t|t-1}(\hat{\theta})$ , and the corresponding PMSE,  $\hat{P}_{t|t-1}(\hat{\theta})$ ,  $t = 1, \dots, T$ .

**Step 2.** Obtain a bootstrap replicate of the series  $\{Y_1^*, \dots, Y_T^*\}$  and of the underlying state  $\{\alpha_1^*, \dots, \alpha_T^*\}$ , by extracting realizations,  $\varepsilon_t^*$  and  $\eta_t^*$ ,  $t = 1, \dots, T$ , from the joint Gaussian distribution of  $\varepsilon_t$  and  $\eta_t$  and using them in model (1) with the parameters substituted by  $\hat{\theta}$ . Estimate the bootstrap parameters,  $\hat{\theta}^*$ .

**Step 3.** Run the Kalman filter with the original observations  $\{Y_1, \dots, Y_T\}$  and the bootstrap parameters estimated in step 2 to obtain a bootstrap replicate of  $\hat{a}_{t|t-1}(\hat{\theta}^*)$  and  $\hat{P}_{t|t-1}(\hat{\theta}^*)$ ,  $t = 1, \dots, T$ .

Steps 2 and 3 are repeated  $B$  times. Then, similarly as in (8), the parametric conditional bootstrap PMSEs are obtained as follows

$$\begin{aligned} \widehat{PMSE}_{t|t-1}^{CB1} &= \frac{1}{B} \sum_{j=1}^B \hat{P}_{t|t-1}^{(j)}(\hat{\theta}^*) \\ &+ \frac{1}{B} \sum_{j=1}^B \left( \hat{a}_{t|t-1}^{(j)}(\hat{\theta}^*) - \hat{a}_{t|t-1}(\hat{\theta}) \right) \left( \hat{a}_{t|t-1}^{(j)}(\hat{\theta}^*) - \hat{a}_{t|t-1}(\hat{\theta}) \right)'. \end{aligned} \quad (10)$$

The first two steps are identical to those proposed by Pfeiffermann and Tiller (2005). However, in step 3, we run the Kalman filter with the bootstrap estimates of the parameters and the original time series, while they run the filter with the bootstrap replicates of the series. In this way, we compute the PMSE conditional on the information contained in the original series, while the  $\widehat{PMSE}_t^{PT}$  in equation (9) are unconditional. Furthermore, by computing the conditional PMSE, we avoid running the filter for each bootstrap replicate as it is done in step 4 of the procedure described in previous section. This simplification implies a large reduction in computing time when estimating the PMSE of the underlying unobserved components.



We also propose a non-parametric bootstrap for estimating the conditional PMSE. Steps 1 and 3 are the same as in the parametric case. However, in step 2, we construct the bootstrap replicates by resampling the standardized one-step-ahead innovations,  $\widehat{V}_t^s$ , and using the IF with the estimated parameters,  $\widehat{\theta}$ , as follows

$$a_{t+1|t}^* = \widehat{T}_{t+1}\widehat{a}_{t|t-1}^* + \widehat{c}_{t+1} + \widehat{K}_{t+1}^*\widehat{F}_t^{*-1}\widehat{V}_t^* \quad (11)$$

$$Y_t^* = \widehat{Z}_t\widehat{a}_{t|t-1}^* + \widehat{d}_t + \widehat{V}_t^*. \quad (12)$$

Then the bootstrap parameters,  $\widehat{\theta}^*$  are estimated. Finally, the conditional PMSE is estimated as in equation (10) and is denoted as  $\widehat{PMSE}_{t|t-1}^{CB2}$ .

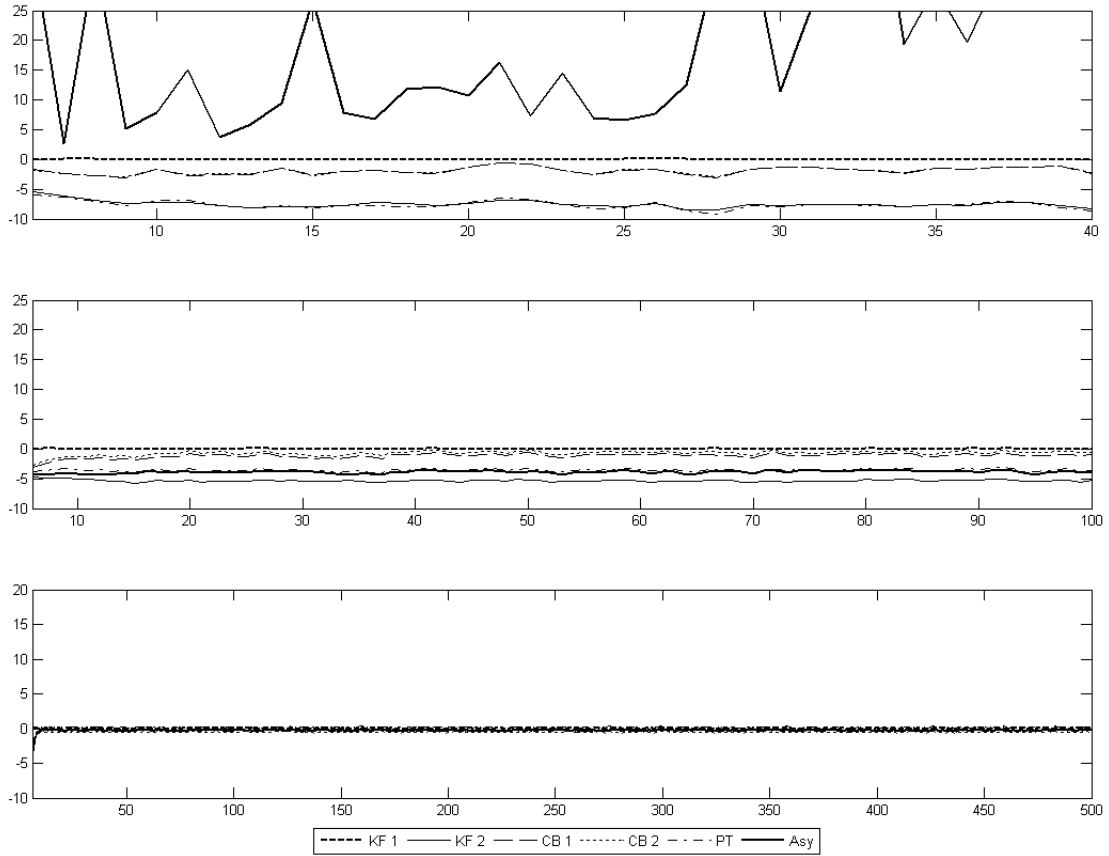
**Table 1:** Averages and standard deviations (Std) through time of the relative biases (in percentage) of PMSE of the underlying level in the RWN models with Gaussian homoscedastic, Gaussian heteroscedastic and non-Gaussian errors.

	Homoscedastic		Heteroscedastic		Non-Gaussian	
	Average	Std	Average	Std	Average	Std
T = 40						
KF1 <sup>a</sup>	0.02	0.04	0.02	0.04	-2.69	9.84
KF2 <sup>b</sup>	-8.02	0.57	-15.44	0.71	-10.85	6.81
Asy <sup>c</sup>	20.53	15.34	20.71	1.23	21.59	19.30
PT	-7.62	0.68	-11.72	1.12	-94.62	8.58
CB1	-1.46	0.61	-1.63	1.28	-3.25	3.75
CB2	-1.21	0.60	-1.87	1.28	-3.42	3.54
T = 100						
KF1	0.02	0.01	0.02	0.04	-1.18	6.52
KF2	-6.82	0.25	-6.24	0.24	-7.02	4.92
Asy	-3.88	0.22	11.03	0.73	-4.32	4.97
PT	-3.55	0.20	-3.20	0.50	-96.53	6.08
CB1	-0.64	0.37	-0.79	0.36	-2.12	2.60
CB2	-0.56	0.37	-2.33	0.36	-2.35	2.58
T = 500						
KF1	0.02	0.05	0.02	0.05	-0.37	3.18
KF2	-0.97	0.14	-1.25	0.10	-1.79	2.74
Asy	-0.29	0.15	-0.23	0.11	-1.65	2.74
PT	0.20	0.15	-0.80	0.34	-99.10	2.35
CB1	-0.18	0.15	-0.96	0.12	-1.86	1.59
CB2	-0.25	0.15	-1.08	0.12	-1.60	1.63

<sup>a</sup> Kalman filter procedure with known parameters.

<sup>b</sup> Kalman filter procedure with estimated parameters.

<sup>c</sup> Asymptotic approximation proposed by Hamilton (1986).

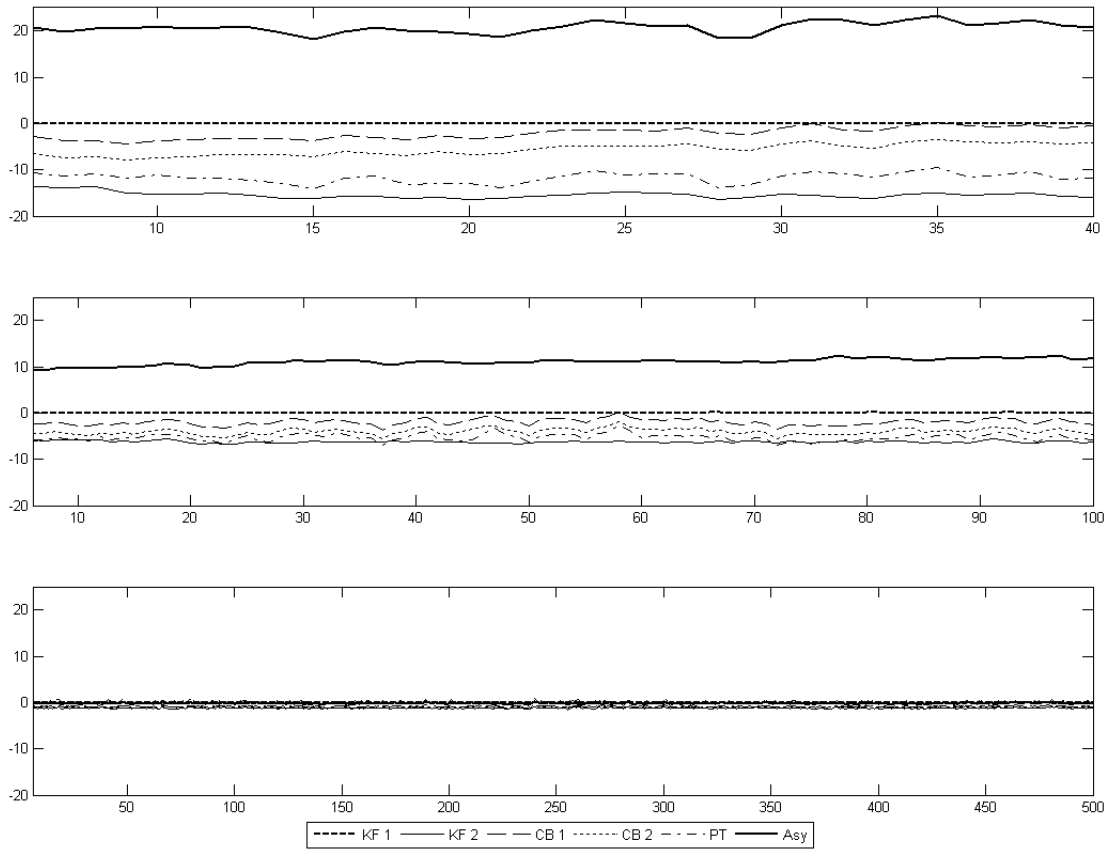


**Figure 1:** Monte Carlo averages of the ratios  $d_t = 100 \times \left( \frac{P_{t|t-1}}{PMSE_t} - 1 \right)$  for the RWN model with homoscedastic Gaussian error and  $T = 40$  (first row),  $T = 100$  (second row) and  $T = 500$  (third row).

In order to analyze the finite sample properties of the two new bootstrap procedures proposed in this paper, we consider the same Monte Carlo experiments described previously. Particularly, for each generated series,  $\{y_1^{(j)}, \dots, y_T^{(j)}\}$ ,  $j = 1, \dots, R$ , and sample sizes,  $T = 40, 100$  and  $500$ , we generate  $B = 1000$  bootstrap replicates, estimate the PMSEs by both procedures, and calculate the corresponding relative biases. Figure 1, that plots the Monte Carlo averages of these relative biases for the homoscedastic RWN model, shows that, regardless of the sample size, the biases of the proposed parametric and non-parametric bootstrap PMSE are very similar. These biases obviously decrease with the sample size and are clearly smaller than those observed when the PMSEs are computed using the Kalman filter with estimated parameters, the asymptotic proposal of Hamilton (1986) or the bootstrap procedure proposed

by Pfeiffermann and Tiller (2005) when the sample sizes are small or moderate. The time averages and standard deviations reported in Table 1 show that the reductions of the relative biases can be very important in the small sample when  $T = 40$ . For example, in the homoscedastic RWN model, the relative bias is -8.02% when using the Kalman filter with estimated parameters, 20.53% when using the asymptotic distribution of the ML estimator, -7.62% when using the bootstrap procedure of Pfeiffermann and Tiller (2005) while they are as small as -1.46% and -1.21% when using the parametric and non-parametric bootstrap procedures proposed in this paper. The reduction of the relative biases is still important when  $T = 100$  while when  $T = 500$  all procedures to compute the PMSEs of the unobserved level  $\mu_t$  are approximately unbiased. It is also remarkable that the relative biases and standard deviations of the parametric and non-parametric bootstrap procedures proposed in this paper are approximately the same when implemented in the Gaussian RWN model regardless of whether the disturbances are homoscedastic or heteroscedastic. The similarity in the behavior of the parametric and non-parametric procedures could be expected given that the model is conditionally Gaussian and in the parametric procedure, we are resampling from the true Gaussian distribution. However, it is comforting to observe that the behavior of the parametric procedure which does not assume any particular distribution is comparable with that of the parametric procedure.

Therefore, our simulation results show that in small and moderate sample sizes the proposed bootstrap procedures to compute the conditional PMSE of  $\hat{a}_{t|t-1}$  have very small biases which are smaller than those of alternative procedures. Furthermore, this reduction of bias is accomplished using procedures which are simpler from a computational point of view. It is also important to point out that we have considered a very simple model in order to illustrate the performance of the CB1 and CB2 procedures. Therefore, it is expected that the simplicity of our procedures when compared with alternatives is going to be even more important when



**Figure 2:** Monte Carlo averages of the ratios  $d_t = 100 \times \left( \frac{P_{t|t-1}}{PMSE_t} - 1 \right)$  for the RWN model with heteroscedastic Gaussian error and  $T = 40$  (first row),  $T = 100$  (second row) and  $T = 500$  (third row).

dealing with more complicated models.

## 4 Robustness against non-Gaussian models

It is important to note that when the conditional Normality assumption is not satisfied, equations (3a) and (3b) do not provide the conditional mean of the unobserved states and their corresponding conditional PMSEs. However, they still provide optimal one-step-ahead estimates of the underlying state in the sense that they have minimum PMSE, given by  $P_{t|t-1}$ , among all estimators which are linear functions of the observations. Taking into account this feature, in this section, we analyze the robustness of the two new bootstrap procedures proposed in this paper to estimate the PMSE of the unobserved components in the context of a particular non-Gaussian model of interest in the context of Stochastic Volatility models.

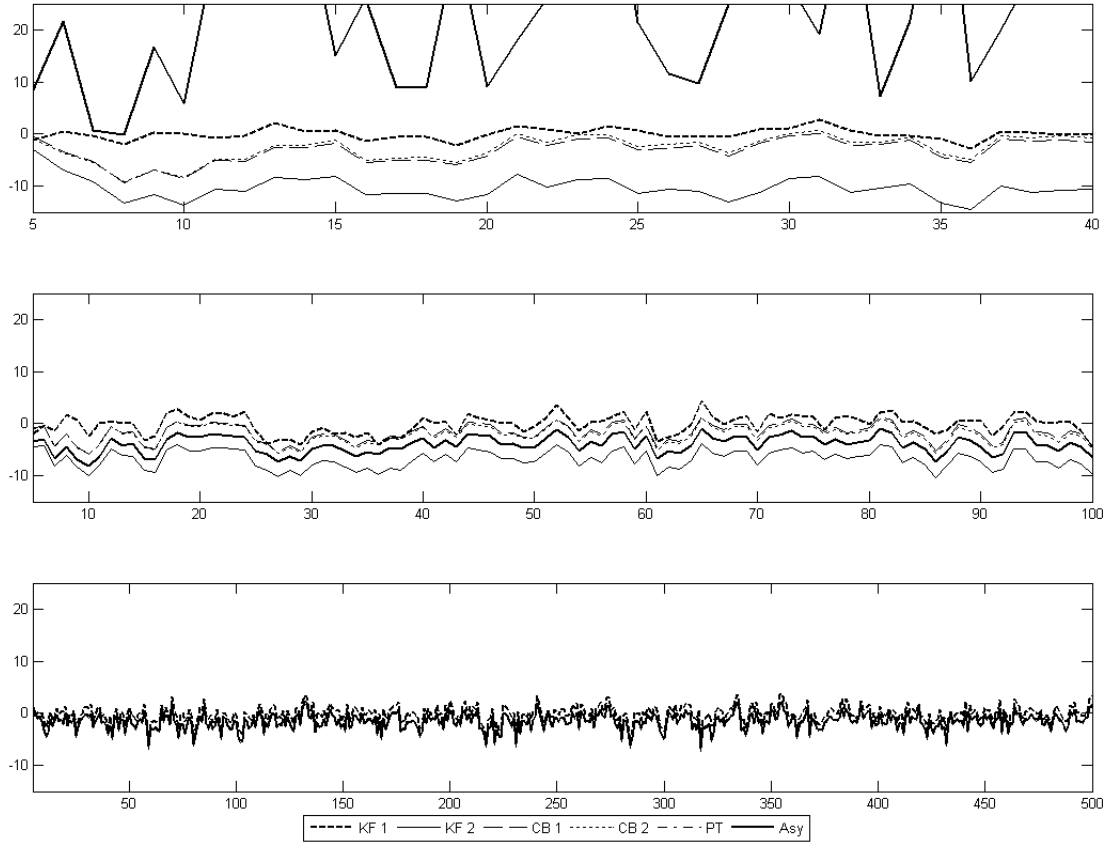
In particular, we consider again the RWN model in (4) but this time the error term of the measurement equation,  $\varepsilon_t$ , has a  $\log(\chi_1^2)$  distribution; see, for instance, Harvey et al. (1994) for the relation between this model and the linear transformation of the Autoregressive Stochastic Volatility Model. In order to guarantee that the variances of the two error terms are equal to those of the homoscedastic Gaussian model considered in the previous sections, we center and re-scaled the  $\log(\chi_1^2)$ . In addition, given that the model is not conditionally Gaussian, the distribution in (2) is not further the true conditional distribution of the vector  $(\alpha_t, Y_t)'$ . Consequently, for each simulated series  $j$  and moment of time  $t$ , we generate 10000 replicates of  $\mu_t^{(j)}$ ,  $\mu_t^{(j,i)}$ ,  $i = 1, \dots, 10000$ , by particle filtering; see, Kitagawa (1996) and Arulampalam et al. (2002) for details about particle filtering procedures. Then, the empirical conditional PMSEs and their corresponding relative biases are computed as in previous sections.

Figure 3 plots the averages through Monte Carlo replicates of the relative biases when the Kalman filter is run with known and estimated parameters. In this case, the parameters have been estimated by QML by maximizing the Gaussian log-likelihood. First of all this figure illustrates that even when the Kalman filter is run with known parameters,  $P_{t|t-1}$  are slightly biased estimates of the true PMSE. This result can also be observed in the fifth column of Table 1 that reports the averages and standard deviations through time of the relative biases plotted in Figure 3. In the small sample, when  $T = 40$ , the relative bias is -2.69% and this bias decrease with the sample size. However, the standard deviations are much larger than those reported for the conditional Gaussian models. These biases can be attributed to the fact that when the model is not conditionally Gaussian,  $m_{t|t-1}$  is not the true conditional mean of  $\mu_t$ . On the other hand, the relative biases reported in the fifth column of Table 1 for the PMSE computed with the Kalman filter with estimated parameters are not very different from those reported for the conditional Gaussian models. However, once more, the standard deviations are much larger. The same result is observed for the asymptotic procedure proposed by Hamilton (1986) which seems to be rather robust to the presence of non-Gaussianity. It is remarkable that in the non-Gaussian RWN model considered in this paper, the parametric

bootstrap procedure proposed by Pfeiffermann and Tiller (2005) has very large biases which do not decrease with the sample size. The parametric bootstrap are based in resampling from the centered and re-scaled  $\log(\chi_1^2)$  distribution to obtain replicates of  $\varepsilon_t$  while the replicates of  $\eta_t$  are obtained by resampling from a  $\mathcal{N}(0, \widehat{\sigma}_\eta^2)$  distribution. It has in average a overestimation of the true PMSE of approximately 95% for all sample sizes. Finally, the two new bootstrap procedures proposed in this paper have an adequate performance even in the small sample size. In particular, the relative biases of the parametric and non-parametric procedures proposed in this paper are -3.25% and -3.42%, respectively when  $T = 40$ , and, -2.12% and -2.35% when  $T = 100$ . They are clearly smaller than the biases of any of the three alternative feasible estimators of the PMSE and very close to those reported for the PMSE computed by the Kalman filter with known parameters. Finally, notice that as in the Gaussian RWN models, the biases and standard deviations of the parametric and non-parametric procedures are very similar. Therefore, we can conclude that the proposed procedures can be implemented in non-Gaussian state space models with adequate performances.

## 5 Empirical application: Estimating the output gap, the NAIRU, the Trend Investment Rate and the Core Inflation in US

In this section, we apply the new proposed bootstrap estimators of the PMSE of the unobserved components to estimate the uncertainty associated with the estimation of the output gap, NAIRU, investment trend and core inflation of the US economy, based on the unobserved components model proposed by Doménech and Gómez (2006). It is important to note that this model is multivariate including four observable variables namely, the logarithm of the GDP,  $y_t$ , the inflation,  $\pi_t$ , the unemployment rate,  $U_t$  and the investment rate,  $x_t$  defined as the ration between investment and GDP. Therefore, as a by product, we illustrate how the bootstrap procedures proposed in this paper to estimate PMSE of the unobserved components



**Figure 3:** Monte Carlo averages of the ratios  $d_t = 100 \times \left( \frac{P_{t|t-1}}{PMSE_t} - 1 \right)$  the RWN model with error term  $\varepsilon_t$  distributed as  $\log \chi_1^2$  and  $T = 40$  (first row),  $T = 100$  (second row) and  $T = 500$  (third row).

can also be implemented in multivariate systems.

The model proposed by Doménech and Gómez (2006) incorporates the following three stylized facts often observed in those macroeconomic variables, namely: (i) Negative correlation between the output gap and the deviations of unemployment from the NAIRU, often known as the Okun's law; (ii) Short run trade-off between inflation and unemployment known as forward looking Phillips curve; and (iii) Co-movement of output and investment called

accelerator-type investment equation. It is given by

$$y_t \equiv y_t^p + z_t, \quad (13a)$$

$$z_{t+1} = 2\theta_1 \cos \theta_2 z_{t-1} - \theta_1^2 z_{t-2} + \omega_{zt}, \quad (13b)$$

$$y_{t+1}^p = \bar{\mu} + y_t^p + \omega_{yt}, \quad (13c)$$

$$\pi_t = \left(1 - \sum_{i=1}^4 \mu_i\right) \bar{\pi}_t + \left(\sum_{i=1}^4 \mu_i \pi_{t-i}\right) + \eta_y z_t + v_{\pi t}, \quad (13d)$$

$$\bar{\pi}_t = \bar{\pi}_{t-1} + \omega_{\pi t}, \quad (13e)$$

$$U_t = \phi_u U_{t-1} + (1 - \phi_u) \bar{U}_t + \phi_0 z_t + v_{ut}, \quad (13f)$$

$$\bar{U}_t = \bar{U}_{t-1} + \omega_{ut}, \quad (13g)$$

$$x_t = \beta_x x_{t-1} + (1 - \beta_x) \bar{x}_t + \beta_{y_0} z_t + \beta_{y_1} z_{t-1} + v_{xt}, \quad (13h)$$

$$\bar{x}_t = \bar{x}_{t-1} + \omega_{xt}, \quad (13i)$$

where  $z_t$  is the unobserved output gap which is assumed to follow cyclical AR(2) process in equation (13b) and,  $y_t^p$  is the logarithm of the potential output represented by a random walk plus drift model in equation (13c). The parameter  $\bar{\mu}$  captures the growth rate of the potential output. The noises  $\omega_{zt}$  and  $\omega_{yt}$  are assumed to be mutually independent Gaussian white noises with zero mean and variances  $\sigma_{\omega_z}^2$  and  $\sigma_{\omega_y}^2$  respectively. The following two equations, (13d) and (13e) describe the dynamic evolution of inflation,  $\pi_t$  and its relation with the output gap.  $\bar{\pi}_t$  is the core inflation which follows a random walk. The noises  $v_{\pi t}$  and  $\omega_{\pi t}$  are Gaussian white noises with variances  $\sigma_{v_{\pi}}^2$  and  $\sigma_{\omega_{\pi}}^2$  respectively. Both noises are mutually independent and independent of  $\omega_{zt}$  and  $\omega_{yt}$ . Equations, (13f) and (13g) describe the Okun's law where  $U_t$  is the unemployment rate and  $\bar{U}_t$  is the NAIRU. Once more, the disturbances associated with the unemployment,  $v_{ut}$  and  $\omega_{ut}$  are Gaussian white noises with variances  $\sigma_{v_u}^2$  and  $\sigma_{\omega_u}^2$ , respectively. They are mutually independent and independent of the rest of disturbances in the model. Finally, the last two equations, (13h) and (13i) describe the dynamic evolution of the investment rate,  $x_t$  where  $\bar{x}_t$  is the long run investment trend. The disturbances  $v_{xt}$  and  $\omega_{xt}$  are Gaussian white noises with zero mean and variances  $\sigma_{v_x}^2$  and  $\sigma_{\omega_x}^2$  and, once more, they are assumed to be mutually independent and independent of all previous disturbances.



Model (13) can be casted into a state space framework as in (1) with  $Y_t = [y_t, U_t - \phi_u U_{t-1}, x_t - \beta_x x_{t-1}, \pi_t - (\sum_{i=1}^4 \mu_i \pi_{t-i})]'$ ,  $\alpha_t = [y_t^p, \bar{U}_t, \bar{x}_t, \bar{\pi}_t, z_{t-2}, z_{t-1}, z_t]'$ ,  $\varepsilon_t = [v_{ut}, v_{xt}, v_{\pi t}]$ ,  $\eta_t = [\omega_{yt}, \omega_{ut}, \omega_{xt}, \omega_{\pi t}, \omega_{zt}]$ ,  $H_t = \text{diag} \{ \sigma_{vu}^2, \sigma_{vx}^2, \sigma_{v\pi}^2 \}$ ,  $Q_t = \text{diag} \{ \sigma_{\omega y}^2, \sigma_{\omega u}^2, \sigma_{\omega x}^2, \sigma_{\omega \pi}^2, \sigma_{\omega z}^2 \}$ ,

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -\theta_1^2 & 2\theta_1 \cos \theta_2 \end{bmatrix},$$

$$Z = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 - \phi_u & 0 & 0 & 0 & 0 & \phi_0 \\ 0 & 0 & 1 - \beta_x & 0 & 0 & \beta_{y1} & \beta_{y0} \\ 0 & 0 & 0 & 1 - \sum_{i=1}^4 \mu_i & 0 & 0 & \eta_y \end{bmatrix},$$

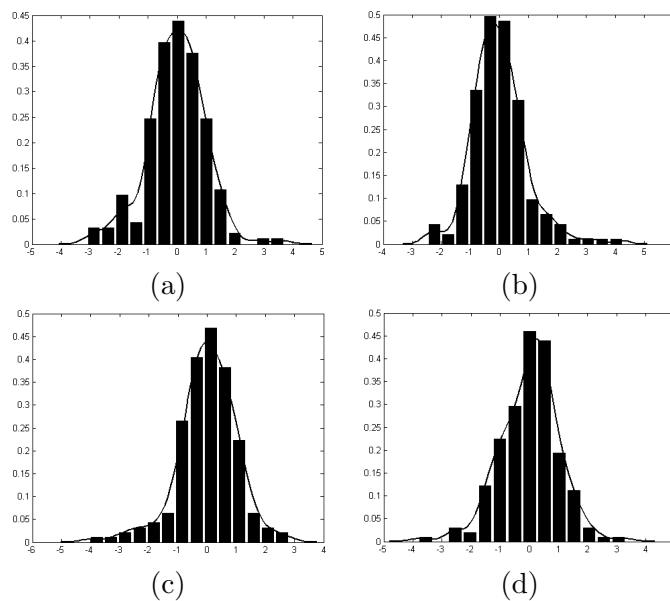
$$R_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \text{ and } R_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where  $\text{diag}\{\cdot\}$  is the diagonal matrix.

In this paper, we fit model (13) to the same data analyzed by Doménech and Gómez (2006). It consists on quarterly observations from 1948:Q1 to 2003:Q1 of the log ( $GDP$ ), the inflation rate, defined as the average inflation over the last four months, the unemployment rate defined as the average of the unemployment rate over the last four months and the nominal investment rate. After a preliminary analysis of the data, several breaks in the marginal variances of inflation and output are detected and incorporated into de model. As in Doménech and Gómez (2006), we estimate parameters by QML by maximizing the one-step-ahead error prediction decomposition of the Gaussian log-likelihood where the innovations and their covariances matrices are obtained by running the Kalman filter. The asymptotic distribution of the QML estimator can be found in, for example, Harvey (1989). Table 2 reports the QML estimates of the parameters which are very close to those reported by Doménech and Gómez (2006). Note that in the output column in Table 2 the estimated

breaks in the output volatility is highly significant, there is a decreasing in the volatility after 1983:Q1. Moreover, the volatility changes in inflation clearly has two significant breaks: substantial increases in 1972:Q1 on the one hand, and decreases in 1983:Q1. The output gap is significant in the three equations that describe the inflation, unemployment and investment. As expected, in the Phillips curve and the investment equation the sign of the coefficients associated with the output gap are positive.

Figure 4 plots the estimated kernel densities of the four one-step-ahead components of the innovations vector. It shows that the estimated innovations of the unemployment and investment seem to have asymmetric distribution. Therefore, it could be expected that in this case, the parametric based on the Normal assumption and non-parametric PMSEs may differ.



**Figure 4:** Histogram and estimated kernel density of the standardized one-step ahead errors of (a) Output, (b) Unemployment, (c) Investment and (d) Inflation.

After estimating the parameters, the Kalman filter is run to obtain one-step-ahead estimates of the underlying components and their PMSE<sup>7</sup>. The estimates of the output gap,

<sup>7</sup>Alternatively, Doménech and Gómez (2006) implement a smoothing algorithm to estimate the unobserved components together with their PMSEs. However, they report very large correlations between smoothed and one-step-ahead estimates of the underlying components. Therefore, their estimates are comparable with those

**Table 2:** Parameter estimates of unobserved components model for output, unemployment, inflation and investment.

Output		Inflation			NAIRU		Investment		
$\theta_1$	0.7710 (15.73)	$\eta_y$	0.3123 (3.52)	$\sigma_{v\pi}$	0.0044 (2.94)	$\phi_0$	-0.3107 (-14.31)	$\beta_{y_0}$	0.6458 (16.24)
$\theta_2$	0.2359 (3.02)	$\mu_1$	0.1427 (2.24)	$\sigma_{\omega\pi_1}$	0.0096 (8.69)	$\phi_U$	0.3653 (6.27)	$\beta_{y_1}$	-0.6102 (-12.89)
$\sigma_{y_1}$	0.0075 (10.81)	$\mu_2$	-0.1332 (-2.67)	$\sigma_{\omega\pi_2}$	0.0163 (4.38)	$\sigma_{vu}$	0.0009 (3.86)	$\beta_x$	0.8253 (15.05)
$\sigma_{y_2}$	0.0069 (11.59)	$\mu_3$	0.2413 (3.13)	$\sigma_{\omega\pi_3}$	0.0047 (3.41)	$\sigma_{\omega u}$	0.0022 (6.41)	$\sigma_{vu}$	0.0033 (12.17)
$\sigma_\gamma$	0.0048 (14.07)	$\mu_4$	-0.1679 (-2.35)					$\sigma_{\omega u}$	0.0023 (3.17)

Note: Estimation were carried out with 221 quarterly observations from 1948:I to 2003:I. In parenthesis is the  $t$ -statistic.

NAIRU, investment trend and the core inflation have been plotted in Figure 5 together with the 90% prediction intervals based on the PMSE estimated by the Kalman filter and the assumption of Normality. We also estimate the PMSEs by using the asymptotic approximation proposed by Hamilton (1986), the bootstrap procedure of Pfeffermann and Tiller (2005) and the two new bootstrap procedures proposed in this paper. Table 3 reports the averages and standard deviations through time of the PMSEs estimated for each of the four underlying components by each of the five procedures. The PMSEs obtained by the Kalman filter run with the estimated parameters and the procedure proposed by Hamilton (1986) are very similar for the NAIRU, Investment and long-run inflation. However, there is a large difference in the PMSE of the output gap which is 0.0143 when estimated by the Kalman filter with estimated parameters but it is 0.0214 when incorporating the parameter uncertainty using the asymptotic distribution of the QML estimator. Furthermore, the PMSEs estimated using the bootstrap procedure proposed by Pfeffermann and Tiller (2005) assuming Gaussian errors are very similar to those obtained by using the asymptotic procedure for all variables but investment. In this case, it is larger using the bootstrap procedure, 0.0093, while it is estimated as 0.0062 using the asymptotic approximation. Finally, the PMSEs obtained using the two

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obtained in this paper.

bootstrap procedures proposed in this paper are clearly larger than those obtained by all the alternative procedures for all four unobserved variables. Note that the parametric bootstrap is based on the assumption of Gaussian errors which seems to be not satisfied in all equations. As expected, given the simulation results in the previous section, the PMSEs estimated using the parametric and non-parametric bootstrap procedures are very similar for all variables. Only in the case of the NAIRU the PMSE estimated using the parametric bootstrap is 0.0063 while it is 0.0089 when the non-parametric procedure is implemented. Note that the unemployment is one of the variables for which the innovations seem to be non-Normal. Also note that, for the investment trend, the bootstrap PMSE is around five times the PMSE computed using the Kalman filter with estimated parameters. The smallest difference between the bootstrap and Kalman filter PMSE is about 30% for the core inflation. Consequently, the 90% prediction intervals based on the PMSEs proposed by Hamilton (1986) and Pfeffermann and Tiller (2005) will be wider than those based on the PMSEs of the Kalman filter with estimated parameters. Furthermore, when the bootstrap PMSEs proposed in this paper are used for constructing prediction intervals, the resulting intervals will be still wider than for the previous procedures. This is reflected in Figure 5, that also plots the 90% prediction intervals for the PMSEs computed the non-parametric bootstrap procedure proposed in this paper. When comparing these intervals with those obtained by using the Kalman filter with estimated parameters, it is clear that the former one is much wider. Therefore, taking into account the parameter uncertainty may change the conclusions about the uncertainty associated to the four unobserved variables estimated. This effect is specially important when estimating the NAIRU and the long-run investment rate. The differences between the prediction intervals constructed for the NAIRU may have important implications as regards to its utility for macroeconomic policy. By looking at the prediction intervals that do not take into account the parameter uncertainty, Doménech and Gómez (2006) conclude that the difference between the NAIRU and the unemployment rate is useful for policy makers in the sense that it can be used for identifying expansions and recessions very accurately. Figure 6, that plots

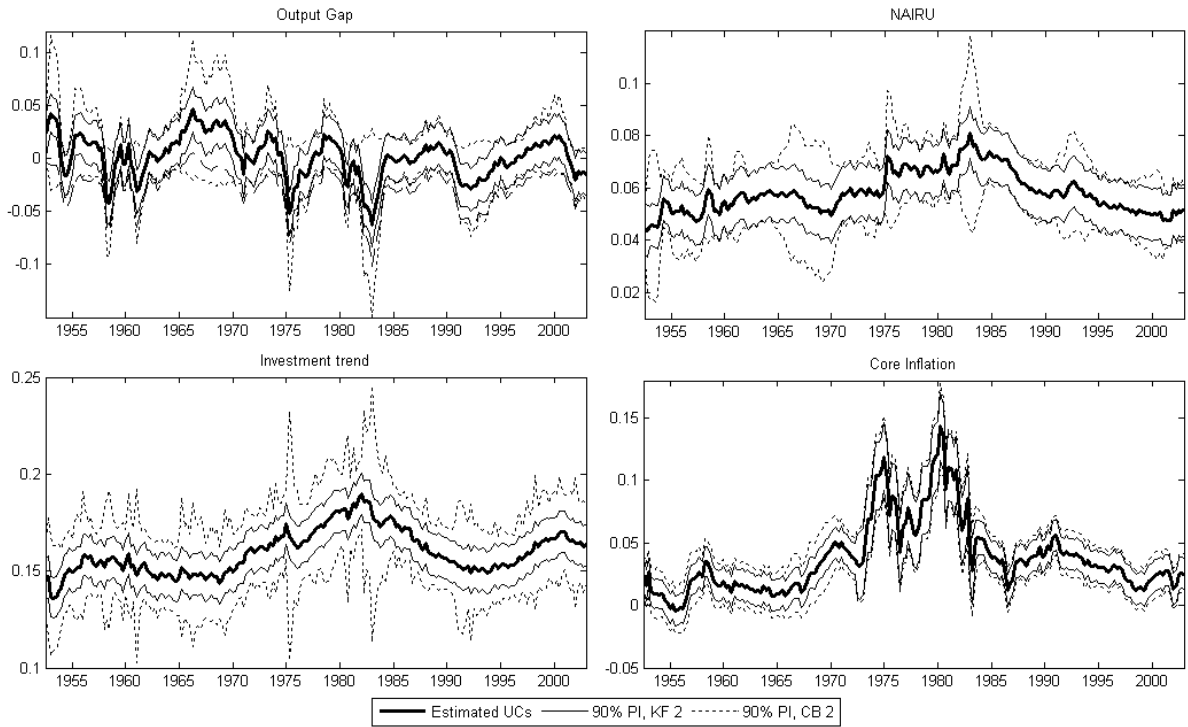
the unemployment rate together with the 90% prediction intervals for the NAIRU, shows that the former is out of the intervals in the second half of the sixties indicating an expansion and in the first half of the eighties suggesting a recession. However, once we construct the intervals be taking into account the parameter uncertainty as suggested in this paper, they are much wider and, consequently, the unemployment is not out of the 90% prediction intervals of the NAIRU in any moment along the sample period considered. Therefore, when taking into account the parameter uncertainty, the conclusion of Staiger et al. (2001) that doubt about the ability of that difference for economy policy is supported.

**Table 3:** Averages and standard deviations (in squared brackets) through time of PMSEs computed using the Kalman filter with estimated parameters (KF2), the asymptotic approximation of Hamilton (1986) (Asy), the bootstrap procedure of Pfeiffermann and Tiller (2005) (PT), and the parametric (CB1) and non-parametric (CB2) bootstrap procedures.

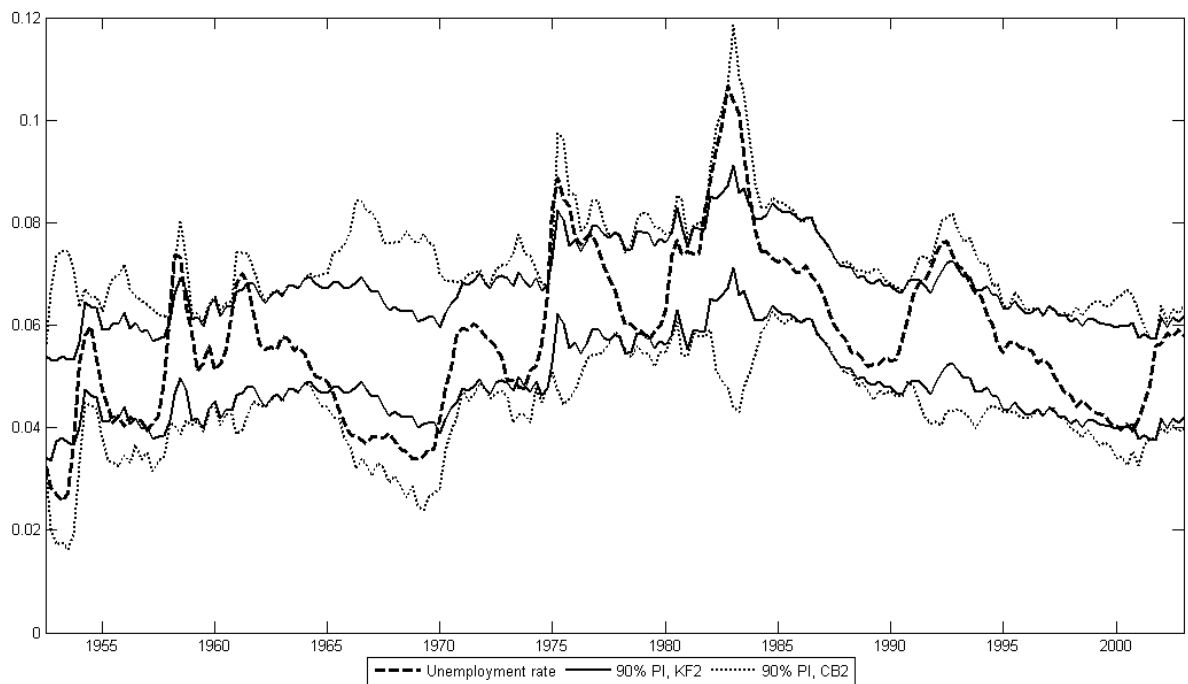
	KF2	Asy	PT	CB1	CB2
Output Gap	0.0143 [0.0012]	0.0214 [0.0013]	0.0217 [0.0016]	0.0238 [0.0078]	0.0283 [0.0095]
NAIRU	0.0050 [0.0004]	0.0051 [0.0004]	0.0051 [0.0004]	0.0063 [0.0021]	0.0089 [0.0074]
Investment	0.0059 [0.0019]	0.0062 [0.0021]	0.0093 [0.0048]	0.0212 [0.0066]	0.0223 [0.0097]
Inflation	0.0137 [0.0116]	0.0140 [0.0119]	0.0142 [0.0121]	0.0179 [0.0132]	0.0173 [0.0110]

## 6 Conclusions

In this paper, we propose two new bootstrap procedures to obtain PMSEs of the Kalman filter estimator of the unobserved states in state space models which take into account the uncertainty attributable to parameter estimation. It has the advantage of being as simple as the procedures based on the asymptotic distribution of the parameters and, at the same time, has the good performance of bootstrap procedures even in small sample sizes.



**Figure 5:** Kalman filter Estimates and 90% prediction intervals for the output gap, NAIRU, Investment trend and Core inflation.



**Figure 6:** Estimated of the NAIRU, the unemployment rate and prediction intervals.

We show that our bootstrap procedures for estimating PMSE of the one-step-ahead estimator of unobserved state for time-invariant and time-variant models have better small sample

properties than alternative bootstrap procedures previously proposed in the literature. The two new bootstrap PMSEs are also more accurate than the asymptotic procedures and that those obtained from the Kalman filter with estimated parameters.

We also show that our bootstrap procedures for estimating PMSE of the one-step-ahead estimator of the underlying components perform very well even when the conditional Normality assumption is not satisfied.

We show the importance of taking into account the parameter uncertainty by implementing the proposed bootstrap procedures to estimate the PMSE for the one-step-ahead estimator of the output gap, NAIRU, trend investment rate and core inflation of the US economy. In this case, with our bootstrap procedures the estimated PMSEs are larger than those obtained with the alternative procedures and, consequently, the prediction intervals will be wider which has consequences for policy makers. We put some doubts on the usefulness of the difference between the unemployment rate and the NAIRU for predicting expansions and recessions of the economy.

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