Modified Ridge Parameters for Seemingly Unrelated Regression Model

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Abstract

In this paper, we modify a number of new biased estimators of seemingly unrelated regression (SUR) parameters which are developed by Alkhamisi and Shukur (2008), AS, when the explanatory variables are affected by multicollinearity. Nine ridge parameters have been modified and compared in terms of the trace mean squared error (TMSE) and (PR) criterion. The results from this extended study are the also compared with those founded by AS. A simulation study has been conducted to compare the performance of the modified ridge parameters. The results showed that under certain conditions the performance of the multivariate ridge regression estimators based on SUR ridge R_{MSmax} is superior to other estimators in terms of TMSE and PR criterion. In large samples and when the collinearity between the explanatory variables is not high the unbiased SUR, estimator produces a smaller TMSEs.

Key words: Multicollinearity; modified SUR ridge regression; Monte Carlo simulations; TMSE

Mathematics Subject classification 62J07.

1. Introduction

The main purpose of this paper is to propose some new ridge regression parameters applied to systems of regression equations, in particular the seemingly unrelated regressions (SUR) model proposed by Zellner (1962). The SUR method has shown to be superior (in term of most efficient) to the more traditional Ordinary Least Squares (OLS) method when the error terms between the equations in the system are highly correlated. In such cases the OLS will not produce Best Linear Unbiased Estimates (BLUE) while the SUR will do. This methodology has applications in many areas, e.g., panel data analysis, allocation models, consumption or demand functions for a number of commodities, investment functions for a number of firms, income distributions between different generations and different countries, consumption functions for subsets of populations or different regions.

Most of the time the exploratory variables for models that are studied in the applications mentioned above are highly correlated. This means there is a linear relationship between some of the exploratory variables. The separate effects of these variables may be confounded. As a result the estimated parameters may not be statistically significant and/or have different signs than expected. This would render misleading statistical inferences. Multicollinearity is a problem that arises from the data itself rather than the model being used for the analysis. A unique solution to the multicollinearity problem does not exist. There are many possible solutions; the most popular one is ridge regession. The study of ridge regression was pioneered by Hoerl and Kennard (1970). Later work may be found in Vinod (1978), Brown and Zidek (1980), Haitovsky (1987), Saleh and Kibria (1993), and Kibria (2003). Simulation studies of the properties of some newly proposed ridge type estimators and the comparison of their mean square errors with popular existing estimators were later done by Khalaf and Shukur (2005), Alkhamisi et. al. (2006), and Muniz and Kibria (2009). All of the results in these studies were for ridge estimators in a single model.

In general, ridge regression estimation is quite uncommon in systems of equations. This may partly be due to the lack of availability of standard methodology. A few exceptions might be found, however, see Srivastava and Giles (1987), Firinguetti, 1997, and Alkhamisi and Shukur (2008), AS hereafter. In AS the authors developed ridge parameters for SUR models and discussed more thoroughly the problems associated with system-wise ridge estimation using different multivariate ridge parameters. As a whole, 9 different parameters were developed and compared in term of Trace MSEs (TMSE). The investigation was done using Monte Carlo simulations for models with sample sizes equal to 30 and 100 observations and systems with 3 and 10 equations. The main results found were that 3 parameters, namely the R_{Sarith}, $R_{Sqarith}$ and R_{Smax} have shown to be superior to other estimators in terms of TMSE and (PR) criterion. The PR is the proportion of replications (out of 1,000) for which the SUR version of the generalised least squares, (SGLS) estimator has a smaller TMSE than the others. The authors also found that the SUR ridge estimators based on R_{SK}, R_{SHK} and R_{Sharm} performed extremely poorly when compared to the other estimators (for formal definition of these parameters we refer to the next section).

The aim of this paper is to modify the SUR ridge estimators mentioned in AS by applying a transformation on these parameters by raising them to a specific power factor given in page 6. We partly produce results according to the same Monte Carlo design as in AS in order to show the merits of our new modified parameters. Then we extend this design to cover a wide range of sample sizes, i.e. 10, 20, 30, 50 and 100 observations. We also extend the dimension of the systems to include 5 and 7 equations. Proceeding in this manner we can get better insight into the performance of these estimators.

The rest of the paper is organized as follows: In Section 2 we present the model, and define our modified SUR ridge regression parameters. Section 3, describes the Monte Carlo experiment together with the factors that can affect the properties of the proposed parameters. In Section 4, we present the results concerning the various ridge parameters in terms of TMSE and PR criterion. Some concluding remarks are presented in Section 5.

2. Methodology

In our analysis and design methodology we use the same model as in AS. Suppose we have a system of M equations, as follows.

$$\mathbf{Y}_i = \mathbf{X}_i\mathbf{B}_i + \mathbf{e}_i , \qquad i = 1,2,\ldots,M , \qquad (1)$$

where $\mathbf{Y}_i$ is a $T \times 1$ vector of observations on the dependent variable, $\mathbf{e}_i$ is a $T \times 1$ vector of random errors with $E(\mathbf{e}_i) = \mathbf{0}$ and $E(\mathbf{e}_i \mathbf{e}_i') = \sigma_i^2 \mathbf{I}_T$ (homoscedastic and non-autocorrelated), $\mathbf{X}_i$ is a $T \times k_i$ matrix of observations on explanatory variables including the intercept, $\mathbf{B}_i$ is a $k_i \times 1$ vector of unknown parameters, M is the number of equations in the system, T is the number of observations per equation and k_i is the number of rows of $\mathbf{B}_i$.

The M equations in (1) can be rewritten compactly as

$$\mathbf{Y} = \mathbf{XB} + \mathbf{e}, \tag{2}$$

where $\mathbf{Y} = (\mathbf{Y}_1', \mathbf{Y}_2', \ldots, \mathbf{Y}_M')'$ and $\mathbf{e} = (\mathbf{e}_1', \mathbf{e}_2', \ldots, \mathbf{e}_M')'$ are both of dimension $TM \times 1$, $E(\mathbf{e}_i \mathbf{e}_j') = \sigma_{ij} \mathbf{I}_T$, $\mathbf{X} = \text{diag}(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_M)$ of dimension $TM \times k$ and $\mathbf{B} = (\mathbf{B}_1', \mathbf{B}_2', \ldots, \mathbf{B}_M')'$ of dimension $k \times 1$, for $k = \sum_{i=1}^{M} k_i$.

The OLS estimator of $\mathbf{B}$ in (2) is

$$\hat{\mathbf{B}} = (\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}'\mathbf{Y}, \text{ with}$$

$$\text{cov}(\hat{\mathbf{B}}) = (\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}'(\mathbf{\Sigma} \otimes \mathbf{I}_T) \mathbf{X} (\mathbf{X}'\mathbf{X})^{-1} \tag{3}$$

where $\mathbf{\Sigma} = [\sigma_{ij}]$ is the matrix of constant contemporaneous variances and covariances of the errors both within and between equations, with $E(\mathbf{ee}') = \mathbf{\Sigma} \otimes \mathbf{I}_T$, since the temporal covariances both within and between equations are zero.

Srivastava and Giles (1987) defined the general ridge estimator of $\mathbf{B}$ in (2) as

$$\hat{\mathbf{B}}_{\text{OR}} = (\mathbf{X}'\mathbf{X} + \mathbf{R})^{-1} \mathbf{X}'\mathbf{Y} \tag{4}$$

where $\mathbf{R}$ is a $k \times k$ matrix of non-negative elements. The ridge estimator in (4) however abandons the information included in the correlation matrix of cross equation errors. The following transformation is more helpful to retain that information (see Srivastava and Giles 1987).

$$\mathbf{Y}^* = (\mathbf{\Sigma}^{-1/2} \otimes \mathbf{I}_T) \mathbf{Y}, \ \mathbf{X}^* = (\mathbf{\Sigma}^{-1/2} \otimes \mathbf{I}_T) \mathbf{X}, \text{ and } \mathbf{e}^* = (\mathbf{\Sigma}^{-1/2} \otimes \mathbf{I}_T) \mathbf{e}.$$

Using this transformation, the model (2) becomes

$$\mathbf{Y}^* = \mathbf{X}^* \mathbf{B} + \mathbf{e}^*. \tag{5}$$

The OLS estimator of $\mathbf{B}$ in (5), which is the GLS estimator of $\mathbf{B}$ in (2), and its ridge estimator as in (4) are respectively as follows,

$$\hat{\mathbf{B}}_{\mathrm{G}} = (\mathbf{X}^{*'}\mathbf{X}^{*})^{-1}\mathbf{X}^{*'}\mathbf{Y}^{*} = (\mathbf{X}'(\boldsymbol{\Sigma}^{-1}\otimes\mathbf{I}_T)\mathbf{X})^{-1}\mathbf{X}'(\boldsymbol{\Sigma}^{-1}\otimes\mathbf{I}_T)\mathbf{Y} \tag{6}$$

$$\hat{\mathbf{B}}_{\mathrm{GR}} = (\mathbf{X}^{*'}\mathbf{X}^{*}+\mathbf{R})^{-1}\mathbf{X}^{*'}\mathbf{Y}^{*} = (\mathbf{X}'(\boldsymbol{\Sigma}^{-1}\otimes\mathbf{I}_T)\mathbf{X}+\mathbf{R})^{-1}\mathbf{X}'(\boldsymbol{\Sigma}^{-1}\otimes\mathbf{I}_T)\mathbf{Y}\,. \tag{7}$$

Generally, $\sum$ is unknown and must be estimated from sampled data. In our simulations we use the most common approach to estimating $\sum$ by the unrestricted residuals obtained from the OLS method.

Let $\boldsymbol{\Lambda}$ be a diagonal matrix of eigenvalues and $\boldsymbol{\Psi}$ a matrix whose columns are eigenvectors of $\mathbf{X}^{*'}\mathbf{X}$. The canonical version of model (5) is

$$\mathbf{Y}^{*} = \mathbf{Z}\boldsymbol{\alpha} + \mathbf{e}^{*}, \tag{8}$$

where $\mathbf{Z} = \mathbf{X}^{*}\boldsymbol{\Psi}$, $\boldsymbol{\alpha} = \boldsymbol{\Psi}'\mathbf{B}$ and $\mathbf{Z}'\mathbf{Z} = (\boldsymbol{\Psi}'\mathbf{X}^{*'}\mathbf{X}^{*}\boldsymbol{\Psi}) = \boldsymbol{\Lambda}$.

The OLS estimator of $\boldsymbol{\alpha}$ in (8) is

$$\hat{\boldsymbol{\alpha}} = (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{Y}^{*} \tag{9}$$

with its associated SUR-ridge regression parameter estimator as,

$$\hat{\boldsymbol{\alpha}}_{\mathrm{SUR}} = (\mathbf{Z}'\mathbf{Z}+\mathbf{R})^{-1}\mathbf{Z}'\mathbf{Y}^{*}, \tag{10}$$

where $\mathbf{R} = \mathrm{diag}(R_1, R_2, \ldots, R_M)$, $R_i = \mathrm{diag}(r_{i1}, r_{i2}, \ldots, r_{ik_i})$ and $k_{ij} > 0$, for $i = 1, 2, \ldots, M$ and $j = 1, 2, \ldots, k_i$.

The bias vector, the mean squared error (MSE) matrix and the trace of the means squared error (TMSE) of $\hat{\boldsymbol{\alpha}}_{\mathrm{SUR}}$ are respectively follows as,

$$E(\hat{\boldsymbol{\alpha}}_{\mathrm{SUR}} - \boldsymbol{\alpha}) = -(\mathbf{Z}'\mathbf{Z}+\mathbf{R})^{-1}\mathbf{R}\boldsymbol{\alpha} \tag{11}$$

$$MSE(\hat{\boldsymbol{\alpha}}_{\mathrm{SUR}}) = (\boldsymbol{\Lambda}+\mathbf{R})^{-1}(\boldsymbol{\Lambda}+\mathbf{R}\boldsymbol{\alpha}\boldsymbol{\alpha}')(\boldsymbol{\Lambda}+\mathbf{R})^{-1} \tag{12}$$

$$TMSE(\hat{\boldsymbol{\alpha}}_{\mathrm{SUR}}(\mathbf{R})) = \sum_{i=1}^{M}\sum_{j=1}^{k_{\mathrm{i}}} \frac{\lambda_{ij} + r_{ij}^2\alpha_{ij}^2}{(\lambda_{ij}+r_{ij})^2} \tag{13}$$

Minimizing the TMSE in (13) with respect to $\boldsymbol{r_{ij}}$ gives us

$$r_{ij} = \frac{1}{\hat{\alpha}_{ij}^2} \quad (14)$$

Moreover, the following conditions ensure the superiority of $\hat{\boldsymbol{\alpha}}_{\text{SUR}}$ over $\hat{\boldsymbol{\alpha}}$ with respect to the MSE matrix.

Result 1.

As a special case of the Bayesian estimators of Gruber (1998) and Gruber (2010) we have the following results for the SUR-ridge regression.

a. MSE($\hat{\alpha}$) – MSE($\hat{\alpha}_{\text{SUR}}(\text{R})$) is a positive semidefinite matrix iff

$$\alpha'(\Lambda^{-1} + 2\text{R}^{-1})^{-1}\alpha \ \leq \ 1 \quad (15)$$

b. Sufficient conditions for (15) to hold are

(i) $\alpha'\Lambda\alpha \leq 1$ (ii) $\alpha'\text{R}\alpha \leq 2$. (16)

c. Set R = rI in (10), to show that MSE ($\hat{\alpha}$) – MSE($\hat{\alpha}_{\text{SUR}}(\text{R})$) is a positive semidefinite matrix if $\text{r} \leq \frac{2}{\alpha'\alpha}$.

Result 2.

For $i = 1, 2, \ldots, M$ and $j = 1, 2, \ldots, k_i$, assume eq. (14) holds, we can modify the SUR ridge parameters presented in AS by imposing our new transformation (i.e. by raising the parameters to the power of $1/\bar{k}$, for $\bar{k} = \sum_{i=1}^{M} k_i / M$) and get the following modified versions.

1. $\mathbf{R}_{\text{MSK}}$. Is a modified version of the *ij*-th component of this matrix given by (14), (see Srivastava and Giles, 1987 and Firinguetti, 1997).
2. $\mathbf{R}_{\text{MSHK}}$. Denotes the modified SUR version of Hoerl and Kennard (1970a) ordinary ridge parameter. $$\text{r}_{\text{ij(MSHK)}} = \left(\frac{1}{\max_{\text{ij}}(\hat{\alpha}_{\text{ij}}^2)}\right)^{1/\bar{k}} \quad (18)$$
3. R_{MSharm}. Designates the modified SUR version to the harmonic mean proposed by Hoerl, Kennard and Baldwin (1975)

$$r_{ij(MSham)} = \left(\frac{n}{\sum_{i=1}^{M}\sum_{j=1}^{n_i}\frac{1}{r_{ij}}}\right)^{1/\bar{k}} = \left(\frac{n}{\sum_{i=1}^{M}\sum_{j=1}^{n_i}\hat{\alpha}_{ij}^2}\right)^{1/\bar{k}} \tag{19}$$

4. $R_{MSarith}$. Is a modified SUR extension to the single equation arithmetic mean proposed by Kibria (2003).
$$r_{ij(MSarith)} = \left(\frac{1}{n}\sum_{i=1}^{M}\sum_{j=1}^{n_i}\frac{1}{\hat{\alpha}_{ij}^2}\right)^{1/\bar{k}} \tag{20}$$

5. R_{MSgeom}. Is a modified generalization to the single equation geometric mean proposed by Kibria (2003).
$$r_{ij(MSgeom)} = \left(\frac{1}{\left(\prod_{i=1}^{M}\prod_{j=1}^{n_i}\hat{\alpha}_{ij}^2\right)^{\frac{1}{n}}}\right)^{1/\bar{k}} \tag{21}$$

6. R_{MSkmed}. The modified median of r_{ij} in (14) is used to define this parameter, (see Kibria, 2003 for a single equation version).
$$r_{ij(MSmed)} = \left(\underset{ij}{\text{median}}\left(\frac{1}{\hat{\alpha}_{ij}^2}\right)\right)^{1/\bar{k}} \tag{22}$$

7. $R_{MSqarith}$. Is a modified version of a new proposed ridge parameter using the arithmetic mean of $\sqrt{r_{ij}}$, with r_{ij} as defined in (14).
$$r_{ij(MSqarith)} = \left(\underset{ij}{\text{mean}}\left(\frac{1}{\sqrt{\hat{\alpha}_{ij}^2}}\right)\right)^{1/\bar{k}} \tag{23}$$

8. R_{MSqmax}. Is a modified version of a new proposed ridge parameter based on the maximization of $\sqrt{r_{ij}}$, with r_{ij} as defined in (14).
$$r_{ij(MSqmax)} = \left(\max_{ij}\left(\frac{1}{\sqrt{\hat{\alpha}_{ij}^2}}\right)\right)^{1/\bar{k}} \tag{24}$$

9. R_{MSmax}. A modification of the generalization to the single equation ridge parameter K_{max}^{HK} proposed by Alkhamisi et. al. (2006).
$$r_{ij(MSmax)} = \left(\max_{ij}\left(\frac{1}{\hat{\alpha}_{ij}^2}\right)\right)^{1/\bar{k}} \tag{25}$$

Clearly all of the ridge estimators defined by eqs. (18) - (22) and eq. (25) are identical to $\mathbf{R}_{MSHK}$ when $\hat{\alpha}_{ij}^2$ is replaced by $\max(\hat{\alpha}_{ij}^2)$. The estimators in eqs. (18) - (19) have already been considered by Firinguetti (1997). In order to assess the performance of multivariate ridge regression estimators defined in terms of the above proposed multivariate ridge estimators we performed a Monte Carlo experiment to compare them in terms of TMSE with the GLS estimator, (see eq. 6) and the general ridge regression estimator defined by eq. (7) and eq. (14).

3. The Monte Carlo Experiment

A number of factors obviously can affect the properties of these parameters in terms of TMSE. The number of equations (*M*), the sample size (*T*), correlation among the explanatory variables and the dependency between equations are four such factors. For computational simplicity, we however hold other factors constant in our investigation, namely, the number of *X* variables, mean of *X* variables, covariance Matrix of *X* variables and the parameters of *X* variables. For more details about these factors, see Tables 1 and 2 below.

Table 1. Values of Factors that Vary for Different Models - Size Calculations

Factor	Symbol	Design			
Number of equations	M	3	5	7	10
Number of observations	T	10, 20, 30, 50, 100			
Correlation among the explanatory variables	ρ_X	0,75, 0.90, 0.97, 0.99			
Dependency between equations	ρ_Σ	0.35, 0.75			

Table 2. Values of Factors Held Constant that Do Not Affect the BG Tests

Factor	Symbol	Value
Constant term		1
Number of *X* variables	$\boldsymbol{k_i}$	4
Mean of *X* variables	$\boldsymbol{\mu}_x$	**0**
Covariance Matrix of *X* variables	$\boldsymbol{\Sigma}_x$	
Parameters of *X* variables	$\boldsymbol{B}$	$\boldsymbol{E}$

X represents the exogenous variables *excluding* the constant term and ***E*** represents the matrix consisting merely of ones.

The Monte Carlo experiment has been performed by generating data according to following algorithm:

a. Generate the explanatory variables from $MVN_4(0, \Sigma_x)$.
b. Set initial value of B either to $(1, 1, 1, 1, 1)'$.
c. Simulate the vector random error e from $MVN_M(0, \Sigma_e)$, M = 3, 5, 7, 10.
d. As outlined earlier, for a given X structure, transform the original model (2) to an orthogonal form given by eq. (8) and calculate the SGLS estimator along with $\hat{\alpha}_{SUR}(R)$, R= R_{MSK}, R_{MSHK}, $R_{MSharm,}$ $R_{MSarith}$, R_{MSgeom}, R_{MSkmed}, $R_{MSqarith}$, R_{MSqmax}. and R_{MSmax}.Then compute the corresponding total mean squared error for the above case respectively.
e. Repeat this process 1,000 times and then calculate the average of the mean squared error and the (PR) for each ridge parameter R, under consideration.

Since the main objective of this study is to evaluate the modified SUR-ridge parameters in a systemwise perspective, the number of equations to be estimated is of central importance. At this stage it is important to mention that as the size of the system increases, the performance of the feasible GLS is likely to deteriorates and loos efficiency, see Fiebig and Kim (2000). Moreover, as the number of equations grows the computation time becomes longer, and we took a system with 10 equations as our largest model when considering the properties of these parameters. This represents a fairly large consumption model of the type that is used in, for example, agricultural economics. Medium size models are represented by 5- and 7-equation systems, while 3-equation systems are typical of the small models.

Another prime factor that affects the performance of these parameters is the number of observations. We have investigated samples typical for small, medium and large sizes with number of observations equal to 10, 20, 30, 50 and 100. Note that in the case when the number of equations in the system is equal to 10, using a number observations equal to 10 will lead to a situation of undersized sample problem. This situation will be avoided in this paper.

Another factor that may affect the performance of the suggested SUR-ridge parameters is the strength and type of dependency among the explanatory variables. The explanatory variables were generated from a multivariate normal distribution, $MVN_4(0, \Sigma_x)$. The variance-covariance matrix Σ_x is defined as $diag(\Sigma_x) = 1$ and off-$diag(\Sigma_x) = \rho_x$. The strength of collinearity among these variables took on these values $\rho_x = 0.75$, 0.90, 0.97 and 0.99, (for moderate to high collinearity).

The random errors were generate from a multivariate normal distribution $MVN_M(0, \Sigma_e)$, where M = 3, 5, 7, or 10 equations. The variance covariance matrix Σ_e is defined as diag(Σ_e) = 1 and off-diag(Σ_e) = ρ_Σ. Two different degrees of interdependency among these equations were considered. These values are $\rho_\Sigma = 0.35 \text{ and } 0.75$, for low and high interdependency respectively.

A final consideration is the criterion to be used when judging the properties of these parameters. In this study we use the same criterion as in AS to compare the performance of the SUR-ridge type estimators of the unknown vector parameter B. The criterion proposed to measure the goodness of an estimator of B, say $\tilde{B}$, are the TMSE and the PR criterion. The total mean square error is defined as

$$\text{TMSE}(\tilde{B}) = \text{Trace}[\, E(\tilde{B} - B)(\tilde{B} - B)' \,].$$

The PR criterion counts the proportion of replications,(out of 1000), for which the SUR version of generalized least square estimator (SGLS) produces a smaller TMSE than the remaining multivariate ridge estimators. In Tables 1-8 these numbers are placed in parenthesis. The performance of the different SUR ridge estimators, under consideration, are examined via Monte Carlo simulations. The Monte Carlo experiment has been performed by generating data in accordance with the following equation

$$y_{ti} = \sum_{j=1}^{5} x_{tij}\beta_{ij} + e_{ti}, \quad t = 1, 2, \cdots, T; i = 1, 2, \cdots, M, \qquad (26)$$

where $x_{ti1} = 1$. The explanatory variables are generated from $MVN_4(0, \Sigma_x)$. The random errors were generated from $MVN_M(0, \Sigma_e)$, M = 3, 5, 7 and 10. For each model we have performed 1,000 replications using the statistical software S-plus version 6.0.

4. Simulation Results

In this section we present the results of our Monte Carlo experiment along with the main dominating factors affecting the properties of the different multivariate ridge parameters

SUR-ridge parameters. Since we are modifying the ridge parameters mentioned in AS, our main intention was to compare our results directly with those obtained by AS. However, when determining the manner of presentation, some account has to be taken to the results obtained. Our study is more extended than that of the AS and hence we will only compare a subset of our results that match those in AS (see Tables A1 and A2 in the appendix for 3 and 10 equations, respectively). Complete results from this study (see Tables 3–6) will be discussed thereafter. We as in AS did not find big differences in the results when ρ_Σ = 0.35 compare with when ρ_Σ = 0.75. We hence only present results for ρ_Σ = 0.35. We however present, in Tables 7 and 8, some results (using our parameters) that match those in AS when ρ_Σ = 0.75. Complete results for all combinations can be ordered from the authors upon request.

Now, when comparing our findings with those are in AS, we find that when M, ρ_X, and ρ_Σ increases the TMSE and PR increases, while when T increases the TMSE decreases and PR increases.

Moreover, the results in AS show a slight increase in the TMSE values for $\hat{\alpha}(R_{Sarith})$, $\hat{\alpha}(R_{Sqarith})$ and $\hat{\alpha}(R_{Smax})$ as the sample size increases. Theses multivariate ridge regression estimators have shown to have the best performance in terms of TMSE and PR criterion when compared with the remaining proposed multivariate ridge regression estimators. In our study, we find that almost the same pattern but that the $\hat{\alpha}(R_{MSmax})$ is superior to the other in terms of TMSE and PR. In large samples and low correlation between the explanatory variables the SGLS has shown to have somewhat smaller TMSE.

On the other hand, the multivariate ridge regression estimators based on R_{SK}, R_{SHK} and R_{Sharm} have produced the highest TMSE and the worst PR values among other estimators in AS. With M = 3 and T = 30, the TMSE for the R_{SK} and R_{SHK} could vary between 16 to 278 for different strength of correlations. When M = 10 and T = 30, the TMSE for the R_{SK} and R_{SHK} vary between 162 to 2715 (see Table A1 and A2 in the appendix). Now when using our modified parameters the TMSEs for the R_{MSK} and R_{MSHK} only vary between 12 -

26. In other words, the TSMEs of these parameters have been considerably modified compared with the R_{SK} and R_{SHK} in the AS. Moreover, the TMSEs for the R_{MSK} and R_{MSHK} shown to be very close to the other parameters in the study and accordingly they can in fact also be considered as useful parameters in empirical works. Unfortunately, for M = 10 and T = 10, the TMSEs are not computable and hence we do not comment this case.

The results show that the TMSEs of almost all of the different parameters are considerably smaller than those of the SGLS. With high degree of collinearity, small samples and large systems, the TMSE of the SGLS can be as high as 7404, while our parameters for the same situations produce TMSEs between 89 to 269. However, our main results is that the R_{MSmax} has shown to be superior, in terms of TMSE and PR criterion, when compared with the remaining proposed multivariate ridge regression estimators, especially when the sample size is small and the strength collinearity is high. In some cases, e.g. large sample and low collinearity, the SGLS produces slightly smaller TMSEs than our parameters. Occasionally, when the number of equations increases, we can see that the M_{Sarith} produces a somewhat smaller TMSE than the R_{MSmax}, but that the differences between them are extremely small.

5. Concluding Remarks

In this paper we modify a number of parameters that are developed in Alkhamisi and Shukur (2008), AS. The modified parameters are then compared with those in AS in terms of TMSEs and PR criteria. The investigation has been done by means of Monte Carlo simulations where 10 multivariate parameters are studied and compared. This investigation used the TMSE and the PR criterion to measure the goodness of SUR ridge-type estimators. The results have shown that our new modified ridge parameters produce smaller TMSEs that those mentioned in AS. Moreover, some parameters like the R_{MSK} and R_{MSHK} have shown to be almost useless in the AS, while our modified counterpart proved that they are still useful and produce TMSEs that almost lay near to our best parameters.

We as in the AS, find three main factors that affect the properties of the SUR ridge estimators, namely, the number of equations, the number of observations per equation and the correlation among explanatory variables. It is noticed that the unbiased estimator, SGLS, has occasionally (in large sample and low correlation among explanatory variables) shown to have the smallest

TMSE when compared with the others. However for high correlation, ρ_X and small samples the SUR ridge estimators based on R_{MSmax} performs better than the remaining estimators.

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TABLE 3. System-wise estimated TMSEs for the different methods, M =3 equations, $\rho_\Sigma = 0.35$.

T = 10

ρ_X	SGLS	MSK	MSHK	MSharm	MSarith	MSgeom	MSkmed	MSqarith	MSqmax	MSmax
0.75	66.42	21.78 (11.6)	23.09 (12.2)	20.37 (10.6)	16.27 (6.9)	18.17 (8.6)	18.40 (8.8)	16.52 (7.2)	17.59 (8.3)	15.44 (6.1)
0.90	169.44	27.17 (2.4)	30.07 (2.7)	25.86 (2.2)	20.08 (1.4)	22.45 (2.1)	22.67 (2.0)	20.20 (1.5)	21.47 (1.8)	19.04 (1.3)
0.97	568.76	33.55 (0.3)	39.51 (0.3)	33.20 (0.1)	24.38 (0)	27.66 (0)	27.86 (0.1)	24.33 (0)	25.84 (0)	23.14 (0)
0.99	1709.22	37.52 (0)	47.55 (0)	38.83 (0)	26.60 (0)	30.73 (0)	30.79 (0)	26.44 (0)	28.01 (0)	25.38 (0)

T = 20

ρ_X	SGLS	MSK	MSHK	MSharm	MSarith	MSgeom	MSkmed	MSqarith	MSqmax	MSmax
0.75	20.04	14.41 (31.4)	14.85 (32.9)	13.90 (30.1)	12.42 (26.4)	13.15 (28.1)	13.24 (28.3)	132.57 (26.8)	13.03 (27.5)	12.12 (25.5)
0.90	46.17	19.78 (16.9)	20.74 (18.2)	19.21 (16.3)	17.15 (14.3)	18.07 (15.0)	18.22 (15.4)	17.25 (14.4)	17.78 (14.9)	16.84 (14.1)
0.97	147.41	25.11 (2.9)	27.33 (3.5)	24.94 (2.8)	21.91 (2.5)	23.09 (2.6)	23.31 (2.6)	21.95 (2.5)	22.54 (2.6)	21.58 (2.5)
0.99	436.53	27.36 (0.5)	30.78 (0.5)	27.70 (0.4)	24.09 (0.4)	25.29 (0.4)	25.48 (0.4)	24.07 (0.4)	24.55 (0.4)	23.87 (0.4)

T = 30

ρ_X	SGLS	MSK	MSHK	MSharm	MSarith	MSgeom	MSkmed	MSqarith	MSqmax	MSmax
0.75	13.39	12.38 (44.1)	12.69 (46.1)	12.01 (43.0)	10.91 (37.1)	11.43 (39.3)	11.48 (39.7)	11.05 (37.6)	11.40 (39.3)	10.69 (36.9)
0.90	28.71	18.72 (33.0)	19.32 (34.4)	18.22 (32.1)	16.75 (28.6)	17.40 (30.7)	17.50 (30.7)	16.84 (29.1)	17.25 (30.2)	16.52 (29.7)
0.97	88.08	23.75 (7.8)	25.27 (8.6)	23.50 (7.5)	21.20 (6.3)	22.14 (7.0)	22.34 (7.1)	21.27 (6.4)	21.77 (6.7)	20.93 (6.1)
0.99	257.64	26.20 (.5)	28.71 (.5)	26.34 (.5)	23.56 (.4)	24.53 (.4)	24.74 (.5)	23.56 (.4)	23.99 (.4)	23.37 (.4)

T = 50

ρ_X	SGLS	MSK	MSHK	MSharm	MSarith	MSgeom	MSkmed	MSqarith	MSqmax	MSmax
0.75	9.08	10.30 (56.1)	10.53 (58.2)	10.13 (55.0)	9.42 (49.9)	9.76 (51.7)	9.79 (52.0)	9.53 (50.5)	9.76 (51.6)	9.30 (49.5)
0.90	17.20	17.06 (55.9)	17.4 (56.6)	16.68 (54.5)	15.57 (49.9)	16.07 (51.8)	16.15 (52.7)	17.69 (50.3)	16.00 (51.4)	15.43 (48.9)
0.97	48.63	22.42 (18.7)	23.43 (19.7)	22.16 (18.7)	20.40 (16.6)	21.17 (17.5)	21.36 (18.1)	20.51 (16.5)	20.92 (17.1)	20.20 (16.5)
0.99	138.39	24.58 (3.0)	26.43 (3.1)	24.62 (2.9)	22.37 (2.6)	23.21 (2.7)	23.45 (2.7)	22.42 (2.6)	22.81 (2.7)	22.22 (2.5)

T = 100

ρ_X	SGLS	MSK	MSHK	MSharm	MSarith	MSgeom	MSkmed	MSqarith	MSqmax	MSmax
0.75	6.29	8.66 (72.8)	8.82 (75.6)	8.61 (71.9)	8.19 (66.2)	8.38 (68.8)	8.39 (69.0)	8.27 (67.5)	8.40 (69.3)	8.14 (65.2)
0.90	9.94	16.31 (85.6)	16.55 (86.2)	16.05 (84.5)	15.23 (81.2)	15.60 (82.3)	15.63 (82.4)	15.35 (81.2)	15.59 (82.3)	15.13 (80.5)
0.97	24.07	21.66 (49.7)	22.22 (52.1)	21.35 (48.9)	19.97 (45.3)	20.62 (46.6)	20.77 (47.2)	20.13 (45.7)	20.49 (46.4)	19.77 (44.7)
0.99	64.41	23.45 (13.5)	24.62 (14.3)	23.32 (12.8)	21.42 (11.2)	22.24 (11.9)	22.51 (12.2)	21.54 (11.2)	21.95 (11.8)	21.25 (10.7)

TABLE 4. Estimated TMSEs and PRS for the different methods, M = 5 equations, ρ_Σ = **0.35**.

T = 10

ρ_X	SGLS	MSK	MSHK	MSharm	MSarith	MSgeom	MSkmed	MSqarith	MSqmax	MSmax
0.75	181.58	56.95 (1.1)	63.00 (1.2)	52.64 (0.8)	38.64 (0.2)	45.283 (0.4)	45.88 (0.5)	39.06 (0.3)	42.74 (0.4)	35.89 (0.1)
0.90	465.51	69.04 (0.1)	80.30 (0.1)	64.99 (0.1)	45.66 (0.1)	53.87 (0.1)	54.33 (0.1)	45.77 (0.1)	49.76 (0.1)	42.73 (0.1)
0.97	1565.88	81.51 (0)	101.98 (0)	80.13 (0)	53.34 (0)	63.33 (0)	63.56 (0)	53.11 (0)	57.18 (0)	50.34 (0)
0.99	4708.47	87.51 (0)	118.04 (0)	90.84 (0)	56.87 (0)	68.50 (0)	69.01 (0)	56.57 (0)	60.41 (0)	54.23 (0)

T = 20

ρ_X	SGLS	MSK	MSHK	MSharm	MSarith	MSgeom	MSkmed	MSqarith	MSqmax	MSmax
0.75	59.50	39.24 (15.4)	41.16 (1775)	37.61 (14.1)	33.26 (10.1)	35.34 (11.5)	35.64 (11.9)	33.50 (10.1)	34.78 (11.3)	32.63 (9.9)
0.90	138.26	47.54 (1.8)	50.80 (1.8)	45.96 (1.7)	40.56 (1.3)	42.80 (1.6)	43.13 (1.6)	40.70 (1.3)	41.92 (1.4)	39.99 (1.3)
0.97	443.46	57.39 (0)	63.13 (0)	57.01 (0)	50.20 (0)	52.70 (0)	53.02 (0)	50.24 (0)	51.33 (0)	49.76 (0)
0.99	1315.05	59.35 (0)	67.46 (0)	60.47 (0)	52.84 (0)	55.26 (0)	55.48 (0)	52.82 (0)	53.65 (0)	52.61 (0)

T = 30

ρ_X	SGLS	MSK	MSHK	MSharm	MSarith	MSgeom	MSkmed	MSqarith	MSqmax	MSmax
0.75	38.45	36.23 (42.2)	37.28 (45.5)	35.04 (39.5)	32.17 (31.6)	33.51 (35.0)	33.71 (35.4)	32.35 (32.4)	33.26 (32.4)	31.86 (30.8)
0.90	82.39	45.77 (8.9)	47.78 (9.8)	44.37 (8.2)	40.74 (7.6)	42.21 (7.9)	42.46 (8.0)	40.84 (7.5)	41.70 (7.8)	40.51 (7.5)
0.97	252.67	54.61 (0.4)	58.27 (0.4)	54.03 (0.4)	49.49 (0.3)	51.15 (0.4)	51.47 (0.4)	49.53 (0.3)	50.31 (0.3)	49.31 (0.3)
0.99	738.94	55.99 (0)	61.09 (0)	56.42 (0)	51.51 (0)	53.04 (0)	53.30 (0)	51.48 (0)	52.03 (0)	51.50 (0)

T = 50

ρ_X	SGLS	MSK	MSHK	MSharm	MSarith	MSgeom	MSkmed	MSqarith	MSqmax	MSmax
0.75	26.17	34.25 (83.6)	34.94 (86.6)	33.51 (81.0)	31.50 (73.0)	32.44 (76.1)	32.58 (77.0)	31.68 (73.5)	32.35 (75.8)	31.34 (72.3)
0.90	49.09	44.22 (40.2)	45.50 (43.1)	43.09 (38.2)	40.36 (33.3)	41.49 (35.5)	41.74 (36.0)	40.48 (33.8)	41.17 (34.6)	40.27 (32.8)
0.97	137.89	52.88 (2.5)	55.32 (2.7)	52.19 (2.3)	48.71 (2.2)	50.05 (2.2)	50.38 (2.2)	48.79 (2.2)	49.47 (2.2)	48.60 (2.0)
0.99	391.48	54.15 (0)	57.71 (0)	54.24 (0)	50.44 (0)	51.71 (0)	52.03 (0)	50.42 (0)	50.95 (0)	50.43 (0)

T = 100

ρ_X	SGLS	MSK	MSHK	MSharm	MSarith	MSgeom	MSkmed	MSqarith	MSqmax	MSmax
0.75	18.58	33.41 (99.93)	33.88 (99.4)	33.05 (99.0)	31.67 (97.5)	32.33 (98.0)	32.40 (98.2)	31.84 (97.7)	32.34 (98.0)	31.60 (97.4)
0.90	28.98	44.19 (94.9)	44.95 (95.8)	43.34 (93.4)	41.19 (88.2)	42.11 (90.7)	42.31 (90.9)	41.35 (88.9)	41.98 (90.2)	41.11 (87.7)
0.97	69.42	52.29 (25.2)	53.82 (27.0)	51.45 (24.5)	48.64 (21.5)	49.81 (22.8)	50.15 (23.5)	48.79 (21.6)	49.43 (22.6)	48.54 (21.3)
0.99	184.90	52.93 (0.9)	55.32 (0.9)	52.64 (0.9)	49.52 (0.7)	50.69 (0.7)	51.03 (0.7)	49.60 (0.7)	50.13 (0.7)	49.51 (0.7)

TABLE 5. Estimated TMSEs and PRS for the different methods, M = 7 equations, $\rho_\Sigma = 0.35$.

T = 10

ρ_X	SGLS	MSK	MSHK	MSharm	MSarith	MSgeom	MSkmed	MSqarith	MSqmax	MSmax
0.75	286.99	100.99 (0.6)	113.85 (0.9)	93.88 (0.3)	66.47 (0)	80.26 (0.1)	81.42 (0.1)	67.07 (0)	74.78 (0)	60.59 (0)
0.90	733.50	127.22 (0)	151.80 (0)	119.40 (0)	77.83 ((0)	96.56 (0)	97.64 (0)	77.87 (0)	87.07 (0)	70.91 (0)
0.97	2463.48	160.54 (0)	207.34 (0)	155.68 (0)	90.51 (0)	117.13 (0)	118.20 (0)	90.01 (0)	100.77 (0)	82.73 (0)
0.99	7404.02	184.97 (0)	260.99 (0)	187.48 (0)	96.82 (0)	130.12 (0)	132.60 (0)	96.16 (0)	107.16 (0)	89.12 (0)

T = 20

ρ_X	SGLS	MSK	MSHK	MSharm	MSarith	MSgeom	MSkmed	MSqarith	MSqmax	MSmax
0.75	99.65	66.39 (10.8)	70.39 (13.2)	63.22 (8.9)	54.52 (6.2)	58.72 (7.0)	59.27 (7.2)	54.94 (6.2)	57.49 (7.0)	53.32 (5.7)
0.90	232.36	79.19 (0.2)	86.14 (0.3)	75.89 (0.1)	65.21 (0.1)	69.61 (0.1)	70.22 (0.1)	65.41 (0.1)	67.71 (0.1)	64.26 (0.1)
0.97	746.55	92.80 (0)	103.81 (0)	91.38 (0)	78.96 (0)	83.40 (0)	83.96 (0)	79.04 (0)	80.86 (0)	78.35 (0)
0.99	2214.99	94.34 (0)	108.24 (0)	95.45 (0)	82.51 (0)	86.55 (0)	86.97 (0)	82.49 (0)	83.77 (0)	82.25 (0)

T = 30

ρ_X	SGLS	MSK	MSHK	MSharm	MSarith	MSgeom	MSkmed	MSqarith	MSqmax	MSmax
0.75	65.02	61.74 (41.4)	64.20 (46.1)	59.55 (37.1)	53.84 (28.5)	56.60 (33.0)	57.03 (33.2)	54.17 (29.1)	55.96 (31.6)	53.28 (28.0)
0.90	139.46	75.47 (4.3)	79.81 (5.5)	73.01 (4)	66.05 (2.9)	68.90 (3.5)	69.45 (3.4)	66.21 (3.0)	67.77 (3.1)	65.74 (3)
0.97	427.88	87.84 (0)	94.62 (0)	86.64 (0)	78.86 (0)	81.62 (0)	82.13 (0)	78.89 (0)	80.08 (0)	78.75 (0)
0.99	1251.55	88.96 (0)	97.61 (0)	89.50 (0)	81.53 (0)	83.96 (0)	84.35 (0)	81.47 (0)	82.27 (0)	81.66 (0)

T = 50

ρ_X	SGLS	MSK	MSHK	MSharm	MSarith	MSgeom	MSkmed	MSqarith	MSqmax	MSmax
0.75	43.72	58.38 (91.0)	59.81 (92.8)	56.96 (87.8)	53.24 (77.1)	55.00 (88.2)	55.30 (83.4)	53.50 (78.8)	54.74 (81.6)	53.07 (76.2)
0.90	82.57	74.05 (135.8)	76.70 (39.8)	72.06 (33.5)	67.24 (26.7)	69.21 (28.8)	69.66 (29.7)	67.39 (26.8)	68.59 (28.2)	67.21 (27.0)
0.97	233.13	84.77 (0.5)	88.99 (0.7)	83.44 (0.5)	78.11 (0.5)	80.00 (0.5)	80.53 (0.5)	78.14 (0.5)	79.05 (0.5)	78.22 (0.5)
0.99	663.12	85.18 (0)	90.75 (0)	85.15 (0)	79.67 (0)	81.37 (0)	81.81 (0)	79.62 (0)	80.25 (0)	79.96 (0)

T = 100

ρ_X	SGLS	MSK	MSHK	MSharm	MSarith	MSgeom	MSkmed	MSqarith	MSqmax	MSmax
0.75	31.04	58.45 (99.6)	59.31 (99.7)	57.71 (99.6)	55.22 (98.7)	56.44 (99.2)	56.62 (99.2)	55.47 (98.9)	56.40 (99.3)	55.17 (98.7)
0.90	48.49	73.55 (96.5)	75.16 (97.7)	72.10 (95.2)	68.35 (91.3)	69.99 (93.8)	70.39 (94.1)	68.58 (91.7)	69.67 (93.5)	68.36 (91.5)
0.97	116.18	83.51 (15.7)	86.36 (17.9)	82.10 (15.3)	77.66 (12.4)	79.40 (13.8)	79.94 (14.3)	77.81 (12.5)	78.75 (13.3)	77.70 (12.2)
0.99	309.51	83.85 (0)	87.62 (0)	83.24 (0)	78.84 (0)	80.36 (0)	80.84 (0)	78.86 (0)	97.53 (0)	79.06 (0)

TABLE 6. Estimated TMSEs and PRS for the different methods, M = 10 equations, ρ_Σ **= 0.35**.

T = 20

ρ_X	SGLS	MSK	MSHK	MSharm	MSarith	MSgeom	MSkmed	MSqarith	MSqmax	MSmax
0.75	161.86	111.36 (7.0)	120.39 (10.4)	105.75 (5.1)	88.30 (2.6)	97.06 (3.5)	98.09 (3.5)	89.05 (2.6)	94.34 (3.3)	85.69 (2.2)
0.90	378.77	134.00 (0.1)	150.34 (0.2)	127.95 (0.1)	105.15 (0.1)	114.86 (0.1)	115.99 (0.1)	105.54 (0.1)	110.47 (0.1)	103.15 (0.1)
0.97	1219.32	152.59 (0)	177.53 (0)	149.37 (0)	123.62 (0)	132.80 (0)	133.80 (0)	123.75 (0)	127.43 (0)	122.37 (0)
0.99	3619.82	153.30 (0)	183.39 (0)	154.26 (0)	128.23 (0)	136.30 (0)	137.21 (0)	128.20 (0)	130.70 (0)	127.65 (0)

T = 30

ρ_X	SGLS	MSK	MSHK	MSharm	MSarith	MSgeom	MSkmed	MSqarith	MSqmax	MSmax
0.75	106.38	100.95 (39.7)	106.17 (47.5)	97.04 (33.5)	86.31 (22.1)	91.53 (26.9)	92.26 (27.5)	86.81 (22.8)	90.18 (25.8)	85.31 (21.3)
0.90	230.55	122.99 (1.7)	132.53 (2.2)	118.58 (1.4)	105.05 (1.0)	110.58 (1.1)	111.51 (1.1)	105.33 (1.0)	108.29 (110)	104.58 (1.0)
0.97	711.65	140.70 (0)	154.75 (0)	138.13 (0)	123.89 (0)	128.68 (0)	129.55 (0)	123.95 (0)	125.93 (0)	123.84 (0)
0.99	2085.61	140.40 (0)	156.71 (0)	140.58 (0)	127.01 (0)	130.86 (0)	131.58 (0)	126.93 (0)	128.09 (0)	127.39 (0)

T = 50

ρ_X	SGLS	MSK	MSHK	MSharm	MSarith	MSgeom	MSkmed	MSqarith	MSqmax	MSmax
0.75	70.94	97.05 (94.4)	100.07 (96.1)	94.54 (92.3)	87.77 (81.8)	91.06 (88.8)	91.59 (89.5)	88.16 (82.6)	90.46 (87.5)	87.53 (81.1)
0.90	134.69	120.05 (31.1)	125.64 (38.3)	116.74 (27.6)	107.99 (19.9)	111.55 (22.9)	112.32 (23.6)	108.16 (20.1)	110.27 (21.9)	108.12 (20.5)
0.97	381.78	134.82 (0)	143.39 (0)	132.53 (0)	123.44 (0)	126.46 (0)	127.24 (0)	123.43 (0)	124.79 (0)	123.85 (0)
0.99	1087.44	134.97 (0)	144.97 (0)	134.61 (0)	126.07 (0)	128.45 (0)	129.09 (0)	125.92 (0)	126.70 (0)	126.76 (0)

T = 100

ρ_X	SGLS	MSK	MSHK	MSharm	MSarith	MSgeom	MSkmed	MSqarith	MSqmax	MSmax
0.75	50	97.98 (99.9)	99.58 (100)	96.57 (99.9)	92.13 (99.7)	94.36 (99.9)	94.72 (99.9)	92.50 (99.8)	94.19 (99.8)	92.18 (99.7)
0.90	78.63	120.66 (98.8)	123.95 (99.3)	118.23 (99.9)	111.74 (94.3)	114.57 (96.4)	115.30 (96.8)	111.99 (94.4)	113.88 (96.0)	111.98 (94.2)
0.97	189.73	132.88 (0)	138.56 (0)	130.54 (0)	123.32 (0)	125.92 (0)	126.76 (0)	123.35 (0)	124.70 (0)	123.80 (0)
0.99	507.02	133.04 (0)	139.79 (0)	131.94 (0)	125.22 (0)	127.30 (0)	128.00 (0)	125.10 (0)	125.97 (0)	125.92 (0)

* Na, means that the results are not computable for ten equations when we only have ten observations.

TABLE 7. System-wise estimated TMSEs for the different methods, M =3 equations, $\rho_\Sigma = 0.75$.

T = 30, $\rho_\Sigma = 0.75$

ρ_X	SGLS	MSK	MSHK	MSharm	MSarith	MSgeom	MSkmed	MSqarith	MSqmax	MSmax
0.75	12.98	13.77 (52.5)	14.09 (53.7)	13.34 (50.9)	11.94 (45.8)	12.56 (48.2)	12.63 (48.3)	12.15 (46.6)	12.50 (48.0)	11.78 (45.6)
0.90	27.63	18.31 (72.2)	18.81 (82.8)	17.76 (36.1)	16.37 (43.3)	16.87 (29.3)	16.99 (29.8)	16.45 (36)	16.75 (30.6)	16.34 (51.9)
0.97	84.4	19.59 (40.1)	20.75 (57.6)	19.26 (6.1)	17.43 (8.6)	18.07 (4.41)	18.27 (4.4)	17.48 (6.8)	17.80 (4.8)	17.40 (12.1)
0.99	246.5	21.96 (25.9)	23.98 (40)	22.11 (1.0)	19.70 (0.9)	20.50 (0.6)	20.76 (0.6)	19.73 (0.6)	20.05 (0.6)	19.68 (1.8)

T = 100, $\rho_\Sigma = 0.75$

ρ_X	SGLS	MSK	MSHK	MSharm	MSarith	MSgeom	MSkmed	MSqarith	MSqmax	MSmax
0.75	6.12	13.10 (89.6)	13.30 (90.8)	13.03 (89.3)	12.46 (86.3)	12.68 (87.3)	12.71 (87.5)	12.55 (86.8)	12.70 (87.4)	12.45 (85.9)
0.90	9.47	20.45 (94.4)	20.68 (94.9)	20.16 (93.8)	19.21 (91.3)	19.57 (92.5)	19.65 (93.1)	19.31 (91.6)	19.54 (92.4)	19.21 (91.1)
0.97	22.46	18.10 (43.6)	18.53 (45.6)	17.75 (42.9)	16.64 (39.6)	17.05 (40.8)	17.22 (41.8)	16.71 (40.0)	16.94 (40.6)	16.63 (39.4)
0.99	59.54	20.78 (11.3)	21.63 (11.7)	20.59 (11.0)	19.06 (9.6)	19.67 (10.6)	19.92 (10.8)	19.16 (9.9)	19.43 (10.1)	19.02 (29.6)

TABLE 8. Estimated TMSEs and PRS for the different methods, M = 10 equations, $\rho_\Sigma = 0.75$.

T = 30, $\rho_\Sigma = 0.75$

ρ_X	SGLS	MSK	MSHK	MSharm	MSarith	MSgeom	MSkmed	MSqarith	MSqmax	MSmax
0.75	104.38	106.39 (51.3)	111.53 (62.1)	102.55 (44.7)	92.12 (31.6)	96.91 (36.4)	97.69 (37.6)	92.98 (31.8)	95.58 (34.8)	91.56 (30.8)
0.90	225.54	127.39 (2.1)	136.62 (3.1)	122.92 (1.7)	110.14 (1.3)	114.90 (1.5)	115.87 (1.5)	110.30 (1.3)	112.80 (1.5)	110.15 (1.5)
0.97	695.00	132.97 (0)	146.05 (0)	130.07 (0)	117.36 (0)	121.11 (0)	121.98 (0)	117.31 (0)	118.76 (0)	117.74 (0)
0.99	2035.7	139.54 (0)	154.68 (0)	139.50 (0)	126.98 (0)	130.30 (0)	130.95 (0)	126.85 (0)	127.78 (0)	127.64 (0)

T = 100, $\rho_\Sigma = 0.75$

ρ_X	SGLS	MSK	MSHK	MSharm	MSarith	MSgeom	MSkmed	MSqarith	MSqmax	MSmax
0.75	49.28	115.87 (100)	117.38 (100)	114.51 (100)	110.41 (100)	112.33 (100)	112.72 (100)	110.71 (100)	112.23 (100)	110.78 (100)
0.90	76.98	140.38 (99.7)	143.50 (99.8)	138.02 (99.6)	132.09 (98.3)	134.37 (99.0)	135.16 (99.1)	132.16 (98.4)	133.73 (98.7)	132.82 (98.6)
0.97	184.40	133.06 (9.0)	138.03 (11.8)	130.65 (8.8)	124.82 (7.9)	126.42 (8.1)	127.16 (8.1)	124.63 (7.9)	125.51 (8.0)	125.81 (8.2)
0.99	491.17	138.60 (0)	144.50 (0)	137.39 (0)	131.93 (0)	133.23 (0)	133.78 (0)	131.65 (0)	132.20 (0)	133.00 (0)

Appendix

TABLE A1. System-wise estimated TMSEs for the different methods, M =3 equations.

T = 30, ρ_Σ = 0.35

ρ_X	SGLS	SK	SHK	Sharm	Sarith	Sgeom	Skmed	Sqarith	Sqmax	Smax
0.75	13.39	15.82 (62)	17.25 (71.5)	12.79 (45.4)	14.50 (56.5)	10.70 (37.3)	10.80 (37.5)	12.25 (46.1)	10.83 (37.9)	18.09 (73.6)
0.90	28.71	34.18 (69)	37.37 (80.2)	22.08 (37.3)	20.09 (38.2)	16.71 (28.1)	17.03 (29.7)	18.08 (32.8)	16.75 (28.9)	23.86 (48.8)
0.97	88.08	92.55 (44.5)	103.96 (61.5)	41.76 (10.2)	23.93 (8.4)	21.96 (6.6)	23.02 (7.3)	22.25 (7)	21.19 (6.3)	27.77 (11.4)
0.99	257.64	242.14 (25.5)	277.65 (42.2)	85.32 (1.0)	26.26 (.5)	25.89 (.4)	27.74 (.5)	24.74 (.4)	23.68 (.4)	30.11 (1.2)

T = 100, ρ_Σ = 0.35

ρ_X	SGLS	SK	SHK	Sharm	Sarith	Sgeom	Skmed	Sqarith	Sqmax	Smax
0.75	6.29	8.79 (74.9)	9.47 (84.2)	8.52 (70.6)	13.19 (91.3)	8.33 (68.1)	8.32 (67.9)	9.89 (79.7)	8.36 (67.7)	17.13 (98.6)
0.90	9.94	18.65 (92.7)	19.85 (95.2)	16.35 (86.2)	19.49 (90)	15.17 (80.1)	15.23 (80.4)	16.63 (84.1)	15.28 (80.9)	23.67 (95.6)
0.97	24.07	35.01 (91.6)	38.18 (96.7)	24.21 (55.6)	23.52 (56.9)	19.86 (44.8)	20.29 (46.4)	20.99 (48.1)	19.86 (44.7)	27.70 (67.8)
0.99	64.41	68.15 (48.8)	77.73 (70.4)	34.16 (16.7)	24.8 (14.3)	21.72 (11.4)	22.93 (12.6)	22.53 (11.8)	21.42 (11)	28.94 (21.2)

T = 30, ρ_Σ = 0.75

ρ_X	SGLS	SK	SHK	Sharm	Sarith	Sgeom	Skmed	Sqarith	Sqmax	Smax
0.75	12.98	17.97 (71.4)	19.62 (81.7)	14.43 (54.5)	16.89 (69.4)	11.79 (46.3)	11.90 (45.9)	14.21 (58.4)	12.07 (47.3)	19.39 (79.3)
0.90	27.63	33.59 (72.2)	36.80 (82.8)	21.62 (36.1)	21.86 (43.3)	16.42 (29.3)	16.62 (29.8)	19.05 (36)	16.76 (30.6)	25.18 (51.9)
0.97	84.4	86.03 (40.1)	97.34 (57.6)	36.23 (6.1)	22.63 (8.6)	17.69 (4.41)	18.59 (4.4)	20.03 (6.8)	17.67 (4.8)	26.37 (12.1)
0.99	246.5	240.75 (25.9)	275.76 (40)	78.75 (1.0)	24.83 (0.9)	20.75 (0.6)	22.62 (0.6)	22.32 (0.6)	20.04 (0.6)	28.78 (1.8)

T = 100, ρ_Σ = 0.75

ρ_X	SGLS	SK	SHK	Sharm	Sarith	Sgeom	Skmed	Sqarith	Sqmax	Smax
0.75	6.12	13.41 (89.9)	14.23 (94.6)	13.0 (88.8)	19.57 (98.6)	12.90 (89)	12.93 (88.9)	15.36 (94.1)	12.99 (88.9)	22.13 (100)
0.90	9.47	22.78 (98)	24.00 (98.6)	20.48 (94.9)	26.49 (97.5)	19.46 (91.6)	19.47 (92)	22.15 (95)	19.73 (92)	29.93 (98.9)
0.97	22.46	30.38 (87.1)	33.57 (94.5)	20.29 (47.5)	23.29 (60.8)	16.68 (39.4)	16.95 (40.4)	19.31 (49.3)	17.07 (40.7)	27.22 (70.9)
0.99	59.54	61.96 (45)	70.58 (64.4)	29.82 (13.4)	25.27 (18)	19.14 (9.6)	20.05 (10.7)	21.49 (12.8)	19.37 (9.8)	29.39 (24.3)

TABLE A2. Estimated TMSEs and PRS for the different methods, M = 10 equations.

T = 30, ρ_Σ = 0.35

ρ_X	SGLS	SK	SHK	Sharm	Sarith	Sgeom	Skmed	Sqarith	Sqmax	Smax
0.75	106.38	162.42 (96.6)	185.13 (99.7)	115.58 (64.1)	92.18 (29.8)	86.32 (22)	87.99 (23.6)	89.83 (26.4)	85.65 (21.5)	97.65 (36.1)
0.90	230.55	329.52 (92.9)	395.34 (99.4)	191.99 (16)	111.77 (1.3)	107.55 (1.1)	111.54 (1.1)	109.53 (1.3)	104.84 (1.0)	118.08 (1.6)
0.97	711.65	961.41 (80.2)	1200.08 (97.9)	436.79 (1.7)	130.63 (0)	131.63 (0)	140.2 (0)	128.62 (0)	124.01 (0)	137.41 (0)
0.99	2085.61	2715.25 (69.4)	3452.26 (95.5)	1086.54 (0.9)	134.33 (0)	142.79 (0)	156.44 (0)	132.42 (0)	127.41 (0)	141.19 (0)

T = 100, ρ_Σ = 0.35

ρ_X	SGLS	SK	SHK	Sharm	Sarith	Sgeom	Skmed	Sqarith	Sqmax	Smax
0.75	50	104.75 (100)	11.47 (100)	96.13 (99.9)	104.04 (100)	92.29 (99.7)	92.02 (99.8)	99.87 (99.9)	92.93 (99.8)	109.14 (100)
0.90	78.63	153.87 (100)	171.89 (100)	125.30 (99.6)	123.60 (97.9)	111.56 (94.3)	112.16 (94.8)	119.58 (97)	112.60 (94.3)	129.33 (98.9)
0.97	189.73	280.26 (99)	344.85 (100)	171.97 (25.2)	134.43 (10.8)	123.59 (5.0)	126.14 (5.3)	130.78 (8.8)	124.31 (5.5)	140.26 (13.3)
0.99	507.02	608.99 (70.2)	808.13 (98.2)	270.10 (0)	136.45 (0)	126.72 (0)	131.04 (0)	132.95 (0)	126.31 (0)	142.61 (0)

T = 30, ρ_Σ = 0.75

ρ_X	SGLS	SK	SHK	Sharm	Sarith	Sgeom	Skmed	Sqarith	Sqmax	Smax
0.75	104.38	168.32 (98.9)	191.13 (99.9)	120.98 (76.7)	99.02 (42.4)	91.92 (31)	93.37 (32.6)	96.64 (39.8)	91.89 (30.8)	103.28 (47.7)
0.90	225.54	334.65 (96.3)	400.03 (99.6)	195.89 (20.2)	118.54 (2.3)	111.68 (1.3)	115.34 (1.4)	115.99 (2.1)	110.41 (1.5)	123.93 (3.1)
0.97	695.00	921.35 (77.0)	1157.39 (98.3)	418.31 (1.4)	125.64 (0)	122.82 (0)	130.63 (0)	123.30 (0)	117.87 (0)	131.62 (0)
0.99	2035.7	2643.89 (69.1)	3366.75 (95.6)	1051.46 (0.8)	135.62 (0)	139.28 (0)	149.91 (0)	133.36 (0)	127.61 (0)	141.87 (0)

T = 100, ρ_Σ = 0.75

ρ_X	SGLS	SK	SHK	Sharm	Sarith	Sgeom	Skmed	Sqarith	Sqmax	Smax
0.75	49.28	122.15 (100)	128.64 (100)	113.80 (100)	125.42 (100)	111.38 (100)	110.75 (100)	120.51 (100)	111.88 (100)	129.77 (100)
0.90	76.98	171.86 (100)	189.41 (100)	144.22 (99.9)	147.17 (99.6)	132.31 (98.3)	132.41 (98.5)	142.47 (99.6)	133.95 (98.7)	152.08 (99.6)
0.97	184.40	270.62 (99.3)	332.40 (99.9)	168.34 (29.4)	139.11 (16.5)	124.72 (7.8)	126.23 (7.9)	134.82 (13.7)	126.67 (9)	144.77 (20.4)
0.99	491.17	595.46 (69.7)	786.30 (97.9)	266.41 (0)	145.98 (0)	132.40 (0)	135.28 (0)	141.65 (0)	133.72 (0)	151.80 (0)