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THE CORE-PERIPHERY MODEL WITH FORWARD-LOOKING EXPECTATIONS

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ABSTRACT

The 'core-periphery model' is vitiated by its assumption of static expectations. That is, migration (inter-regional or intersectoral) is the key to agglomeration, but migrants base their decision on current wage differences alone--even though migration predictably alters wages and workers are (implicitly) infinitely lived. The assumption was necessary for analytic tractability. The model can have multiple stable equilibria, so allowing forward-looking expectations would have forced consideration of the very difficult—perhaps even intractable—issues of global stability in non-linear dynamic systems. This paper's main contribution is to present a set of solution techniques—partly analytic and partly numerical—that allow us to consider forward-looking expectations. These techniques reveal a startling result. If quadratic migration costs are sufficiently high, allowing forward-looking behaviour has no impact on the main results, so static expectations are truly an assumption of convenience. If migration costs are lower, however, forward-looking behaviour creates history-vs-expectations considerations. In this case, agglomeration can be a self-fulfilling prophecy.

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1. Introduction

The economics of trade and agglomeration have recently been the subject of intensive theoretical and empirical research. While many models have been proposed—see surveys by Ottaviano (1998), Ottaviano and Puga (1997), and Fujita and Thisse (1996)—most researchers rely on the so-called 'core-periphery model' introduced by Krugman (1991a,b) and elaborated, *inter alios*, by Krugman and Venables (1990, 1995), Venables (1996), and Puga (1998); see Fujita, Krugman and Venables (1998) for a précis on the state of this literature.

The standard core-periphery model adopts an impressive number of simplifying assumptions in order to focus tightly on agglomeration forces created by pecuniary externalities. Perhaps the least attractive of these simplifying assumption concerns migrants' behaviour. Migration, after all, is the key to agglomeration in the model, yet migrants are assumed to be extraordinarily naïve. To wit, migrants are assumed to ignore the future entirely, basing their migration choices on current wage differences alone. This is awkward since migration alters wages in a predictable manner and the model implicitly assumes an infinite time horizon and infinitely lived workers.[†]

While these shortcomings were abundantly clear to the model's progenitors, they were necessary for analytic tractability. Because the core-periphery model can have multiple stable equilibria, allowing forward-looking expectations would have forced consideration of the very difficult—perhaps even intractable—issues of global stability in non-linear dynamic systems. (See Matsuyama (1991), Krugman (1991c) and Fukao and Benabou (1993) for explorations of such issues in much simpler models). Recent

[*] I would like to thank Pierre-Olivier Gournichas, Xavier Gabaix, Paul Krugman, Rikard Forslid, Jess Gaspar and Federica Sbergami for comments and suggestions.

[†] The core-periphery model has two main variants. The footloose-labour variant (Krugman 1991a) has agglomeration driven by inter-regional labour migration within a single sector; here regional real wage differences motivate the migration. The vertical-linked-industries variant (Venables 1996) has agglomeration driven by intersectoral migration within each region; here intersectoral nominal wage differences motivate migration. These two variants can be shown to have identical local stability properties under certain conditions. Our paper works exclusively with the footloose-labour variant although we conjecture that our findings would also hold for the vertical-linked-industries variant.

advances in numerical techniques (see Judd 1989) and computing power, however, make it fairly simple to deal with such problems.

This paper's main contribution is to introduce a set of solution techniques—partly analytic and partly numerical—that allow us to tackle the difficulties raised by forward-looking expectations in the core-periphery model.

Using these techniques we find a very startling result. The inclusion of forward-looking expectations leads to absolutely no change in the main results of the standard core-periphery model. In particular, the 'break' and 'sustain' points are not affected (these points identify, respectively, the levels of trade cost where symmetry becomes unsustainable and where full agglomeration becomes sustainable). Moreover, we show that under certain parameter values (mainly the level of migration costs), even the model's global stability properties are unaffected by the inclusion of forward-looking expectations. Under these conditions the assumed myopia of migrants—upon which the model relies—is truly an assumption of convenience.

We also show that when migration costs are sufficiently low, allowing for forward-looking dynamics radically complicates the model's global stability properties. In particular, interesting 'history versus expectations' considerations arise and this permits us to address formally the possibility that the agglomeration can be a self-fulfilling prophecy.

The paper also makes a number less important and less novel points that essentially serve to validate the literature's current practices. The first and most obvious of these is that the standard core-periphery model (CP model henceforth) is a dynamic model, and that its main results are propositions about local stability of various equilibria. That is, the main questions addressed by the CP model are, "When will agglomerations occur when they have not yet occurred?" and "When will they be reversed when they have?"[*] These, of course, are nothing more than questions about the local stability of the symmetric and core-periphery equilibria. Second, in order to evaluate the impact of allowing forward-looking expectations, we re-derive the main results (i.e. local stability results) using formal methods for judging local stability. This *per se* is not new; Puga (1996) was the first to do it. What is demonstrated here is the slightly more general and slightly less obvious result that the informal methods commonly used in the literature for evaluating stability are mathematically equivalent to formal methods. Furthermore, we evaluate global stability properties of the standard CP model using Liaponov's direct method. The literature typical ignores global stability properties, but when it does, it relies on informal methods (e.g. the economy moves to the nearest stable equilibria). Again, we show that these informal methods can be validated using Liaponov's method.

The rest of the paper is in six parts. The next section presents the standard CP model. The subsequent section (section 3) studies the model with static and forward-looking expectations, showing that the *ad hoc* migration equation of the standard CP model is consistent with quadratic adjustment costs and static expectations. The next

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[*] In the words of Fujita, Krugman and Venables (1998, Chapter 5), the model's principle concern is "how the interactions among increasing returns at the firm level, transport costs, and factor mobility can cause spatial economic structure to emerge and change."

section (section 4) uses formal stability analysis to show that the CP model's informal stability-checking techniques are equivalent to formal techniques. Section 5 shows that introducing forward-looking expectations changes none of the basic results from the standard model. The only new result concerns the global stability properties of the model, namely that under certain parameter values the model is subject to history-versus-expectations considerations. The last section presents some concluding remarks.

2. The Standard Core-Periphery Model

Krugman, Fujita and Venables (1998 Chapter 5)—henceforth FKV—provides a definitive exposition of the standard CP model. This standard model assumes two initially symmetric regions (north and south), two factors of production (workers L and agriculturists A) and two sectors (manufactures X and agriculture Z). The monopolistically competitive X-sector employs only L to produce output and faces increasing returns with a linear cost function. In particular, production requires a fixed cost of F units of L in addition to a_X units of L per unit of output. The Z-sector produces a homogeneous good under perfect competition and constant returns using only A. Z and X are traded. Z trade is costless, but trade in X is inhibited by frictional (i.e. iceberg) trade costs. Specifically, $\tau \geq 1$ units of an X-variety must be shipped to sell one unit in the other region. As usual, τ is viewed as capturing all the costs of selling to distant markets, not just transport costs, and τ-1 is the tariff-equivalent of these costs.

Preferences of the representative consumer are:

$$U_\tau = C,\ C \equiv C_X^{\mu} C_Z^{1-\mu},\ C_X \equiv \left(\int_{i=0}^{n+n^*} c_i^{\,1-1/\sigma} \right)^{\frac{1}{1-1/\sigma}}; \qquad \sigma > 1,\, 0 < \mu < 1 \qquad \textbf{(1)}$$

where C_X and C_Z are consumption of the X-variety composite and Z (respectively), μ (a mnemonic for manufacturing) is the expenditure share on X-varieties, n and n* are the number (mass) of north and south varieties, and σ is the constant elasticity of substitution between X-varieties.

Regional supplies of A as well as the global supply of L are fixed, but the inter-regional distribution of L is endogenous with L flowing in response to real wage differences. As in FKV, migration is governed by the *ad hoc* migration equation:

$$\dot{s}_L = (\omega - \overline{\omega}) s_L; \qquad \omega \equiv \frac{w}{P},\ \ \overline{\omega} \equiv s_L \omega + (1 - s_L)\omega^*,\ \ s_L \equiv \frac{L}{L^w},\ P \equiv p_A^{1-\mu} \left(\int_{i=0}^{n+n} p_i^{1-\sigma} di \right)^{\frac{\mu}{1-\sigma}} \quad \textbf{(2)}$$

where L is the northern labour supply, L^w is the world labour supply, ω, ω* and $\overline{\omega}$ are the northern, southern and average real wages, w is the northern wage, P is the north's region-specific perfect price index. Analogous definitions hold for w* and P*. Observe that migration occurs unless labour is concentrated in a single region or real wages are equalised.

2.1. *Intermediate Results and Normalisations*

Utility optimisation yields a constant division of expenditure between X and Z, and CES demand functions for X varieties, which may be written as:

$$c_j \equiv \frac{p_j^{-\sigma}\mu E}{\int_{i=0}^{K+K^*} p_i^{1-\sigma}di} \quad \textbf{(3)}$$

where E is region-specific expenditure.

On the supply side, free trade in Z equalises northern and southern agriculturists' wage rates, viz. w_A and w_A*, as long as both nations make some Z. Under all circumstances, both countries produce some Z as long as $\mu<1/2$. We maintain this assumption throughout, so taking Z as numeraire $p_Z=w_A=w_A{}^*=1$. In the X-sector, 'milling pricing' is optimal, so measuring X in units such that $a_X=(1-1/\sigma)$, the price of a northern variety in its local and export markets are (respectively):

$$p = w, \qquad p^* = w\tau \quad \textbf{(4)}$$

Similar pricing rules hold for southern firms.

Since mill pricing is optimal, operating profit (call this π) is the value of sales divided by σ, so*:

$$\pi = \frac{w(L-nF)}{(\sigma-1)n} \quad \textbf{(5)}$$

The free entry condition is that the number of northern firms rises to the point where π=wF. Using (5), the equilibrium number and scale of firms are:

$$n = L/\sigma F, \quad \bar{x} = \sigma F \quad \textbf{(6)}$$

where $\bar{x}$ is the equilibrium output of a typical firm. Similar expressions define the analogous southern variables.

The market for northern X-varieties must clear at all moments. The equilibrium output per northern firm is σF and the producer price is w, so using (3) and rearranging, the North's aggregate market clearing condition is:

$$w\bar{x} = R; \qquad R \equiv \frac{w^{1-\sigma}\mu E}{nw^{1-\sigma} + \phi n^* w^{*1-\sigma}} + \frac{\phi w^{1-\sigma}\mu E^*}{\phi n w^{1-\sigma} + n^* w^{*1-\sigma}} \quad \textbf{(7)}$$

where R is a mnemonic for 'retail sales' and $\phi=\tau^{1-\sigma}$ measures 'free-ness' (phi-ness) of trade.† Note that the free-ness of trade rises from ϕ=0 (with infinite trade costs) to ϕ=1 with zero trade costs. This expression and its southern equivalent are often called the

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* Total northern production of X is $(L-nF)/a_X$. Using our choice of units and symmetry of varieties yields the expression for π.

† Due to markup price and iceberg trade costs, the value of a typical firm's retail sales at consumer prices always equals its revenue at producer prices. Thus R is also a mnemonic for revenue.

wage equations since using (6), (7) can be written in terms of w and w* and the equilibrium w, w* must satisfy the pair of market-clearing conditions.

For convenience, we follow Krugman (1991) and choose units of the world endowments, L^w and A^w, such that w=w*=1 in the symmetric equilibrium. From the wage equations, this holds when $L^w=1$ and $A^w \equiv 2A = \mu/(1-\mu)$. Note that these choices also imply that $L \equiv s_L \equiv L/L^w$ and that w=1 in the core-periphery outcome.*

3. Static and Forward-Looking Expectations

The main purpose of this paper is to consider the theoretical implications of expanding the CP model to include forward-looking expectations. To this end, this section shows that the standard CP model is a special case of a more fully specified dynamic model. An unintentional dividend of this demonstration is to provide more rigorous foundations for the *ad hoc* migration equation used in the standard model.†

3.1. The Dynamic Problem and Optimal Migration Behaviour

Since the standard CP model is a dynamic model, intertemporal preferences of agents need to be identified. Specifically, assume that the instantaneous utility function (sometimes called the felicity function) is as in (1) and intertemporal preferences are:

$$U_t = \int_{s=t}^{\infty} e^{-\rho(s-t)} C(s) ds \qquad \textbf{(8)}$$

where ρ is the constant, subjective discount rate.

Optimal migration behaviour is simple to derive. Consider a southern household that divides its labour between north and south to maximise the real earnings of its members less some adjustment cost. Specifically, migration involves a cost that is quadratic in the flow as a proportion of the sending and receiving region labour forces, viz. $\gamma(m^2/L(1-L))/2$. Note that migration costs rise as the flow becomes a large share of the receiving or sending population.

Observing that the real wage is an index for worker's instantaneous utility (i.e. P is a perfect price index), utility optimisation requires the household (which owns one unit of labour) to choose the optimal migration time path to solve:

$$\max_m \int_0^{\infty} e^{-\rho t} \left(\omega L + \omega^* (1-L) - \frac{\gamma}{2} \left(\frac{m^2}{L(1-L)} \right) \right) dt; \quad m = \dot{L} \qquad \textbf{(9)}$$

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* KFV takes L^w as μ and A^w as 1-μ, but wages are unity as long as L^w/A^w equals μ/(1-μ).

† FKV (chapter 5) assert that the *ad hoc* migration behaviour in the standard model has no deep justification, but note that it is equivalent to 'replicator dynamics' (a concept routinely used in evolutionary game theory). This section shows that *ad hoc* migration equation does have a deep justification; it is consistent with dynamic optimisation, quadratic adjustment costs and static expectations.

This allows for almost any sort of migration behaviour, including the possibility that migrants will return and re-migrate.* Taking $m \equiv \dot{L}$ as the control variable, the current valued Hamiltonian is $\left(L\omega + (1-L)\omega* - \gamma(m^2/L(1-L))/2\right) + Wm$ where W is the co-state variable that captures the asset value of migration. The necessary conditions are $m = \dot{L} = WL(1-L)/\gamma, \dot{W} = \rho W - (\omega - \omega*)$ which must hold at all moments and the endpoint, or transversality condition $\lim_{t\to\infty} e^{-\rho t} Wm = 0$. Using our units-convention (L^w=1 and L=s_L) the migration equation is:

$$\dot{s}_L = Ws_L(1-s_L)/\gamma \qquad \textbf{(10)}$$

where W is the asset value of migrating now. As usual, W is governed by an asset-pricing-like expression:

$$\dot{W} = \rho W - (\omega - \omega*) \qquad \textbf{(11)}$$

3.2. The Role of Expectations

If migrants assume that the current real wage gap will persist forever, then (11) can be solved to yield $W = (\omega - \omega*)/\rho$. Using this in (10) implies:

$$\begin{aligned}\dot{s}_L &= (\omega - \omega*)s_L(1-s_L)/\gamma\rho \\ &= (\omega - \overline{\omega})s_L\end{aligned} \qquad \textbf{(12)}$$

The second expression, which is identical to FKV's assumed migration equation, follows from the first by choice of time units (such that ργ=1) and the definition of $\overline{\omega}$. Summarising this result, we have:

Proposition 1:

The migration equation of the standard core-periphery model is consistent with optimal migration behaviour subject to quadratic adjustment costs and static expectations. When migrants' expectations are forward-looking and model-consistent (i.e. migrants use the model's relationships to predict future real wages), the model's behaviour depends on the two non-linear differential equations (10) and (11).

4. Stability Analysis with Static Expectations

The standard CP model is a dynamic model and its main results concern the model's local stability properties. This simple fact, however, can usually be ignored since the standard approach to stability analysis (i.e. finding break and sustain points) relies on a highly intuitive, informal method for checking stability. Unfortunately, it is not obvious

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* The main restriction is that we rule out an infinite number of migrations in a finite period. Moreover since the co-state variable must be a continuous function of time, the migrants cannot expect to change their migration time path in the future.

that the informal method (introduced by Krugman 1991a,b and extended by Puga 1996, and FKV) carries over to the model with forward-looking expectations. Thus, in order to evaluate the impact that forward-looking expectations have on the standard model's main results, we must first establish the mapping between well-known formal stability-evaluation methods and the informal methods used in the CP literature.

4.1. Informal Stability Analysis: Symmetric Equilibrium

Stability of equilibria in the core-periphery model is typically established using the following informal method. Starting from an equilibrium point (symmetry or full agglomeration), the allocation of L is perturbed by exogenously moving a small mass of labour between regions. Firms are assumed to instantaneously enter or exit until pure profits in both regions are restored to zero. For the symmetric equilibrium, one checks the change in the real wage gap ω-ω*. If the change is negative, the perturbation creates self-correcting forces, i.e. the displaced workers would wish to move back to their original location. Otherwise the equilibrium is unstable since additional workers would be attracted to the receiving region. For the full agglomeration equilibrium, which we sometimes call the CP outcome, the stability test is based on the level of regional real wages rather than on the change in the real wage gap. Namely, if the perturbation produces a level of the real wage in the periphery that exceeds the real wage in the core, the equilibrium is said to be unstable. Otherwise, it is stable.

These two informal stability tests can be summarised symbolically as follows:

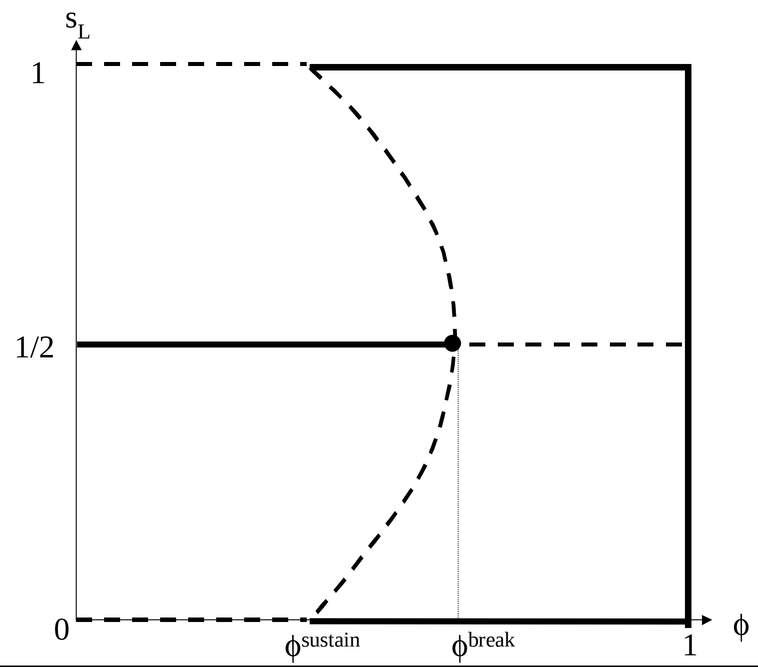
$$\left.\frac{d(\omega-\omega^*)}{dL}\right|_{sym} < 0, \qquad \omega_{CP} > \omega^*_{CP} \tag{13}$$

where 'sym' and 'CP' indicate that the variables are evaluated at the symmetric and core-periphery outcomes, respectively. The level of trade free-ness where the first expression in (13) holds with equality is called the 'break' point, while the level where the second expression holds with equality is called the 'sustain' point.

Using this method, one can establish (see FKV chapter 5) that: (i) the symmetric equilibrium is stable only for sufficiently low levels of trade free-ness, (ii) that CP outcomes are stable only for sufficiently high levels of trade free-ness, and (iii) that there is a range of ϕ for which both the symmetric and core-periphery outcomes are stable. This is often illustrated with the so-called 'tomahawk' diagram (see figure 1). The diagram shows the stable equilibria with heavy solid lines and unstable equilibria with dashed lines. The 'tomahawk' moniker comes from viewing the stable-part of the symmetric equilibrium as the handle of a double-edged axe.

Intuition for how such a complicated configuration can be generated from such a simple model is gained by studying the impact of trade costs on the agglomeration and dispersion forces. While garnering such intuition is important, the analysis is tangential to the main thrust of the paper. The reasoning, therefore, is relegated to Appendix 1.

Figure 1: The tomahawk diagram.



4.2. *Formal Local Stability Analysis*

As far as stability is concerned, the core-periphery model can be reduced to a single non-linear ordinary differential equation (ODE), namely:

$$\dot{s}_L = s_L(1-s_L)\Omega[s_L]; \qquad \Omega \equiv \omega - \omega^* \tag{14}$$

The function $\Omega[s_L]$ cannot be written explicitly since the wage equations cannot be solved for w and w* (this is why so many applications of CP model rely on numerical simulations).

The formal approach to studying the local stability properties of (14) at various points is simple. One approximates the non-linear ODE with a linear ODE and then checks the sign of the coefficient on s_L. If the coefficient is negative, the system is locally stable (with a negative coefficient the state variable shrinks towards its steady-state level as time passes). Otherwise it is locally unstable. The optimal linear approximation is given by the first-order Taylor expansion, so the best linear approximation and corresponding stability test are:

$$\begin{aligned} \dot{s}_L &= \left(s_L(1-s_L)\frac{d\Omega[s_L^o]}{ds_L} + (1-2s_L)\Omega[s_L^o] \right)(s_L - s_L^o) \\ & s_L(1-s_L)\frac{d\Omega[s_L^o]}{ds_L} + (1-2s_L)\Omega[s_L^o] < 0 \end{aligned} \tag{15}$$

where s_L^o is the equilibrium point under investigation. Note that at the symmetric equilibrium (i.e. s_L=1/2) the necessary and sufficient condition for local stability is that $d(\omega-\omega^*)/ds_L<0$. At the core-periphery outcome (i.e. s_L=1) the necessary and sufficient condition is $(\omega-\omega^*)<0$. These line up exactly with the informal stability test proposed in (13). To summarise this finding we write:

Proposition 2:

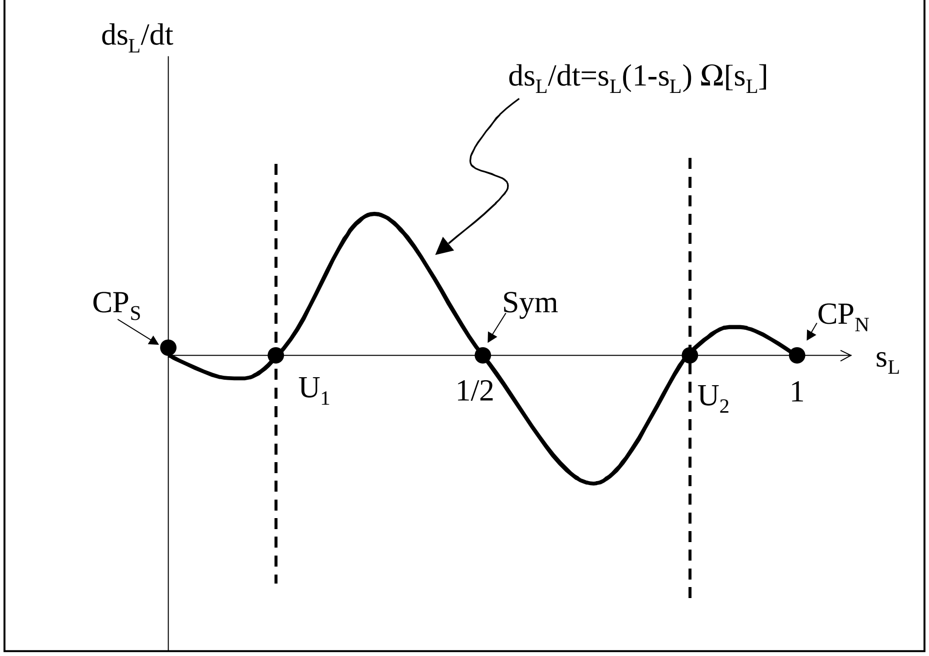
The informal stability test of the CP model corresponds exactly to formal, local stability analysis of the CP model.

4.3. Global Stability Analysis: Liaponov's Direct Method

Local stability analysis is fine for most uses (such as finding the break and sustains points). It is not sufficient, however, for fully characterising the model's behaviour when s_L is away from a long-run equilibrium (e.g. when the process of agglomeration is 'en route'). The economic geography literature typically avoids discussing what happens between long-run equilibria, but where is does it relies on a heuristically approach. Namely, it is asserted that the system approaches the nearest stable equilibrium that does not require crossing an unstable equilibrium. As with the local stability analysis, this heuristic approach can be justified formally.

One simple approach to global stability analysis of non-linear ODEs is called

Figure 2: Global Stability Analysis



Liaponov's direct method. Instead of working with a potentially complicated function of the state variable (the solution to the non-linear ODE for s_L, in this case), one works with a simple function—defined on a specific region—that attains its minimum at the long-run level of s_L. If the simple function (called the Liaponov function) and its domain are chosen judiciously, one can show that the value of the function continuously approaches its minimum as time passes and that this implies that the state variable also approaches its long-run equilibrium as time passes. This is sufficient for showing that the system is globally stable in the region (see Beavis and Dobbs, 1990 p.167 for details).

As is well known, the dynamics of the CP model depends upon the level of trade free-ness. There are three qualitative cases (see figure 1). When trade free-ness is very low (high trade costs), only the symmetric equilibrium is stable. When trade free-ness is very high, only the CP equilibria are stable. For an intermediate level of trade costs the model has five equilibria, three of which are stable (the symmetric and the two CP outcomes) and two of which, U_1 and U_2, are unstable.

The most interesting case in terms of global stability analysis is third. Figure 2 shows the model's ODE in this case. What we wish to show is that the system is globally stable in the sense that the system always converges to one steady state or another regardless of initial conditions.

Consider first stability in the open set $s_L \in (U_1, U_2)$. The Liaponov function we choose is $(s_L-1/2)^2/2$. This satisfies the regularity conditions of Theorem 5.24 in Beavis and Dobbs (1990), namely the equilibrium point and initial point are in the set, the function is always positive on the set and the value of the function is zero at the equilibrium. Most importantly, $\dot{V} = (s_L - 1/2)\dot{s}_L < 0$ for all t and for all non-equilibrium values of s_L in the set. To see this, note that s_L is increasing when s_L is less than ½, but decreasing when s_L exceeds ½. Since V is always decreasing and attains its minimum at the symmetric steady state, we know that s_L converges to the symmetric steady state whenever the initial value is in the $s_L \in (U_1, U_2)$ range. This range is sometimes called the symmetric equilibrium's 'basin of attraction'.

Next consider stability in the $s_L \in (U_2,1]$ interval with the Liaponov function $(s_L-1)^2/2$. This function meets all the regularity conditions and time-derivative condition, so we know that $s_L \in (U_2,1]$ is the basin of attraction for the core-in-the-north CP outcome. Similar reasoning implies that $s_L \in [0,U_1)$ is the basin of attraction for the core-in-the-south CP outcome.

Finally, analogous reasoning can show that the CP model is globally stable in the two simpler cases when only the symmetry outcome is stable and when only the CP outcome is stable. Moreover, in the latter case, it is straightforward to establish that (1/2,1] and [0,1/2) are, respectively, the basins of attractions for the core-in-the-north and core-in-the-south CP outcomes.

We summarise these results as follows.

Proposition 3:

The CP model is globally stable in the sense that regardless of initial conditions, migration drives the system to one steady state or another. Furthermore, the heuristic approach to determining the system's destination is confirmed by formal methods.

5. Forward-Looking Workers: History vs Expectations Revisited

One of the most persistent critiques of the CP model turns on its assumption that worker care only about the current wage difference in making their migration decision. This section relaxes this assumption by allowing for forward-looking behaviour. Importantly, it shows that doing so implies absolutely no changes in the local stability analysis (i.e. the break and sustain points) derived in the standard model. In this sense, the assumption of static expectations can be viewed as a convenient simplification.

The section also shows that true usefulness of forward-looking expectations lies in its impact on the models global stability properties. In particular, we show that with forward-looking expectations the dynamics becomes much richer, but radically more difficult. In particular, we find that the CP model displays ‘history versus expectations’ behaviour.

5.1. Formal Local Stability Analysis

The first task is to map out local stability of the symmetric and CP equilibria. As before the procedure is to use a linear approximation to the non-linear system given by (10) and (11). The linearised system is $\dot{x} = J\,(x - x^{ss})$ where $x \equiv (s_L, W)^T$ and J is the Jacobian matrix (i.e. matrix of own and cross partials) evaluated at a particular steady state. Specifically, J is:

$$J = \begin{bmatrix} W(1-2s_L)/\gamma & s_L(1-s_L)/\gamma \\ -\dfrac{d\Omega}{ds_L} & \rho \end{bmatrix}; \qquad \Omega \equiv \omega - \omega^* \qquad \textbf{(16)}$$

Local stability is determined by checking J’s eigenvalues at the symmetric and CP outcomes. As usual, we can establish saddle path stability by finding one negative eigenvalue and one positive eigenvalue (if the eigenvalues are complex, then the test involves the signs of the real parts).*

One useful fact reduces our work. A standard result from linear algebra is that the determinant of J equals the product of the eigenvalues (Beavis and Dobbs, 1990 p.161). Thus the system is saddle-path stable, if and only if det(J)<0. The determinant det(J) is equal to $(d\Omega/ds_L)s_L(1-s_L)/\gamma-\rho W(1-2s_L)/\gamma$, so for the symmetric equilibrium, the stability test is $(d\Omega/ds_L)/4\gamma<0$ and in the CP outcome it is $\rho W/\gamma<0$, where the expressions and derivatives are evaluated at the appropriate steady state. Noting that W in the CP equilibrium equals $(\omega_{CP} - \omega^*_{CP})/\rho$, we see that the informal local stability test for the CP model with static expectations—viz. expression (13)—is equivalent to the formal local stability test for the CP model with forward-looking expectations. To summarise this striking result we write:

* See the appendix to Barro and Sala-i-Martin (1995) for details and an excellent exposition of local stability and phase diagram analysis.

Proposition 4:

The informal stability test of the CP model with static expectations is exactly equivalent to formal, local stability analysis of the CP model with forward-looking expectations.

In particular, the break and sustain points in the model with forward-looking expectations are identical to those in the model with static expectations.

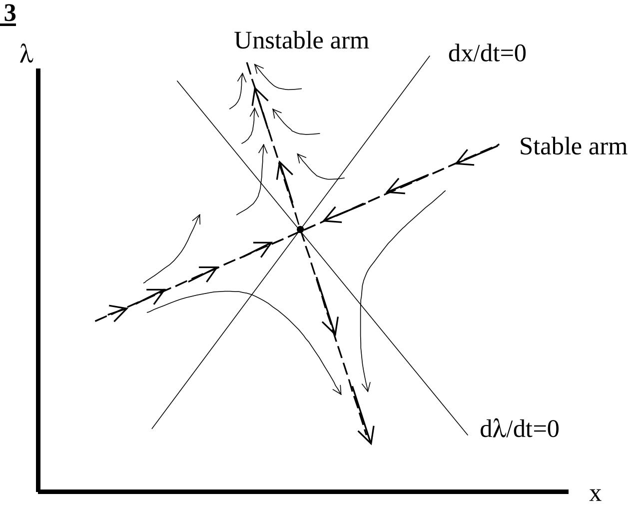
5.2. Global Stability Analysis with Multiple Equilibria

When trade costs are such that the CP model has a unique stable equilibrium, local analysis is sufficient. After any shock, W will jump to put the system on the saddle path leading to the unique stable equilibrium (if it did not, the system would diverge and thereby violate a necessary condition for intertemporal optimisation). The same cannot be said when trade costs are such that the model has multiple stable steady states. The point is very simple. With multiple stable equilibria, we will have multiple saddle paths. In principle, these may overlap so it is not clear which path the system must jump to. In other words, the interesting possibility of history versus expectations may arise. We turn now to exploring this possibility.

5.2.1. Making Progress by Reversing Time

Dealing with non-linear differential equations is difficult, generally requiring

Figure 3



qualitative analysis using a phase diagram. While this works well with unique steady states, it is not sufficient for dealing with the interesting case of multiple steady states. Recent advances in computing speed and simulation software, however, have made it possible to tackle such problems with a desktop computer. Before using these techniques, we provide a simple example to illustrate the logic of the technique.

Consider the simple system of two linear ODEs depicted in figure 3. This system is saddle path stable as drawn, so if it starts out on the stable arm, which corresponds to the negative eigenvector, it converges to the steady state. If it starts out anywhere else, it diverges. Importantly, the system moves towards the unstable arm (which is the positive eigenvector) for any initial condition that is off of the stable arm. In this sense, the instability is very stable. That is to say, the unstable arm is like a downward sloped trough into which all motion is attracted. More to the point, this feature makes it easy to find the unstable arm via numerical simulation.

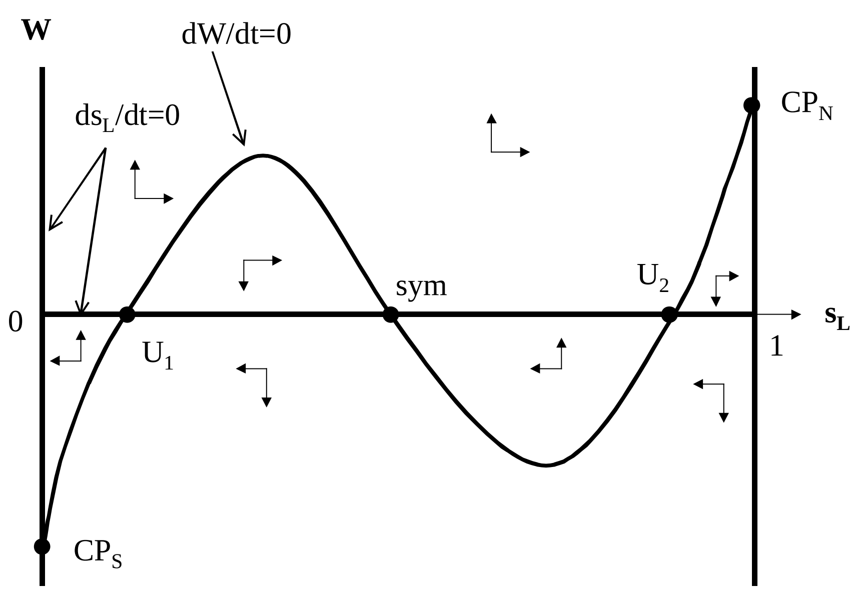
The ease of identifying the unstable arm becomes interesting when we reverse time. When time is run backward, the stable arm becomes the unstable arm and vice versa.* As a result, we can identify the stable arm (normally a very difficult task in a non-linear model) by looking for the unstable arm in reverse time. Mechanically, we start slightly off the steady state and then run time backward. The resulting path traces out what will be the stable arm in normal time.

With non-linear ODEs we still need to rely on numerical simulation to find the saddle path. Yet given the speed of recent PCs, the numerical approximation to the true path can be made to be very accurate (an algorithm showing how the find these paths is available from the author upon request).

5.2.2. Phase Portraits and Overlaps

When trade costs are such that the model has multiple stable steady states, a number of interesting possibilities arise (this corresponds to the overlapping heavy lines

Figure 4



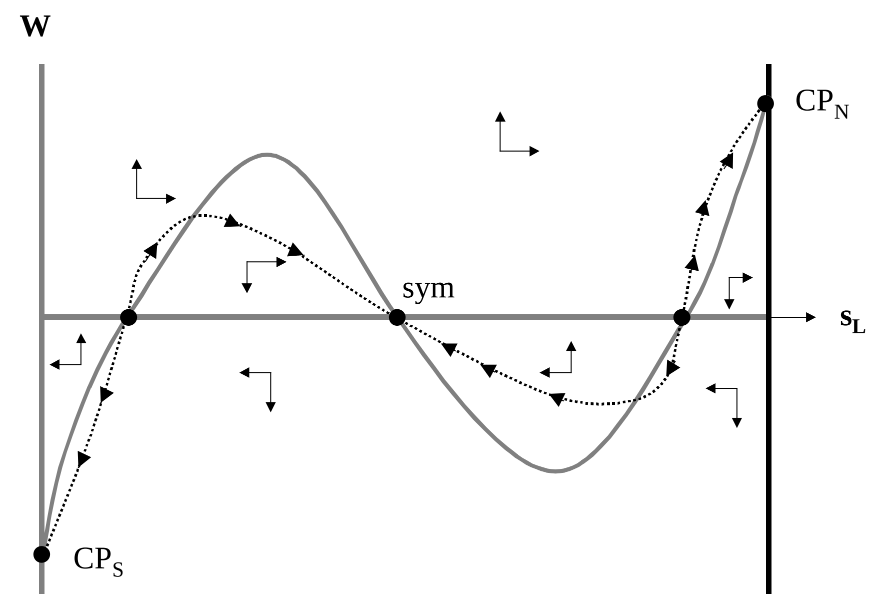
in figure 1). The phase portrait for this case is shown in figure 4. Note that the $\dot{s}_L = 0$ schedule consists of the H-shape line made up of the horizontal axis between 0 and 1, and

* To see what we mean by reversing time, consider the differential equation $\dot{x} = x_0 e^{at}$. If a>0 to ODE is unstable in normal time, i.e. t∈[0,∞). In reverse time, i.e. t∈(-∞,0], the system becomes stable since $\dot{x} = x_0 e^{-as}$ for s∈[0,∞).

the s_L=1 and s_L=0 lines. The $\dot{W} = 0$ schedule is the wave-like curve that crosses the x-axis three times. The arrows qualitatively indicate the direction of motion off of the isokines.

Before turning to the computer, a few facts should be pointed out. First, in normal time, the U_1 and U_2 equilibrium are locally unstable (i.e. the real parts of their eigenvalues are both positive), so in reverse time they are globally stable (i.e. have two negative eigenvalues). What this means is that in reverse time they are 'sinks' in the sense that a particle starting from anywhere in the diagram will eventually end up either at U_1 or U_2.[*] In particular, the saddle paths that end up at CP_N, CP_S, or 'sym' in normal time, must originate in either U_1 or U_2. Second, horizontal motion limits to zero as the system gets close to either s_L=0 or 1, given (10). As a result, the system will never run into the s_L boundaries except when it starts on the saddle path leading to one of the CP

Figure 5: γ large



outcomes (even then, the system requires infinite time to reach equilibrium).

Using computer simulation (in reverse time), it is possible to find the saddle paths for various parameter values. Consider three qualitatively different cases (in all simulations we take σ=5, μ=4/10, ρ=1/10 and ϕ=1/10).

The first case is when γ, the migration cost parameter, is very large, so horizontal movement is very slow. This is shown in Figure 5. Importantly, there is no overlap of saddle paths in this case, so the global stability analysis with static expectations is exactly right. That is, the basins of attraction for the various equilibria are the same with static and forward-looking expectations. We summarise this as follows:

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[*] This is not quite airtight since there is still the possibility of a limit cycle.

Proposition 5:

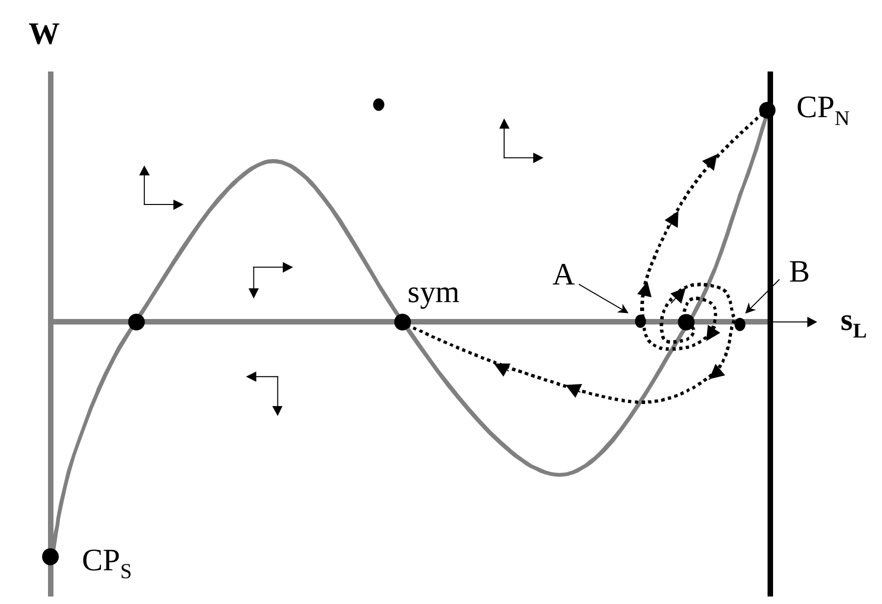
When migration costs are sufficiently high, the global stability properties of the forward-looking CP model are qualitatively the same as those of the static CP model.

Taking this together with proposition 4 implies that the standard assumption of static expectations is really an assumption of convenience when migration costs are sufficiently high.

5.2.3. History versus Expectations

The second case, shown in Figure 6, is for an intermediate value of migration

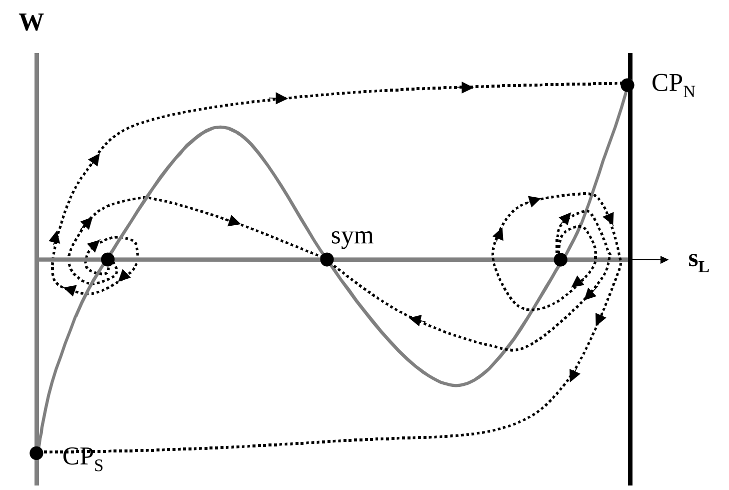
Figure 6: γ intermediate



costs. Here there is some overlap since the Jacobian evaluated at either unstable equilibrium has complex eigenvalues—this means that the system spirals out from U_1 and U_2 in normal time. (The figure shows only the saddle paths in the right side of the diagram since the left side is the mirror image of the right).

The existence of overlapping saddle paths changes things dramatically. If the economy finds it itself with a level of s_L that lays in the overlap, namely the interval (A,B) shown in the figure, then a fundamental indeterminacy exists. Both saddle paths provide perfectly rational adjustment tracks. That is, forward-looking workers who are fully aware of how the economy works could adopt the path leading to the symmetric outcome. It would, however, be equally rational for them to jump on the track that will take them to the CP_N outcome. Which track is taken cannot be decided in this model. However, stepping slightly outside the model, one can believe that workers chose the path that they think other workers will take. In the words of Matsuyama (1991) and Krugman (1991c), expectations, rather than history, can matter.

Figure 7: γ low



The final case is the most spectacular. Here migration costs are very low, so horizontal movement is quite fast. As a result, the saddle path for CP_N originates from U_1 rather than U_2. Interestingly, the overlap of saddle paths includes the symmetric equilibrium. This raises the possibility that the economy could jump from the symmetric equilibrium onto a path that leads it to a CP outcome mere because all the workers expected that everyone else was going to migrate. Plainly, this raises the possibility of a big-push drive by a government having some very dramatic effects.

Appendix 2 shows the computer generated output used to draw figures 6 and 7.

Finally note that the region of overlapping saddle paths will never include a CP outcome. Thus, although one may 'talk the economy' out of a symmetric equilibrium, one can never do the same for an economy that is already agglomerated.

5.2.4. Necessary Conditions for Overlap

While it is difficult to characterise the constellation of parameters which corresponds to each of the three cases, we can easily find a necessary and sufficient condition for there to be some overlap of saddle paths. If the eigenvalues of the Jacobian evaluated at the unstable equilibria are complex, then there must be some overlap, and if there is some overlap the eigenvalues must be complex. The eigenvalues at U_2 are $\left(\rho \pm \sqrt{\rho^2 - 4(d\Omega/ds_L)s_L(1-s_L)/\gamma}\right)/2$, so we get complex roots when migration costs are sufficiently low, namely when:

$$\gamma < \frac{4(d\Omega/ds_L)s_L(1-s_L)}{\rho^2} \qquad (17)$$

Consequently, we can write:

Proposition 6:

The possibility of history-versus-expectations arises whenever the costs of migration are low relative to the patience of workers (i.e. $1/\rho^2$) and the impact that migration has on the real wage gap (i.e. $d\Omega/ds_L$) is large.

6. Summary and Concluding Remarks

This paper shows that informal stability-checking techniques widely employed in the ‘economic geography’ literature can be validated using formal methods. It also show that the standard model’s assumption of static expectations is not as bad is it seems. In particular, allowing forward-looking expectations changes none of the standard results, since these essentially concern local stability (i.e. finding break and sustain points). Adding forward-looking expectations, however, does enrich the model by opening the door to history-versus-expectations considerations. To demonstrate this, the paper introduces a simulation technique that permits full characterisation of the transitional dynamics of the CP model’s non-linear differential equations.

This paper shows that putting forward-looking behaviour into the CP model and addressing out-of-steady-state dynamics is quite simple. It seems, therefore, that it should be easy to formally model the rich set of dynamic forces emphasised by classic growth scholars such as Perroux (1955) and Hirschman (1958)

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Appendix 1: Decomposing Effects (Circular Causalities and Local Competition)

There are three distinct forces governing stability in this model. Two of them—demand-linked and cost-link circular causality (also called backward and forward linkages)—are forces for agglomeration forces, i.e. they are de-stabilising. The third—the local competition effect—is a force for dispersion, i.e. it is stabilising. The level of trade costs changes the magnitude of each of the three forces. We turn now to illustrating the forces and their dependence on trade costs.

The forces affecting the real wage gap are easily demonstrated at the symmetric equilibrium due to a very handy fact. Starting from symmetry, all effect will be equal and opposite. For instance, if the perturbation raises the northern wage, then it will lower the southern wage with identical magnitudes. The real wage gap created by a migration shock is therefore exactly twice the change in ω, so we limit our study to $d\omega$.

The proportional change in the northern wage $d\omega/\omega$ equals dw/w minus dP/P. Consider first dw/w; w and w* are determined by the two market-clearing conditions—expression (7) and its southern analogue. Finding dw/w would—in general—require us to manipulate the total derivatives of the two market-clearing conditions (also known as the wage equations). The equal-and-opposite fact, however, provides a shortcut. Using dw = -dw*, dn=-dn*, dL=-dL* and dE=-dE*, dw/w can be found from (7) alone. Noting that E and E* are themselves functions of the w's and L's, and re-writing (7), we get:

$$w = R\left[w, w^*, n, n^*, E[w, L], E^*[w^*, L^*]\right] \qquad \textbf{(18)}$$

Differentiating this yields:

$$\frac{dw}{w} = \frac{R_{\Delta n}}{b}dn + \frac{R_{\Delta E}}{b}\frac{\partial E}{\partial L}dL; \quad b \equiv 1 - R_{\Delta w} - R_{\Delta E}\frac{\partial E}{\partial w} \qquad \textbf{(19)}$$

where the notation $R_{\Delta n}$, $R_{\Delta w}$ and $R_{\Delta E}$ is shorthand for $(\partial R/\partial n - \partial R/\partial n^*)$, $(\partial R/\partial w - \partial R/\partial w^*)$ and $(\partial R/\partial E - \partial R/\partial E^*)$.* Note that 'b' is always positive.

Using the definition of the perfect price index P, we get:

$$\frac{dP}{P} = \frac{P_{\Delta n}}{P}dn + \frac{P_{\Delta w}}{P}dw \qquad \textbf{(20)}$$

where again we have used implicit function notation, viz. P[w,w*,n,n*], for P and the delta-notation to simplify the algebra. Combining (19) and (20), and using the free entry condition to write $dn=(\partial n/\partial L)dL$ (NB $\partial n/\partial L=1$), we find:

$$\frac{d\omega}{\omega} = (1-\frac{P_{\Delta w}}{P})\frac{R_{\Delta n}}{b}\frac{\partial n}{\partial L}dL + (1-\frac{P_{\Delta w}}{P})\frac{R_{\Delta E}}{b}\frac{\partial E}{\partial L}dL - P_{\Delta n}\frac{\partial n}{\partial L}dL \qquad \textbf{(21)}$$

This equation shows a three-way decomposition of agglomeration and dispersion forces.

* More fully, $R_{\Delta n} = (\partial R/\partial n)dn + (\partial R/\partial n^*)dn^*$, but dn=-dn*.

The first term is unambiguously negative since 1-$P_{\Delta w}$/P and ∂n/∂L are positive and $R_{\Delta n}$—which is the local competition effect—is negative.* It is called the local competition effect since raising the number of northern firms and lowering the number of southern firms intensifies competition in the north. This lowers sales per northern firm and thus it is a force that tends to make firms like dispersion.

The second term in (21) captures the demand-linked circular causality. The terms 1-$P_{\Delta w}$/P and ∂E/∂L are positive. It is also true that $R_{\Delta E}$ is positive since raising the size of the northern market while simultaneously lowering the size of the southern market (demand shifting) tends to boost the sales of north-based firms. The circularity comes from the fact that migration shifts demand, shifting demand tends to raise the northern wage gap and raising the gap tends to foster further migration. The final term in (11) reflects cost-linked circular causality, i.e. forward linkages. $P_{\Delta n}$ is positive since shifting the production of varieties from south to north tends to lower the northern price index. This is sometimes called the Venables effect, after Venables (1987). The circularity here is that migration tends to lower the price index in the receiving region due the Venables effect. This tends to boost the real wage, which in turns tends to attract more migrants.†

Using the functional forms assumed, it is straightforward to show evaluate the various derivatives in (21). The result is:

$$\frac{d\omega}{\omega} = -(s-s^*)^2\left(\frac{b'\mu}{2}\right)\left(\frac{\partial n}{\partial L}\right)dL + (s-s^*)(b'\mu)\left(\frac{\partial E}{\partial L}\right)dL + (s-s^*)\left(\frac{\mu^2}{2(\sigma-1)}\right)\frac{\partial n}{\partial L}dL;$$
$$b' \equiv \frac{1-\mu^2(s-s^*)}{1-\mu^2(s-s^*)-2\mu(1-\sigma)ss^*} \qquad \textbf{(22)}$$

Observe that the difference between a firm's share in its local market and its export market, namely (s-s*), enters each of the three terms. This market-share gap obviously shrinks to zero as trade costs fall.‡ Clearly then, the magnitude of both the agglomeration and dispersion forces diminish as trade cost fall. This point is made graphically in Figure A-1.

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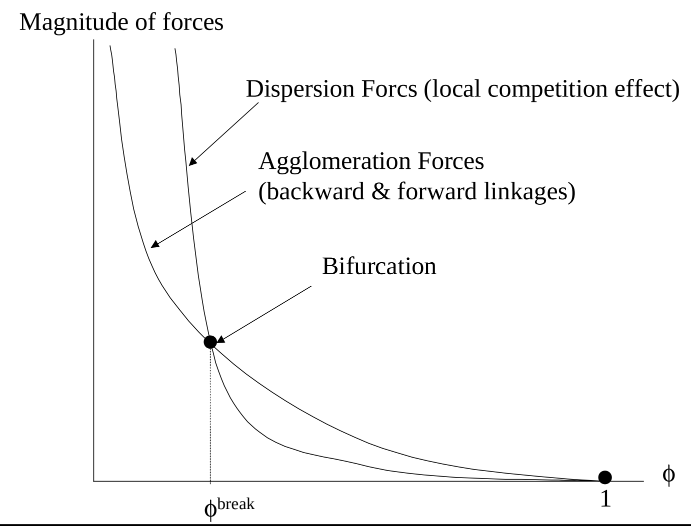
* Note that (1-$P_{\Delta w}$/P)>0 since $P_{\Delta w}$/P=nμs and 1-nμs =μns* by the adding up constraint on market shares.

† The terminology 'backward and forward linkages' is somewhat awkward here since they are generally applied to the attractiveness of a location to firms. In this version of the model, however, we are considering the attractiveness of a location to workers.

‡ The term (s-s*) is proportional to the ubiquitous 'Z' factor in FKV, namely Z≡μ(s-s*)/2.

Figure A-1: Agglomeration and Dispersions Erode with ϕ



It is well known that the symmetric equilibrium is stable for very high trade costs and unstable for very low trade costs. This fact implies that two things must happen. First, at very high trade costs the local competition effect must outweigh the backward and forward linkages. As KFV show, this is true as long as the 'no black hole' condition, $\mu<(1-1/\sigma)$, holds. Second, falling trade costs must erode the local competition effect more rapidly than the agglomeration forces. The figure illustrates both of these facts. The bifurcation point (i.e. the level of trade costs where the nature of the model's stability changes) is where the agglomeration and dispersion forces are equally strong. This is obvious from the figure, but more precisely, note that the market share difference is squared in the local competition term, so the magnitude of the dispersion force falls more rapidly as s approaches s*.

Appendix 2: Computer Output for Figures 6 and 7 (with normal time).