Short-Run Analysis of Income Fluctuation and Optimum Foreign Borrowing in the Bardhan-Pitchford Model

By
Heng-fu Zou
Public Economics Division
The World Bank
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ABSTRACT

This paper utilizes the technique developed by Kenneth Judd to quantify the short-run effects of temporal income falls on the current account. It is found that (a) for both a fixed time discount rate and an endogenous discount rate, a future, temporal fall in income improves the initial current account; (b) with a current income fall and with no future income change, the initial current account deteriorates; (c) but when a current income fall is combined with a future income fall, the effect on the initial current account is ambiguous.

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I. Introduction

In a recent paper in this journal, Pitchford (1989) examines the effects of temporary income fluctuations on foreign borrowing and the current account for a Bardhan-Pitchford economy whose cost of foreign borrowing is an increasing function of its accumulated total borrowing; see also Bardhan (1967) and Pitchford (1970). Pitchford presents his results for two basic cases: the case of the endogenous time discount rate *a la* Uzawa (1968) and Epstein and Hynes (1983), and the case of a fixed time discount rate. According to Pitchford's analysis, there exist two different types of possible response of the short-run consumption and current account to temporary income changes. "If the income fall is short term, in all cases there will be a smoothing response whereby consumption falls less than income so resulting in a worsening current account. If it is long, there may be an adjustment response whereby the current account improves" (p.346) as the short-run consumption adjusts downward far below its long-run equilibrium level. But this "adjustment response works for long-term temporary income changes when the discount rate is endogenous, but when the discount rate is fixed, all responses are of the smoothing types" (p.346).

In this paper, we will see that the Pitchford's findings regarding the effects of temporary income fall are basically misleading, and we will show that (a) for both a fixed time discount rate and an endogenous discount rate, a future, temporary (both long-term and short-term) fall in income will result in only adjustment response whereby the initial current account improves; (b) for both a fixed time discount rate and an endogenous discount rate, a present or current fall in income will worsen the initial current account dollar by dollar with no effect on the initial consumption; and (c) a combination of a current income shock with a future shock has ambiguous effect on the initial current account.

In proving our finding, we utilize the Judd approach developed in a series of paper by Kenneth Judd (1983, 1985 and 1987) to quantify the short-run effects of income fluctuations on the current account or foreign borrowing. The Judd approach not only offers definite, quantitative answers to many problems [see Judd's papers and Dixit

(1990)] in dynamic economic analysis, it also shows that the use of the phase diagrams in predicting the short-run responses of endogenous variables is sometimes misleading and inaccurate.

We present our results for the cases of a fixed time discount rate and an endogenous time discount rate in section II and section III, respectively. We conclude this study in section IV.

II. The Case of a Fixed Time Discount Rate

In this section, we briefly present the model set-up. The details are spelled out in Pitchford (1989). A representative agent in this economy maximizes a discounted utility stream subject to a dynamic budget constraint

$$Max\int_0^\infty u(c)e^{-\rho t}dt \tag{1}$$

subject to:

$$\dot{a} = R(a) + y - c, A(0) = 0. \tag{2}$$

where c is consumption, $u(c)$ is a twicely differentiable utility function, ρ is the fixed time discount rate, a is the foreign asset holding (net foreign borrowing) if a is positive (negative), $\dot{a}$ is the current account, $R(a)$ is interest income (debt service) if a is positive (negative), y is the other income which will be treated as a parameter subject to fluctuations in this model. For convenience, y will be referred to as income and $R(a)$ as net interest. In (2), $A(0)$ is the initial foreign asset holding, which is set to be zero here.

The specialty about this model is the Bardhan-Pitchford assumption about the net interest or debt service $R(a)$. It is assumed that:

$$R'(a) > 0, \text{ and } R''(a) < 0. \tag{3}$$

That is to say, the more a country borrows, the higher the marginal cost. For more explanations on this point, see Bardhan (1967), Pitchford (1970, 1989).

The present value Hamiltonian function for this optimization problem is:

$$H(c,a,\lambda) = u(c) + \lambda(y + R(a) - c)$$

here λ is the marginal utility (disutility) of one unit more of foreign asset holding (foreign borrowing).

The necessary conditions for maximization are:

$$\dot{c} = -u'(c)R'(a) - \rho)/u''(c), \qquad (4)$$

$$\dot{a} = R(a) + y - c, A(0) = 0, \qquad (2)$$

$$\lim_{t\to\infty} u'(c)ae^{-\rho t} = 0. \qquad (5)$$

The steady state conditions are:

$$R'(a^*) - \rho = 0 \qquad (6)$$

$$R(a^*) + y - c^* = 0 \qquad (7)$$

here a^* and c^* are the steady state values of foreign asset and consumption, respectively. Since the analysis of the steady state and the impacts of permanent shocks on the steady state is done in Pitchford (1989), we will focus on the impacts of temporary income shocks on the current account and consumption. The technique for this kind of short-run analysis has been developed by Kenneth Judd in quite a few papers. It has also been used in Dixit (1990) and Zou (1991). Here we follow Judd (1985, 1987).

Suppose that today, i.e., $t = 0$, the foreign asset holding (or foreign borrowing) and consumption are at the steady state level corresponding to the exogenous income y. The steady state values are given in equations (6) and (7). Now at time $t = 0$, let the exogenous variable y change as follows:

$$y'(t) = y + \varepsilon h_y(t). \qquad (8)$$

where ε is a parameter, $h_y(t)$ represents the intertemporal structure of income change. For example, let the income fall for an amount of $\varepsilon(\varepsilon > 0)$ at time $t \subset (T_1, T_2)$. Then $h_y(t) = -1$ for $t \subset (T_1, T_2)$ and $h_y(t) = 0$ otherwise. Thus $y' = y - \varepsilon$ for $t \subset (T_1, T_2)$ and $y' = y$ otherwise. It is assumed that $h_y(t)$ will be a constant for sufficient large t.

Substituting $y'(t)$ for y in equation (2):

$$\dot{c} = -u'(c)R'(a) - \rho)/u''(c), \qquad (4)$$

$$\dot{a} = R(a) + y + \varepsilon h_y(t) - c. \qquad (9)$$

The solution to the these two differential equations will depends on both time t and ε: $c(t, \varepsilon)$ and $a(t, \varepsilon)$. For $\varepsilon = 0$, the system will remain at the initial steady state. As ε changes from zero to nonzero value, namely, as income changes, the impact on consumption and the foreign asset holding can be shown from the partial differentiation of c and a with respect to ε. To see this, we first define:

$$\partial c(t,0)/\partial\varepsilon = c_\varepsilon(t), \partial a(t,0)/\partial\varepsilon = a_\varepsilon(t)$$

$$\partial[\partial c(t,0)/\partial t]/\partial\varepsilon = \dot{c}_\varepsilon(t), \partial[\partial a(t,0)/\partial t]/\partial\varepsilon = \dot{a}_\varepsilon(t).$$

Differentiating (4) and (9) with respect to ε and evaluating the derivatives at $\varepsilon = 0$, we have

$$\begin{bmatrix} \dot{c}_\varepsilon \\ \dot{a}_\varepsilon \end{bmatrix} = \begin{bmatrix} 0 & -u'(c^*)R''(a^*)/u''(c^*) \\ -1 & -R'(a^*) \end{bmatrix} \begin{bmatrix} c_\varepsilon \\ a_\varepsilon \end{bmatrix} + \begin{bmatrix} 0 \\ h_y(t) \end{bmatrix} \tag{10}$$

where the 2x2 matrix is the Jacobian evaluated at the initial steady state. Thus (10) is a system of differential equations with constant coefficients. It can be easily shown that the initial equilibrium is saddle-point stable under the Bardhan-Pitchford assumption (3) about the net interest: $R''(a) < 0$. From (10), the eigenvalues for the Jacobian matrix are given by:

$$\omega = 2^{-1}[R'(a^*) - \sqrt{(R'(a^*)^2 + 4u'(c^*)R''(a^*)/u''(c^*)}] \tag{11}$$

$$\mu = 2^{-1}[R'(a^*) + \sqrt{(R'(a^*)^2 + 4u'(c^*)R''(a^*)/u''(c^*)}] \tag{12}$$

here ω is the negative eigenvalue and μ is the positive eigenvalue. So the system is saddle-point stable.

As in Judd (1987), the Laplace transformation can be utilized to solve (10). The Laplace transformation of a function $f(t)$ ($t > 0$) is another function $F(s)$ defined for sufficient large s:

$$F(s) = \int_0^\infty f(t)e^{-st}dt.$$

Let $C_\varepsilon(s)$, $A_\varepsilon(s)$ and $H_y(s)$ be the Laplace transformations of $c_\varepsilon(t)$, $a_\varepsilon(t)$ and $h_y(t)$, respectively, and apply the Laplace transformation to (10) [the technical details are in the appendix of Judd (1987)]:

$$\begin{bmatrix} C_\varepsilon(s) \\ A_\varepsilon(s) \end{bmatrix} = \begin{bmatrix} s & u'(c^*)R''(a^*)/u''(c^*) \\ 1 & s - R'(a^*) \end{bmatrix}^{-1} \begin{bmatrix} c_\varepsilon(0) \\ H_y(s) \end{bmatrix} \tag{13}$$

Upon solving the inverse matrix, (13) can be written as

$$\begin{bmatrix} C_\varepsilon(s) \\ A_\varepsilon(s) \end{bmatrix} = (s-\mu)^{-1}(s-\omega)^{-1}\begin{bmatrix} s-R'(a^*) & -u'(c^*)R''(a^*)/u''(c^*) \\ -1 & s \end{bmatrix}\begin{bmatrix} c_\varepsilon(0) \\ H_y(s) \end{bmatrix} \tag{14}$$

In (14), $c_\varepsilon(0)$ represents the initial jump in consumption responding to any change in both the present and future income. This jump has been typically illustrated in the phase diagrams and it is necessary for the convergence of the endogenous variables along the perfect foresight path. To determine $c_\varepsilon(0)$ quantitatively, not only qualitatively as in the phase diagram, we will use the fact that the existence of a saddle-point equilibrium in our model implies a finite foreign asset holding or foreign borrowing for any ε. Therefore $A_\varepsilon(s)$ is finite for $s>0$. In particular, $A_\varepsilon(\mu)$ is finite for s equaling the positive eigenvalue of the dynamic system μ. However, when $s=\mu$, the inverse matrix in (14) is singular because the denominator is zero. Thus the only way to move this singularity is to set the numerator in (14) to zero [again see the appendix of Judd (1987) for technical details]:

$$-c_\varepsilon(0)+\mu H_y(\mu)=0.$$

or, $$c_\varepsilon(0)=\mu H_y(\mu). \tag{15}$$

Here it becomes very clear that any future fall in the income will reduce the initial consumption. In fact, the effect on the initial consumption of any future income fall will be the present value of future income discounted at the rate of the positive eigenvalue μ, also weighted by the positive eigenvalue μ. To give an example, let future income fall in the way illustrated by Pitchford (1989, p.356):

$$y'(t)=y_0 \text{ for } t=0,$$

$$y'(t)=y_1 \text{ for } t\subset(0,T), \text{and } y_1<y_0,$$

$$y'(t)=y_0 \text{ for } t\subset[T,\infty).$$

Then,

$$H_y(\mu)=\int_0^T (y_1-y_0)e^{-\mu t}dt=(y_1-y_0)(1-e^{-\mu T})/\mu<0, \text{for } y_1<y_0,$$

and the initial consumption is reduced by

$$c_\varepsilon(0)=\mu H_y(\mu)=(y_1-y_0)(1-e^{-\mu T})<0.$$

To see how the income shock affects the initial current account, we substitute (15) into (10) and evaluate (10) at time zero:

$$\dot{a}_{\varepsilon}(0) = -c_{\varepsilon}(0) + h_y(0),$$

or,

$$\dot{a}_{\varepsilon}(0) = -\mu H_y(\mu) + h_y(0). \tag{16}$$

Therefore, any future income fall, no matter how long it lasts, will improve the initial current account by $-\mu H_y(\mu)$. But any initial fall in income, i.e., income falls at $t = 0$, will worsen the current account dollar by dollar as it is represented by the term $h_y(0)$. To illustrate this, if income falls today by one, i.e., $h_y(0) = -1$, the initial current account deteriorates by one: $\dot{a}_{\varepsilon}(0) = -1$ if there is no change in the future income. If no income change today but the future income changes in the way specified in Pitchford (1989) as cited above, the initial current account will improve by

$$\dot{a}_{\varepsilon}(0) = -(y_1 - y_0)(1 - e^{-\mu T}) > 0\,.$$

To sum up, we have the following proposition:

Proposition 1: When the time discount rate is fixed, (a) any fall in the future income, no matter how long it lasts, reduces the initial consumption and improves the initial current account; (b) when income falls by one today, the initial consumption does not change and the initial current account balance worsens by one.

As a current income shock works in the opposite direction of a future income shock in influencing the initial current account, an income shock happening today and continuing to the future can improve or worsen the initial current account. For example, let income fall by one from $t = 0$ to $t =$ T, then the initial current account changes by

$$\dot{a}_{\varepsilon}(0) = -1 + 1 - e^{-\mu T} = -e^{-\mu T} < 0.$$

But if income falls by different amount today and in the future, then the effect on the initial current account is ambiguous. To see this, suppose income falls today by $\alpha(>0)$ and in the future by $\beta(>0)$:

$$\dot{a}_{\varepsilon}(0) = -\alpha + \beta(1 - e^{-\mu T})\,.$$

If α is larger than or equal to β, the initial current account goes to deficit. If β is much greater than α, the initial current account is likely to improve. Thus,

Preposition 2: For a fixed time discount rate, if income falls by an equal amount both toady and in the future, the initial current account deteriorates; if income falls less today than in the future, the initial current account may improve.

Comparing these two propositions to the Pitchford's finding for the case of a fixed time discount rate, we can see that the initial response of consumption and the current account is not limited to the smoothing type claimed by Pitchford whereby the current account worsens. In fact, the Pitchford's assertion holds only for a current income shock without any future shock or an equal amount of income shock happening both today and in the future. If the income shock happens only in the future or if the future shock is much greater than today, there will be adjustment response whereby the initial current account improves.

III. The Case of Endogenous Time Discount Rate

Pitchford (1989) has mainly focused on the effects of temporary income shock for the case of an endogenous time discount rate. The model used by Pitchford is taken from Epstein and Hynes (1983) and the details are presented in the appendix of Pitchford's paper. To save space, we only write the objective function and the first-order conditions here. Throughout the remainder of this paper, all shocks are assumed to be temporary and they will have no effect on the long-run values of consumption and the foreign asset holding.

The representative agent now tries to maximize

$$\max -\int_0^\infty e^{-q(t)}dt \ \text{ and } \ q(t)=\int_0^t u(c(\tau))d\tau. \qquad (17)$$

subject to the same dynamic constraint (2):

$$\dot{a}=R(a)+y-c, A(0)=0. \qquad (2)$$

The first order conditions are:

$$\dot{c}=\{u(c)+u'(R(a)+y-c)-R'(a)\}u'(c)/u''(c) \qquad (18)$$

and (2).

Introduce the same income shock as in (8) into equations (18) and (2):

$$\dot{c} = \{u(c) + u'(R(a) + y + \varepsilon h_y(t) - c) - R'(a)\}u'(c)/u''(c) \qquad \text{(19a)}$$

$$\dot{a} = R(a) + y + \varepsilon h_y(t) - c \qquad \text{(19b)}$$

Again, the solution to (19) will depend on both time t and the parameter $\varepsilon : c(t,\varepsilon)$ and $a(t,\varepsilon)$. Differentiate (19) with respect to ε and evaluate the derivatives at the initial steady state, i.e., $\varepsilon = 0$:

$$\begin{bmatrix} \dot{c}_\varepsilon \\ \dot{a}_\varepsilon \end{bmatrix} = \begin{bmatrix} (u'-u'')u'/u'' & (u''R'-R'')u'/u'' \\ -1 & R' \end{bmatrix} \begin{bmatrix} c_\varepsilon \\ a_\varepsilon \end{bmatrix} + \begin{bmatrix} u'h_y(t) \\ h_y(t) \end{bmatrix} \qquad (20)$$

where all derivatives are evaluated at the initial steady state. The system is saddle-point stable because the determinant of the Jacobian matrix in (20) is equal to $[u'R'-R'']u'/u''$, which is negative. As the determinant is the product of the two eigenvalues, a negative determinant means that one of the two eigenvalues is positive and one negative. Thus the equilibrium is saddle-point stable.

Denote the positive eigenvalue as μ' and the negative eigenvalue as ω', and take the Laplace transformation in (20):

$$\begin{bmatrix} C_\varepsilon(s) \\ A_\varepsilon(s) \end{bmatrix} = (s-\mu)^{-1}(s-\omega')^{-1} \begin{bmatrix} s-R' & (u''R'-R'')u'/u'' \\ -1 & s-(u'-u'')u'/u'' \end{bmatrix} \begin{bmatrix} c_\varepsilon(0) + u'H_y(s) \\ H_y(s) \end{bmatrix} \qquad (21)$$

Again the existence of a saddle-point equilibrium for the dynamic system (20) implies a bounded foreign asset holding or foreign borrowing for any ε. Thus the Laplace transformation $A_\varepsilon(s)$ is finite for any $s(>0)$. When $s = \mu'$ (the positive eigenvalue), a finite $A_\varepsilon(\mu')$ requires that:

$$-[c_\varepsilon(0) + u'H_y(\mu')] + H_y(u')[\mu'-(u'-u'')u'/u''] = 0,$$

or,

$$c_\varepsilon(0) = \mu' H_y(\mu') - u'^2 H_y(\mu')/u''. \qquad (22)$$

From (22), the effect of future income change on the initial consumption is very similar to the expression (15) with one extra, positive term in (22). If future income temporarily falls, no matter how long it lasts, the initial consumption will fall.

To see the impact on the initial current account, we substitute (22) into (20) and set $t = 0$:

$$\dot{a}_{\varepsilon}(0) = -c_{\varepsilon}(0) + h_{y}(0).$$

$$= -[\mu' H_{y}(\mu') - u'^{2} H_{y}(\mu')/u''] + h_{y}(0). \qquad (23)$$

Therefore, any future income fall improves the initial current account and any current income fall worsens the initial current account dollar by dollar. Hence, we obtain:

Proposition 3: When the time discount is endogenous, any fall in the future income, no matter how long it lasts, reduces the initial consumption and improves the initial current account. But when income falls by one today, the initial consumption does not change and the initial current account balance worsens by one.

Proposition 3 stands in sharp contrast to Table 1 in Pitchford's paper (p.355). According to table 1, when the time discount rate is endogenous, a short-term temporary fall in income will worsen the initial current account while a long-term temporary fall in income will improve the initial current account. As shown in proposition 3 above, the claim made by Pitchford is not true. Since Pitchford has used the previously cited example to derive this claim, we had better reproduce the example here. Let income change as follows:

$$y'(t) = y_0 \text{ for } t = 0,$$

$$y'(t) = y_1 \text{ for } t \subset (0, T), \text{and } y_1 < y_0,$$

$$y'(t) = y_0 \text{ for } t \subset [T, \infty).$$

With this time profile of income, the initial current account responds to a temporary future income fall by:

$$\dot{a}_{\varepsilon}(0) = -[\mu' - u'^{2}/u''](y_1 - y_0)(1 - e^{-\mu T})/\mu' > 0.$$

which is always positive no matter the temporary fall is long-term or short-term.

Furthermore, unlike the case of a fixed time discount rate, equation (23) shows:

Proposition 4: For an endogenous time discount rate, an equal amount fall in both today's income and the future income has ambiguous impact on the initial current account.

This is true because, comparing to proposition 2, equation (23) has one extra term, $u'^2 H_y(\mu')/u''$, whose size cannot be determined without specific forms of the utility function and the steady state values of consumption and the foreign asset holding. Therefore, if an income fall starts today and lasts for a long time in the future, the initial current account may not improve as claimed by Pitchford.

IV. Conclusions

We conclude our study by summing up our main findings here. For both a fixed time discount rate and an endogenous discount rate, a future, temporal fall in income, no matter how long it lasts, reduces the initial consumption and improves the initial current account. But with a current income fall and with no future income change, the initial current account deteriorates in both cases of time discount rate. When a current income fall is combined with a future income fall, the differentiation between a fixed time discount rate and the endogenous time discount rate becomes very important. With the fixed time discount rate, the initial current account always deteriorates if the current income falls more than or as the same as the future income, and the initial current account is likely to improve if the current income falls much less than the future income. With an endogenous discount rate, the effect on the initial current account is always ambiguous.

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