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Three-Candidate Spatial Competition When Candidates Have Valence: Stochastic Voting

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1 Introduction

In this paper we study the effects of stochastic (probabilistic) voting on the equilibrium location, equilibrium vote shares and comparative statics in a model of unidimensional political competition among three candidates who differ in valence.[1] It is well known that (in models of both two-candidate and multi-candidate spatial competition) one can restore the existence of PSNE by adding a stochastic component to voter preferences.[2] Naturally, this point can be illustrated by using models under which there is no equilibrium without stochastic voting. The model we study, instead, has (local or global) Nash equilibria without stochastic voting. Comparing the equilibria of the model with and without stochastic voting, allows us to note the effects of stochastic voting on the equilibrium location, equilibrium vote shares and comparative statics.

Evrenk (2009a;b and 2010) studies several versions of our model *without* stochastic voting (below, we refer to all these versions as the deterministic model). He notes that the equilibria of the deterministic model has several non-plausible features. More specifically, when the voter density is symmetric and the candidates are vote-maximizers, (i) a pure strategy Nash equilibrium (PSNE) exists only in two non-generic cases, and (ii) in the local Nash Equilibrium (LNE), the candidates with the second–and the third–highest valence receive the same vote share even when their valence differs significantly. Although these

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[1]Coined by Stokes (1963), the term valence refers to non-policy characteristics of a candidate such as honesty, competency, charisma and campaign ability.

[2]The seminal work in Hinich (1977) on two-candidate competition is extended to a multi-candidate setup by Lin, Enelow, and Dorussen (1999) and Adams (1999a).

two features can be eliminated by considering an asymmetric density, the following three non-plausible features remain in the deterministic model: (iii) the highest-valence candidate is always located between the other two candidates and he always receives a majority of votes; (iv) unless a candidate's valence is higher than that of the others, in equilibrium all his supporters prefer policies that are more extreme than that of the candidate; and (v) when a lower-valence candidate's valence (appeal) increases, the other lower-valence candidate's vote share increases.[3]

Given an appropriate degree of uncertainty about the stochastic component of voter preferences, we find that all of the non-plausible features mentioned above will be eliminated when the voter preferences have a stochastic component (voting is stochastic). Under stochastic voting PSNE exists quite regularly; a voter of a lower-valence candidate does not necessarily prefer a policy that is more "extreme" than that of the candidate (a centrist voter may also vote for a candidate on the left due to a favorable preference shock); the highest-valence candidate still positions himself at the center, but he does not necessarily receive a majority of votes; and, always, if a candidate's appeal (valence) increases while the others do not change, this candidate's vote share increases while the other candidates' vote shares decrease.

It should also be noted that the equilibria of the stochastic voting has non-plausible features of its own. First, although the center candidate does not necessarily receive a majority of votes, his vote share is still larger than that of the other candidates. It seems that this feature of the model can be modified by adding further assumptions.[4] Second, when the variance of the preference shock is large, there is an agglomerated PSNE (an equilibrium in which all candidates adopt the same policy). Sometimes this is the only equilibrium and sometimes the model has both an agglomerated and dispersed equilibria. We also find that when each candidate maximizes his plurality, some equilibria are supported by a paradoxical candidate-behavior: by deviating from his PSNE location, the candidate with the highest valence can receive a majority of votes; he does not do so in equilibrium, because such a move reduces his plurality. Comparing the stochastic and deterministic models helps us to see that this non-plausible feature is due to the plurality maximization assumption, and not due to stochastic voting: Evrenk (2010) reports the same result (and, for a larger set of parameters) for the deterministic model.

In our analysis, we first derive some comparative statics analytically for the case in which the level of uncertainty about the voting behavior is almost zero (when the stochastic voting model is in an epsilon–neighborhood of the deterministic model). But, in general the equilibria under stochastic voting cannot be calculated analytically; it must be calculated through numerical simulations.[5]

Due to the constraints on computing power, an equilibrium search in models of multi-candidate competition is typically done as this: starting from an initial strategy profile (specified by the researcher) the code produces a chain of best-responses to see if this chain converges (a fixed point of the best-responses is a PSNE of the game). If it does not converge

[3]This is non-plausible as it implies that the latter lower-valence candidate's vote share increases when his (relative) appeal decreases.

[4]Simulating models using data from the UK recently Adams (2001) and Schofield (2005) show that one can obtain equilibria that are surprisingly close to the actual situation (in which the center party does *not* receive the highest vote share). The models these authors simulate allow for valence differences, stochastic voting (and activist valence in the former and party loyalty in the latter).

[5]See Groseclose (2001) for a similar approach (theoretical results for a convex loss function satisfying a weak restriction on the curvature; then, simulations for a specific loss function to uncover the comparative statics that cannot be derived analytically) in the context of two-party competition.

after a certain number of steps (specified by the researcher), then the code stops looking for an equilibrium and the researcher concludes that no equilibrium exists. Our method is quite different. As we discuss in Appendix C in more detail, our code checks for *all* possible strategy profiles. Thus it finds all the equilibria there are. We simulate the model under both symmetric and asymmetric densities (we consider different degrees of asymmetry). For each case, we also simulate the model under different candidate-objectives. All the equilibria we found (as well as our codes used to find them) are available as online supplements.

In the literature, Adams (1999b) uses numerical simulations to study equilibria of three-candidate spatial competition under valence differences and stochastic preferences. Our paper differs from Adams (1999b) and complements it in several ways. He studies political competition when the voter density is uniform, while we study political competition when voter density is Triangular (always unimodal and possibly asymmetric). Under uniform density, the model has no Nash equilibrium when voting is deterministic. Thus, Adams (1999b) compares equilibria with and without valence differences given stochastic voting while we compare equilibria with and without stochastic voting given that there are valence differences among the candidates. His benchmark is the less dispersed (or, usually, completely agglomerated) equilibria of the stochastic model without any valence differences; thus, he focuses on centrifugal incentives.[6] Our benchmark is the more dispersed LNE of the deterministic model, and, therefore, we focus on centripetal incentives (Cox, 1987). He studies comparative statics with respect to the weight voters put on the utility from candidate's policy (the policy salience parameter) while we study comparative statics with respect to the degree of uncertainty about voting decision.[7] Finally, Adams (1999b) focuses on (and reports only) equilibrium locations but not on vote shares or comparative statics.

2 The Model

Consider a setup with a continuum of voters and three candidates, $j \in \{1, 2, 3\}$. Let i denote the voter whose most preferred policy platform is $i \in \mathcal{I} \subset \mathbb{R}$, where $\mathcal{I}$ is compact, and let $f(i)$ denote the density of i. For now, only assume that the density is atomless and $f(i) > 0$ at the interior of $\mathcal{I}$ (we later simulate the model for a Triangular density with support $\mathcal{I} = [-1, 1]$). Let $v_j \in \mathbb{R}$ denote valence (Stokes, 1963) of candidate j, e.g., his competency, integrity, charisma or campaigning ability. We normalize candidate valences as

$$v_2 > v_1 \geq v_3 = 0$$

and let $r = \frac{v_1}{v_2}$.

The preferences of voter i over candidate j are represented by the utility function[8]

$$U_i^j(p_j, v_j) = -L(i - p_j) + v_j + \sigma\varepsilon_{ij}. \tag{1}$$

[6] Using the agglomerated LNE of the stochastic model as the benchmark, Schofield (2007) studies centrifugal incentives analytically in a much more general setup.

[7] de Palma, Hong and Thisse (1990) also uses simulations to study the effect of stochastic voting on PSNE of a stylized model of spatial (unidimensional) competition among two to six candidates. They, too, focus on how equilibria change as one varies the level of uncertainty, but, as they do not consider candidates with different valences, in their model, too, there is no equilibrium when the number of competing candidates is three and the voting is deterministic.

[8] It is worth noting that the additive form used in (1) assumes that (i) each voter's valuation of candidate valence is independent of the policy of the candidate (or, the most preferred policy of that voter); and (ii) each candidate's non-policy characteristics have an identical influence on each voter.

In (1), $L(x) : \mathbb{R} \to \mathbb{R}$ is the "loss function" representing the voter's policy preferences. Again while simulating the model we specify $L(.)$ further, but for now only assume that $L(.)$ is (i) symmetric around zero, $L(x) = L(-x)$; *(ii)* twice continuously differentiable with $L'(0) = 0$, $\lim_{x\to\infty} L'(x) = \infty$, and for $x > 0$, $L'(x) = -L'(-x) > 0$; and, *(iii)* strictly convex. The voting is sincere: i votes for the j who provides the highest $U_i^j(p_j, v_j)$.

In (1), $\sigma\varepsilon_{ij} \in \mathbb{R}$, denotes a preference shock for voter i. More specifically, $\sigma \geq 0$ is a scaling constant and ε_{ij} is a Type I extreme value (also known as Gumbel or log-Weibull) random variable with c.d.f. $\Psi(x) = \exp(-\exp(-x))$. Since a Type I random variable has variance $\pi^2/6$, the variance of the preference shock is equal to $(\sigma\pi)^2/6$. Below, we focus on the case in which the preference shock is independently and identically distributed. Thus, for any two voters i and h, where $h \neq i$, and any two candidates j and k, where $k \neq j$, we assume that $\mathbf{E}[\varepsilon_{ij}|\varepsilon_{ik}] = \mathbf{E}[\varepsilon_{ij}|\varepsilon_{hj}] = \mathbf{E}[\varepsilon_{ij}]$.

The candidates cannot observe individual preference shocks; they only know the distribution of $\sigma\varepsilon_{ij}$. Given this information structure, each j simultaneously chooses his policy platform to maximize his (expected) vote share $V_j(p_j, p_{-j})$, where $p_{-j} = (p_k, p_l)$. (In addition, in Section 4 we simulate the model when each candidate's objective is to maximize his plurality.)

Let us define some of the terms that are used throughout the paper. If for each j, $V_j(p_j, p^*_{-j})$ has a global maximum at p^*_j, then policy profile $\mathbf{p}^* = (p^*_1, p^*_2, p^*_3)$ is a (pure strategy Nash Equilibrium) PSNE. If for each j, $V_j(p_j, p^*_{-j})$ has (at least) a local maximum at p^*_j, then policy profile $\mathbf{p}^* = (p^*_1, p^*_2, p^*_3)$ is a (local pure strategy Nash Equilibrium) LNE.[9] In an *agglomerated* equilibrium, all the candidates adopt the same policy platform. If at least one candidate's equilibrium policy differs, then it is a *dispersed* equilibrium. If the only equilibria of the game are $(p^*_1, p^*_2, p^*_3) \neq \mathbf{0}$ and its mirror image, i.e., $(-p^*_1, -p^*_2, -p^*_3)$, then we say that the game has a (modulo symmetry) unique equilibrium. We also call (p^*_1, p^*_2, p^*_3) as a *left*-PSNE if $p^*_1 < p^*_2$ or $p^*_3 > p^*_2$. Naturally, a *right*-PSNE, then, is simply the PSNE policy platform in which we have $p^*_1 > p^*_2$ or $p^*_3 < p^*_2$.

This paper focus on the performance of the model when the voting is *stochastic*, $\sigma > 0$. To see the changes in voting pattern, comparative statics and other features of equilibria due to stochastic voting, we compare the results with those under the *deterministic* model, $\sigma = 0$. In these comparisons $(p^{*D}_1, p^{*D}_2, p^{*D}_3)$ denotes the equilibria of the *deterministic* model.

Equilibrium locations for the stochastic model cannot be identified analytically although those for the deterministic model can be. Therefore, we first use the fact that deterministic model is a special (or, degenerate) case of the stochastic model, and analyze the stochastic model at an $\varepsilon-$neighborhood of the deterministic model, i.e., when σ is infinitesimally small. Then, we simulate the game for larger values of σ, and compare the resulting PSNE with the equilibria of the deterministic model.

2.1 The deterministic model and its non-plausible features

The LNE locations and vote shares for the deterministic model can be calculated analytically when one specifies the voter density, $f(i)$. Evrenk (2009a;b) analyze the equilibria under a symmetric voter density, while without imposing a specific voter density Evrenk (2010, Theorem 2) shows that in any LNE the lower-valence candidates differentiate their policies from each other and from the policy of the higher-valence candidate. More specifically,

[9] Note that, by definition, each PSNE is an LNE.

in any LNE, Candidate 2 positions himself between the two lower-valence candidates; the distance between Candidate 2 and Candidate $j \in \{1, 3\}$ is equal to $\delta_j = L^{-1}(v_2 - v_j)$, where, by abusing the notation, we define $L^{-1}(y) = \{x \geq 0 | L(x) = y\}$.

Motivating the current paper, equilibria in the deterministic model have several non-plausible features. As we note in Section 1, when the voter density is symmetric: (i) the model does *not* have a PSNE, (except for a non-generic case in which $r = 0$) only an LNE exists;[10] (ii) in this LNE, even when there is a significant valence difference between Candidates 1 and 3, they always locate *symmetrically* around the mean and each receives the *same* vote share; (iii) when the PSNE exists, Candidate 2 (the candidate with the highest valence) receives the majority of votes in equilibrium; (iv) a lower-valence candidate's voters are always more "extreme" than the candidate himself; (v) when a lower-valence candidate's appeal increases, the other lower-valence candidate's vote share increases by an equal amount.

Evrenk (2010) shows that the first two features listed above can be eliminated by considering an asymmetric voter density and that the third feature, too, can be eliminated by considering plurality-maximizing candidates. This latter result holds, however, only because plurality maximization implies paradoxical candidate-behavior.

The fourth feature directly (and, the fifth feature indirectly) follows from the voting pattern in the LNE of the deterministic model. To see this consider a *left*-LNE ($p_1^{*D} < p_2^{*D} < p_3^{*D}$). For a given $\mathbf{p}^{*D}$, let B_j^* denote the set of voters for whom j is the best candidate: B_j^* is j's voter base. Similarly, let W_j^* denote the set of voters for whom j (located at p_j^{*D}) is the worst candidate. Given that the distance between Candidate 2 and Candidate $j \in \{1, 3\}$ is equal to $\delta_j = L^{-1}(v_2 - v_j)$ in LNE, we have $B_1^* = \{i | i < p_1^{*D}\}$, $B_3^* = \{i | i > p_3^{*D}\}$, and B_2^* is equal to the open interval (p_1^{*D}, p_3^{*D}), i.e., each B_j^* is separated by the policies of the lower-valence candidates.[11] To see the boundaries of W_j^*'s let $I(p_j, p_k)$ denote the voter indifferent between candidates j and k. As we prove in Appendix A,

Lemma 1 *When $\sigma = 0$, in equilibrium Candidate 2 is never the worst candidate for any voter ($W_2^* = \varnothing$) while $W_1^* = \{i | i > I(p_1, p_3)\}$ and $W_3^* = \{i | i < I(p_1, p_3)\}$.*

Figure 1 demonstrates the voter utilities, resulting LNE locations, B_j^*'s and W_j^*'s in an LNE (for now, ignore the specific numbers in Figure 1). To relate this to a stochastic model, let ρ_{ij} denote the probability that voter i votes for candidate j. In the deterministic model when candidates are located at $\mathbf{p}^{*D}$, we have

$$\rho_{ij} = \begin{cases} 1 \text{ if } i \in B_j^*, \\ 1/2 \text{ if } i = p_j^{*D} \text{ and } j \in \{1, 3\}, \\ 0 \text{ otherwise.} \end{cases} \tag{2}$$

In contrast with (2), for any policy platforms and candidate valences, in the stochastic

[10] In the other non-generic case, $r = 1$, too, a PSNE exists (Evrenk 2009a;b). But, as this LNE has a different pattern, and, as this case is not generic, we do not consider it here.

[11] Simply note that when $p_1 = p_2 - L^{-1}(v_2 - v_1)$ we have for any $i < (>) p_1$,

$$U_i(p_2, v_2) = -L(i - p_2) + v_2 < (>) U_i(p_2, v_2) = -L(i - p_1) + v_1.$$

Similarly, when $p_3 = p_2 + L^{-1}(v_2 - v_3)$, all the voters to the right-hand-side of Candidate 3 prefer this candidate strictly to Candidate 2.

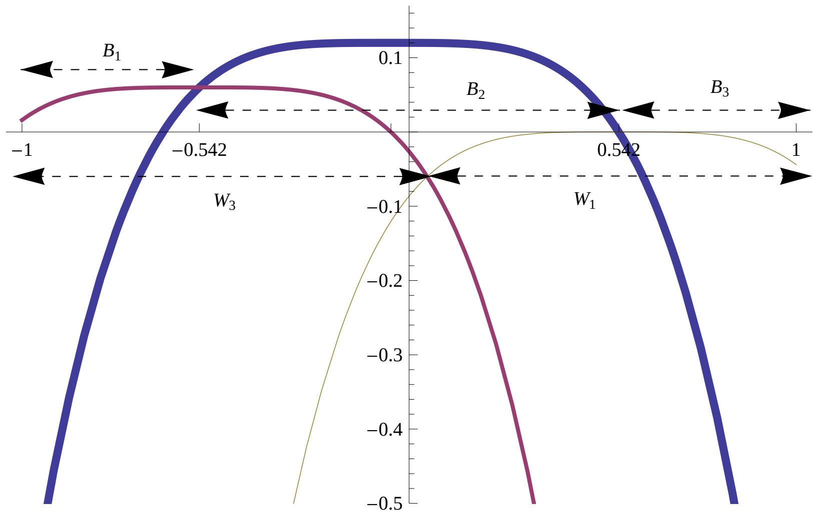


Figure 1: Voter utilities, LNE locations, candidate bases (B_j^*'s) and W_j^*'s in the deterministic model. The thickly (thinly) drawn blue (yellow) parabola depicts utility from Candidate 2 (Candidate 3) while the third parabola depicts utility from Candidate 1.

model[12]

$$\rho_{ij} = \frac{\exp((-L(i-p_j)+v_j)/\sigma)}{\sum_{k=1}^{3}\exp((-L(i-p_k)+v_k)/\sigma)}, \tag{3}$$

and $V_j(p_j, p_{-j}) = \int_{\mathcal{I}} \rho_{ij} f(i)di$. When candidate locations are fixed at $\mathbf{p}^{*D}$, as σ converges to zero from the right, (3) converges to (2). That is, at $\sigma = 0$ and $\mathbf{p} = \mathbf{p}^{*D}$, the functions ρ_{ij} are continuous over B_j^*'s and W_j^*'s in σ from the right. Using this continuity we have the following result.

Proposition 1 *When candidates are located at their LNE locations under the deterministic model ($\mathbf{p}^{*D}$), for any lower-valence candidate ($j \in \{1,3\}$), any voter i, and any $\sigma \geq 0$, if $i \in B_j^*$, then $\frac{\partial \rho_{ij}}{\partial \sigma} < 0$, and if $i \in W_j^*$, then $\frac{\partial \rho_{ij}}{\partial \sigma} > 0$. For $\sigma = 0$, if $i \notin B_j^*$, $\frac{\partial \rho_{ij}}{\partial \sigma} > 0$.*

Proved in Appendix A, Proposition 1 shows that if we perturb the deterministic model so that voter preferences become slightly stochastic, then the voting patterns change in a specific way: ρ_{ij} decreases over the voter base of j and increases over the voter bases of other candidates. As we keep increasing the level of uncertainty further, an increase in σ still reduces ρ_{ij} over the voter base of j *but* does not necessarily increase all over $\mathcal{I}\backslash B_j^*$. We only know that ρ_{ij} increases over the set of voters who find j as the worst candidate in the deterministic model.[13]

Proposition 1 indicates the source of centripetal incentives (Cox 1990) to the lower-valence candidates under stochastic voting. Moreover, it notes that such incentives exist at the LNE locations of the deterministic model. Comparing the resulting equilibria under

[12]This is simply the probability that $U_i^j(p_j, v_j)$ is larger than both $U_i^m(p_m, v_m)$ and $U_i^n(p_m, v_m)$ where $m \neq n$ are both from $\{1,2,3\}\backslash j$. See Train (2003) for derivation.

[13]For σ large enough, one can show that there are voters close to p_j^{*D} but still in $I\backslash(B_j^* \cup W_j^*)$ for whom $\frac{\partial \rho_{ij}}{\partial \sigma} < 0$. Their voting behavior, however, does not contradict the relationship identified in Proposition 1. As we show in the next section, when σ is large, candidate j locates closer to the center, so these voters are not likely to be in his voter base in the stochastic model anyway.

stochastic voting with equilibria under deterministic voting, we note the effects of these incentives in sections 3 and 4. Further, in Appendix B we provide some numerical calculations that measure the strength of these incentives. But, intuitively one can explain the source as follows.

When $\sigma = 0$, moving from p_j^{*D} towards the electoral center reduces the vote share of a lower-valence candidate j: as he moves towards the center, the indifferent voter between him and Candidate 2 moves further away from the center, his vote share decreases among B_j^*, but he cannot get any new votes from the voters in $\mathcal{I}\backslash B_j^*$. When $\sigma > 0$, however, with a positive probability each $i \in \mathcal{I}\backslash B_j^*$ will receive a preference shock and vote for j even though (by definition of B_j^*) there is another candidate k who would be preferred to j had there been no preference shock. The probability that i will receive a sufficiently high preference shock and will switch to j decreases in (i) the valence difference between j and k, and (ii) the distance between i and p_j. That is, for $i \in B_k^*$, as $-L(i-p_k)+v_k-(-L(i-p_j)+v_j)$ increases, i will need a higher ε_{ij} to vote for j. But the higher the needed preference shock, the less likely that it will occur. As valence is exogenous in our model, there is nothing that candidate j can do to reduce $v_k - v_j$. Yet, j can reduce $-L(i-p_k)+L(i-p_j)$ by coming closer to the voters in B_k^*, and, thus, increasing the probability that an $i \in B_k^*$ switches to j.

3 Equilibria

We describe our simulation model in detail in Appendix C. To summarize in a nutshell, our method differs from that of earlier simulation studies in the number of strategy profiles checked. More specifically, a typical simulation study starts with an initial strategy profile, then checks if one of the candidates has any profitable deviation from this profile. If he has, then this candidate is located to this profitable deviation and a new strategy profile is obtained. Then the code checks if the next candidate has a profitable deviation at this new strategy profile. This process will stop if it converges to a strategy profile under which none of the candidates has a profitable deviation or if it carries this search a certain number of steps and gets no convergence. When the latter happens, the researcher concludes that there is no PSNE or repeats the search with another initial strategy profile. In our method, the code checks all possible strategy profiles to see if any PSNE exists. Since it checks all the strategy profiles, the method finds all the PSNE and when it finds none, one knows that it is because there is none. It has a disadvantage: it can take quite some time.[14]

When simulating the model we set $\mathcal{I}$ equal to $[-1,1]$. To be able to calculate the vote shares and the best-responses, we discretize the strategy space, approximating the interval $[-1,1]$ by 201 equidistant locations from minus one to one.[15] That is, for each j, we have $p_j \in \mathcal{I}^{\mathcal{S}} = \{-\frac{100}{100}, -\frac{99}{100}, ...0, \frac{1}{100}, ...\frac{100}{100}\}$. As the voter density, we use a discrete approximation of a Triangular density with base $[-1,1]$ and mode m, that is

$$f(i,m) = \begin{cases} (1+i)K/(1+m) \text{ for } i \in \mathcal{I}^{\mathcal{S}} \text{ and } i < m \\ (1-i)K/(1-m) \text{ for } i \in \mathcal{I}^{\mathcal{S}} \text{ and } i \geq m \end{cases}$$

[14] For more details about the actual times, see Appendix C. Note, however, that with the advances in the computing power commonly available, this disadvantage should soon disappear.

[15] There are cases in which we use a finer approximation, see Appendix C.

where $K = (\sum_{i \in \mathcal{I}^S} f(i,m))^{-1}$. As a result

$$V_j(p_j, p_{-j}) = \sum_{i \in \mathcal{I}^S} \rho_{ij}(p_j, p_{-j}) f(i,m).$$

In all simulations we use a quartic loss function,[16] $L(i - p_j) = (i - p_j)^4$. In Section 4.3, we discuss this choice. In all simulations we set $r = 1/2$. We vary several other factors. We simulate the model both under symmetric and asymmetric densities: we consider three different values of m (0, 1/4, and 3/4). For each value of m, we simulate the model under two different values of σ (0.12 and 0.25). For each of these cases, we simulate the model under two different candidate objective functions: vote share maximization and plurality maximization. In this section the results under the symmetric density ($m = 0$) and vote-share maximization are discussed. In Section 4 we report some of the results under asymmetric density and plurality maximization to note how these different assumptions affect the equilibria when voting is stochastic. Data (MS Excel) files provided as online supplements present all the equilibria of all the variations mentioned above.

3.1 Results for symmetric density and vote maximization

Figure 2 presents equilibrium locations and vote shares for three different levels of σ: the deterministic case ($\sigma = 0$) as the benchmark, and the two stochastic cases at different levels of uncertainty, $\sigma = 0.12$ and $\sigma = 0.25$. In all these cases we set $r = 1/2$. Therefore, when $\sigma = 0$ no PSNE exists under a symmetric density. Thus, for $\sigma = 0$ we plot the LNE (calculated analytically in Evrenk (2009a;b)). Note that when the valence differences are small, the necessary condition in Evrenk (2009b) is violated; then the LNE does not exist.

Under symmetric density (and, any σ), if (p_1^*, p_2^*, p_3^*) is an equilibrium, so is $(-p_3^*, -p_2^*, -p_1^*)$. Since the *right*-PSNE is more common in the cases we study in next section (when $m > 0$), to avoid messy figures we plot only the *right*-PSNE in Figure 2.

Below, we discuss how all non-plausible features of PSNE of the deterministic model disappear when $\sigma = 0.12$.

First, a PSNE does exist as long as the valence differences between the candidates are not too large. More specifically, when $\sigma = 0.12$, a (modulo symmetry) unique, dispersed PSNE exists for any $0.01 \leq v_2 \leq 0.35$.[17] An agglomerated PSNE at the electoral mean

[16] Quartic loss functions are not commonly used in the literature. Although Proposition 1 holds under any (strictly) convex loss function, when simulating the model we use a quartic loss function because, in the deterministic (benchmark) case, the quartic loss function performs better than the commonly used quadratic loss function: the set of parameters under which PSNE exists is significantly larger when $L(x) = x^4$. Yet, a quartic loss function has a higher curvature than the quadratic one, so one may think that we impose too much risk aversion on the voter's preferences over policy lotteries by using a quartic loss function. We do not: in our model the voters are *not* deciding under certainty (the policy platforms of candidates are certain). In our model the uncertainty exists only from the point of view of the candidates. Still, it is straightforward to modify our code and calculate all the equilibria under any other loss function (simply replace all expressions of the form $()^4$ with the alternative loss function).

[17] Since there is no analytical solution, to investigate why PSNE fails to exist when it does, we studied the vote share of each candidate when he deviates from his PSNE platform for all values of v_2. We found that as v_2 increases, Candidate 1's vote share from a policy platform between Candidates 2 and 3 increases and at the highest valence value under which the equilibrium exists ($v_2 = 0.35$), by deviating to this location he can get a vote share very close to his PSNE vote share. Without any analytical solution one cannot know for sure, but it seems to us that the equilibrium fails to exists because of this deviation. Note that the standard criticism of vote maximization as an objective function applies to this deviation as well: although it increases his vote share, such a deviation also increases the vote share of Candidate 2 reducing plurality

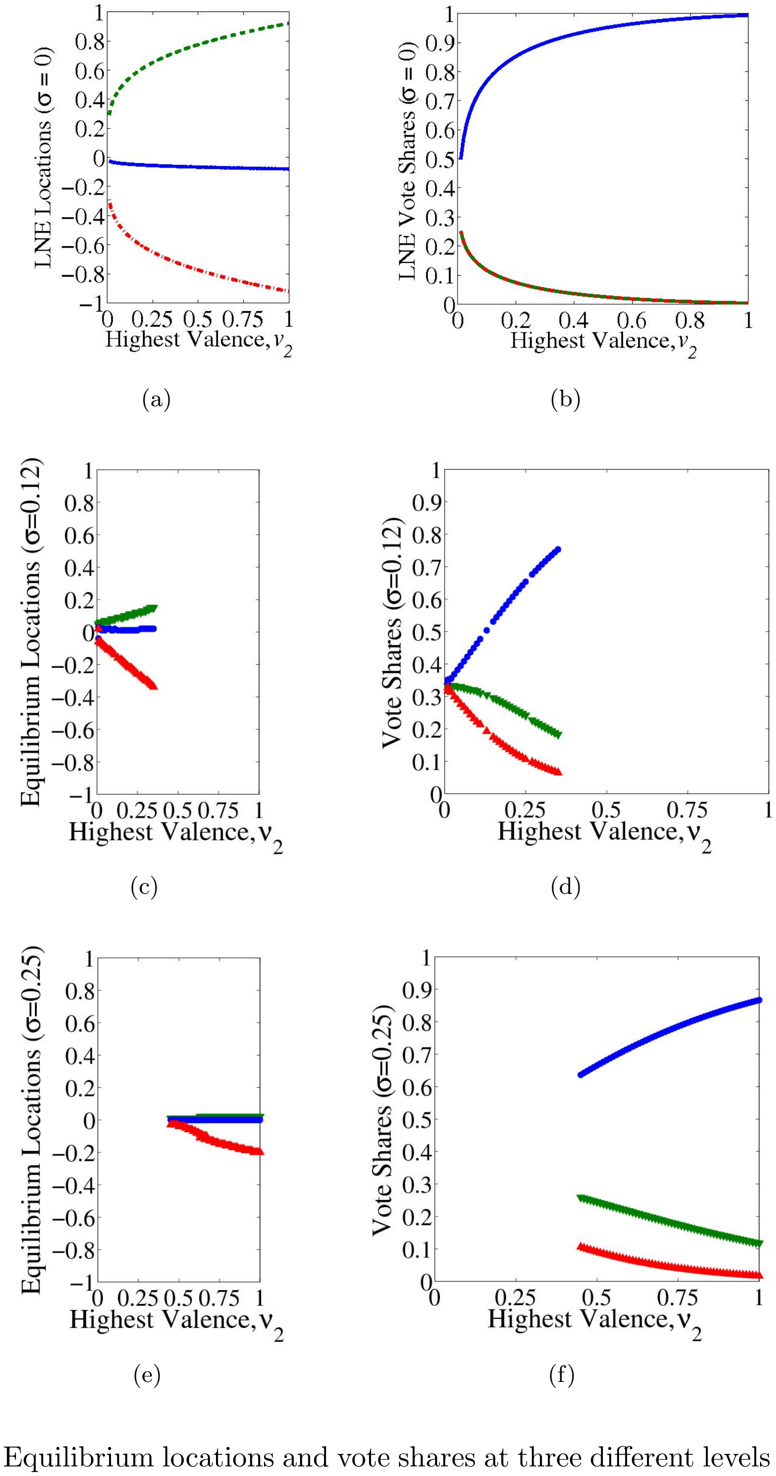


Figure 2: Equilibrium locations and vote shares at three different levels of σ. Panels (a) and (b) depict the LNE of the deterministic model. The rest depict PSNE obtained through simulations. In all panels $r = 1/2$, the blue curve, the green curve, and the red curve represents (respectively) Candidates 2, 1, and 3.

($\mathbf{p} = \mathbf{0}$), too, exists for very small valence differences ($0.01 \leq v_2 \leq 0.03$).[18]

Second, the lower-valence candidates are not located symmetrically around the mean, nor do they receive the same vote share. Instead, the one with a higher valence is located closer to the electoral mean ($|p_1^*| < |p_3^*|$) and receives a larger vote share in the equilibrium, $V_1(\mathbf{p}^*) > V_3(\mathbf{p}^*)$.

Third, for given valence values, the highest-valence candidate receives a smaller vote share than $V_2(\mathbf{p}^{*D})$. More important, for $v_2 \leq 0.12$, Candidate 2 receives less than a majority of votes in the PSNE of the stochastic model. This, however, does not mean that under stochastic voting the center candidate's vote share varies completely. In all the simulations we mention in this paper, (that is, under any $\sigma > 0$) if a dispersed PSNE exists, then, in this equilibrium the candidate with the highest valence is located between the two lower-valence candidates and he always receives the largest vote share. Thus, to capture political competition in countries such as the UK, one has to extend the model further.[19]

Fourth, in equilibrium if i is to the left of Candidate 1, then it is still more likely that she will vote for Candidate 1 (the left-most candidate). Yet, in the stochastic model i will vote for Candidate 2 (and even for Candidate 3) with some non-zero probability. Similarly, now, some of the voters with most preferred policies on the right-hand side of the left-most candidate will vote for him.

Fifth, the comparative statics with respect to a lower-valence candidate's valence are opposite to these of the deterministic model. In the deterministic model, when v_1 increases, Candidate 3 comes *closer* to the center, and, thus, he ends up with a *higher* vote share.[20] Perhaps more surprising, the increase in both candidate's vote shares are the same although only the valance of Candidate 1 increases. In the stochastic model, however, when v_1 increases, Candidate 3 moves *away* from the center and his vote share *decreases*. The comparative statics under stochastic voting is more plausible: why should we expect Candidate 3 to gain votes when his appeal (*relative* to Candidate 1) decreases.

Intuitively, we have the non-plausible comparative statics in the deterministic model, because the voting pattern implied by the deterministic preferences means that in the LNE of the deterministic model the two lower-valence candidates are not *directly* competing with each other. That is, if Candidate 1 (Candidate 3) changes his policy slightly, then there will be no change in the vote share of Candidate 3 (Candidate 1); see Figure 1.

In both the deterministic and the stochastic models, when v_1 increases, Candidate 1 moves towards the center and Candidate 2 moves towards Candidate 1 (and, thus, further away from the center). As there is no direct competition between Candidates 1 and 3, Candidate 3 does not respond to a change in p_1; he only responds to the change in p_2. As Candidate 2 moves further away from Candidate 3 and the center, Candidate 3 comes closer to the center to increase his vote share (those voters who were in B_2^* but close to p_3^{*D} will switch to Candidate 3 when Candidate 2 moves further away from them). Therefore, in the deterministic model an increase in v_1 results in a higher vote share for Candidate 3.

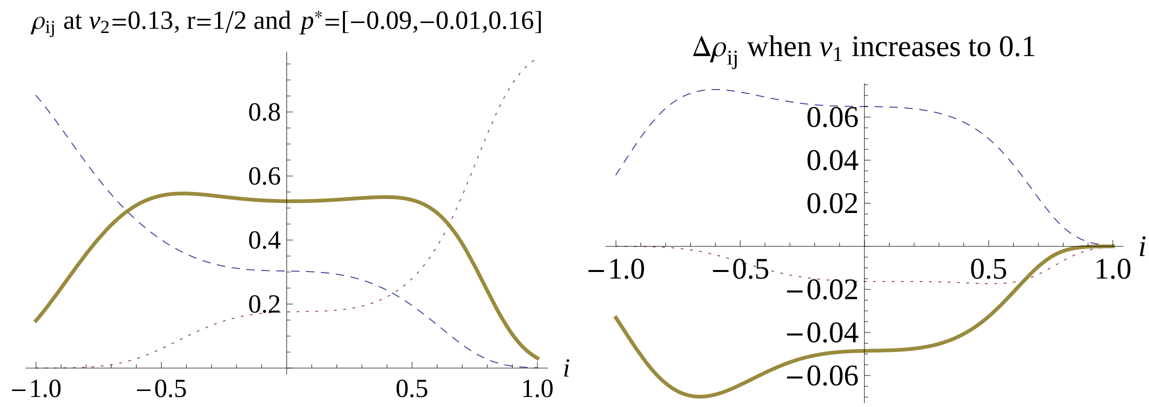
In the stochastic model, all candidates are directly competing with each other: ρ_{ij} is

of Candidate 1.

[18] We present and discuss agglomerated PSNE in Section 4.2.

[19] When modelling multi-party competition in UK, Schofield (2005) considers *activist valence* and Adams (2001, p 136-143) considers *partisan voters*.

[20] See Proposition 1 in Evrenk (2009a;b). There is no comparative statics with respect to v_3 since we set $v_3 = 0$. This normalization is innocuous: Evrenk (2009a) shows that the LNE of the deterministic model depends only on the valence difference and equation (3) shows that ρ_{ij}, and, thus, PSNE of the stochastic model depends only on these differences.



(a) (b)

Figure 3: In both (a) and (b) $\sigma = 0.12$, $v_2 = 0.13$, $v_1 = 0.065$ and $\mathbf{p} = [-0.09, -0.01, 0.16]$. Panel (a) plots the probabilities that voter i votes for each candidate when $v_1 = 0.065$ and panel (b) plots the changes in these probabilities when v_1 increases to 0.1. The solid, the dashed and the dotted curves represent the corresponding variables for candidates (respectively) 2, 1, and 3.

always strictly positive for any i and j. Differentiating (3) with respect to v_k, we can see that an increase in v_k leads to an increase in ρ_{ik} and to a decrease in ρ_{ij} for any $i \in \mathcal{I}$ and $j \neq k$. The exact magnitude of these changes are presented in Figure 3. Panel (a) plots ρ_{ij}'s for a given PSNE. Panel (b) plots the change in ρ_{ij}'s when r increases from 1/2 to 10/13 (when v_1 increases from 1/2 of v_2 to 10/13 of v_2) without changing candidate locations or σ. Note that the changes in ρ_{ij}'s predict the directions in which the candidates will move after the increase in v_1. After becoming more popular with every voter, Candidate 1 comes closer to the center, while now (relatively) less popular Candidates 2 and 3 move further away from the center to appeal to voters who prefer extreme policies. Candidate 2 moves towards Candidate 1, picking up some of the supporters of Candidate 1 among the farthest left part of the spectrum (now, these are less likely to vote for Candidate 1 who becomes more centrist), while Candidate 3 moves further to the right and increases his (already high) support among the extreme right. The new PSNE is at $[-0.06, -0.025, 0.165]$. The candidates who move away from the center end up with a lower vote share as $V_1(\mathbf{p}^*)$ increases.

As the reader may note, the last PSNE locations have precision beyond the 201 point approximation of the interval $[-1, 1]$: neither -0.025 nor 0.165 can be written as $\frac{N}{100}$ where N is an integer between -100 and 100. To calculate the new PSNE we had to divide $[-1, 1]$ into 401 equidistant locations; because, when $\sigma = 0.12$, $v_2 = 0.13$, and $v_1 = 0.1$, a PSNE does not exist under a 201 point approximation. As we discuss in more detail in Appendix C, the holes among the set of equilibria (no PSNE) as well as multiple equilibria are not due to stochastic voting per se, but rather they are due to the discrete approximation necessary for the simulations.[21] The next issue we discuss, however, is completely due to stochastic voting.

In addition to the several plausible features of the PSNE of the stochastic model dis-

[21] Typically the objective function is quite flat and quite symmetric around its maximum at PSNE location, evaluating it at only a few points (discretizing the policy space) leads to such problems

cussed, it should also be noted that stochastic voting gives rise to a more plausible PSNE only under an *appropriate* level of uncertainty (σ). As we run simulations under different values of σ, we find that when σ increases, (i) the set of valence differences under which a PSNE exists becomes larger; (ii) in the dispersed PSNE, equilibrium locations become *less* dispersed, leading to an increase in the set of parameters under which $V_2(\mathbf{p}^*) < 1/2$; and (iii) the set of parameters under which an agglomerated PSNE exists increases (see Section 4.2 and Figure 9.b). For instance, when $\sigma = 0.08$ no PSNE exists. On the other hand, when $\sigma = 0.25$, a PSNE always exists. Under small valence differences, there is only an agglomerated equilibrium and under moderate valence differences both type of equilibria coexist: if $0.01 \leq v_2 \leq 0.44$, then only an agglomerated equilibrium exists; if $0.44 \leq v_2 \leq 0.61$, then both an agglomerated equilibrium and a dispersed one exist; and if $0.62 \leq v_2 \leq 1$, then only a dispersed equilibrium exists. Although PSNE is more likely to exists under larger σ, a plausible PSNE is less likely: as panels (e) and (f) make clear under $\sigma = 0.25$, there is no dispersed PSNE in which $V_2(\mathbf{p}^*) < 1/2$.

4 Equilibria under alternative assumptions and the alternative (agglomerated) equilibria

In the previous section, we discussed several features of the equilibrium under stochastic voting using a stylized model (symmetric density and vote share maximizing candidates). Actual voter densities are not symmetric. Nor is there agreement on the most appropriate objective function in models of multi-candidate competition; still many argue that it is not vote-maximization, see Shepsle (1991 p.24) for a review. In this section we discuss how the results change when one simulates a model in which voter density is asymmetric or each candidate maximize his plurality. In short, under such extensions we find that all the main points noted in Section 3 still hold. Still these extensions help us to illustrate some non-plausible features of equilibria under stochastic voting (as well as those under plurality-maximization). One such non-plausible feature is the agglomerated equilibria. Presented at the end of this section, agglomerated equilibria exist under all the variations.

4.1 Alternative voter densities and candidate objectives

Figure 4 presents the PSNE of the model under different values of σ when the mode of the voter density is on the right-hand-side of the center ($i = 0$) with $m = 1/4$. The first row presents LNE locations under deterministic voting; (Evrenk, 2010 Proposition 3). The second and the third rows are obtained under stochastic voting. Evrenk (2010) shows that some of the non-plausible features of the deterministic model can be eliminated by considering an asymmetric voter density. Under an asymmetric density a PSNE exists; in panels (a) and (b), when v_2 is in the shaded region the LNE is a PSNE. When it exists, the PSNE is unique (its mirror image is not a PSNE anymore). Due to the voting pattern under the deterministic model (presented in Figure 1 above), when the majority of voters are on the right-hand-side of the center, in the PSNE of the deterministic model the candidate with the second-highest valence is located to the left of Candidate 2 (and the center). Intuitively, Candidate 1 positions himself to the left of Candidate 2 as under the voter density we consider the left tail is longer (and, thus, there are more voters there).

As the reader can see from the online supplementary files, the set of equilibria under stochastic voting is richer than what is depicted in Figure 4 (and, thus, the set of equilibria

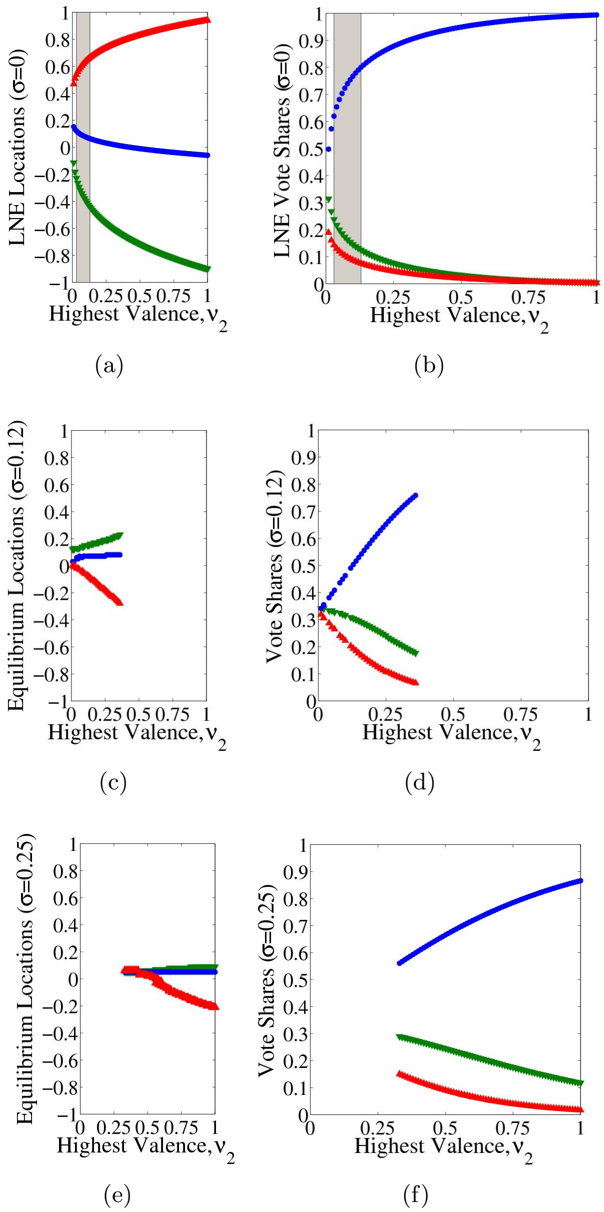


Figure 4: PSNE under vote maximization. The first row presents equilibrium under deterministic voting while the last two rows present equilibrium under stochastic voting. In all panels $m = 1/4$, $r = 1/2$, the blue curve, the green curve, and the red curve represents (respectively) Candidates 2, 1, and 3.

in the deterministic model); this is especially true when σ is large. Further, under stochastic voting, too, PSNE that are the mirror image of another PSNE are eliminated. The stochastic and the deterministic voting models differ in the location of lower-valence candidates when the density is asymmetric. As discussed above, when $m > 0$, in the deterministic model Candidate 1 is always located to the left of Candidate 2 while in the stochastic model there are more *right*-PSNE (for this reason, we plot only the *right*-PSNE in Figure 4). Again, this difference is due to the different voting patterns. In the deterministic model Candidate 1 positions himself on the longer tail because he will be supported by voters whose most preferred policies are more "extreme" than his policy. Under the stochastic model the lower-valence candidates receive votes from those voters whose most preferred policies are close to the candidate's policy. Then, for $m > 0$, the right-hand side of Candidate 2 provides proximity to many voters as the majority of the electorate prefers policies to the righ-hand-side of the center. Although the locations of the lower-valence candidates differ between the stochastic and deterministic models, these models are similar in that all equilibrium locations move to the right (left) as the mode of the density moves to the right (left).

The asymmetric voter density underlying Figure 4 also helps us to emphasize our earlier point that the stochastic voting model performs better only under an appropriate voter density. Note that under stochastic voting the size of the set of values of v_2 under which a PSNE exists is larger; but when $\sigma = 0.25$, a PSNE does not exists for small values of v_2 under which a PSNE exists in the deterministic model.

The second extension we consider is competition among plurality maximizing candidates. To keep the discussion short we do not present the LNE and PSNE under deterministic voting. Evrenk (2010) finds that under deterministic voting the LNE of the model is the same as the LNE under vote-maximization if the voter density is symmetric. If the density is asymmetric, the LNE's will differ. Under stochastic voting, we find that the PSNE of these two models differ whether the density is symmetric or not. To note this point Figure 5 presents the PSNE under a symmetric voter density and $\sigma = 0.12$ when each candidate's objective is plurality maximization; compare panels (a) and (b) of Figure 5 with panels (c) and (d) of Figure 2. One point we want to note about the model under plurality-maximization is the paradoxical behavior that supports some equilibria.[22] Figure 6 illustrates this anomaly through a numerical example. When each candidate maximizes his plurality, for $m = 0$, $\sigma = 0.12$, $v_2 = 0.13$ and $r = 1/2$, there exists a (modulo symmetry) unique PSNE at $[0.12, 0.08, -0.14]$. At this PSNE, Candidate 2 receives slightly less than 50 percent of the votes. As Figure 6 shows, however, Candidate 2 could win more than the majority of the votes by simply moving to -0.01. But, he does not do so, as such a move would reduce his plurality.

4.2 Agglomerated Equilibrium

It is well known that under stochastic voting an agglomerated PSNE exists, see Lin, Enelow and Dorussen (1999) and Adams (1999a;b). In our simulations, too, we find agglomerated equilibria under certain valence and variance parameters. Figure 7 presents the vote shares in the agglomerated equilibria for different levels of m and σ. Here, we plot only the agglomerated equilibrium under vote share maximization. The agglomerated equilibrium under plurality maximization is provided in the supplementary files. Under either objective function, the agglomerated PSNE shares certain characteristics first noted by Schofield

[22]Evrenk (2010) notes that when candidates are plurality-maximizers the same type of equilibria exist under the deterministic model as well.

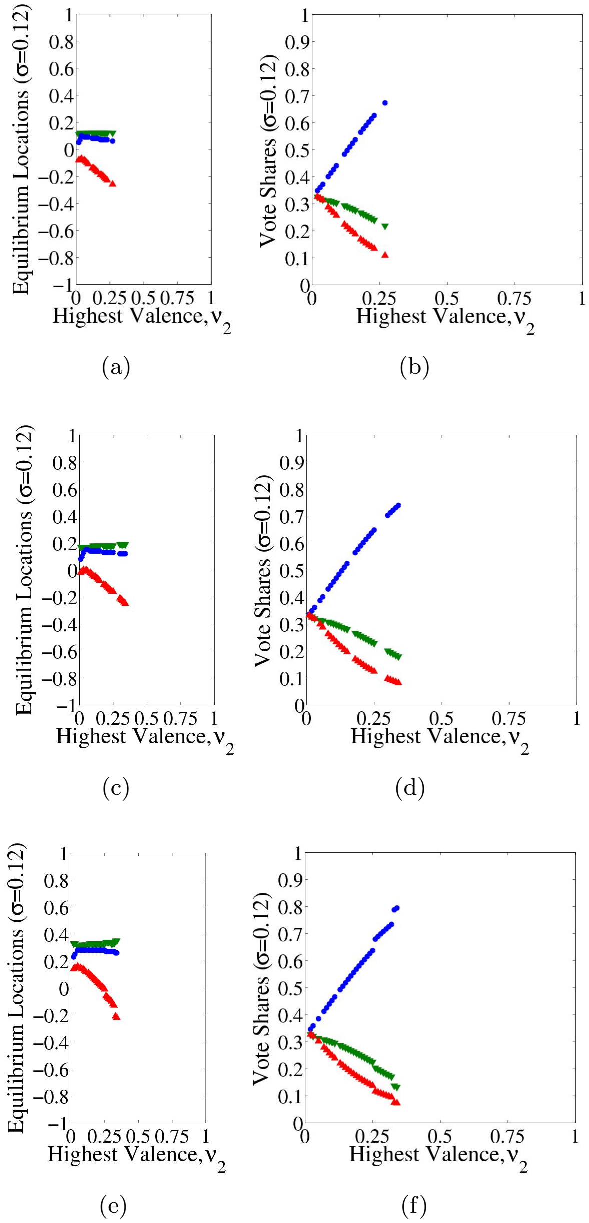


Figure 5: PSNE under plurality maximization. In all panels $\sigma = 0.12$ and $r = 1/2$. In the first, middle, and last row m is equal to (respectively) 0, 0.25, and 0.75. The blue curve, the green curve, and the red curve represents (respectively) Candidates 2, 1, and 3.

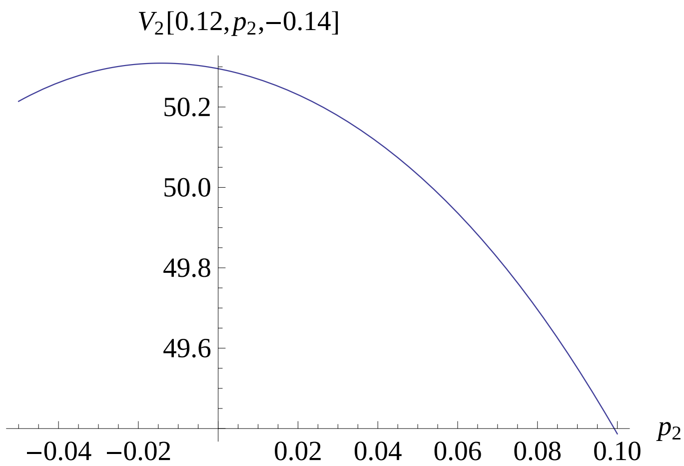


Figure 6: Under plurality-maximization, in PSNE Candidate 2 position himself at 0.08. Yet, by deviating to -0.02, he can receive more than a majority of all votes.

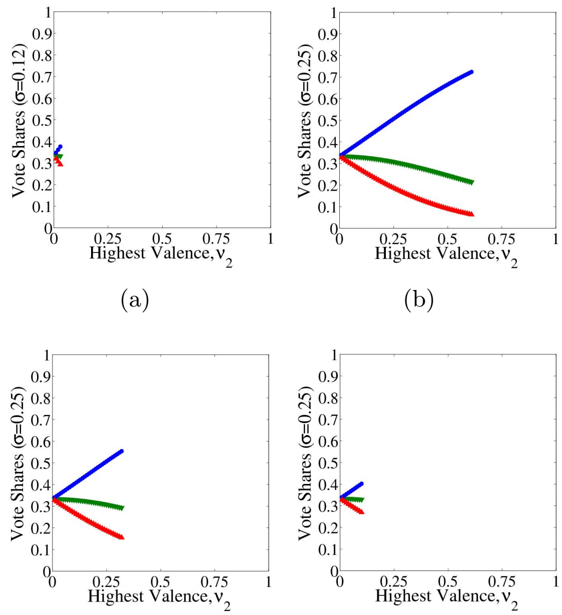
(2005) in his analysis of conditions under which an LSNE exists at the joint mean: the larger is σ and the larger is the variance of voter density, the larger is the set of parameters under which an agglomerated PSNE exists.

5 Discussion and Conclusion

In this paper we note how stochastic voting affects the voting patterns, comparative statics, equilibrium location and equilibrium vote shares using a stylized model in which equilibrium exists when voting is deterministic. When one assumes that voting is deterministic, the equilibria of the three-candidate spatial competition have several non-plausible features. We find that these features can be eliminated by assuming that voting is stochastic and by choosing an appropriate degree of uncertainty about voting behavior. Still, one cannot control all the aspects of the equilibria by choosing an appropriate degree of uncertainty. We find, for instance, in the equilibrium the center candidate always receives more votes than the other two candidates in equilibrium. Thus, to model political competition in countries where this is not the case, one has to extend the model further. We also find that when candidates are plurality-maximizing, the equilibria under stochastic voting is supported by a paradoxical behavior (Figure 6).

The setup we study is stylized and it is not meant to capture political competition in a specific country. Yet, it helps us examine both the potential and the limits of a model that assumes only candidate heterogeneity (valence differences) and stochastic voting. This paper is a part of a project in which several extensions of this model will be studied. The simplicity of the model becomes an advantage; within this framework the model can be easily extended (as extensions are easier to simulate and, more important, are analytically more tractable compared to more general models of multi-candidate competition).

One such extension is to study the role of electoral uncertainty when competition is among three candidates with policy preferences (and, valence differences). It is possible that electoral uncertainty may create "centrifugal" incentives for candidates who prefer a policy at the margin. The other extension involves strategic voting. For both extensions, one needs to model which policy will be implemented when neither of the three candidates



(a) (b)

(c) (d)

Figure 7: Agglomerated PSNE vote shares under vote maximization. In both (a) and (b), $m = 0$ and all candidates are located at 0; in (c) $m = 0.25$ and all candidates are located at 0.05; and, in (d) $m = 0.75$ and all candidates are located at 0.17. In all panels $r = 1/2$, the blue curve, the green curve, and the red curve represents (respectively) Candidates 2, 1, and 3.

receives a majority of votes. There is, however, no agreement on how to model this issue (coalition formation) in the literature. Further, this raises another question that has not received much attention in models of political competition with valence differences: how the valence value of a coalition of different politicians with different valences is determined. These issues are left for future research.

6 Appendix A: Proofs

Proof of Lemma 1. Simply note that for a voter located in B_1^* or B_3^*, Candidate 2 is never the worst candidate: there is always another candidate whose valence is lower and whose policy is further from this voter's most preferred policy, thus $W_2^* = \varnothing$. Then for each voter either one of the remaining candidates is the worst candidate. As the locations of these candidates differ, $I(p_1, p_3)$ exists and it is unique (Groseclose, 2001, Appendix III), and it is closer to Candidate 1 when $v_2 > v_1$ (Evrenk, 2009a, Lemma 1). ■

Proof of Proposition 1. Dividing both sides by $u_i^j = -L(i - p_j) + v_j$, we can rewrite ρ_{ij} as

$$[1 + \sum_{k \in \{1,2,3\} \backslash j} \exp((u_i^k - u_i^j)/\sigma)]^{-1}$$

where $u_i^k = -L(i - p_k) + v_k$. Taking the derivative with respect to $\sigma > 0$, we find that

$$\frac{\partial \rho_{ij}}{\partial \sigma} = \frac{\sum_{k \in \{1,2,3\} \backslash j} (\exp((u_i^k - u_i^j)/\sigma))(u_i^k - u_i^j)}{\sigma^2 [1 + \sum_{k \in \{1,2,3\} \backslash j} \exp((u_i^k - u_i^j)/\sigma)]^2}$$

Since the denominator (and, $\exp(x)$) is always positive, we have $\frac{\partial \rho_{ij}}{\partial \sigma} \lesseqgtr 0$ if and only if $(\exp((u_i^k - u_i^m)/\sigma))(u_i^k - u_i^j) + (u_i^m - u_i^j) \lesseqgtr 0$, where k, m, and j are distinct candidates. If $i \in B_j^*$, then $(u_i^k - u_i^j) < 0$ and $(u_i^m - u_i^j) < 0$. Hence, $\frac{\partial \rho_{ij}}{\partial \sigma} < 0$. Similarly, if $i \in W_j^*$, then $(u_i^k - u_i^j) > 0$ and $(u_i^m - u_i^j) > 0$. Hence, $\frac{\partial \rho_{ij}}{\partial \sigma} > 0$. Finally, when $\sigma = 0$, consider $i \in (B_j^*)^C \backslash W_j^*$. If k is the higher-valence candidate, then for any $i \in (B_j^*)^C \backslash W_j^*$ we have $u_i^k - u_i^m > 0$, and thus $\exp((u_i^k - u_i^m)/\sigma))(u_i^k - u_i^j)$ converges to infinity, so no matter the magnitude of the negative term $(u_i^m - u_i^j)$, we have $\frac{\partial \rho_{ij}}{\partial \sigma}|_{(\sigma=0)} > 0$. If k is the other lower-valence candidate, then for any $i \in (B_j^*)^C \backslash W_j^*$, we have $u_i^k - u_i^m < 0$, and thus $\exp((u_i^k - u_i^m)/\sigma))(u_i^k - u_i^j)$ converges to zero with σ. Therefore, $\frac{\partial \rho_{ij}}{\partial \sigma}|_{(\sigma=0)}$ has the same sign as $(u_i^m - u_i^j)$: we have $\frac{\partial \rho_{ij}}{\partial \sigma}|_{(\sigma=0)} > 0$. ■

7 Appendix B: Centripetal incentives: numerical calculations

To provide a basis for our conjecture, Figure 8 demonstrates some calculations. Consider a symmetric voter density over the interval $[-1, 1]$. Then, we have $p_1^{*D} = -\frac{\delta_1 + \delta_3}{2} = -p_3^{*D}$, and $p_2^{*D} = \frac{\delta_1 - \delta_3}{2}$. When, $v_2 = 0.12$, $r = 1/2$, and $L(x) = x^4$, the LNE is given by $\mathbf{p}^{*D} = [-0.542, -0.047, 0.542]$ (In Figure 1, candidates are at these LNE locations). Panels (a), (b), and (c) of Figure 2 shows ρ_{ij}'s under different levels of σ while keeping candidates at $\mathbf{p}^{*D} = [-0.542, -0.047, 0.542]$. As long as the candidates are at $\mathbf{p}^{*D}$, under *any* σ, we have $\rho_{ij} > \rho_{ik}$ for all $i \in B_j^*$ and any $j \neq k$. That is, under stochastic voting, too, a candidate is more popular in his (deterministic) voter base. But, as σ increases, the difference between ρ_{ik} and ρ_{ij} decreases, resulting in an increase in the number of voters who vote for k although they are in B_j^*. Consider, for instance, the voting behavior of the voter located at the center ($i = 0$). When σ increases from 0.01 to 0.12 and then to 0.25, the probability that she votes for the lowest-valence candidate (Candidate 3) who is located at 0.542, increases from 1×10^{-9} to 0.12 to 0.22. This follows from Proposition 1, as $0 \in W_3^*$. Since $0 \notin W_1^*$, Proposition 1 does not specify how ρ_{ij} changes for $i \in \mathcal{I} \backslash (W_j^* \cup B_j^*)$; we find that the probability that the center voter votes for Candidate 1 (located at -0.542) increases from 4×10^{-7} to 0.20 to 0.28. Panels (b) and (c) show that both ρ_{i1} and ρ_{i3} increase around the electoral center. This increase is the basis for our conjecture on centripetal incentives for the lower-valence candidates.

Panels (d), (e) and (f) in Figure 2 calculate the net effect of a move towards the center on the vote share of a lower-valence candidate. In panels (d) and (e), we calculate the change in his vote share, $\Delta \rho_{i3} f(i)$, as the lowest-valence candidate moves 0.01 units towards the center (from $p_3^{*D} = 0.542$ to $p_3' = 0.541$).[23] In panel (e), we plot $\Delta \rho_{i3} f(i)$ for $\sigma \in [0, 1]$, where panel (d) presents a slice from the three dimensional graph in panel (e). As one can see from these two panels, by moving towards the center, Candidate 3 loses some of his supporters from the right side of p_3^{*D}, but he gains far more supporters from the left of p_3^{*D}.

Panel (f) shows how the expected number of votes he receive changes when Candidate 1 moves 0.01 units towards the center. Note that the incentives of the other lower-valence candidate, Candidate 1, is similar and even stronger: the net change in his vote share is larger. As a result, when $\sigma = 0.12$, he, too, moves towards the center, resulting in a dispersed PSNE at $[-0.08, -0.02, 0.15]$. Intuitively, Candidate 1 has stronger centripetal incentives; with his relatively higher valence, it takes a smaller preference shock to convince the voters around the electoral center to vote for Candidate 1.

8 Appendix C: Simulation technique and the resulting problems

The program code(s) used to calculate the PSNE are provided as online supplementary files. Both are MATLAB files posted with a txt extension (they would work in several freely available MATLAB clones as well): one calculates the PSNE under vote share maximization and the other one calculates PSNE under plurality maximization. Below we describe how these codes work.

Under both codes, the program works as follows. For a $2N + 1$ point approximation, first one sets N (so, to repeat our calculations with 201-point approximation one needs to set $N = 100$ in these codes). Then, by setting r, σ, and v_2 the parameters of the game are determined. With, say, a 201 point approximation, there is a total of 201^3 (slightly more than eight million) possible strategy profiles. The program starts from the policy profile $[-1, -1, -1]$ and checks all the way until $[1, 1, 1]$ each of these strategy profiles.

To check if a given single policy profile (say, $[-1, -1, -1]$) is PSNE the code first checks if Candidate 1 can increase his payoff by deviating from $p_1 = -1$ given that $p_3 = p_2 = -1$. If he can, it moves to the next strategy profile (the next profile would be $[-0.99, -1, -1]$ when we have a 201-point approximation). If he cannot, then the program calculates if Candidate 2 can increase his payoff by deviating from $p_2 = -1$ given that $p_1 = p_3 = -1$. Similarly, if

[23] In both panels, Candidates 1 and 2 are still located at p_1^{*D} and p_2^{*D}, $v_2 = 0.12$ and $r = 1/2$.

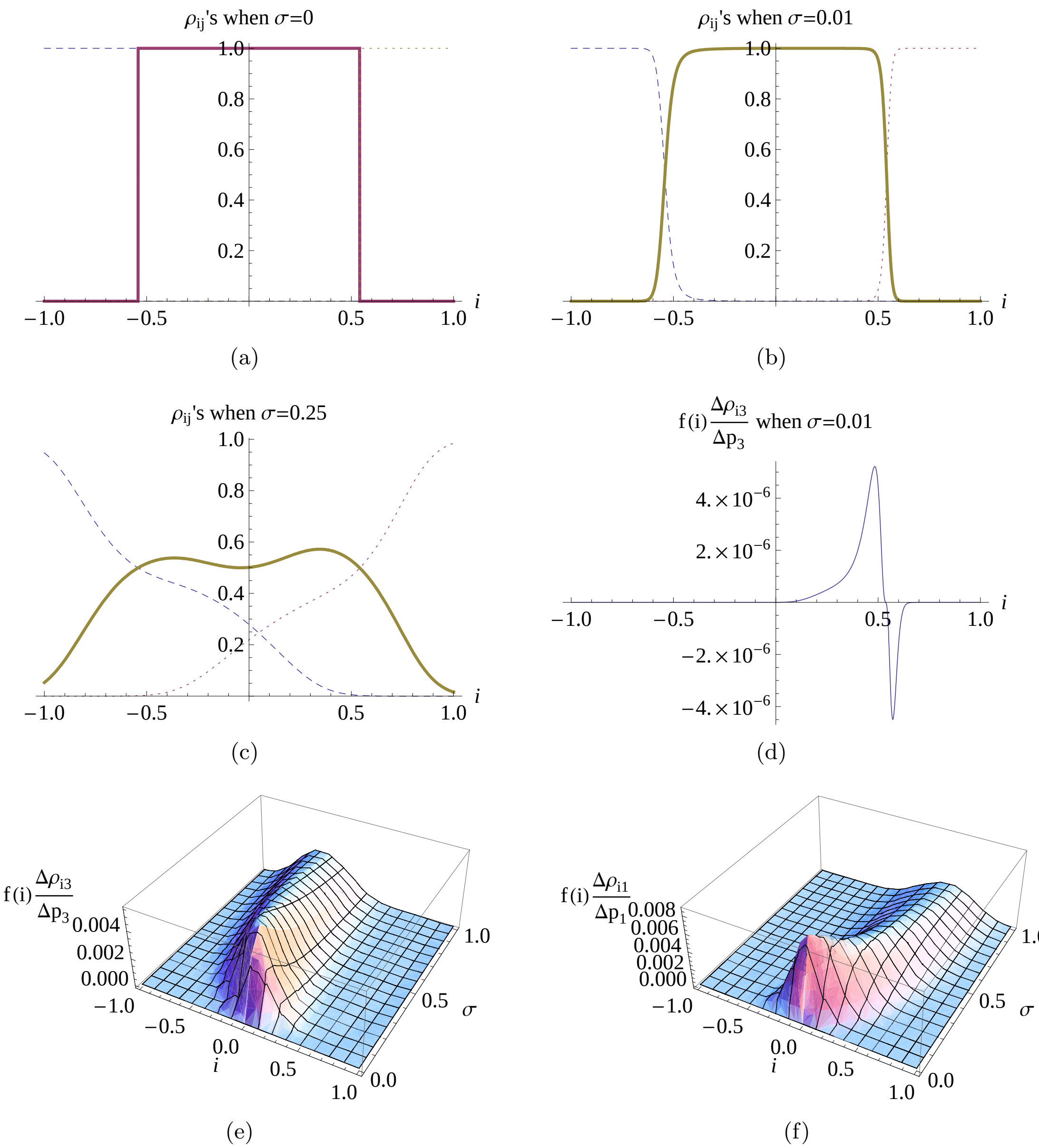


Figure 8: In (a), (b), and (c), we have $v_2 = 0.12$, $v_1 = 0.06$, $\mathbf{p} = \mathbf{p}^{*D} = [-0.542, -0.047, 0.542]$, and the solid, dashed and dotted curves represent the corresponding variable for candidates (respectively) 2, 1, and 3. In (d) and (e) (in panel (f)), Candidate 3 (Candidate 1) moves to 0.532 (-0.532).

he can, then it moves to the next strategy profile $[-0.99, -1, -1]$, and if Candidate 2 cannot, then it checks if Candidate 3 can increase his payoff by deviating from $p_3 = -1$ given that $p_1 = p_2 = -1$. Only if none of these candidates can increase his payoff by deviating from $[-1, -1, -1]$, it identifies this profile as a PSNE.[24]

After checking these 201^3 strategy profiles for a given valence vector $[rv_2, v_2, 0]$, the program moves to next value of v_2 and checks the resulting 201^3 strategy profiles. Each graph presenting the PSNE of the stochastic model is the result of this process applied to 100 different values of v_2 (from 0.01 to 1). As of year 2010, when repeating the calculations, one should be aware of the required time (especially if access to a supercomputer is not possible): in a personal computer with a 2.4 Ghz processor and 3 GB RAM, calculating the PSNE under vote-maximization takes about three days, while calculating the PSNE under plurality-maximization takes almost 12 days.

When we began simulating the PSNE of the stochastic model, we followed the most common method: start the game at an initial strategy profile and see if it converges to anything. As Merrill and Adams (2001) notes, this method works fine when the best-response correspondences constitute a contraction mapping. But, trying different starting points, we found that for some values there is a neighboring PSNE. As a result, we switched to the current method which identifies all the PSNE. These neighboring equilibria are due to the discrete approximation we used. Another result due to the discrete approximation are the holes among the set of equilibria.

While discussing the equilibrium locations underlying Figure 3 we note that under a 201-point approximation, a PSNE does not exist for $\sigma = 0.12$, $v_2 = 0.12$, and $v_1 = 0.06$. One can see that in the supporting documents listing all the PSNE that the computer detected, there are several holes, i.e., values of v_2 with no corresponding PSNE. To study why we have these holes at certain N (and, why they disappear later) another such hole at $\sigma = 0.12$, $v_2 = 0.14$, and $v_1 = 0.07$ is a particularly good example. (In this discussion we always have $\sigma = 0.12$ and $r = 1/2$; so, to keep it short we refer to a case by the value of v_2.)

The lack of PSNE at $v_2 = 0.14$ is particularly illuminating because a PSNE exists both when $v_2 = 0.13$ and when $v_2 = 0.15$. These two PSNE differ from each other only slightly: in both PSNE $p_1^* = 0.09$, $p_2^* = 0.01$ while $p_3^* = -0.16$ when $v_2 = 0.13$, and $p_3^* = -0.17$ when $v_2 = 0.15$. Note that the distance between the two lower-valence candidates is equal to 0.25 when $v_2 = 0.13$ and that this distance is equal to 0.26 when $v_2 = 0.15$. As this distance increases in the valence difference, we would expect an equilibrium in which $|p_1^* - p_3^*|$ is between 0.25 and 0.26 when v_2 is between 0.13 and 0.15. Yet, when one divides $[-1, 1]$ into 201-equidistant points, it is impossible to have $0.25 < |p_1^* - p_3^*| < 0.26$. What exactly happens when the program searches for an equilibrium is illustrated in Figure 9. When $v_2 = 0.14$, for $p_1 = 0.09$, $p_2 = -0.01$, the best response of Candidate 3 is around -0.165. Yet, in a 201 point approximation this location is not available; among the available points $V_3[0.09, 0.01, p_3]$ is highest at $p_3 = -0.17$. But, when Candidate 3 moves to -0.17, Candidate 2 has incentives to deviate to 0.02 (see Figure 3(b)). But, when $p_2 = 0.02$, Candidate 3 prefers to move closer to the center, i.e., among the available points $V_3[0.09, 0.02, p_3]$ is highest at $p_3 = -0.16$. This is not an equilibrium either. Given that Candidate 3 came closer to the center, Candidate 2 moves back to 0.01 closing the full cycle.[25] When we consider a finer approximation[26] ($N = 400$), it is possible to have

[24]So, when Candidate 3 can increase his payoff by deviating, the program, again, goes to the next strategy profile, $[-0.99, -1, -1]$.

[25]Throughout this whole cycle, Candidate 1 has no incentives to deviate from 0.09.

[26]We don't we carry out all simulations under a very fine approximation because this is quite costly in

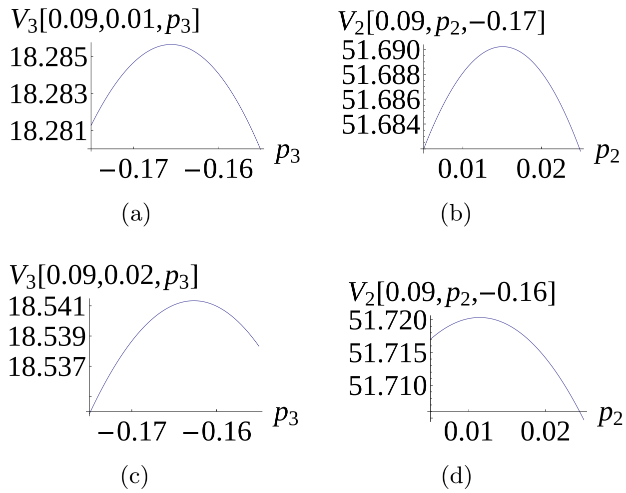


Figure 9: The cycling best-responses. For $\sigma = 0.12$, $v_2 = 0.14$, $v_1 = 0.07$ and $p_1 = 0.09$, panels (a) and (c) show the best-responses of Candidate 3, and panels (b) and (d) show the best-responses of Candidate 2.

$0.25 < |p_1^* - p_3^*| < 0.26$. The PSNE exists at $[0.09, 0.015, -0.165]$ with $|p_1^* - p_3^*| = 0.255$. The holes among the set of PSNE are not the only effect of discrete density used. As we discuss next, for large values of σ we may have two neighboring equilibria.

At higher levels of σ, there is an increase in the number of dispersed PSNE as well. At $\sigma = 0.25$, for instance, there is more than one (or, if you count the mirror images, more than two) dispersed PSNE when v_2 is high. More specifically, for $0.62 \leq v_2 \leq 0.67$, there are two PSNE (and, their mirror images). In all of these PSNE the distance between Candidates 1 and 2 is less than or equal to (2/100). As a result, between two PSNE's the vote shares of the same candidate are almost the same (most of the time the difference is equal to one hundredth of a percent). This multiplicity of equilibria, too, we believe, is due to the discrete approximation. We do not have any analytical argument to support this result, but when we choose a finer approximation by dividing $[-1, 1]$ into 801 equidistant points, we find that now for three values of v_2 $(0.64, 0.65, 0.67)$, there is a (modulo symmetry) unique PSNE (while for the other three values of v_2 there are still two symmetric PSNE).

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terms of the computer time needed. Note that under this finer approximation, the code must check a little more that 64 million strategy profiles to find the PSNE of a single case.

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