ISSN 1825-0211

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Quaderno n. 63 — Dicembre 2008

QUADERNI DEL DIPARTIMENTO
DI ECONOMIA, FINANZA
E STATISTICA

Limit results for discretely observed stochastic volatility models with leverage effect*

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Abstract

In this note we generalize the limit results in [Genon-Catalot, Jeantheau, Laredo, 2000, *Bernoulli*] for simple stochastic volatility models to the case where a non zero correlation is allowed between the Brownian motion driving the main diffusion process and the Brownian motion driving the dynamics of the instantaneous variance. We also extend the results to the case where the main diffusion admits a non zero drift which is linear in the variance process. The main motivation for such an extension is the application of these limit results in order to perform statistical inference in some of the stochastic volatility models introduced in the financial mathematics literature. In this framework it is of relevance the so called "leverage effect" between the stock log-price and its volatility, which is indeed explained by a negative correlation between the Brownian motions driving the log-price process and its instantaneous variance respectively. Moreover a linear term in the variance appears in the drift of the log-price diffusion.

1 The model setting

In the paper by Genon-Catalot et al. (2000) some limit results are proved for the simple stochastic volatility model, when discretely observed, described by the following bivariate diffusion:

$$\begin{aligned} dY_t &= \sqrt{V_t}d\widetilde{W}_t, & Y_0 = 0, \\ dV_t &= b(V_t)dt + a(V_t)dW_t, & V_0 = \eta, \end{aligned} \tag{1}$$

where a and b are suitable functions in order to guarantee the existence of a strong solution for the second diffusion in (1) and where $(\widetilde{W}, W)$ is a standard Brownian motion in $\mathbb{R}^2$. Similar results are also obtained in Sørensen (2000). We want to generalize the results of Genon-Catalot et al. (2000) by allowing a

*Revised version (October 2009)

non zero correlation for the bivariate Brownian motion and for a non zero drift in the first equation of (1). Our motivation is essentially given by the possible application of these limit results for the stochastic volatility specifications available in the financial mathematics literature. In this context a negative correlation in the Brownian motion (the so-called *leverage effect*) could explain the asymmetry in the empirical distribution of historical data and in the implied volatility curve, obtained plotting the implied volatility of European options written on the stock with respect to their strike price, as evidenced, among others, in Cont (2001).

Define, for $x_0, x \in (l, r)$, the scale and speed densities of V_t respectively as

$$\begin{aligned} s(x) &= \exp\left(-2\int_{x_0}^{x} \frac{b(u)}{a^2(u)}du\right), \\ m(x) &= \frac{1}{a^2(x)s(x)} \end{aligned}$$

and the stationary density of V_t as

$$\pi(x) = \frac{m(x)}{M}\mathbf{1}_{\{x\in(l,r)\}}.$$

In Genon-Catalot et al. (2000) (GC hereafter) the model defined in (1) is considered with the following assumptions:

(A0) $(\widetilde{W}, W)$ is a standard Brownian motion in $\mathbb{R}^2$ defined on a probability space $(\Omega, \mathcal{F}, P)$ and η is a random variable defined on Ω, independent of $(\widetilde{W}, W)$.

(A1) The functions $a(x)$ and $b(x)$ are defined on $(l, r) \subset (0, +\infty)$ and satisfy

i) $b \in C^1(l, r), a^2 \in C^2(l, r), a(x) > 0, \forall x \in (l, r)$
ii) $\exists K > 0$ such that, $\forall x \in (l, r)$, $|b(x)| \leq K(1 + |x|)$ and $a^2(x) \leq K(1 + x^2)$.

(A2) $\int_l s(x)dx = +\infty$, $\int^r s(x)dx = +\infty$ and $\int_l^r m(x) = M < +\infty$.

(A3) The initial random variable v has distribution $\pi(dx) = \pi(x)dx$.

Let $\mathcal{A}$ and $\mathcal{B}$ be two σ-algebras included in $\mathcal{F}$. A measure of dependence between $\mathcal{A}$ and $\mathcal{B}$ can be defined as

$$\alpha(\mathcal{A}, \mathcal{B}) = sup_{A\in\mathcal{A},B\in\mathcal{B}} \left|P(A \cap B) - P(A)P(B)\right|.$$

Given a process $\{S_t\}_t$ the $\alpha - mixing$ coefficient of the process is defined as

$$\alpha_S(\Delta) = \sup_{s\geq 0} \alpha(F_{-\infty}^s, F_{s+\Delta}^{+\infty}),$$

with $F_{-\infty}^s = \sigma(V_u,\ -\infty < u \leq s)$ and $F_{s+\Delta}^{+\infty} = \sigma(V_u,\ s + \Delta \leq u < +\infty)$ and represents a measure of weak dependence of the process.

Definition: A process $\{S_t\}_t$ is $\alpha-mixing$ (or strongly mixing) if $\alpha_S(\Delta) \to 0$ as $\Delta \to +\infty$.

The strongly mixing condition was firstly introduced in Rosenblatt (1956) as a dependence condition under which a central limit result for stationary process can be obtained. Other weak dependence measures can also be defined.

A detailed analysis on weak dependence measures, on mixing properties and on limit results for mixing processes can be found in Doukhan (1994). A brief review on the results that we need is given in CG (Section 2).

When necessary, the following properties are also assumed to hold.

(A4) $\lim_{x\to l+} a(x)m(x) = \lim_{x\to r-} a(x)m(x) = 0$

(A5) $\lim_{x\to l+} \frac{1}{\gamma(x)}$ and $\lim_{x\to r-} \frac{1}{\gamma(x)}$ exist where $\gamma(x) = a'(x) - 2\frac{b(x)}{a(x)}$.

Notice that assumptions (A1) to (A3) guarantee that the instantaneous variance V_t is a positive recurrent diffusion on an interval and a strictly stationary ergodic and time reversible process. Assumptions (A4) and (A5) are in order when studying the mixing properties of the instantaneous variance process.

In our setting we leave assumptions (A1) to (A5) unchanged while the assumption (A0) is replaced by

(A0') $(\widetilde{W}, W)$ is a Brownian motion in $\mathbb{R}^2$ defined on a probability space $(\Omega, \mathcal{F}, P)$ with $\langle d\widetilde{W}, dW\rangle = \rho dt$ and η is a random variable defined on Ω, independent of $(\widetilde{W}, W)$.

Under the modified set of assumptions the process in (1) can be written as

$$\begin{aligned} dY_t &= \sqrt{V_t}\left(\rho dW_t + \sqrt{1-\rho^2}dB_t\right), \qquad & Y_0 = 0, \\ dV_t &= b(V_t)dt + a(V_t)dW_t, \qquad & V_0 = \eta, \end{aligned} \tag{2}$$

where (B, W) is a standard bi-dimensional Brownian Motion. By using the results in GC (Section 2.6) we know that if assumptions (A1) to (A5) are fulfilled then the process V_t is strictly stationary, ergodic, time reversible and $\alpha-mixing$ and that the discretely observed process $V_{i\Delta}$, for $\Delta > 0$ and $i \geq 1$, is also ergodic and $\alpha - mixing$.

In what follows we will focus on the Heston volatility specification (Heston, 1993)

$$dV_t = \alpha(\beta - V_t)dt + c\sqrt{V_t}dW_t, \tag{3}$$

for which Assumption A1 to A5 are fullfilled if $2\alpha\beta > c^2$.

2 Properties of the discretely sampled process

Let us define, for $i \geq 1$, the discrete processes

$$\begin{aligned} X_i &= \frac{1}{\sqrt{\Delta}} \int_{(i-1)\Delta}^{i\Delta} \sqrt{V_s}(\rho dW_s + \sqrt{1-\rho^2} dB_s), \qquad (4) \\ \overline{V_i} &= \frac{1}{\Delta} \int_{\Delta(i-1)}^{\Delta i} V_s ds, \quad \text{and} \\ U_i &= (V_{\Delta(i-1)}, V_{\Delta i}, \overline{V_i}). \end{aligned}$$

In financial applications the process X_i is the log-return of the stock during the time interval $[(i-1)\Delta, i\Delta)$ (suitably scaled) and $\overline{V_i}$ is the mean (integrated) variance during the same period.

Definition (Leroux (1992)): A stochastic process $X_i, i \geq 1$, with state space $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$, is a Hidden Markov Chain if the following conditions hold:

i) (U_i) is a strictly stationary non observable Markov chain with state space $(\mathcal{U}, \mathcal{B}(\mathcal{U}))$.

ii) For all i, given $(U_1, U_2, ..., U_i)$ the X_i are conditionally independent and the conditional distribution of X_i depends only on U_i

iii) The conditional distribution of X$_i$ given $U_i = u$ does not depend on i.

where $\mathcal{X}$ and $\mathcal{U}$ are Polish spaces and $\mathcal{B}(\mathcal{X})$ and $\mathcal{B}(\mathcal{U})$ are the corresponding Borel $\sigma-$algebras. In the classical definition of Leroux (1992) the state space $\mathcal{U}$ is assumed to be finite; in GC this assumption is relaxed and the hidden process U_i is called Hidden Markov Model.

Theorem 1: If assumptions (A0') to (A3) hold then:

- $(U_i, i \geq 1)$ is a strictly stationary Markov chain with state space $(l, r)^3$;
- $(X_i \ , i \geq 1)$ is a Hidden Markov model with hidden chain $(U_i, i \geq 1)$.

Proof: we proceed as in GC, Theorem 3.1.

Let $\mathcal{G}_t = \sigma(V_s, s \leq t)$, $E = C([0, \Delta], (l, r))$ the space of continuous functions defined on $[0, \Delta]$ with values in (l, r), and B the Borel $\sigma-$algebra associated with the uniform topology, and write

$$\begin{aligned} V_{(i-1)\Delta} &= V_{(i-2)\Delta+\Delta} \\ V_{i\Delta} &= V_{(i-2)\Delta+2\Delta}, \quad \text{and} \\ \overline{V_i} &= \frac{1}{\Delta} \int_{\Delta}^{2\Delta} V_{(i-2)\Delta+s} ds. \end{aligned}$$

More generally set, for $s \in [0, \Delta]$, $i \geq 1$,

$$Z_i(s) = V_{(i-2)\Delta+s}$$

and define function $T : E \to (l,r)^3$ as

$$T(z) = \left(z(\Delta), z(2\Delta), \frac{1}{\Delta}\int_{\Delta}^{2\Delta} z(s)ds, \right).$$

Let $\varphi : (l,r)^3 \to \mathbb{R}$ be a bounded Borel function and $H_i = G_{(i-1)\Delta}$; we have

$$\begin{aligned}
E[\varphi(U_i)|\mathcal{H}_{(i-1)\Delta}] &= E\left[\varphi(V_{(i-1)\Delta}, V_{i\Delta}, \overline{V_i})|V_t, t \le (i-2)\Delta\right] \\
&= E\left[\varphi(Z_i(\Delta), Z_i(2\Delta), \frac{1}{\Delta}\int_{\Delta}^{2\Delta} Z_i(s)ds)|V_t, t \le (i-2)\Delta\right] \\
&= E\left[\varphi(V_{(i-2)\Delta+\Delta}, V_{(i-2)\Delta+2\Delta} \cdot \frac{1}{\Delta}\int_{\Delta}^{2\Delta} V_{(i-2)\Delta+s} ds)|V_t, t \le (i-2)\Delta\right] \\
&= E\left[\varphi(V_{(i-2)\Delta+\Delta}, V_{(i-2)\Delta+2\Delta}, \frac{1}{\Delta}\int_{\Delta}^{2\Delta} V_{(i-2)\Delta+s} ds)|V_{(i-2)\Delta}\right] \\
&= \psi(V_{(i-2)\Delta})
\end{aligned}$$

where

$$\psi(v) = E[\varphi(V_\Delta, V_{2\Delta}, \overline{V_2})|V_0 = v]$$

proving that $(U_i, i \ge 1)$ is a Markov chain with respect to H_i.

The process Z_i has state space (E, B) and it inherits markovianity, strictly stationarity, ergodicity from the process V_t. Besides, $U_i = T(Z_i)$ where T is a continuous function on E, hence the process $(U_i)_i$ is also strictly stationary and property i) of Definition 1 holds true.

Let us denote

$$\begin{aligned}
A_i &= \frac{1}{\sqrt{\Delta}}\rho \int_{(i-1)\Delta}^{i\Delta} \sqrt{V_s} dW_t, \quad \text{and} \\
B_i &= \frac{1}{\sqrt{\Delta}}\sqrt{1-\rho^2} \int_{(i-1)\Delta}^{i\Delta} \sqrt{V_s} dB_t.
\end{aligned}$$

Conditionally on $\mathcal{G}_{i\Delta}$, A_i is known and and B_i is a stochastic integral of a deterministic function with respect to a Brownian motion and thus is a martingale with zero mean. Hence,

$$\begin{aligned}
E(X_i|\mathcal{G}_{n\Delta}) &= E(A_i|\mathcal{G}_{n\Delta}) + E(B_i|\mathcal{G}_{n\Delta}) \\
&= A_i,
\end{aligned}$$

$$\begin{aligned}
Var(X_i|\mathcal{G}_{n\Delta}) &= Var(A_i|\mathcal{G}_{n\Delta}) + Var(B_i|\mathcal{G}_{n\Delta}) \\
&= \frac{1-\rho^2}{\Delta}\int_{(i-1)\Delta}^{i\Delta} V_s ds \\
&= \left(1-\rho^2\right)\overline{V_i}
\end{aligned}$$

and, for $i \neq j$,

$$Cov(X_i, X_j | \mathcal{G}_{n\Delta}) = Cov(B_i, B_j | \mathcal{G}_{n\Delta}) = 0.$$

Thus, conditionally on $\mathcal{G}_{n\Delta}$, the random variables $(X_1, X_2, ..., X_n)$ are independent and X_i has distribution $N(A_i, \overline{V_i})$.

Notice that in our model setting we can obtain by integration of (3) that

$$A_i = \frac{1}{c\sqrt{\Delta}}\rho\left(V_{i\Delta} - V_{(i-1)\Delta} - \alpha(\beta - \overline{V}_i)\Delta\right)$$

and thus it is completely known when U_i is known.

To demonstrate properties ii) and iii) in the definition of HMM we have to show that the above distributional results are valid when conditioning with respect to $\sigma(U_1, U_2, ..., U_n)$.

Using conditional independence on $\mathcal{G}_{n\Delta}$, the joint characteristic function of $(X_1, X_2, ..., X_n)$ is given by

$$\Phi(\lambda_1, \lambda_2, ..., \lambda_n) = E[\exp\sum_{j=1}^{n} i\lambda_j X_j | \mathcal{G}_{n\Delta}] = \exp\left(\sum_{j=1}^{n} i\lambda_j A_j - \frac{1}{2}\sum_{j=1}^{n} i\lambda_j^2(1-\rho^2)\overline{V_j}\right). \tag{5}$$

Since the last expression in (**??**) is measurable with respect to $\sigma(U_1, U_2, ..., U_n)$ we have

$$E[\exp\sum_{j=1}^{n} i\lambda_j X_j | U_1, U_2, ...U_n] = \exp\left(\sum_{j=1}^{n} i\lambda_j A_j - \frac{1}{2}\sum_{j=1}^{n} i\lambda_j^2(1-\rho^2)\overline{V_j}\right)$$

which finally gives both property ii) and iii) of HMM.

In GC (Proposition 3.1) it is proved, extending a result in Leroux (1992), that if Y_i is a HMM with hidden chain U_i then Y_i is strictly stationary. Moreover if U_i is ergodic then Y_i is ergodic and if U_i is $\alpha - mixing$ then Z_i is $\alpha - mixing$ with $\alpha_Y(k) \leq \alpha_U(k)$. It is then proved (GC. Prop. 3.2) that U_i is $\alpha - mixing$ with $\alpha_U(k) \leq \alpha_V((k-1)\Delta)$. Theorem 2.3 in GC gives the ergodicity of U_i.

Since we have proved in Theorem 1 that X_i is a HMM with respect to U_i,we get the following outcome

Proposition 1: Under assumptions (A0') to (A3) the process X_i is strictly stationary, ergodic and $\alpha - mixing$.

3 Limit Results

Suppose we are given with a Borel function $g : \mathbb{R}^d \to \mathbb{R}$, where d is a positive integer, and define $G_i = g(X_{i+1}, X_{i+2}, ..., X_{i+d})$, for $i = 1, 2, ...n$. Denote $\varphi_k = \sqrt{1-\rho^2}\epsilon_k$ where ϵ_k, for $k = 1, 2, ..., d$, are standard Gaussian i.i.d random variables . Since A_j is $\sigma(U_j)$-measurable we can write, for $j = 1, 2, ...n$, $A_j = A(u_j)$ for a suitable function A.

Consider the function $H_g : (\mathbb{R}^3_+)^d \to \mathbb{R}_+$ defined as

$$H_g(u_1, u_2, ..., u_d) = E\left[g\left(A(u_1) + \sqrt{v_1}\varphi_1, A(u_2) + \sqrt{v_2}\varphi_2, ..., A(u_d) + \sqrt{v_d}\varphi_d\right)\right]$$

where for the sake of simplicity we set $v_j = u_{j3}$. We generalize Theorems 3.2 and 3.3 of GC as follows:

Theorem 2: Under assumptions (A0') to (A3) and if g is such that

$$E\left|H_g(U_1, U_2, ..., U_d)\right| < +\infty$$

then

$$\frac{1}{n}\sum_{i=0}^{n-d} G_i \underset{n\to+\infty}{\overset{a.s.}{\longrightarrow}} E[H_g(U_1, U_2, ..., U_d)]. \tag{6}$$

Proof. From Proposition 1 the process X_i is ergodic so it suffices to check that $E\left|G_0\right|$ is finite and that $E\left|G_0\right| = E[H_g(U_1, U_2, ..., U_d)]$. This is obtained by conditioning on $\mathcal{G}_{d\Delta}$.

Theorem 3: Under assumptions (A0') to (A5), if it exist $\delta > 0$ such that $E\left|G_0\right|^{2+\delta} < +\infty$ and $\sum_{k\geq 1}\alpha_V^{\frac{2}{2+\delta}}(k\Delta) < +\infty$

$$\Sigma_\Delta(g) = Var(G_0) + 2\sum_{i=1}^{\infty} Cov(G_0, G_i),$$

is well defined and non negative: if it is positive then

$$\frac{1}{\sqrt{n}}\sum_{i=0}^{n-d}\left(G_i - E[H_g(U_1, U_2, ..., U_d)]\right) \underset{n\to+\infty}{\overset{Law.}{\longrightarrow}} N\left(0, \Sigma_\Delta(g)\right). \tag{7}$$

Proof. The proof follows that of Theorem 3.3 in GC and it is based on the application of Ibragimov Central Limit Theorem for strictly stationary $\alpha-$ *mixing* sequences (see chapter 18 in Ibragimov, Linnik, 1971, and chapter 5 in Hall and Heyde 1980).

The $\alpha - mixing$ coefficient of the sequence (G_i) satisfies

$$\alpha_G(k) \leq \alpha_X((k+1-d)) \leq \alpha_V((k-d-1)\Delta).$$

Therefore, the quantity

$$\Sigma_\Delta(g) = \lim \frac{Var(G_0 + G_1 + ... + G_{n-d})}{n}$$

exists and it is non negative. If it is also positive the thesis holds.

Theorem 3 can also be stated in a multivariate setting. Given an integer d and a set of Borel functions $g_1, g_2, ..., g_m$ with $g_j : \mathbb{R}^d \to \mathbb{R}$, for $j = 1, 2, ..., m$, denote

$$G_{i,j} = g_j(X_{i+1}, X_{i+2}, ..., X_{i+d}).$$

Theorem 4: Under the assumptions (A0') to (A5), if it exist $\delta > 0$ such that $E|G_{0,j}|^{2+\delta} < +\infty$, for $j = 1, 2, ...m$, and $\sum_{k\geq 1} \alpha_V^{\frac{2}{2+\delta}}(k\Delta) < +\infty$ then

$$\Sigma_\Delta(g_j, g_l) = Cov(G_{j,0}, G_{l,0}) + \sum_{i=1}^{\infty} Cov(G_{j,0}, G_{l,i}) + \sum_{i=1}^{\infty} Cov(G_{j,i}, G_{l,0})$$

is well defined for $j, l = 1, 2, ...m$.

If $\mathbf{\Sigma}(\mathbf{g}, \Delta) = (\Sigma_\Delta(g_j, g_l))_{j,l}$ is a positive definite matrix then

$$\frac{1}{\sqrt{n}} \sum_{i=0}^{n-d} \begin{pmatrix} (G_{i,1} - E[H_{g_1}(U_1, U_2, ..., U_d)]) \\ (G_{i,2} - E[H_{g_2}(U_1, U_2, ..., U_d)]) \\ ... \\ (G_{i,m} - E[H_{g_m}(U_1, U_2, ..., U_d)]) \end{pmatrix} \underset{n\to\infty}{\overset{law}{\longrightarrow}} N(0, \mathbf{\Sigma}_\Delta(\mathbf{g})). \quad (8)$$

4 A further generalization

Let us consider the following generalized dynamics for Y_t:

$$dY_t = \mu(V_t)dt + \sqrt{V_t}\left(\rho dW_t + \sqrt{1-\rho^2}dB_t\right).$$

A natural question arises whether the theory developed in Genon-Catalot et al. (2000) and in this paper might be applied to this more general setting. Let us restrict our attention to the case of a linear function $\mu(x) = \xi + \kappa x$ which is indeed of great interest in financial applications.

Define the discrete process

$$R_i = \frac{1}{\sqrt{\Delta}} \int_{(i-1)\Delta}^{i\Delta} \mu(V_t)dt + \frac{1}{\sqrt{\Delta}} \int_{(i-1)\Delta}^{i\Delta} \sqrt{V_s}\left(\rho dW_t + \sqrt{1-\rho^2}dB_t\right)$$

Theorem 1': If assumptions (A0') to (A3) hold then:

- $(U_i, i \geq 1)$ is a strictly stationary Markov chain with state space $(l, r)^3$;
- $(R_i\ , i \geq 1)$ is a Hidden Markov model with hidden chain $(U_i, i \geq 1)$.

Proof: All the previous results on U_i are still valid so, in order to show that R_i is a HMM we only have to demonstrate ii) and iii) of Definition 1.

We remark that $R_i = C_i + X_i$ where $C_i = \frac{1}{\sqrt{\Delta}} \int_{(i-1)\Delta}^{i\Delta} \mu(V_t)dt$ is, conditionally on $\mathcal{G}_{n\Delta}$, a deterministic function; conditionally on $\mathcal{G}_{n\Delta}$, the random variables $(R_1, R_2, ..., R_n)$ are independent and

$$\begin{aligned} E[R_i|\mathcal{G}_{n\Delta}] &= \\ &= E[C_i + X_i|\mathcal{G}_{n\Delta}] \\ &= A_i + C_i \\ Var(R_i|\mathcal{G}_{n\Delta}) &= Var(X_i|\mathcal{G}_{n\Delta}) \\ &= (1-\rho^2)\overline{V_i}. \end{aligned}$$

Hence, R_i, for $i = 1, 2, ...n$, has distribution $\mathcal{N}(A_i + C_i, \overline{V_i})$. To prove properties ii) and iii) of HMM for this new process M_i we have to show that the above distributional also hold when conditioning with respect to a $\sigma(U_1, U_2, ..., U_n) \subset \mathcal{G}_{n\Delta}$. This latter condition may fail for a generic drift function $\mu(V_t)$ since the integral defining C_i may depend on the whole path of V_t in the interval $[(i-1)\Delta, i\Delta)$.

By using the conditional independence on $\mathcal{G}_{n\Delta}$, the joint characteristic function of $(R_1, R_2, ..., R_n)$ is

$$\Phi_Z(\lambda_1, \lambda_2, ..., \lambda_n) = E[\exp \sum_{j=1}^{n} i\lambda_j R_j | \mathcal{G}_{n\Delta}] = \exp \sum_{j=1}^{n} (\sqrt{\Delta}(\xi + \kappa \overline{V_j}) + A_j) i\lambda_j - \frac{1}{2}\lambda_j^2 (1-\rho^2)\overline{V_j}. \tag{9}$$

The expression in the right hand side of (**??**) is measurable with respect to $\sigma(U_1, U_2, ...U_n)$, then

$$E[\exp \sum_{j=1}^{n} i\lambda_j R_j | U_1, U_2, ...U_n] = \exp \sum_{j=1}^{n} (\sqrt{\Delta}(\xi + \kappa \overline{V_j}) + A_j) i\lambda_j - \frac{1}{2}\lambda_j^2 (1-\rho^2)\overline{V_j}.$$

so that property ii) and iii) of HMM are fulfilled.

In order to extend the above limit theorems to this more general framework define $c(u) = A(u) + \sqrt{\Delta}\mu(v)$ and, when it exists, the function $\widetilde{H}_g : (\mathbb{R}_+^3)^d \to \mathbb{R}_+$,

$$\widetilde{H}_g(u_1, u_2, ...u_n) = E\left[g\left(c(u_1) + \sqrt{v_1}\varphi_1, c(u_2) + \sqrt{v_2}\varphi_2, ..., c(u_d) + \sqrt{v_d}\varphi_d\right)\right]$$

where φ_k and v_k are as defined in the previous section and denote $\widetilde{G}_i = g(R_{i+1}, R_{i+2}, ..., R_{i+d})$.

Since R_i is a HMM with respect to U_i and having in mind the properties of U_i from the previous section, it is straightforward to prove the following results:

Theorem 2': Under assumptions (A0') to (A3) and if g is such that

$$E\left|\widetilde{H}_g(U_1, U_2, ..., U_d)\right| < +\infty$$

then

$$\frac{1}{n}\sum_{i=0}^{n-d} \widetilde{G}_i \underset{n\to+\infty}{\overset{a.s.}{\longrightarrow}} E[\widetilde{H}_g(U_1, U_2, ..., U_d)]. \tag{10}$$

Theorem 3': Under assumptions (A0') to (A5), if it exist $\delta > 0$ such that $E\left|\widetilde{G}_0\right|^{2+\delta} < +\infty$ and $\sum_{k\geq 1} \alpha_V^{\frac{2}{2+\delta}}(k\Delta) < +\infty$ then

$$\widetilde{\Sigma_\Delta}(g) = Var(\widetilde{G}_0) + 2\sum_{i=1}^{\infty} Cov(\widetilde{G}_0, \widetilde{G}_i),$$

is well defined and non negative. If it is non zero then

$$\frac{1}{n}\sum_{i=0}^{n-d}\left(\widetilde{G}_i - E[\widetilde{H}_g(U_1,U_2,...,U_d)]\right) \underset{n\to+\infty}{\overset{Law.}{\longrightarrow}} N\left(0,\widetilde{\Sigma}_\Delta(g)\right). \tag{11}$$

An multivariate extension of (11) can also be derived.

5 Asymptotic variance for polynomial functions

Assume at first that $\mu(V_t)=0$. By conditional independence, we have, for $i\geq d$

$$Cov(G_0,G_i) = Cov\left(H_g(U_1,U_2,...,U_d), H_g(U_{i+1},U_2,...U_{i+d}\right).$$

Define, for $j=1,2,...n$,

$$F(p,u_j) = E_U[(A(u_j)+\sqrt{v}\varphi_j)^{2p}]$$

where we denote $v_j = u_{j3}$.

Proposition 3.4 of GC can be generalized as follows.

Proposition 2: Assume (A0')-(A3) to hold. If it exist $\delta>0$ such that $E\left|G_0\right|^{2+\delta}<+\infty$ and $\sum_{k\geq1}\alpha_V^{\frac{2}{2+\delta}}(k\Delta)<+\infty$ the following properties hold

i) if $g_1(x_1,...,x_{d_1}) = x_1^{2p}$ with $d_1=1$ and $E[V_0^{2p(1+\frac{\delta}{2})}]<+\infty$, then

$$\Sigma_\Delta(g_1,g_1) = E\left[F(2p,U_1)\right] - E\left[F(p,U_1)\right]^2 + 2\sum_{i=1}^{\infty}\left(E[F(p,U_1)F(p,U_{1+i})] - E[F(p,U_1)]^2\right).$$

ii) if $g_2(x_1,...,x_{d_2}) = x_1^{2q}x_{1+h}^{2r}$ with $d_2=h+1$, $z=\max\{r,q\}$ and $E[V_0^{4z(1+\frac{\delta}{2})}]<+\infty$, then

$$\begin{aligned}\Sigma_\Delta(g_2,g_2) &= E\left[F(2q,U_1)F(2r,U_{1+h})\right] - 2E\left[F(q,U_1)F(r,U_{1+h})\right]^2 + \\ &\quad +E\left[F(q,U_1)F(q+r,U_{1+h})F(r,U_{1+2h})\right] + \\ &+2\sum_{i=1,i\neq h}^{\infty}\left(E\left[F(q,U_1)F(r,U_{1+h})F(q,U_i)F(r,U_{1+h+i})\right] - E\left[F(q,U_1)F(r,U_{1+h})\right]^2\right).\end{aligned}$$

Moreover, if g_1 and g_2 are defined as above, $g_3(x_1,...,x_{d_3}) = x_1^{2u}$ with $d_3=1$ and $g_4(x_1,...,x_{d_4}) = x_1^{2t}x_{1+k}^{2s}$ with $d_4=k+1$, then:

iii) if $z=\max\{p,u\}$ and $E[V_0^{4z(1+\frac{\delta}{2})}]<+\infty$

$$\begin{aligned}\Sigma_\Delta(g_1,g_3) &= E\left[F(p+u,U_1)\right] - E\left[F(p,U_1)\right]E\left[F(u,U_1)\right] + \\ &\quad +\sum_{i=1}^{\infty}(E\left[F(p,U_1)F(u,U_{i+1})\right] - 2E\left[F(p,U_1)\right]E\left[F(u,U_1)\right] + E\left[F(p,U_{i+1})F(u,U_1)\right]).\end{aligned}$$

iv) if $z = \max\{p, q, r\}$ and $E[V_0^{3z(1+\frac{\delta}{2})}] < +\infty$,

$$\Sigma_\Delta(g_1, g_2) = E\left[F(p+q, U_1)F(r, U_{1+h})\right] - E\left[F(p, U_1)\right] E\left[F(q, U_1)F(r, U_{1+h})\right] + \\ +E\left[F(p+r, U_{1+h})F(q, U_1)\right] - E\left[F(p, U_{1+h})\right] E\left[F(q, U_1)F(r, U_{1+h})\right] + \\ \sum_{i=1, i\neq h}^{\infty} \left(E\left[F(p, U_{i+1})F(q, U_1)F(r, U_{1+h})\right] - E\left[F(p, U_1)\right] E\left[F(q, U_1)F(r, U_{1+h})\right]\right) \\ \sum_{i=1}^{\infty} \left(E\left[F(p, U_1)F(q, U_{1+i})F(r, U_{1+h+i})\right] - E\left[F(p, U_1)\right] E\left[F(q, U_1)F(r, U_{1+h})\right]\right).$$

Proof: See Appendix A.

In the case of a linear drift $\mu(x) = \xi + \kappa x$, conditional independence gives, for $i \geq d$,

$$Cov(\widetilde{G}_0, \widetilde{G}_i) = Cov\left(\widetilde{H}_g(U_1, U_2, ..., U_d), \widetilde{H}_g(U_1, U_2, ..., U_d)\right).$$

As it is shown in Appendix A, Proposition 2 can be generalized to this more general case by simply replacing function A with function c defined by:

$$c(u) = A(u) + \sqrt{\Delta}\mu(v).$$

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6 Appendix A

Proof of Proposition 2 (in the more general setting): assume that

$$\begin{aligned} c(u) &= A(u) + \sqrt{\Delta}\mu(v), \\ F(p, u_j) &= E_U[(c(u_j) + \sqrt{v_j}\varphi_j)^{2p}]. \end{aligned}$$

Simple computations give

$$\begin{aligned} F(p, u_j) &= E_U\left[\sum_{s=0}^{p} \binom{2p}{2s} c(u_j)^{2(p-s)} \sqrt{v_j^{2s}} \varphi_j^{2s}\right] \\ &= \sum_{s=0}^{p} \binom{2p}{2s} c(u_j)^{2(p-s)} v_j^s E_U[\varphi_j^{2s}] \\ &= \sum_{s=0}^{p} \binom{2p}{2s} c(u_j)^{2(p-s)} v_j^s (1-\rho^2) m_{2s}. \end{aligned}$$

where we denote m_{2k} the $2k-th$ moment of a standard Gaussian distribution.

i)

$$\begin{aligned} Var(\widetilde{G}_0) &= Var[R_1^{2p}] = E[R_1^{4p}] - E[R_1^{2p}]^2 = \\ &= E[E_U[(c(U_1) + \sqrt{V_1}\varphi_1)^{4p}]] - E[E_U[(c(U_1) + \sqrt{V_1}\varphi_1)^{2p}]]^2 \\ &= E[F(2p, U_1)] - E[F(p, U_1)]^2, \end{aligned}$$

$$\begin{aligned}
Cov(\widetilde{G}_0,\widetilde{G}_i) &= Cov(R_1^{2p},R_{i+1}^{2p}) = E[R_1^{2p}R_{i+1}^{2p}]-E[R_1^{2p}]E[R_{i+1}^{2p}] = \\
&= E[E_U[(c(U_1)+\sqrt{V_1}\varphi_1)^{2p}(c(U_{i+1})+\sqrt{V_{i+1}}\varphi_{i+1})^{2p}]] \\
&\quad -E[E_U[(c(U_1)+\sqrt{V_1}\varphi_1)^{2p}]]E[E_U[(c(U_{i+1})+\sqrt{V_{i+1}}\varphi_{i+1})^{2p}]] \\
&= E[F(p,U_1)F(p,U_{1+i})]-E[F(p,U_1)]E[F(p,U_{1+i})]
\end{aligned}$$

Then

$$\Sigma_\Delta(g_1,g_1) = E\left[F(2p,U_1)\right]-E\left[F(p,U_1)\right]^2+2\sum_{i=1}^{\infty}\left(E[F(p,U_1)F(p,U_{1+i})]-E[F(p,U_1)]^2\right).$$

ii) By conditional independence

$$\begin{aligned}
&Var(\widetilde{G}_0) = Var[R_1^{2q}R_{1+h}^{2r}] = E[R_1^{4q}R_{1+h}^{4r}]-E[R_1^{2q}R_{1+h}^{2r}]^2 = \\
&= E[E_U[(c(U_1)+\sqrt{V_1}\varphi_1)^{4q}(c(U_{1+h})+\sqrt{V_{1+h}}\varphi_{1+h})^{4r}]] \\
&-E[E_U[(c(U_1)+\sqrt{V_1}\varphi_1)^{2q}(c(U_{1+h})+\sqrt{V_{1+h}}\varphi_{1+h})^{2r}]]^2 \\
&= E[E_U[(c(U_1)+\sqrt{V_1}\varphi_1)^{4q}]E_U[(c(U_{1+h})+\sqrt{V_{1+h}}\varphi_{1+h})^{4r}]] \\
&-E[E_U[(c(U_1)+\sqrt{V_1}\varphi_1)^{2q}]E_U[(c(U_{1+h})+\sqrt{V_{1+h}}\varphi_1)^{2r}]]^2 \\
&= E\left[F(2q,U_1)F(2r,U_{1+h})\right]-E\left[F(q,U_1)\right]^2E\left[F(r,U_{1+h})\right]^2.
\end{aligned}$$

For $i \neq h$,

$$\begin{aligned}
&Cov(\widetilde{G}_0,\widetilde{G}_i) = Cov(R_1^{2q}R_{1+h}^{2r},R_{1+i}^{2q}R_{1+h+i}^{2r}) \\
&= E[R_1^{2q}R_{1+h}^{2r}R_{1+i}^{2q}R_{1+h+i}^{2r}]-E[R_1^{2q}R_{1+h}^{2r}]E[R_{1+i}^{2q}R_{1+h+i}^{2r}] \\
&= E[E_U[R_1^{2q}]E_U[R_{1+h}^{2r}]E_U[R_{1+i}^{2q}]E_U[R_{1+h+i}^{2r}]] \\
&-E[E_U[R_1^{2q}]E_U[R_{1+h}^{2r}]]E[E_U[R_{1+i}^{2q}]E_U[R_{1+h+i}^{2r}]] \\
&= E\left[F(q,U_1)F(r,U_{1+h})F(q,U_i)F(r,U_{1+h+i})\right]-E[F(q,U_1)F(r,U_{1+h})]^2,
\end{aligned}$$

while, for $i = h$

$$\begin{aligned}
&Cov(\widetilde{G}_0,\widetilde{G}_h) = Cov(R_1^{2q}R_{1+h}^{2r},R_{1+h}^{2q}R_{1+h+h}^{2r}) \\
&= E[R_1^{2q}R_{1+h}^{2(q+r)}R_{1+2h}^{2r}]-E[R_1^{2q}R_{1+h}^{2r}]E[R_{1+h}^{2q}R_{1+2h}^{2r}] \\
&= E[E_U[R_1^{2q}]E_U[R_{1+h}^{2(q+r)}]E_U[R_{1+2h}^{2r}]] \\
&-E[E_U[R_1^{2q}]E_U[R_{1+h}^{2r}]]E[E_U[R_{1+h}^{2q}]E_U\left[R_{1+2h}^{2r}\right]] \\
&= E\left[F(q,U_1)F(q+r,U_{1+h})F(r,U_{1+2h})\right]-E\left[F(q,U_1)F(r,U_{1+h})\right]^2.
\end{aligned}$$

Hence,

$$\Sigma_\Delta(g_2,g_2) = E\left[F(2q,U_1)F(2r,U_{1+h})\right] - 2E\left[F(q,U_1)F(r,U_{1+h})\right]^2 + \\ +E\left[F(q,U_1)F(q+r,U_{1+h})F(r,U_{1+2h})\right] +$$

$$+2\sum_{i=1,i\neq h}^{\infty}\left(E\left[F(q,U_1)F(r,U_{1+h})F(q,U_i)F(r,U_{1+h+i})\right] - E\left[F(q,U_1)F(r,U_{1+h})\right]^2\right).$$

iii) By using similar arguments,

$$\begin{aligned} Cov(\widetilde{G}_0^1,\widetilde{G}_0^3) = Cov(R_1^{2p},R_1^{2u}) &= E[R_1^{2(p+u)}] - E[R_1^{2p}]E[R_1^{2u}] \\ &= E[(c(U_1)+\sqrt{V_1}\varphi_1))^{2(p+u)}] \\ &-E[(c(U_1)+\sqrt{V_1}\varphi_1))^{2p}]E[(c(U_1)+\sqrt{V_1}\varphi_1))^{2u}] \\ &= E\left[F(p+u,U_1)\right] - E\left[F(p,U_1)\right]E\left[F(u,U_1)\right], \end{aligned}$$

$$\begin{aligned} Cov(\widetilde{G}_0^1,\widetilde{G}_i^3) = Cov(R_1^{2p},R_{i+1}^{2u}) &= E[R_1^{2p}R_{i+1}^{2u}] - E[R_1^{2p}]E[R_{i+1}^{2u}] \\ &= E[(c(U_1)+\sqrt{V_1}\varphi_1)^{2p}(c(U_{i+1})+\sqrt{V_{i+1}}\varphi_{i+1})^{2u}] \\ &-E[(c(U_1)+\sqrt{V_1}\varphi_1)^{2p}]E[(c(U_{i+1})+\sqrt{V_{i+1}}\varphi_{i+1})^{2u}] \\ &= E\left[F(p,U_1)F(u,U_{i+1})\right] - E\left[F(p,U_1)\right]E\left[F(u,U_{i+1})\right] \\ &= E\left[F(p,U_1)F(u,U_{i+1})\right] - E\left[F(p,U_1)\right]E\left[F(u,U_1)\right] \end{aligned}$$

Similarly

$$\begin{aligned} Cov(\widetilde{G}_i^1,\widetilde{G}_0^3) &= Cov(R_{i+1}^{2p},R_1^{2u}) = \\ &= E\left[F(p,U_{i+1})F(u,U_1)\right] - E\left[F(p,U_{i+1})\right]E\left[F(u,U_1)\right] \\ &= E\left[F(p,U_{i+1})F(u,U_1)\right] - E\left[F(p,U_1)\right]E\left[F(u,U_1)\right]. \end{aligned}$$

Hence,

$$\begin{aligned} \Sigma_\Delta(g_1,g_3) &= E\left[F(p+u,U_1)\right] - E\left[F(p,U_1)\right]E\left[F(u,U_1)\right] + \\ &+\sum_{i=1}^{\infty}(E\left[F(p,U_1)F(u,U_{i+1})\right] - 2E\left[F(p,U_1)\right]E\left[F(u,U_1)\right] + E\left[F(p,U_{i+1})F(u,U_1)\right]). \end{aligned}$$

iv) Again, conditioning on $\sigma(U_1,U_2,...,U_n)$,

$$\begin{aligned} Cov(\widetilde{G}_0^1, \widetilde{G}_0^2) &= Cov(R_1^{2p}, R_1^{2q} R_{1+h}^{2r}) = E[R_1^{2(p+q)} R_{1+h}^{2r}] - E[R_1^{2p}] E[R_1^{2q} R_{1+h}^{2r}] = \\ &= E\left[F(p+q, U_1) F(r, U_{1+h})\right] - E\left[F(p, U_1)\right] E\left[F(q, U_1) F(r, U_{1+h})\right], \end{aligned}$$

$$\begin{aligned} Cov(\widetilde{G}_0^1, \widetilde{G}_i^2) &= Cov(R_1^{2p}, R_{i+1}^{2q} R_{i+1+h}^{2r}) = E[R_1^{2p} R_{1+i}^{2q} R_{i+1+h}^{2r}] - E[R_1^{2p}] E[R_{i+1}^{2q} R_{i+h+1}^{2r}] = \\ &= E\left[F(p, U_1) F(q, U_{1+i}) F(r, U_{1+h+i})\right] - E\left[F(p, U_1)\right] E\left[F(q, U_1) F(r, U_{1+h})\right]. \end{aligned}$$

For $i \neq h$

$$\begin{aligned} Cov(\widetilde{G}_i^1, \widetilde{G}_0^2) &= Cov(R_{1+i}^{2p}, R_1^{2q} R_{1+h}^{2r}) = E[R_{1+i}^{2p} R_1^{2q} R_{1+h}^{2r}] - E[R_{1+i}^{2p}] E[R_1^{2q} R_{h+1}^{2r}] = \\ &= E\left[F(p, U_{i+1}) F(q, U_1) F(r, U_{1+h})\right] - E\left[F(p, U_1)\right] E\left[F(q, U_1) F(r, U_{1+h})\right], \end{aligned}$$

while

$$\begin{aligned} Cov(\widetilde{G}_h^1, \widetilde{G}_0^2) &= Cov(R_{1+h}^{2p}, R_1^{2q} R_{1+h}^{2r}) = E[R_{1+h}^{2(p+r)} R_1^{2q}] - E[R_{1+h}^{2p}] E[R_1^{2q} R_{1+h}^{2r}] = \\ &= E\left[F(p+r, U_{1+h}) F(q, U_1)\right] - E\left[F(p, U_{1+h})\right] E\left[F(q, U_1) F(r, U_{1+h})\right]. \end{aligned}$$

Hence,

$$\begin{aligned} &\Sigma_\Delta(g_1, g_2) = E\left[F(p+q, U_1) F(r, U_{1+h})\right] - E\left[F(p, U_1)\right] E\left[F(q, U_1) F(r, U_{1+h})\right] + \\ &\quad + E\left[F(p+r, U_{1+h}) F(q, U_1)\right] - E\left[F(p, U_{1+h})\right] E\left[F(q, U_1) F(r, U_{1+h})\right] + \\ &\sum_{i=1, i\neq h}^{\infty} \left(E\left[F(p, U_{i+1}) F(q, U_1) F(r, U_{1+h})\right] - E\left[F(p, U_1)\right] E\left[F(q, U_1) F(r, U_{1+h})\right]\right) \\ &\sum_{i=1}^{\infty} \left(E\left[F(p, U_1) F(q, U_{1+i}) F(r, U_{1+h+i})\right] - E\left[F(p, U_1)\right] E\left[F(q, U_1) F(r, U_{1+h})\right]\right). \end{aligned}$$

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ISSN 1722-618X

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