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ABSTRACT

Human beings want to believe that good outcomes in the future are more likely, but also want to make good decisions that increase average outcomes in the future. We consider a general equilibrium model with complete markets and show that when investors hold beliefs that optimally balance these two incentives, portfolio holdings and asset prices match six observed patterns: (i) because the cost of biased beliefs are typically second-order, investors typically hold biased assessments of probabilities and so are not perfectly diversified according to objective metrics; (ii) because the costs of biased beliefs temper these biases, the utility costs of the lack of diversification are limited; (iii) because there is a complementarity between believing a state more likely and purchasing more of the asset that pays off in that state, investors over-invest in only one Arrow-Debreu security and smooth their consumption well across the remaining states; (iv) because different households can settle on different states to be optimistic about, optimal portfolios of ex ante identical investors can be heterogeneous; (v) because low-price and low-probability states are the cheapest states to buy consumption in, overoptimism about these states distorts consumption the least in the rest of the states, so that investors tend to overinvest in the most skewed securities; (vi) finally, because investors with optimal expectations have higher demand for more skewed assets, ceteris paribus, more skewed asset can have lower average returns.

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This paper studies portfolio holdings and asset prices in an economy in which people's natural tendency to be optimistic about the payout from their investments is tempered by the ex post costs of basing their portfolio decisions on incorrect beliefs. We show that this model can generate the following three stylized facts.

First, households' portfolios are not optimally diversified according to various theoretical-based measures (Marshall E. Blume et al. (1974), William N. Goetzmann and Alok Kumar (2001) , Laurent E. Calvet et al. (2006), Stephanie Curcuru et al. (forthcoming)). The costs of this lack of diversification appear to be modest. Most households hold a well-diversified portfolio of mutual funds and also a significant amount of one or two additional stocks.[1]

Second, and part of the evidence for the first fact, household portfolios are tilted towards stocks with identifiable attributes, and in particular towards holdings of individual stocks with positive skewness. Further, undiversified households hold individual stocks that have relatively high idiosyncratically skewed returns and their portfolios have relatively high idiosyncratically skewed returns (Todd Mitton and Keith Vorkink (forthcoming)).

Finally, positively skewed assets tend to have lower returns. This is true for stocks in the US stock market in general (Yijie Zhang (2005)) as well as for specific well-studied examples, such as the value-growth premium and the long-run underperformance of IPOs.[2]

This paper argues that these three patterns are observed because human beings both want to believe what makes them happier and want to make good decisions that lead to good outcomes in the future. We consider an exchange economy with two periods and complete markets in which households with log utility invest in the first period and consume in the second period. We show that these patterns arise in this economy when investors hold beliefs that optimally trade-off the ex ante benefits of anticipatory utility against the

[1] Further, and complementary evidence for our purposes, household income risk is not fully-insured by households across groups of households, where moral-hazard would seem an implausible reason for this failure (Orazio Attanasio and Steven J. Davis (1996)).

[2] There is also complementary evidence from gambling behavior. Lotteries are highly skewed assets, and the demand for lottery tickets rises with probability controlling for expected return. And, in parimutuel betting on races, in which the bettors determine returns in equilibrium, long-shots have lower expected returns than favorites.

ex post costs of basing investment decisions on biased beliefs.

Our model of beliefs follows the optimal expectations framework of Markus K. Brunnermeier and Jonathan A. Parker (2005). We assume that people behave optimally given their beliefs, choosing portfolios that maximize their expected present discounted value of utility flows.[3] Because investors care about expected future utility flows, they are happier if they overestimate the probabilities of states of the world in which their investments pay off well. But such optimism would lead to suboptimal decision making, and lower levels of utility on average ex post. Optimal beliefs trade-off these competing forces: people's beliefs maximize the objective expectation of their well-being, the average of their expected present discounted value of utility flows. This economic model of beliefs balances the anticipatory benefits of optimism against the costs of basing actions on distorted beliefs. Because the costs of small deviations from optimal behavior are of second order, and the anticipatory benefits of biases in probabilities typically are of first-order, optimal subjective and objective probabilities differ. Christian Gollier (2005) and Brunnermeier and Parker (2005) study portfolio choice and asset prices in incomplete markets (and a two-state complete market example). This paper derives a general characterization in a complete markets economy.

In terms of portfolios, we show in Section I that an investor with optimal expectations does not fully diversify its portfolio but instead biases upwards (a lot) its subjective beliefs about the likelihood of one state and biases downward (a little) its subjective beliefs about the likelihood of all the remaining states. It does this because there is a natural complementarity between believing a state more likely and purchasing more of the asset that pays off in that state. Once a state is perceived as more likely, one wants more consumption in that state, and once one has more consumption in that state, one wants even more to believe that that state is more likely. We further show that an investor chooses to be optimistic about the states associated with the most skewed Arrow-Debreu securities: either the least expensive state

[3] Annette Vissing-Jorgensen (2004) shows that differences in investor's self-reported beliefs about future market returns (or internet stock returns) are highly significantly correlated with the share of their portfolio in equities (or in internet stocks).

(when states are equally likely) or the least likely state (when state prices are actuarially fair) or the least expensive and least likely state when these coincide (in general). This happens because low-price and low-probability states are the cheapest states to buy consumption in, and so distort consumption in the rest of the states the least (for a given bias). Thus portfolios are not perfectly diversified, households overinvest in the most skewed assets, and household portfolios have positively skewed returns.

In general equilibrium, we show in Section II that investors tend to be optimistic about different states. Thus investors' portfolios have idiosyncratically skewed returns and consumption insurance appears to be incomplete. In terms of asset prices, this preference for skewed returns has price effects. Ceteris paribus, states with relatively small probabilities tend to have relatively low expected returns.[4]

All proofs are contained in the appendix.

I. Portfolio choice with optimal expectations

The economy has two periods. There are S possible states of the world in period 2, with state s having objective probability $\pi_s > 0$. An investor with subjective beliefs $\hat{\pi}_s$ allocates his wealth among a complete set of Arrow-Debreu securities in the first period and consumes the payoff from this portfolio in the second period. A person's investment choices, $\mathbf{c} = \{c_1, c_2, .., c_S\}$, maximize his expected utility given his subjective beliefs, $\hat{\boldsymbol{\pi}} = \{\hat{\pi}_1, \hat{\pi}_2, .., \hat{\pi}_S\}$:

$$V_1 = \max_{\mathbf{c}} \sum_{s=1}^{S} \hat{\pi}_s \ln(c_s) \text{ subject to } \sum_{s=1}^{S} p_s c_s = 1 \text{ and } c_s \geq 0, \tag{1}$$

[4]Nicholas C. Barberis and Ming Huang (2005) show that exogenous belief distortion as proposed by Prospect Theory can lead to similar investment and price patterns. Gollier (2005) shows that in an incomplete markets investment problem with a stock and a bond, optimal expectations imply that the investor biases up the probability of the states in which the risky asset's returns are the highest and the lowest, as implied by Prospect Theory. Thus, in this situation, overweighting extreme events is an endogenous outcome of the trade-off between the benefits of anticipatory feelings and the suboptimality actual outcomes, rather than being an exogenous characteristic of human beings.

where $p_s > 0$ is the price of the Arrow-Debreu security yielding one unit in state s, and initial wealth is normalized to unity.[5] Optimal portfolio choices exist and are unique:

$$c_s^* (\hat{\boldsymbol{\pi}}) = \frac{\hat{\pi}_s}{p_s}. \tag{2}$$

Optimal beliefs. But what are the investor's subjective beliefs? One assumption is that people hold rational expectations, an extreme assumption typically made both for its tractability and for the discipline it provides. Further, the argument goes, since objective beliefs lead to the best decisions and thus the highest average present discounted value of utility, people have the incentive to acquire information and learn rationally so that their beliefs should have a general tendency to converge to objective probabilities.

But in fact, rational beliefs do not lead to the highest expected present discounted value of utility flows. An investor can increase V_1 by holding quite distorted beliefs, trading on these, and then anticipating high average future utility. But biased beliefs come at a cost. A person that makes objectively poor investment decisions has lower utility ex post, $V_2 = \ln c$, on average. Our theory balances these effects – it trades off the anticipatory benefits of optimism against the utility losses caused by decisions based on optimistic beliefs. Further, this approach provides discipline: biases in beliefs are determined endogenously by the economic environment.

Formally, each investor's beliefs maximize his well-being, defined as the average expected utility across periods 1 and 2 when actions are optimal given subjective beliefs. That is, $\hat{\boldsymbol{\pi}}$ maximizes $\frac{1}{2}E[V_1 + V_2]$ subject to the constraints that the $\hat{\pi}_s$ are probabilities and that portfolio choices are optimal given $\hat{\boldsymbol{\pi}}$. This wellbeing function is similar to that proposed in Andrew J. Caplin and John Leahy (2000), and analogous arguments support our use of this

[5] While here we assume $p_s > 0$, Section II endogenizes prices and derives this as a result.

function. Optimal beliefs maximize the Lagrangian

$$\mathcal{L} = \textstyle\sum_{s=1}^{S} \hat{\pi}_s \ln c_s^* (\hat{\boldsymbol{\pi}}) + \textstyle\sum_{s=1}^{S} \pi_s \ln c_s^* (\hat{\boldsymbol{\pi}}) - \mu \left[\sum_{s=1}^{S} \hat{\pi}_s - 1 \right] \tag{3}$$

(and subject to $\hat{\pi}_s \geq 0$). Beliefs impact well-being directly through anticipation of future flow utility and indirectly through their effect on portfolio choice.[6]

Because $c_s^* (\hat{\boldsymbol{\pi}})$ is continuous in subjective probabilities, $\mathcal{L}$ is also; and since probability spaces are compact, optimal beliefs exist. Further, if $\hat{\pi}_s = 0$, $c_s^* (\hat{\boldsymbol{\pi}}) = 0$ and the investor would get infinite negative utility if state s is realized.

Proposition 1: (Existence of interior optimal beliefs)
Optimal subjective probabilities, $\hat{\boldsymbol{\pi}}^*$, exist and are positive: $0 < \hat{\pi}_s^* < 1$ for all s.

Turning to the characterization of behavior, the first-order conditions for beliefs are

$$\frac{\pi_s}{\hat{\pi}_s} - \ln \frac{\pi_s}{\hat{\pi}_s} = \mu - 1 + \ln \frac{p_s}{\pi_s} \text{ for all } s. \tag{4}$$

And the second order conditions (reorganized) are

$$\hat{\pi}_s \left[1 - \frac{\pi_{s'}}{\hat{\pi}_{s'}} \right] \leq \hat{\pi}_{s'} \left[\frac{\pi_s}{\hat{\pi}_s} - 1 \right] \text{ for all } s \neq s'. \tag{5}$$

The first-order conditions are displayed in Figure I.1, which plots the left-hand-side and the right-hand-side of the equations against $\pi_s / \hat{\pi}_s$. The left-hand-sides of the first-order conditions are all identical convex curves with minima at objective beliefs, $\hat{\pi}_s = \pi_s$; the right-hand-sides are horizontal lines, independent of beliefs, that are higher for states that are more expensive per unit of probability. By Proposition 1, we know that μ is such that the left-hand-side of each first-order condition intersects the right at least once (with $0 < \hat{\pi}_s < 1$

[6]This approach is a 'frictionless extreme' in the sense that only ex-post costs limit biases in beliefs. Additional factors might also constrain biases.

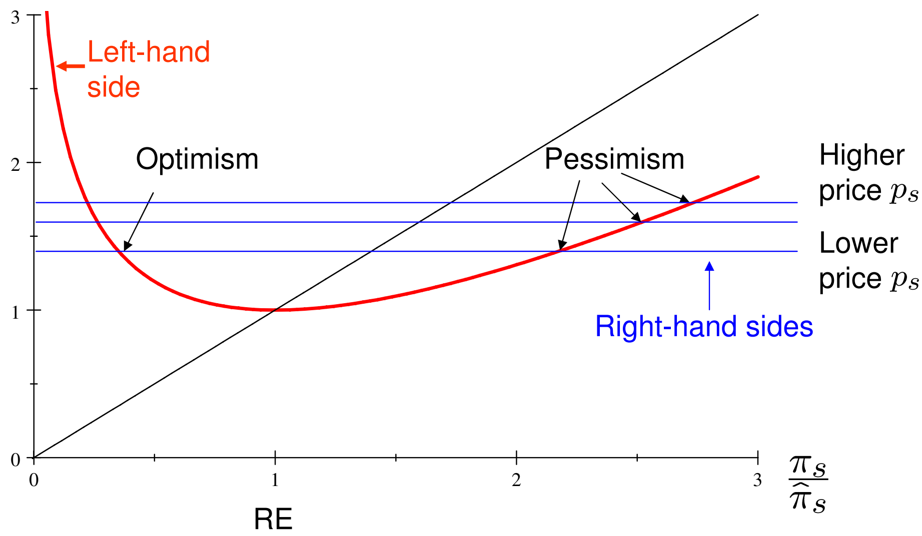

Figure I.1: First-order conditions for optimal beliefs

for all s). Thus, each first-order condition has one or two solutions.[7] If for state s, the right-hand side equals one, then objective beliefs are the only possible solution. Otherwise, the right-hand side is greater than one, and, by the concavity and linearity of the two sides, there are two solutions to the first-order condition, one with a positive bias and one with a negative bias. From the second-order condition if beliefs about the probability of s' are biased upwards, so that $\pi_{s'}/\hat{\pi}_{s'} < 1$, then $\pi_s/\hat{\pi}_s > 1$ for all $s \neq s'$ so that beliefs about the probabilities of all other states are biased downwards. Further analysis of the program shows that objective beliefs are optimal beliefs only if $S = 2$ and $\pi_1 = \pi_2$ and $p_1 = p_2$.

Proposition 2: If $S = 2$ and $\pi_1 = \pi_2$ and $p_1 = p_2$, objective beliefs are optimal. Otherwise:

(i) one and only one state has upward-biased subjective probability, all other states have downward-biased subjective probability: $\exists\ s'$ such that $\hat{\pi}^*_{s'} > \pi_{s'}$ and $\hat{\pi}^*_s < \pi_s$ for all $s \neq s'$;

(ii) among states with downward-biased subjective probabilities, states with larger price-probability ratios (economy-wide stochastic discount factors) are biased down by larger fac-

[7] Equations (4) are Lambert W functions in $\pi_s/\hat{\pi}_s$ and in general no closed-form solution exists.

tors: for s'', $s' \in \{s : \hat{\pi}^*_s < \pi_s\}$, $\pi_{s'}/\hat{\pi}^*_{s'} > \pi_{s''}/\hat{\pi}^*_{s''}$ iff $p_{s'}/\pi_{s'} > p_{s''}/\pi_{s''}$ and $\pi_{s'}/\hat{\pi}^*_{s'} = \pi_{s''}/\hat{\pi}^*_{s''}$ iff $p_{s'}/\pi_{s'} = p_{s''}/\pi_{s''}$.

The result that the investor biases upward the probability of only one state comes from a natural complementarity between the subjective belief about a state and the level of consumption in that state. Once a state is perceived as more likely, one wants more consumption in that state, and once one has more consumption in that state, there are greater benefits to believing that that state is more likely. The second part of the proposition is driven by the same force. An investor purchases less consumption in a more expensive state, and so has a greater incentive to believe that the more expensive state is unlikely to occur.

For the remainder of this section, we rule out the knife-edge case that delivers rational expectations.

Assumption 1: Either $S > 2$ or $\pi_s \neq 1/2$ or $p_1 \neq p_2$.

We now characterize which state an investor is optimistic about. The benefits of optimism about a state are related to the consumption purchased in that state and the costs are related to the objective misallocation of consumption across states. The costs are second-order, so for an infinitesimal change in beliefs a person should bias upwards the probability of the state in which they have the most consumption. Starting from objective beliefs, this is the cheapest state in terms of price-probability ratio.

Analogously, optimal expectations, which are not infinitesimal deviations from rational expectations, tend to bias upward the probability of the cheapest state because extra consumption in that state requires the least decrease in consumption in the remaining states, where 'cheap' refers to a combination of low price and low ratio of price to probability. If all states have the same ratio of price to probability, the investor biases upwards the probability of the lowest price (and probability) state. If states have equal objective probabilities but vary in price, then the investor overestimates the probability of the least-expensive state because this requires the smallest reduction in consumption in the other states.[8]

Proposition 3: (i) If all states have the same price-probability ratio, $p_s/\pi_s = m$ for all s, the investor overestimates the probability of (one of) the state(s) with the lowest probability.

(ii) If all states are equally likely, $\pi_s = \pi$ for all s, then the investor overestimates the probability of (one of) the state(s) with the lowest price-probability ratio.

(iii) If one state has both the lowest probability and the lowest price-probability ratio, then the investor overestimates the probability of this state.

(iv) For any state, there exist $\bar{m}$ and $\underline{m}$, such that for a sufficiently low price $p_s \leq \underline{m}\pi_s$ optimal beliefs overestimate the probability of this state, $\hat{\pi}_s^* > \pi_s$, and for a sufficiently high price $p_s \geq \bar{m}\pi_s$ optimal beliefs underestimate the probability of this state, $\hat{\pi}_s^* < \pi_s$.

The labels of 'optimism' and 'pessimism' in Figure I.1 denote the actual optimal beliefs when states have equal probability: the state with the lowest price per unit of probability is viewed with optimism and the remaining states are viewed with pessimism (Proposition 3(ii)). Figure I.2, discussed subsequently, displays the first-order conditions when states are priced fairly.

Optimal portfolio choice. While beliefs are interesting, our ultimate interest is in explaining prices and quantities, that is, returns and portfolios.

Consider first the case of actuarially fair prices, $p_s/\pi_s = m$ for all s. Under rational expectations, the optimal portfolio is risk-free. For optimal beliefs, Equations (2) and (4) imply first-order conditions $1/mc_s + \ln mc_s = \mu - 1 + \ln m$ for all s, where the Lagrange multiplier μ is such that $\bar{c}\pi_{s'} + \underline{c}\sum_{s \neq s'} \pi_s = 1/m$ and $(\underline{c}, \bar{c})$ are the two solutions to this equation: $\bar{c}$ is the consumption level in the state with positively-biased subjective probability, and $\underline{c}$ is the consumption level in the remaining states. This can be seen in Figure I.2, which displays the first-order conditions when prices are actuarially fair. Because the right-

[8] Other effects are present – the elasticity of consumption to beliefs, and the curvature of the utility function matter – but this is the strongest effect here.

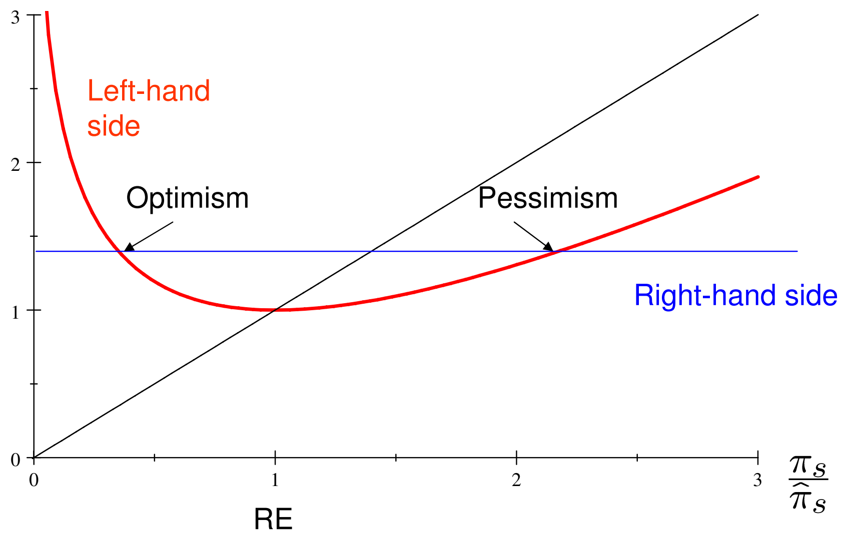

Figure I.2: First-order conditions when states are priced fairly

hand-sides are identical, all pessimistic biases are identical. The following corollaries follow directly.

Corollary 1: **(Preference for skewness)** If $p_s/\pi_s = m$ for all s, then the investor prefers the most skewed assets: the investor buys $\bar{c}$ of one of the Arrow-Debreu securities that pays off with the smallest probability and $\underline{c} < \bar{c}$ of each of the remaining securities.

Corollary 2: **(Two-fund separation)** If $p_s/\pi_s = m$ for all s, then the investor holds a portfolio consisting of the risk-free asset (an equal amount of all Arrow-Debreu securities) and an additional positive amount of one and only one of the most skewed securities.

These Corollaries match two of the empirical findings described at the start of the paper. First, investors are well diversified except for investing in one asset. Second, both the return on the additional asset they hold and the return on their portfolios are positively skewed.

When prices are not actuarially fair, investors still do not optimally diversify and invest more than the investor with rational expectations in securities with skewed returns. The latter occurs both because investors tend to be optimistic about states with low probabilities and prices (Proposition 3) and because pessimism is more severe for states with high prices (Proposition 2(ii)). In general, diversification, preferred by an agent with rational expectations, would destroy skewness, preferred by an agent with optimal expectations.

As we now show, equilibrium prices tend to make different investors optimistic about different states, and so portfolios in equilibrium tend to be heterogeneous and have idiosyncratically skewed returns.

II. Asset pricing in an exchange economy with optimal expectations

We consider an exchange economy with a unit mass of investors, $S > 2$, and aggregate per capita endowment in each state of C_s. Due to space constraints, we consider an example that illustrates the general characteristics of optimal expectations equilibria. In this economy, portfolios are heterogeneous across investors, portfolio returns are idiosyncratically skewed, and securities with positively skewed returns have lower expected returns.

Definition: An *optimal expectations equilibrium* is a portfolio $\mathbf{c}^i$ and beliefs $\hat{\boldsymbol{\pi}}^i$ for each agent i and prices $\mathbf{p}$ such that: (i) each agent's portfolio is optimal given his beliefs and prices; (ii) each agent's beliefs maximize his well-being; (iii) the market for each asset clears.

Before analyzing a more complex environment, consider first an economy with equally probable states, $\pi_s = \pi$, and no aggregate risk, $C_s = C = 1$. Suppose that prices are actuarially fair, $p_s = p$. Each investor biases upward the subjective probability of one state, purchases $\bar{c}$ of the Arrow-Debreu security associated with this state, and purchases $\underline{c} < \bar{c}$ of the Arrow-Debreu security associated with the remaining downwards-biased states (where $\bar{c}$ and $\underline{c}$ are as defined in the previous section). This is an equilibrium if an equal share of agents are optimistic about each state, so that demand for consumption is equal across states, and each asset's price is $p = 1/S$. This equilibrium is locally stable, in the sense that a small change in prices would lead all investors to bias up the subjective probabilities of the cheapest states (Proposition 3(ii)), which would lead to excess demand for consumption in these states and a (relative) increase in price for the cheapest states.

Consider now similar economies in which the variation in the aggregate endowment across states is 'not too large.' An equilibrium with actuarially fair prices exists as long as there exist different shares of agents that are optimistic about each state so that the demand for each asset matches the supply. Thus, in economies with equally probable states and low aggregate risk, prices are fair and agents hold heterogeneous beliefs, overinvest in different skewed assets, and thus hold portfolios with idiosyncratically skewed returns. In the corresponding rational expectations equilibrium, investors' portfolios would be homogeneous and perfectly diversified, $c_s = C_s$. Further, also unlike in the rational expectations equilibrium, aggregate risk, in limited amounts, is not priced.[9] People have an interest in risk, and a small amount of aggregate risk satisfies this desire without changing prices. Finally, as aggregate endowment risk increases beliefs become less heterogeneous.

Proposition 4: **(Heterogenous portfolios and idiosyncratic skewness)** Suppose that $\pi_s = \pi$ for all s. For any vector $(C_1, ..., C_S)$ of aggregate endowment such that $\underline{c} \leq C_s \leq \bar{c}$ and $\sum_{s=1}^{S} C_s = \bar{c} + (S-1)\underline{c}$, there exists an optimal expectations equilibrium with the following characteristics:

(i) prices are actuarially fair: $p_s = p$ for all s;

(ii) for all s, a fraction $\lambda_s = (C_s - \underline{c})/(\bar{c} - \underline{c})$ of investors buys $\bar{c}$ of the Arrow-Debreu security associated to state s and $\underline{c}$ of the security for every other state;

where $\underline{c}$ and $\bar{c}$ are defined in Section I.

Having established this result, we now construct our example that matches all three stylized facts discussed in the introduction. Consider an economy with some unlikely states and some likely states. At actuarially fair prices, each investor would bias upward his probability

[9] The equality of probabilities across states is key for this results. With unequal probabilities, aggregate risk is typically priced.

of one of the unlikely states. Analogously to Proposition 4, this is an equilibrium if there exists shares of investors that are optimistic about each unlikely state such that the market clears. For example, if $\pi_s = \pi^A$ and $C_s = C^A := \frac{1}{\underline{s}}\bar{c} + \left(1 - \frac{1}{\underline{s}}\right)\underline{c}$ for $s \leq \underline{s}$ and $\pi_s = \pi^B > \pi^A$ and $C_s = C^B := \underline{c}$ for $s > \underline{s}$, then an equilibrium with fair prices exists in which $1/\underline{s}$ investors bias up their probabilities for states $s \leq \underline{s}$. But if the endowments across states are not so different, then prices in the unlikely states must be relatively higher so that demand for output is also relatively lower in these states, and hence the expected returns of the most skewed Arrow-Debreu securities are lower.

Proposition 5: **(Underperformance of skewed assets)** For a small reduction in $C^A - C^B$ such that p_s does not change for $s > \underline{s}$, p_s increases for $s \leq \underline{s}$ so that:

(i) the securities with the more skewed returns have lower expected returns than in a rational expectations equilibrium;

(ii) the securities with the more skewed returns have relatively lower expected returns, $\pi^A/p_s < \pi^B/p_{s'}$ for all $s \leq \underline{s}$ and $s' > \underline{s}$.

This equilibrium fits all three stylized facts: (a) portfolios are heterogeneous and not perfectly diversified; (b) each investor overinvests in one security that is more skewed than the average security and his portfolio return is more skewed than the market return; (c) more positively skewed securities have lower returns. These results relate to the use of co-skewness as a pricing factor. As C^A varies (and in richer environments), the relative importance of idiosyncratic skewness and aggregate skewness for asset prices varies. Finally, consumption insurance appears incomplete, but not because of missing markets or moral hazard, but rather because households optimally choose to hold risk.

How important are the assumptions of our example? First, if the low-probability states have much larger endowments than the other states ($C_s > C^A$), then we still match (a) and (b), but the expected returns on the most skewed assets are higher for the usual reason that investors discount payouts in states with high aggregate endowment. However, even in this case, Proposition 5(i) implies that the returns on the most skewed assets are lower than in a rational expectations equilibrium. Second, the assumption that some probabilities are equal is not essential. If probabilities differ among high-probability states, nothing changes. If probabilities differ among low-probability states, prices would have to be higher for lower probability states for investors to remain indifferent among (perhaps a subset) of states and for portfolios to be heterogeneous.

The desire for skewness can also impact the market return. If bad aggregate states have low probabilities, as for disasters or Peso problems, then it is possible for the desire for skewness to increase the equity premium, as investors seek to avoid negative skewness.

In conclusion, the natural human tendency towards optimism tempered by the real costs of poor decisions, implies that people hold heterogeneous, underdiversified portfolios to attain skewed returns, and that this behavior reduces the returns of positively skewed assets.

References

Attanasio, Orazio and Steven J. Davis, "Relative Wage Movements and the Distribution of Consumption," *Journal of Political Economy*, December 1996, *104* (6), 1227–62.

Barberis, Nicholas C. and Ming Huang, "Stocks as Lotteries: The Implications of Probability Weighting for Security Prices," *Mimeo, Yale University*, March 2005.

Blume, Marshall E., Jean Crockett, and Irwin Friend, "Stock ownership in the United States: Characteristics and trends," *Survey of Current Business*, 1974, *54* (11), 16–40.

Brunnermeier, Markus K. and Jonathan A. Parker, "Optimal Expectations," *American Economic Review*, September 2005, *95* (4), 1092–1118.

Calvet, Laurent E., John Y. Campbell, and Paolo Sodini, "Down or Out: Assessing the Welfare Costs of Household Investment Mistakes," 2006. NBER Working Paper 12030.

Caplin, Andrew J. and John Leahy, "The Social Discount Rate," October 2000. NBER Working Paper 7983.

Curcuru, Stephanie, Deborah J. Lucas, John C. Heaton, and Damien Moore, "Heterogeneity and Portfolio Choice: Theory and Evidence," in Yacine Ait-Sahalia and Lars P. Hansen, eds., *The Handbook of Financial Econometrics*, Elsevier, forthcoming.

Goetzmann, William N. and Alok Kumar, "Equity Portfolio Diversification," December 2001. NBER Working Paper 8686.

Gollier, Christian, "Optimal Illusions and Decision under Risk," 2005. Mimeo, Toulouse.

Mitton, Todd and Keith Vorkink, "Equilibrium Underdiversification and the Preference for Skewness," *Review of Financial Studies*, forthcoming.

Vissing-Jorgensen, Annette, "Perspectives on Behavioral Finance: Does "Irrationality" Disappear with Wealth? Evidence from Expectations and Actions," in Mark Gertler and

Ken Rogoff, eds., *N.B.E.R. Macroeconomics Annual 2003*, Cambridge: MIT Press 2004.

Zhang, Yijie, "Individual Skewness and the Cross-Section of Average Stock Returns," November 2005. Mimeo, Yale.

Appendix: Proofs

A. Proof of Proposition 1

Define the set $\Pi = \left\{ \hat{\boldsymbol{\pi}} : 0 \leq \hat{\pi}_s \leq 1, \sum_{s=1}^{S} \hat{\pi}_s = 1 \right\}$. Optimal consumption choices given beliefs are continuous in probabilities on Π (Equation (2)). The objective for beliefs (Equation (3)) is thus continuous in beliefs on Π since L is linear in $\hat{\boldsymbol{\pi}}$ and continuous in c. Since Π is compact, a maximum exists on Π. To see that the optimum requires $0 < \hat{\pi}_s < 1$ for all s, note that for $\hat{\boldsymbol{\pi}}$ such that $\hat{\pi}_s = 0$ for at least one s, $c_s^* = 0$, and thus the objective for beliefs has value negative infinity and this cannot be optimal since the objective is finite on the interior of Π.■

B. Proof of Proposition 2

To establish the results, we first prove four lemmas. Without loss of generality, we choose units so that

$$\sum_{s=1}^{S} p_s = 1.$$

This implies that $p_s/\pi_s = m = 1$ for all s for actuarially fair prices.

Lemma 1: The subjective belief of at most one state is biased upwards.

Proof of Lemma 1: The second-order condition, Equation (5), implies

$$\hat{\pi}_s^2 \left[\hat{\pi}_{s'} - \pi_{s'} \right] \leq \hat{\pi}_{s'}^2 \left[\pi_s - \hat{\pi}_s \right] \text{ for all } s \neq s'. \tag{B.1}$$

Thus if beliefs about the likelihood of s' are biased upwards, so that $\hat{\pi}_{s'} - \pi_{s'} > 0$, then $\hat{\pi}_s - \pi_s < 0$ for all $s \neq s'$.■

The proof of Lemma 1 also directly implies Lemma 2.

Lemma 2: If the subjective belief of one state is biased upwards, then the subjective beliefs of all other states are biased downwards.

Lemma 2 corresponds to result (i) in Proposition 2, except that we still need to prove that rational expectation cannot be optimal except in the case in which prices are actuarially fair, probabilities are equal, and there are only two states. We first examine the case of actuarially unfair prices.

Lemma 3: If prices are not actuarially fair, rational expectations cannot be optimal.

Proof of Lemma 3: The first-order conditions (4) for rational expectations are

$$1 = \mu - 1 + \ln \frac{p_s}{\pi_s} \text{ for all } s$$

which cannot be satisfied for all s if $p_s/\pi_s \neq p_{s'}/\pi_{s'}$ for some s and s'.■

Combining lemmata 2 and 3 implies that there is one, and only one, state whose probability is biased upwards when prices are actuarially unfair. In the remaining proof of result (i), we assume that prices are actuarially fair.

Lemma 4: If prices are actuarially fair, $p_s/\pi_s = 1$ for all s, then the investor biases his beliefs for all states with downward-biased subjective probabilities by a common factor: For one state, s', $c_{s'} = \bar{c} \geq 1$ and $\hat{\pi}_{s'} = \bar{c}\pi_{s'} \geq \pi_{s'}$, and for all other states, $s \neq s'$, $c_s = \underline{c} \leq 1$ and $\hat{\pi}_s = \underline{c}\pi_s \leq \pi_s$, where $(\underline{c}, \bar{c})$ are the two solutions to

$$\frac{1}{c} - \ln \frac{1}{c} = \mu - 1 \tag{B.2}$$

with μ such that $\bar{c}\pi_{s'} + \underline{c}\sum_{s \neq s'} \pi_s = 1$.

Proof of Lemma 4: Equation (B.2) is the first-order conditions for optimal beliefs written in terms of consumption using Equation (2). If $\mu = 2$, $c_s = 1$ for all states, in which case beliefs are objective and the Lemma holds trivially ($\underline{c} = \bar{c}$). If $\mu > 2$, every first-order condition has the same two possible solution, so that $c_s = \hat{\pi}_s/p_s = \hat{\pi}_s/\pi_s$ is the same among all states whose likelihood is biased down. For these states let $\underline{c} := c_s^*(\hat{\boldsymbol{\pi}})$. Then Lemma 1 implies that there is at most one state perceived as more likely than it is, and let its consumption level be $\bar{c}$. We note $\mu < 2$ is not possible by Proposition 1.■

We now prove that the one specific case has rational expectations and part (i) for actuarially fair prices by using these Lemmas to analyze the program given by Equation (3).

Let π denote the probability of the state whose probability is biased upwards, and let $\bar{c}$ be the consumption level in that state, which is denoted s'. By Lemma 3, we know that the consumption level is a constant $\underline{c}$ in all other states. Using the fact that $\widehat{\pi} = \bar{c}\pi$, we can rewrite the Lagrangian objective for beliefs as a function of only $\bar{c}$ and π:

$$\begin{aligned}
\mathcal{L} &= \textstyle\sum_{s=1}^{S} c_s \pi_s \ln c_s + \sum_{s=1}^{S} \pi_s \ln c_s - \mu \left[\displaystyle\sum_{s=1}^{S} c_s \pi_s - 1 \right] \\
&= \pi \bar{c} \ln \bar{c} + (1-\pi)\, \underline{c} \ln \underline{c} + \pi \ln \bar{c} + (1-\pi) \ln \underline{c} - \mu \left[\bar{c}\pi + (1-\pi)\, \underline{c} - 1 \right] \\
&= \pi (\bar{c}+1) \ln \bar{c} + (2 - \pi(\bar{c}+1)) \ln \left(\frac{1-\pi\bar{c}}{1-\pi} \right) := U(\bar{c}, \pi)
\end{aligned}$$

where the last step imposes the constraint by substituting in $\underline{c} = \frac{1-\pi\bar{c}}{1-\pi}$. $U(\bar{c},\pi)$ is maximized over the choice of state to bias upward and consumption level in that state. We proceed in two steps.

Suppose that we know the state, and thus its objective probability π, whose belief is biased upwards. We determine the optimal bias $\bar{c}$ in the following way. Notice first that $U(1,\pi) = 0$ and that

$$\frac{\partial U}{\partial \bar{c}}(\bar{c},\pi) = \pi \ln \frac{\bar{c}(1-\pi)}{1-\pi\bar{c}} + \pi \frac{1-\bar{c}}{\bar{c}(1-\pi\bar{c})}.$$

Evaluating $U_{\bar{c}}$ at $\bar{c} = 1$ and $\bar{c} = (1-\pi)/\pi$, we obtain that

$$\frac{\partial U}{\partial \bar{c}}(1,\pi) = 0,$$

and

$$\frac{\partial U}{\partial \bar{c}}(\frac{1-\pi}{\pi},\pi) = 2\pi \ln \frac{1-\pi}{\pi} + \frac{2\pi-1}{1-\pi}.$$

The second derivative of U with respect to $\bar{c}$ is:

$$\frac{\partial^2 U}{\partial \bar{c}^2}(\bar{c},\pi) = -2\pi^2 \frac{(\bar{c}-1)(\bar{c}-\frac{1}{2\pi})}{\bar{c}^2(1-\pi\bar{c})^2}. \tag{B.3}$$

When $\pi = 1/2$, the right-hand side of the above equality is negative, implying that U is concave in $\bar{c}$. Combining this with $U_{\bar{c}}(1, 1/2) = 0$ directly implies that $\bar{c} = 1$ (and so $\widehat{\pi} = \pi$) is optimal if $\pi = 1/2$. This sheds light on the special case with only two states that are equally likely. In that case, the choice of the state whose probability would be biased upwards is arbitrary, and we have just shown that it is optimal not to distort beliefs. We now argue that in all other cases, $\hat{\pi} \neq \pi$.

First, we show that if we consider $\pi < 1/2$, then $\bar{c} > 1/2\pi$ so that $\hat{\pi} > 1/2 > \pi$. When π is less than 1/2, we see from (B.3) that U is alternatively concave, convex and concave (in $\bar{c}$) over the intervals $]0,1]$, $[1,1/2\pi]$ and $[1/2\pi,1/\pi]$. Combining this with $U(1,1/2) = U_{\bar{c}}(1,1/2) = 0$ implies that the optimal solution has a $\bar{c}$ larger than $1/2\pi$.

Second, we show that given π, the optimal $\bar{c}$ has $\bar{c} \leq (1-\pi)/\pi$, or equivalently that $\widehat{\pi} \leq 1-\pi$. This follows from

$$q(\pi) = \frac{\partial U}{\partial \bar{c}}(\frac{1-\pi}{\pi},\pi) = 2\pi \ln \frac{1-\pi}{\pi} + \frac{2\pi-1}{1-\pi}$$

being negative when π is less than 1/2.[10] This implies that $\bar{c} \leq (1-\pi)/\pi$ or $\hat{\pi} \leq 1-\pi$ which

[10] This is because $q(1/2) = 0$, and

$$q'(\pi) = 2\ln\frac{1-\pi}{\pi} + \frac{2\pi-1}{(1-\pi)^2},$$

$$q''(\pi) = \frac{2(2\pi-1)}{\pi(1-\pi)^3},$$

so that $q'(1/2) = 0$ and q'' has the same sign than $\pi - 0.5$. This implies that q has the same sign than $\pi - 0.5$.

implies that the state perceived as more likely than it is necessarily has $\pi \leq 1/2$. Combined with the first result, we know that if the optimal $\pi \neq 1/2$, we have $\hat{\pi}_s \neq \pi_s$ for all s ("for all s" follows from Lemma 2).

What we still need to show is that $\pi \neq 1/2$ when there are more than 2 states with one state having probability 1/2. We do this by showing that the function $V(\pi) = \max_{\bar{c}} U(\bar{c}, \pi)$ is symmetric and U-shaped with a minimum at 1/2. Thus, as long as there exists a state with probability different from 1/2, $\hat{\pi} \neq \pi$. V is symmetric around $\pi = 1/2$ from the definition of V. By the envelop theorem, we have that

$$V'(\pi) = \frac{\pi \left(\bar{c} - \frac{1-\pi}{\pi}\right)(1-\bar{c})^2}{\bar{c}(1-\pi)(1-\pi\bar{c})},$$

where $\bar{c}$ maximizes $U(\bar{c}, \pi)$. Suppose that π is less than 1/2. We have seen above that it implies that $\bar{c}$ is larger than 1/2 but smaller $(1-\pi)/\pi$, yielding $V'(\pi) < 0$. This shows that the optimal state to bias upwards is the objectively least likely one. Except the case $\pi_1 = \pi_2 = 1/2$, this state has a π less than 1/2, which implies that $\underline{c} < 1 < \bar{c}$. This concludes the proof of result (i) in Proposition 2.

The above proof also directly implies Lemma 5, used in the proof of Lemma 6.

Lemma 5: If prices are actuarially fair, $p_s/\pi_s = 1$ for all s, then $\hat{\pi}_{s'} \geq 1/2$ where s' is the state whose probability is biased upwards.

Proposition 2(ii) follows directly from the first-order conditions, Equation (4). Given that one selects the solution with $\pi_s/\hat{\pi}_s > 1$, the left-hand side is increasing in $\pi_s/\hat{\pi}_s$ and the right-hand side is increasing in price-probability ratio, p_s/π_s.■

C. Proof of Proposition 3

Proposition 2 implies the pattern of belief distortion but does not specify which is the state that has its probability biased upwards.

The proof of part (i) relies on the function $V(\pi)$ defined in the proof of Proposition 2(i) as the value of holding beliefs that are optimistic about state s' where π is the probability associated with this state. The proof of Proposition 2(i) shows that this function, $V(\pi)$, is symmetric and U-shaped with a minimum at 1/2. Further, the proof of Proposition 2(i) shows that the optimal s' is necessarily such that $\pi \leq 1/2$. Thus, $V(\pi)$ is maximized by choosing to bias upwards the probability of (one of) the smallest probability state(s).

To prove part (ii), we show that the local maximum in which subjective probabilities satisfy the first-order conditions and the budget constraint with an arbitrary optimistic state is dominated by the same set of subjective probabilities in which the subjective probability of this state and the cheapest state are interchanged.

Let states 1 to $\bar{s}$ be the least expensive states, $p_s = p_{s'} < p_{s''}$ for $s', s \leq \bar{s}$ and $s'' > \bar{s}$. Let $\hat{\boldsymbol{\pi}}^*(s'')$ denote the vector of subjective probabilities that satisfy the first-order conditions, sum to one, and have $\hat{\pi}_{s''} > \pi_{s''}$ and $\hat{\pi}_s < \pi_s$ for all $s \neq s''$ and let $\mathcal{L}(\hat{\boldsymbol{\pi}}^*(s''), \mathbf{p})$ denote the associated value of the objective for beliefs:

$$\mathcal{L}(\hat{\boldsymbol{\pi}}^*(s''), \mathbf{p}) := \textstyle\sum_{s=1}^{S} \hat{\pi}_s^*(s'') \ln\left(\dfrac{\hat{\pi}_s^*(s'')}{p_s}\right) + \textstyle\sum_{s=1}^{S} \pi_s \ln\left(\dfrac{\hat{\pi}_s^*(s'')}{p_s}\right)$$

where $\mathbf{p}$ denotes the vector of prices.

Consider taking $\hat{\boldsymbol{\pi}}^*(s'')$ for some $s'' > \bar{s}$ and switching the subjective probability for state s'' $(\hat{\pi}_{s'}^*(s''))$ with the subjective probability of some state $s' \leq \bar{s}$ $(\hat{\pi}_{s'}^*(s''))$,

$$(\hat{\pi}_1^*(s''), ..., \hat{\pi}_{s''}^*(s''), ..., \hat{\pi}_{\bar{s}}^*(s''), ..., \hat{\pi}_{s'}^*(s''), ..\hat{\pi}_S^*(s'')) := \hat{\boldsymbol{\pi}}^{Switch}(s'')$$

This is feasible because it is still the case that probabilities sum to one and as a result the budget constraint is also satisfied. For notational simplicity, for the moment, let $\hat{\pi}_s^* = \hat{\pi}_s^*(s'')$.

Since well-being differs only in states s' and s'',

$$\begin{aligned}\mathcal{L}\left(\hat{\boldsymbol{\pi}}^{Switch}\left(s''\right), \mathbf{p}\right) - \mathcal{L}\left(\hat{\boldsymbol{\pi}}^{*}\left(s''\right), \mathbf{p}\right) &= \left(\hat{\pi}^{*}_{s''} + \pi\right) \ln \frac{\hat{\pi}^{*}_{s''}}{p_{s'}} - \left(\hat{\pi}^{*}_{s'} + \pi\right) \ln \frac{\hat{\pi}^{*}_{s'}}{p_{s'}} \\ &\quad + \left(\hat{\pi}^{*}_{s'} + \pi\right) \ln \frac{\hat{\pi}^{*}_{s'}}{p_{s''}} - \left(\hat{\pi}^{*}_{s''} + \pi\right) \ln \frac{\hat{\pi}^{*}_{s''}}{p_{s''}} \\ &= \left(\hat{\pi}^{*}_{s''} - \hat{\pi}^{*}_{s'}\right) \ln \frac{p_{s''}}{p_{s'}} > 0\end{aligned}$$

where the sign follows from the initial assumptions that $p_{s'} < p_{s''}$ and that optimism is focussed on state s' so that $\hat{\pi}^{*}_{s''}\left(s''\right) > \pi > \hat{\pi}^{*}_{s'}\left(s''\right)$.

Since $\hat{\boldsymbol{\pi}}^{Switch}\left(s''\right)$ is not optimally chosen for the situation in which the investor biases upward his beliefs about state s', these probabilities may not satisfy the first-order conditions and so are weakly worse than those that are optimally chosen conditional on being optimistic about state s':

$$\mathcal{L}\left(\hat{\boldsymbol{\pi}}^{*}\left(s'\right), \mathbf{p}\right) \geq \mathcal{L}\left(\hat{\boldsymbol{\pi}}^{Switch}\left(s''\right), \mathbf{p}\right)$$

Thus we have,

$$\mathcal{L}\left(\hat{\boldsymbol{\pi}}^{*}\left(s'\right), \mathbf{p}\right) \geq \mathcal{L}\left(\hat{\boldsymbol{\pi}}^{Switch}\left(s''\right), \mathbf{p}\right) > \mathcal{L}\left(\hat{\boldsymbol{\pi}}^{*}\left(s''\right), \mathbf{p}\right) \text{ for all } s'' > \bar{s} \text{ and } s' \leq \bar{s}$$

which completes the proof of part (ii).

For part (iii), we make a similar argument. Let states 1to $\bar{s}$ be the least expensive and lowest probability states, $p_s = p_{s'} < p_{s''}$ and $\pi_s = \pi_{s'} < \pi_{s''}$ for $s', s \leq \bar{s}$ and $s'' > \bar{s}$. Consider taking $\hat{\boldsymbol{\pi}}^{*}\left(s''\right)$ for some $s'' > \bar{s}$ and switching the subjective probability for state s'' $\left(\hat{\pi}^{*}_{s'}\left(s''\right)\right)$ with the subjective probability of some state $s' \leq \bar{s}$ $\left(\hat{\pi}^{*}_{s'}\left(s''\right)\right)$ and let this vector be denoted

$\hat{\boldsymbol{\pi}}^{Switch}(s'')$. Since well-being differs only in states s' and s'',

$$\begin{aligned}
\mathcal{L}\left(\hat{\boldsymbol{\pi}}^{Switch}(s''),\mathbf{p}\right)-\mathcal{L}\left(\hat{\boldsymbol{\pi}}^{*}(s''),\mathbf{p}\right) &= \left(\hat{\pi}^{*}_{s''}+\pi_{s''}\right)\ln\frac{\hat{\pi}^{*}_{s''}}{p_{s'}}-\left(\hat{\pi}^{*}_{s'}+\pi_{s'}\right)\ln\frac{\hat{\pi}^{*}_{s'}}{p_{s'}} \\
&\quad+\left(\hat{\pi}^{*}_{s'}+\pi_{s'}\right)\ln\frac{\hat{\pi}^{*}_{s'}}{p_{s''}}-\left(\hat{\pi}^{*}_{s''}+\pi_{s''}\right)\ln\frac{\hat{\pi}^{*}_{s''}}{p_{s''}} \\
&= \left(\hat{\pi}^{*}_{s''}-\hat{\pi}^{*}_{s'}\right)\ln\frac{p_{s''}}{p_{s'}}+\left(\pi_{s''}-\pi_{s'}\right)\ln\frac{p_{s''}}{p_{s'}}
\end{aligned}$$

where the sign follows from the initial assumptions that $\pi_{s''} > \pi_{s'}$, $p_{s'} < p_{s''}$ and that optimism is focussed on state s' so that $\hat{\pi}^{*}_{s''}(s'') > \pi > \hat{\pi}^{*}_{s'}(s'')$.

Analogously to part (ii), since $\hat{\boldsymbol{\pi}}^{Switch}(s'')$ is not optimally chosen for the situation in which the investor biases upward his beliefs about state s':

$$\mathcal{L}\left(\hat{\boldsymbol{\pi}}^{*}(s'),\mathbf{p}\right)\geq\mathcal{L}\left(\hat{\boldsymbol{\pi}}^{Switch}(s''),\mathbf{p}\right)$$

Thus we have,

$$\mathcal{L}\left(\hat{\boldsymbol{\pi}}^{*}(s'),\mathbf{p}\right)\geq\mathcal{L}\left(\hat{\boldsymbol{\pi}}^{Switch}(s''),\mathbf{p}\right)>\mathcal{L}\left(\hat{\boldsymbol{\pi}}^{*}(s''),\mathbf{p}\right)\text{ for all } s''>\bar{s}\text{ and } s'\leq\bar{s}$$

which completes the proof of part (iii).

For part (iv) of Proposition 3, we first note that, for any problem, there is a lower bound placed on μ by the requirement of real solutions to the first-order conditions (Proposition 1). Thus,

$$\mu\geq\underline{\mu}:=\max_{s}\left\{2-\ln\frac{p_s}{\pi_s}\right\}\tag{C.1}$$

Second, for any problem, there is an upper bound placed on μ by the requirement that the $\hat{\pi}_{s'}<1$ in the solution to the first-order condition for the probability that is positively-biased state. Thus, for state s' to have $\hat{\pi}_{s'}>\pi_{s'}$, we require

$$\mu<\bar{\mu}(s'):=1+\pi_{s'}-\ln p_{s'}.$$

Consider first $\underline{m}$. As one decreases $p_{s'}$, $\underline{\mu}$ increases (at least once $p_{s'}/\pi_{s'}$ is the minimum), $\bar{\mu}(s')$ increases, and $\bar{\mu}(s)$ for $s \neq s'$ does not change. Thus, there is an $\underline{m}_{s'}$ such that for $p_{s'} = \underline{m}_{s'}\pi_{s'}$, $\bar{\mu}(s') > \underline{\mu}$ and $\bar{\mu}(s') < \underline{\mu}$ for all $s \neq s'$. Thus, the agent must be optimistic about state s'. Then $\underline{m} = \min_{s'}\{\underline{m}_{s'}\}$.

In terms of $\bar{m}$, as one increases $p_{s'}$, there is an $\bar{m}$ such that for $p_{s'} = \bar{m}\pi_s$, $\bar{\mu}(s') < \underline{\mu}$ and $\bar{\mu}(s') > \underline{\mu}$ for some $s \neq s'$. Thus, the agent must be pessimistic about state s'. Then $\bar{m}$ $= \max_{s'}\{\bar{m}_{s'}\}$.■

D. Proof of Proposition 4

Proof by construction in the text.

E. Proof of Proposition 5

This Proposition is not trivial because, while an increase in the price of a given state decreases the demand for the asset that pays off in that state, an increase in the price of a state for which $\hat{\pi}^*_s < \pi_s$ decreases the demands for the assets that pay off in all other states that are viewed with pessimism. Following the proof of Proposition 5, we prove two Lemmata that characterize these price effects in Section F.

We start with the following Lemma.

Lemma 6: In any equilibrium with actuarially fair prices:

$$\sum_{s=1}^{S}\left(\frac{\hat{\pi}_s^2}{\hat{\pi}_s - \pi_s}\right) > 0. \tag{E.1}$$

or equivalently

$$\pi_{s'}\left(\frac{\bar{c}^2}{\bar{c}-1}\right) + \sum_{s \neq s'} \pi_s \left(\frac{\underline{c}^2}{\underline{c}-1}\right) > 0 \tag{E.2}$$

Proof: By Lemma 4, there are two consumption levels such that $\hat{\pi}_s/\pi_s = \underline{c}$ for all $s \neq s'$, and $\hat{\pi}_{s'}/\pi_{s'} = \bar{c}$. Since subjective probabilities sum to one, $\sum_{s\neq s'}\hat{\pi}_s = 1-\hat{\pi}_{s'}$, and the budget constraint at fair prices implies

$$\begin{aligned}
1 &= \bar{c}\pi_{s'} + \sum_{s\neq s'} \underline{c}\pi_s \\
\underline{c} &= \frac{1-\pi_{s'}\bar{c}}{1-\pi_{s'}} \\
\underline{c} &= \frac{1-\hat{\pi}_{s'}}{1-\pi_{s'}}.
\end{aligned}$$

Using these relationships, our term of interest becomes:

$$\begin{aligned}
\sum_{s=1}^{S}\frac{\hat{\pi}_s^2}{\hat{\pi}_s-\pi_s} &= \frac{\hat{\pi}_{s'}}{1-\frac{\pi_{s'}}{\hat{\pi}_{s'}}} + \sum_{s\neq s'}\hat{\pi}_s\left(\frac{1}{1-\underline{c}^{-1}}\right) \\
&= \frac{\hat{\pi}_{s'}^2}{\hat{\pi}_{s'}-\pi_{s'}} + (1-\hat{\pi}_{s'})\frac{1}{1-\frac{1-\pi_{s'}}{1-\hat{\pi}_{s'}}} \\
&= \frac{\hat{\pi}_{s'}^2}{\hat{\pi}_{s'}-\pi_{s'}} + \frac{(1-\hat{\pi}_{s'})^2}{1-\hat{\pi}_{s'}-1+\pi_{s'}} \\
&= \frac{2\hat{\pi}_{s'}-1}{(\hat{\pi}_{s'}-\pi_{s'})}
\end{aligned}$$

which is positive if $\hat{\pi}_{s'} > 1/2$. This follows from from Lemma 5.■

For notational simplicity, let a index the states $s \leq \underline{s}$ so that $\pi_s = \pi_a$ and $C_s = C_a$ for $s \leq \underline{s}$, and let b index the $S-\underline{s}$ states with $\pi_s = \pi_b$ and $C_s = C_b$. Let a' be the state, less than or equal to $\underline{s}$, that is viewed as more likely than it is; this state differs across investors but all a-states are symmetric so we can impose that $p_{a'} = p_a$.

To prove Proposition 5, we write the conditions for the initial equilibrium with fair prices, totally differentiate the system for a small increase in the aggregate endowment in the a-states, dC_a, imposing that $dp_b = 0$, show that $dp_a > 0$, and check that $dC_b < 0$. Because there is a discrete interval between π_a and π_b and because the wellbeing functions evaluated for different choices of s'(the state such that $\hat{\pi}_s > \pi_s$) are continuous in prices, small enough changes in p_a and p_b do not change the relative rankings of the wellbeing as a function of s'.

Thus, locally the pattern of a' across investors remains unchanged.

E1. Equilibrium conditions at fair prices

The exogenous variables are S, $\underline{s}$, π_a. Given these three, π_b is given by the fact that probabilities sum to one

$$\underline{s}\pi_a + (S - \underline{s})\,\pi_b = 1.$$

For actuarially fair prices that (normalized) sum to one, we have

$$\begin{aligned} p_a &= \pi_a \\ p_b &= \pi_b. \end{aligned}$$

Optimal beliefs are given by the investor first-order conditions,

$$\begin{aligned} \frac{\pi_{a'}}{\hat{\pi}_{a'}} - \ln\frac{\pi_{a'}}{\hat{\pi}_{a'}} &= \mu - 1 + \ln\frac{p_a}{\pi_a} \\ \frac{\pi_a}{\hat{\pi}_a} - \ln\frac{\pi_a}{\hat{\pi}_a} &= \mu - 1 + \ln\frac{p_a}{\pi_a} \text{ for } a \neq a' \\ \frac{\pi_b}{\hat{\pi}_b} - \ln\frac{\pi_b}{\hat{\pi}_b} &= \mu - 1 + \ln\frac{p_b}{\pi_b} \end{aligned} \tag{E.3}$$

(note that at fair prices, the last two imply and $\frac{\pi_a}{\hat{\pi}_a} = \frac{\pi_b}{\hat{\pi}_b}$), and the fact that subjective probabilities sum to one,

$$\hat{\pi}_{a'} + (\underline{s} - 1)\,\hat{\pi}_a + (S - \underline{s})\,\hat{\pi}_b = 1. \tag{E.4}$$

These equations can be used to solve for $\hat{\pi}_{a'}$, $\hat{\pi}_a$, $\hat{\pi}_b$ and μ, which we know exist and are unique. Note that $\underline{c} = \frac{\hat{\pi}_a}{\pi_a} = \frac{\hat{\pi}_b}{\pi_b}$ and $\bar{c} = \frac{\hat{\pi}_{a'}}{\pi_{a'}}$.

Finally, the two remaining exogenous variables that deliver fair prices in equilibrium, C_a

and C_b, are calculated from market clearing conditions:

$$\begin{aligned} C_a &= \frac{1}{\underline{s}} \frac{\hat{\pi}_{a'}}{p_a} + \left(1 - \frac{1}{\underline{s}}\right) \frac{\hat{\pi}_a}{p_a} \\ C_b &= \frac{\hat{\pi}_b}{p_b}. \end{aligned} \tag{E.5}$$

We have three exogenous variables that are chosen to generate an initial fair-prices equilibrium, π_b, C_a, and C_b, six endogenous variables (μ, $\hat{\pi}_{a'}$, $\hat{\pi}_b$, $\hat{\pi}_a$, p_a, p_b), and nine equations.

E2. Totally differentiated equilibrium conditions

We totally differentiate the system (E.3), (E.4), and (E.5) allowing C_a and C_b to vary and imposing $dp_b = 0$:

$$\begin{aligned} \frac{\hat{\pi}_{a'} - \pi_{a'}}{\hat{\pi}_{a'}^2} d\hat{\pi}_{a'} &= d\mu + \frac{1}{p_a} dp_a \\ \frac{\hat{\pi}_a - \pi_a}{\hat{\pi}_a^2} d\hat{\pi}_a &= d\mu + \frac{1}{p_a} dp_a \text{ for } a \neq a' \\ \frac{\hat{\pi}_b - \pi_b}{\hat{\pi}_b^2} d\hat{\pi}_b &= d\mu + 0 \\ d\hat{\pi}_{a'} + (\underline{s} - 1)\, d\hat{\pi}_a + (S - \underline{s})\, d\hat{\pi}_b &= 0 \end{aligned} \tag{E.6}$$

$$\begin{aligned} p_a^2 dC_a &= \frac{1}{\underline{s}} (p_a d\hat{\pi}_{a'} - \hat{\pi}_{a'} dp_a) + \left(1 - \frac{1}{\underline{s}}\right) (p_a d\hat{\pi}_a - \hat{\pi}_a dp_a) \\ dC_b &= \frac{d\hat{\pi}_b}{p_b} - 0 \end{aligned} \tag{E.7}$$

Now we want to study dp_a, dC_b, and dC_b around an equilibrium with fair prices, i.e., with $\frac{p_a}{\pi_a} = \frac{p_b}{\pi_b} = 1$.

E3. Signing dp_a, dC_a and dC_b around a fair price equilibrium

Set $dC_a = 1$ (so that implicitly dx is dx/dC_a), and re-write the equations using $\bar{c} = \frac{\hat{\pi}_{a'}}{\pi_{a'}}$ and $C_b = \frac{\hat{\pi}_b}{p_b}$ and replace prices with probabilities:

$$\begin{aligned}
\frac{\bar{c}-1}{\bar{c}^2}\frac{1}{\pi_a}d\hat{\pi}_{a'} &= d\mu + \frac{1}{\pi_a}dp_a \\
\frac{C_b-1}{C_b^2}\frac{1}{\pi_a}d\hat{\pi}_a &= d\mu + \frac{1}{\pi_a}dp_a \text{ for } a \neq a' \\
\frac{C_b-1}{C_b^2}\frac{1}{\pi_b}d\hat{\pi}_b &= d\mu \\
d\hat{\pi}_{a'} + (\underline{s}-1)\,d\hat{\pi}_a + (S-\underline{s})\,d\hat{\pi}_b &= 0 \\
\frac{1}{\underline{s}}\left(d\hat{\pi}_{a'} - \bar{c}dp_a\right) + \left(1-\frac{1}{\underline{s}}\right)\left(d\hat{\pi}_a - C_b dp_a\right) &= \pi_a \\
p_b dC_b &= d\hat{\pi}_b
\end{aligned}$$

Now using the first three equations (the first-order conditions) to eliminate beliefs in the third-to-last and the second-to-last equations gives:

$$\begin{aligned}
\left(\pi_a\alpha + \pi_b\beta\right)d\mu + \alpha dp_a &= 0 \\
\alpha\pi_a d\mu + \left(\frac{\bar{c}}{\bar{c}-1} + (\underline{s}-1)\frac{C_b}{C_b-1}\right)dp_a &= \underline{s}\pi_a
\end{aligned} \tag{E.8}$$

where

$$\begin{aligned}
\alpha &= \frac{\bar{c}^2}{\bar{c}-1} + (\underline{s}-1)\frac{C_b^2}{C_b-1} \\
\beta &= (S-\underline{s})\frac{C_b^2}{C_b-1}
\end{aligned}$$

Note that $\beta < 0$ since $C_b - 1 = \hat{\pi}_b/\pi_b - 1$. In the current notation, the inequality of Lemma 6 is

$$\pi_a\left(\frac{\bar{c}^2}{\bar{c}-1}\right) + (\underline{s}-1)\,\pi_b\left(\frac{\underline{c}^2}{\underline{c}-1}\right) + (S-\underline{s})\,\pi_b\left(\frac{\underline{c}^2}{\underline{c}-1}\right) = \pi_a\alpha + \pi_b\beta > 0$$

Together these two inequalities imply that $\alpha > 0$. Thus $d\mu$ and dp_a have opposite signs. Combining equations (E.8) gives

$$dp_a = \frac{\underline{s}\pi_a}{\frac{\bar{c}}{\bar{c}-1} + (\underline{s}-1)\frac{C_b}{C_b-1} - \frac{\alpha^2\pi_a}{\pi_a\alpha+\pi_b\beta}}$$

To show that $dp_a < 0$, we note that the numerator is positive and that

$$\frac{\bar{c}}{\bar{c}-1} + (\underline{s}-1)\frac{C_b}{C_b-1} = \alpha - \bar{c} - (\underline{s}-1)\, C_b$$

so that the sign of the denominator is given by the sign of:

$$\begin{aligned} & \left(\frac{\bar{c}}{\bar{c}-1} + (\underline{s}-1)\frac{C_b}{C_b-1}\right)(\pi_a\alpha + \pi_b\beta) - \alpha^2\pi_a \\ = \; & (\alpha - \bar{c} - (\underline{s}-1)\, C_b)(\pi_a\alpha + \pi_b\beta) - \alpha^2\pi_a \\ = \; & -\bar{c}\,(\pi_a\alpha + \pi_b\beta) - (\underline{s}-1)\, C_b\,(\pi_a\alpha + \pi_b\beta) + \pi_b\beta\alpha < 0 \end{aligned}$$

From equation (E.8), we have that $d\mu > 0$, and from equations (E.6), we have that $d\hat{\pi}_b < 0$, and so from equation (E.7), we have $dC_b < 0$.

Thus, reversing signs, for a small reduction in aggregate risk – a decrease in C_a and an increase in C_b – such that the expected returns on the less-skewed assets do not change, $dp_b = 0$, the expected returns on the more skewed assets rises, $dp_a < 0$.■

F. Lemmata on the effects of prices on demands

Lemma 7: **(Law of demand for fair prices)** For $\pi_s/p_s = 1$ for all s, $c_s^*(\hat{\boldsymbol{\pi}}^*(\mathbf{p}), \mathbf{p}) = c_s^*(\hat{\pi}_s^*(\mathbf{p}), p_s)$ is decreasing in p_s.

Proof of Lemma 7: Let t be the (only) state for which price increases. If $\hat{\pi} > \pi$ and switches to $\hat{\pi} < \pi$, then we have our result. Otherwise, using Equation (2), the change in portfolio for a small change in p_t is

$$\frac{dc_t}{dp_t} = \frac{1}{p_t}\frac{d\hat{\pi}_t}{dp_t} - \frac{\hat{\pi}_t}{(p_t)^2}. \tag{F.1}$$

We thus want to show that

$$\frac{d\hat{\pi}_t}{dp_t} < \frac{\hat{\pi}_t}{p_t}.$$

Totally differentiating each first-order condition (Equations (4)), gives

$$\frac{d\hat{\pi}_t}{dp_t} = \left(\frac{\hat{\pi}_t^2}{\hat{\pi}_t - \pi_t}\right)\left(\frac{d\mu}{dp_t} + \frac{1}{p_t}\right) \tag{F.2}$$

$$\frac{d\hat{\pi}_s}{dp_t} = \left(\frac{\hat{\pi}_s^2}{\hat{\pi}_s - \pi_s}\right)\frac{d\mu}{dp_t} \text{ for all } s \neq t. \tag{F.3}$$

Summing across all states and imposing that $\sum_{s=1}^{S} d\pi_s/dp_t = 0$ gives

$$\frac{d\mu}{dp_t} = -\frac{\left(\frac{\hat{\pi}_t^2}{\hat{\pi}_t - \pi_t}\right)}{\sum_{s=1}^{S}\left(\frac{\hat{\pi}_s^2}{\hat{\pi}_s - \pi_s}\right)}\frac{1}{p_t}$$

which can be plugged into equation (F.2) to give

$$\frac{d\hat{\pi}_t}{dp_t} = \frac{\hat{\pi}_t}{p_t}\left(\frac{\hat{\pi}_t}{\hat{\pi}_t - \pi_t}\right)\left(1 - \frac{\left(\frac{\hat{\pi}_t^2}{\hat{\pi}_t - \pi_t}\right)}{\sum_{s=1}^{S}\left(\frac{\hat{\pi}_s^2}{\hat{\pi}_s - \pi_s}\right)}\right).$$

Thus, we have our result iff

$$1 > \left(\frac{\hat{\pi}_t}{\hat{\pi}_t - \pi_t}\right)\left(\frac{\sum_{s\neq t}\left(\frac{\hat{\pi}_s^2}{\hat{\pi}_s - \pi_s}\right)}{\sum_{s=1}^{S}\left(\frac{\hat{\pi}_s^2}{\hat{\pi}_s - \pi_s}\right)}\right).$$

If $\hat{\pi}_t > \pi_t$, then $\sum_{s\neq t}\left(\frac{\hat{\pi}_s^2}{\hat{\pi}_s - \pi_s}\right) < 0$, and our result follows if

$$\sum_{s=1}^{S}\left(\frac{\hat{\pi}_s}{1 - \frac{\pi_s}{\hat{\pi}_s}}\right) > 0$$

which is true by Lemma 6.

If $\hat{\pi}_t < \pi_t$, then our result also follows if this inequality is satisfied because that implies $\sum_{s\neq t}\left(\frac{\hat{\pi}_s^2}{\hat{\pi}_s-\pi_s}\right) > 0.$■

Lemma 8: **(Cross-price effects for fair prices)** For $\pi_s/p_s = 1$ for all s and t such that $t \neq s$, $c_s^*\left(\hat{\pi}_s^*\left(\mathbf{p}\right), p_s\right)$ is increasing in p_t if $\hat{\pi}_s^* > \pi_s$ or $\hat{\pi}_t^* > \pi_t$, otherwise it is decreasing in p_t as long as it remains true that $\hat{\pi}_s^* < \pi_s$.

Proof of Lemma 8: If the investor switches to become optimistic about state s as we increase the price of the state previously viewed with optimism, then we have our result. Increasing the price of another state cannot cause the investor to switch their optimism from the state viewed with optimism.

So under the assumption that the investor remains optimistic about the same state, using Equation (2), for any $s \neq t$, the change in portfolio for a small change in p_t is

$$\frac{dc_s}{dp_t} = \frac{1}{p_s}\frac{d\hat{\pi}_s}{dp_t}.$$

We thus the sign of $\frac{dc_s}{dp_t}$ is the same as that of $\frac{d\hat{\pi}_s}{dp_t} > 0$.

Totally differentiating each first-order condition (equations (4)), gives

$$\begin{aligned} \frac{d\hat{\pi}_t}{dp_t} &= \left(\frac{\hat{\pi}_t^2}{\hat{\pi}_t-\pi_t}\right)\left(\frac{d\mu}{dp_t}+\frac{1}{p_t}\right) \\ \frac{d\hat{\pi}_s}{dp_t} &= \left(\frac{\hat{\pi}_s^2}{\hat{\pi}_s-\pi_s}\right)\frac{d\mu}{dp_t} \text{ for all } s \neq t. \end{aligned} \tag{F.4}$$

Summing across all states and imposing that $\sum_{s=1}^{S} d\pi_s/dp_t = 0$ gives

$$\frac{d\mu}{dp_t} = -\frac{\left(\frac{\hat{\pi}_t^2}{\hat{\pi}_t-\pi_t}\right)}{\sum_{s=1}^{S}\left(\frac{\hat{\pi}_s^2}{\hat{\pi}_s-\pi_s}\right)}\frac{1}{p_t}$$

which can be plugged into equation (F.4) to give

$$\frac{d\hat{\pi}_s}{dp_t} = -\frac{\left(\frac{\hat{\pi}_s^2}{\hat{\pi}_s-\pi_s}\right)\left(\frac{\hat{\pi}_t^2}{\hat{\pi}_t-\pi_t}\right)}{\sum_{s=1}^{S}\left(\frac{\hat{\pi}_s^2}{\hat{\pi}_s-\pi_s}\right)}\frac{1}{p_t}.$$

From Lemma 6, we have that $\sum_{s=1}^{S}\left(\frac{\hat{\pi}_s^2}{\hat{\pi}_s-\pi_s}\right) > 0$, thus $\frac{d\hat{\pi}_s}{dp_t} > 0$ if $\hat{\pi}_s > \pi_s$ or $\hat{\pi}_t > \pi_t$ otherwise it is negative.■