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On the link between the Bonferroni index and the measurement of inclusive growth.

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Abstract

In a recent paper Ali and Son (2007) suggested measuring the concept of "inclusive growth" via the use of what they called a "social opportunity function". The latter was assumed to depend on the average opportunities available in the population and to give greater weight to the opportunities enjoyed by the poor. On the basis of this approach Ali and Son (2007) then defined an "opportunity index" and an "opportunity curve". The present paper derives the link which exists between these concepts of "opportunity index" and "opportunity curve" and what is known in the literature as the Bonferroni index and the Bonferroni curve. It also defines what could be called a Bonferroni concentration index, a Bonferroni concentration curve, a Generalized Bonferroni curve and a Generalized Bonferroni concentration curve.

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1. Introduction

In a recent paper Ali and Son (2007) suggested measuring the concept of "inclusive growth" via the use of what they called a "social opportunity function". The latter was defined as a function of both the average opportunities available in the population and of the way opportunities are shared in the population. More precisely the social opportunity function was assumed to give greater weight to the opportunities enjoyed by the poor: the poorer an individual is, the greater the weight given to this individual. On the basis of this approach Ali and Son (2007) then defined an "opportunity index" and an "opportunity curve".
The purpose of the present paper is to show the link which exists between the concepts of "opportunity index" and "opportunity curve" and what is known in the literature as the Bonferroni index and the Bonferroni curve. The paper is organized as follows. Section 2 below recalls the definition of the Bonferroni index and curve. Section 3 then shows how these two concepts may be extended to derive a Bonferroni concentration index, a Bonferroni concentration curve, a Generalized Bonferroni curve and a Generalized Bonferroni concentration curve. Section 4 finally shows how to apply these concepts to measure inequality in human opportunities and indicates their link with what Ali and Son (2007) called "opportunity index" and "opportunity curve". Section 5 concludes.

2. The Bonferroni Index

This index was originally proposed by Bonferroni (1930) who derived also what is called the Bonferroni curve. This curve is defined as follows. Assume n individuals whose shares in total incomes are defined as $\{s_1,...,s_i,...,s_n\}$ with $s_1 \leq ... \leq s_i \leq ... \leq s_n$, where n is the number of individuals. On the horizontal axis plot, like for the Lorenz curve, the cumulative population shares $\{(1/n),(2/n),...,(i/n),...((n-1)/n),1\}$. On the vertical axis however do not plot the cumulative income shares (as in the case of the Lorenz curve) but the ratio of the cumulative income shares over the cumulative population shares. In other words plot the following values:

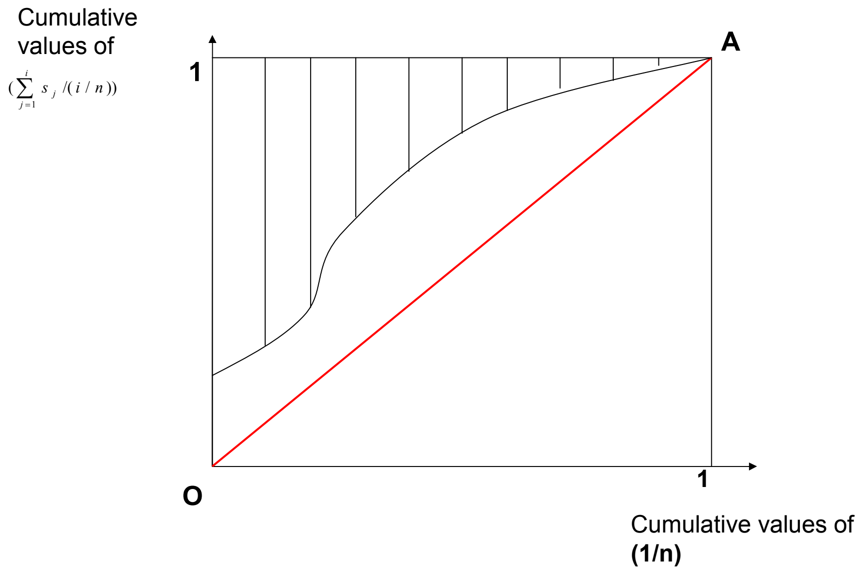
$$\{(\frac{s_1}{(1/n)}),(\frac{(s_1+s_2)}{(2/n)}),...,(\frac{(s_1+s_2+...+s_i)}{(i/n)}),...,(\frac{(s_1+...+s_i+...+s_n)}{(n/n)}=\frac{1}{1}=1)\}$$

Note that if x_i is the income of individual i, with $s_i = (x_i / n\bar{x})$, $\bar{x}$ being the average income in the total population, the ratio $(\frac{(s_1+s_2+...+s_i)}{(i/n)})$ may be also expressed as $(\frac{((x_1+x_2+...+x_i)/i)}{\bar{x}})$, that is, as the ratio of a conditional mean (the mean income of the first i individuals, ranked by increasing income) over the mean $\bar{x}$ in the whole population.
The Bonferroni index is then defined as the area lying between the Bonferroni curve and the horizontal line at height 1 (see Figure 1). The Bonferroni index I_B is hence defined as

$$I_B = [(1/n)(1-(\frac{s_1}{(1/n)}))] + [(1/n)(1-(\frac{(s_1+s_2)}{(2/n)}))] + ...[(1/n)(1-(\frac{(s_1+...+s_i)}{(i/n)}))] + ... \\ + [(1/n)(1-(\frac{s_1+...+s_{n-1})}{((n-1)/n)}))] + [(1/n)(1-(\frac{s_1+...+s_{n-1}+s_n)}{(n/n)}))] \tag{1}$$

From this definition of the Bonferroni index various algorithms have in fact been proposed in the literature to compute it (see, for example, Tarsitano, 1990, Chakravarty, 2007, and Bárcena and Imedio, 2008)

Figure 1:
The Bonferroni Curve

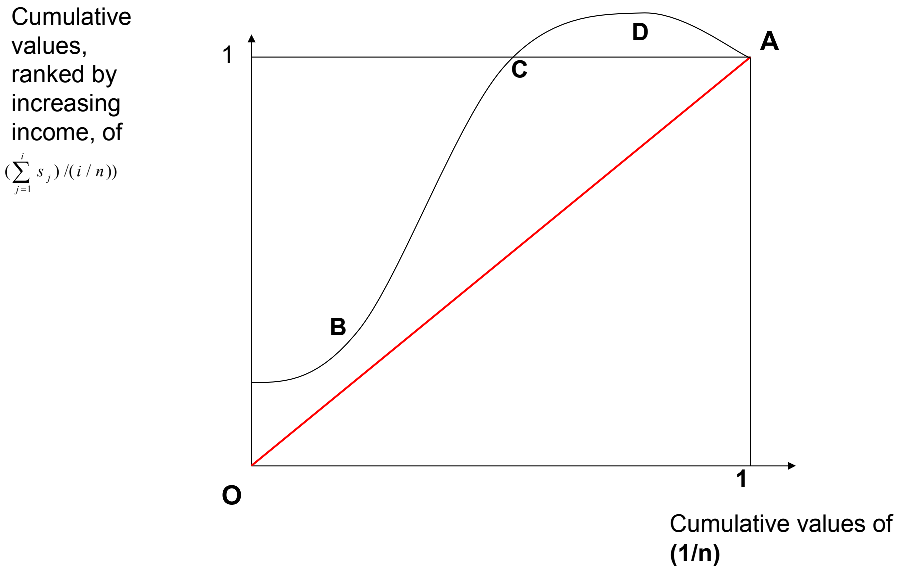


3. Extensions of the Bonferroni index and curve

Whereas the Bonferroni index and the Bonferroni curve which have just been defined may be used to measure income inequality, it is also possible to apply the concepts of Bonferroni index and Bonferroni curve to derive what could be called a Bonferroni concentration index and a Bonferroni concentration curve, in the same way as the Gini concentration index and the concentration curve were derived (see, Kakwani, 1980) from the Gini index.
Let us, for example, assume that we want to take what has been called a "bivariate approach to health inequality measurement" (see, O'Donnell et al., 2008) in order to analyze the link between health and income. We can then derive a "Bonferroni concentration index" C_B by computing the ratios of the "conditional means" of the health variable over the mean value of the health variable in the whole population, that is, the expressions $(\frac{((h_1+h_2+...+h_i)/i)}{\bar{h}})$ where the health variable h_i (e.g. the body mass index) and its mean $\bar{h}$ replace respectively the income variable x_i in the definition of the conditional mean and the average income $\bar{x}$ in the definition of the

index. Note however that here we rank the health variable h_i by increasing income x_i rather than by increasing values of the health variable itself. We can in fact derive such a "Bonferroni concentration index" C_B from a graph that could be called a "Bonferroni concentration curve". Such a curve is constructed like a regular Bonferroni curve, the only difference being that the health variable h_i is ranked by increasing income x_i . Note that this "Bonferroni concentration curve" may at time lie above the equality line (horizontal line at height 1) and in such a case it can be shown that the area above such an equality line will be given a negative sign.

Figure 2:
The Bonferroni Concentration Curve



Finally a third graphical tool may be derived from the Bonferroni index. We know that in the case of income inequality analysis the Lorenz curve is obtained by plotting on the horizontal axis the cumulative population shares and on the vertical axis the cumulative income shares. If, on the vertical axis, we multiply the product of the cumulative income shares by the average income, we obtain what has been called a *Generalized Lorenz curve* (see, Shorrocks, 1983). This curve will therefore start at point (0,0) and end at point $(1,\bar{x})$ where $\bar{x}$ is the average income.

We can similarly derive a Generalized Bonferroni curve. On the horizontal axis plot, as previously, the cumulative population shares and on the vertical axis we now plot the cumulative values $\{(x_1),((x_1+x_2)/2),...,((x_1+x_2+...+x_n)/n)\}$. Such a Generalized Bonferroni curve, like the Generalized Lorenz curve, will start at point (0,1) and end at point $(1,\bar{x})$. Since the Bonferroni index I_B is equal to the area lying above the Bonferroni curve, the area lying above the Generalized Bonferroni curve will be equal to $I_B\bar{x}$, where I_B is the Bonferroni index, and therefore the area lying below the Generalized Bonferroni curve will be equal to $x_{EB} = \bar{x} - I_B\bar{x} = \bar{x}(1-I_B)$. x_{EB} can therefore be considered a measure of welfare similar to the index $x_{EG} = \bar{x}(1-I_G)$ defined by Sen (1974).

One can naturally apply the concepts of Generalized Lorenz or Bonferroni curves to measure the welfare derived from some health attainment. Such welfare measures x_{EG} and x_{EB} would in fact give a greater weight to an individual, the lower the level of his health.
One may however think of an alternative approach, one where the weight of an individual, when measuring health related welfare, would be higher, not the lower the level of his/her health, but the lower his/her income. This is in fact the approach taken by Wagstaff (2002) in his definition of health achievement. We can call such an approach the pro-poor approach to the measurement of health achievements.
Using the concept of "Bonferroni concentration index" C_B which was defined previously, we may therefore define the level of health achievement A_B as

$$A_B = \bar{h}(1 - C_B) \tag{2}$$

In fact, in the same way as we derived previously the concept of Generalized Bonferroni curve, we can now derive the concepts of Generalized Bonferroni Concentration curve. We simply have to order the vertical coordinates of the Generalized Bonferroni curve not by increasing values of the health variable, but by increasing income. It is then easy to derive that the area under such a Generalized Bonferroni concentration curve will be equal to half the product $\bar{x}(1 - C_B) = A_B$.

4. The Bonferroni index and the measurement of inequality in human opportunities

Let p_i be the probability for group i of having access to some public service (e.g. a hospital) and let $\bar{p}$ be the average probability for an individual in the population to have access to this service. Let w_i be the weight of population subgroup i in the total population. The weights w_i may therefore represent the "prior" probability of having access to the service, while the expression $(w_i(p_i / \bar{p}))$ would represent the "posterior" probability of having access to the service.
If one wishes to use the Bonferroni index to measure the degree of inequality in the opportunity to have access to this service (hospital), we may apply expression (1) but the "a priori" weights w_i will replace the population shares $(1/n)$ and the "a posteriori" weights $\sigma_i = (w_i(p_i / \bar{p}))$ will replace the income shares s_i . In other words the Bonferroni index of inequality of opportunity would in such a case be defined as

$$\begin{aligned} I_B = {} & [(w_1)(1 - (\frac{\sigma_1}{(w_1)}))] + [(w_2)(1 - (\frac{(\sigma_1 + \sigma_2)}{(w_1 + w_2)}))] + ...[(w_i)(1 - (\frac{(\sigma_1 + ... + \sigma_i)}{(w_1 + ... + w_i)}))] + ... \\ & + [(w_{m-1})(1 - (\frac{\sigma_1 + ... + \sigma_{m-1})}{(w_1 + ... + w_{m-1})}))] + [(w_m)(1 - (\frac{\sigma_1 + ... + \sigma_{m-1} + \sigma_m)}{(w_1 + ... + w_{m-1} + w_m)}))] \end{aligned} \tag{3}$$

where m is the number of population subgroups.

The following graphical interpretation may be given to this approach. Let us plot on the horizontal axis the cumulative values of the "a priori" probabilities $(w_1, w_2, ..., w_i, ...1)$ and on the vertical axis the cumulative values of the ratios $\{[(w_1(p_1/\overline{p}))/w_1]\}, ..., \{[(w_1(p_1/\overline{p})) + ... + (w_i(p_i/\overline{p}))]/[w_1 + ... + w_i]\}, ...$. If we multiply these vertical coordinates by the average probability $\overline{p}$, we will obtain new coordinates which will be expressed as $\{[(w_1(p_1))/w_1]\}, ..., \{[(w_1 p_1) + ... + (w_i p_i)]/[w_1 + ... + w_i]\}, ...$
$..., \{[(w_1 p_1) + ... + (w_n p_n)]/[w_1 + ... + w_n]\}$.
Since $[(w_1 p_1) + ... + (w_n p_n)] = \overline{p}$ and $[w_1 + ... + w_n] = 1$, it is then clear that we end up with a curve which starts also at point (0,0) at which ends at point $(1, \overline{p})$. As was done previously when we analyzed health achievements, we will call this curve a Generalized Bonferroni curve. It is easy to derive that the area lying under such a curve is in fact equal to $\overline{p}(1 - I_B)$, an index which can be called the "Bonferroni-related Human Opportunity Index" HOI_B.
Note that since the Bonferroni index (like the Gini index) gives a higher weight to the categories who have a lower probability of accessing the service, we can call the "Bonferroni-related Human Opportunity Index HOI_B" a Human Opportunity Index which is "welfare-related", in the sense that it favors groups with low probabilities of accessing the service.
The approach which has just been described is based on a univariate approach to the measurement of inequality in opportunities. In other words we measured the inequality in the access to a hospital, no matter what the socioeconomic background of the individual is. Let us however assume that we want to analyze the link which exists between this access to a hospital and the socioeconomic background. In such a case we would classify in (3) the "a priori" probabilities w_i and the "a posteriori" probabilities σ_i, not by increasing ratios $(\sigma_i / w_i) = [w_i(p_i/\overline{p})]/w_i = (p_i/\overline{p})$, but by increasing socioeconomic background. As a consequence we would not compute the Bonferroni index but the Bonferroni Concentration index C_B and derive a "Human Opportunity Index" on the basis of the Bonferroni concentration index, rather than of the Bonferroni inequality index. We will call such an index a "Bonferroni-related Pro-Poor Human Opportunity Index" $O_{B,pp}$, and it will evidently be expressed as

$$O_{B,pp} = \overline{p}(1 - C_B) \qquad (4)$$

The graphical interpretation of such an index is very simple. We rank the cumulative values that are plotted on the horizontal and vertical axes to derive a Generalized Bonferroni curve, not by increasing values of the probabilities p_i of accessing the service, but by increasing values of the socioeconomic background of the individuals. Such a curve is clearly a "Generalized Bonferroni Concentration curve". It is easy to prove that the area lying under such a curve is to equal the product $\overline{p}(1 - C_B) = O_{B,pp}$.
It is interesting to note that what we have just called the "Bonferroni-related Pro-Poor Human Opportunity Index" is in fact what Ali and Son (2007) called "Opportunity Index ", while what we called "Generalized Bonferroni Concentration curve" is identical to what Ali and Son (2007) called "Opportunity Curve". The proof is simple. Ali and Son (2007) defined their opportunity curve (or generalized concentration

curve of opportunity) as follows. In terms of the notations previously used, their approach amounts in fact to plotting on the horizontal axis the cumulative values of the "a priori" population shares w_i, these probabilities being ranked by increasing socioeconomic background of the individuals, and on the vertical axis the cumulative values of the conditional means of the variable analyzed which in our case are the ratios:

$$[((w_1 p_1)/w_1),((w_1 p_1 + w_2 p_2)/(w_1 + w_2)),...,((w_1 p_1 + w_2 p_2 + ... + w_i p_i)/(w_1 + w_2 + ... + w_i)),$$
$$...,((\sum_i w_i p_i)/(\sum_i w_i))]$$

Ali and Son (2007) then define their opportunity index $\bar{y}^*$ as the area lying below this opportunity curve and this index is clearly identical to what we previously called the "Bonferroni-related Pro-Poor Human Opportunity Index" $O_{B,pp}$.

5. Conclusion

Ali and Son (2007) derived their opportunity index and opportunity curve using the framework of a general social opportunity function, implicitly arguing that society should focus on expanding or maximizing this social opportunity function. The opportunity index provides an operational way to implement policies that would maximize the social opportunity function. Moreover, Ali and Son introduced the idea of equity of opportunity index, which measures how equitably or inequitably opportunities are distributed across the population. The two indices are not only useful in assessing average access to the public services available to the people, but also in evaluating the equity of access to such services across different income groups. More importantly, their study has demonstrated that while the analysis can be done at a point of time, it is also possible to assess the changes in access to and equity of opportunities over time.

The Bonferroni index was proposed in 1930 and was not mentioned very often until recently. Ali and Son's paper demonstrates how useful Bonferroni's approach may be to measure and analyze inclusive growth. Our paper linking the Bonferroni index and Ali and Son's opportunity index contributes thus to the revival of Bonferroni's important contribution.

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