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Competitive Risk Sharing Contracts with One-Sided Commitment
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ABSTRACT

This paper analyzes dynamic equilibrium risk sharing contracts between profit-maximizing intermediaries and a large pool of ex-ante identical agents that face idiosyncratic income uncertainty that makes them heterogeneous ex-post. In any given period, after having observed her income, the agent can walk away from the contract, while the intermediary cannot, i.e. there is one-sided commitment. We consider the extreme scenario that the agents face no costs to walking away, and can sign up with any competing intermediary without any reputational losses. Contrary to intuition, we demonstrate that not only autarky, but also partial and full insurance can obtain, depending on the relative patience of agents and financial intermediaries. Insurance can be provided because in an equilibrium contract an up-front payment effectively locks in the agent with an intermediary. We then show that our contract economy is equivalent to a consumption-savings economy with one-period Arrow securities and a short-sale constraint, similar to Bulow and Rogoff (1989). From this equivalence and our characterization of dynamic contracts it immediately follows that without cost of switching financial intermediaries debt contracts are not sustainable, even though a risk allocation superior to autarky can be achieved.

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1 Introduction

This paper analyzes dynamic equilibrium risk sharing contracts between profit-maximizing financial intermediaries (which we also shall call principals) and agents that face idiosyncratic income uncertainty. In any given period, the agent can walk away from the contract and sign with a competing principal, while the principal itself cannot, i.e. there is one-sided commitment.

The paper is motivated by a common feature of a number of long-term relationships between principals and agents such as those between firms and workers, between international lenders and borrowing countries, between car or health insurers and their clients, or between countries and their citizens. They all have in common that the agents have the option to quit the relationship and engage in a relationship with a competing party. Quitting and turning to a competing party involves two types of costs. First, there are direct costs like the loss of relationship-specific human capital or giving up some agreed-upon collateral or time delays between the request to quit and the legal dissolution of the contract. Second, there is the indirect cost of losing the relationship capital ("goodwill"), which may have been built up over time due to past good behavior, and cannot be taken along.

We analyze the relationships listed above from the perspective of providing insurance against unfavorable agent-specific income shocks. In this paper, we take the most extreme perspective and study whether and to what degree relationship capital alone can support risk-sharing arrangements, when the income process is perfectly observable, when there is perfect competition between the intermediaries and no costs of switching for the agent.

Without the ability of agents to move between competing principals, the existing "endogenous" incomplete markets literature (e.g. see Atkeson (1991), Kehoe and Levine (1993, 2001), Kocherlakota (1996), Krueger (1999), Krueger and Perri (1999), Alvarez and Jermann (2000, 2001) or Ligon, Thomas and Worrall (2000)) has demonstrated that goodwill can be built up and that substantial risk sharing may be achieved. In this literature it is commonly *assumed* that the only alternative to the risk sharing contract the agent has available is financial autarky, and the threat of autarky sustains the risk sharing arrangement.

In this paper we *endogenize* the outside option of agents as being determined by the best possible deal that can be obtained from a competing principal. With such competition, intuition would suggest that in equilibrium no risk sharing is feasible and the resulting allocation is autarky. Competition between intermediaries dictates that the expected net present value of a contract for a newly arrived agent equals the expected net present value of her income stream. Since for all possible income realization the continuation value from the long-term risk sharing contract has to be at least as high as the outside option, the cost for the continuation contract is at least as high as the net present value of the individual's income process upon continuation. Since the intermediary wants to at least break even, he will not offer the agent anything more for consumption than her current income: he would be unable to recuperate this loss later on. There is also no point in offering less because of competition between intermediaries.

This implies that the unique equilibrium allocation is the autarkic allocation. Somewhat surprisingly, perhaps, it turns out that this intuition is flawed.

We now summarize our main results, which depend crucially on the relative size of the time discount factor of the agent and the time discount factor of the principal (the inverse of which can be interpreted as the gross real interest rate in our economy). If principals and agents discount the future at the same rate, the equilibrium dynamic risk sharing contract necessarily entails full consumption insurance for the agent in the long run. If the intermediary is somewhat more patient than the agent (that is, for lower interest rates), partial insurance will result. Only if the intermediary is very patient and thus interest rates are extremely low, the intuitive autarky result obtains. What happens in the other cases is the following. Assume for the sake of the argument that income is iid. The agent with the strongest incentive to leave the current contract is the agent with high income. A comparatively impatient principal will not mind to extract some resources from this agent now against generous promises of insurance later on. As the agent ages on the contract, she turns into a liability for the intermediary: the intermediary has received an initial up-front payment, and is now liable to let the agent consume more than the net present value of her future income. The intermediary, at this stage of the contract, has an incentive to default, but is not allowed to do so by our crucial assumption of one-sided commitment. The agent, on the other hand, will not want to walk away (even though she could), since she would be worse off at the beginning of any new contract, in which she again is asked to deliver an up-front payment. That is, the pre-payment feature of the contract, sunk after the contact has been "signed" provides the necessary glue between intermediaries and agents that enables some risk-sharing to occur.

We then show that our economy with competitive contracts is equivalent to a consumption-savings problem with one-period Arrow securities and state-dependent short-sale constraints, as in Alvarez and Jermann (2000). Without any costs of moving between principals, the associated short-sale constraint in the consumption-savings economy rules out borrowing altogether. This result is very similar in spirit to Bulow and Rogoff's (1989) no-lending result in the context of a model of sovereign debt. Nonetheless, in their as well as in our environment some insurance may be possible against payment (opening an account with positive balance with an intermediary) "up front". The characterization of the optimal contract is also reminiscent of observed features e.g. of health insurance or car insurance: such insurance can typically only be obtained (or only be obtained for "reasonable" premia), if the agent is currently healthy or the driving record is currently clean: the insurance continues, if conditions worsen.

The paper at hand builds on the recent endogenous incomplete markets literature with ex-post heterogeneous agents, as discussed above. The paper most closely related to our work is Phelan (1995), who also considers an environment where agents can leave the current contract and sign up with another principal. Phelan, too, shows that autarky will not result. However, he assumes that agents can only leave the contract at the beginning of the period, without knowledge of their period income: this amounts to a one-period waiting time for exiting the contract. In further contrast to our paper, Phelan assumes that the principal does not observe the endowment of the agent, and thus needs to elicit it from the agent. In fact, if endowments were observable in his environment, full insurance would prevail: there would be no reason for an agent to exit a full insurance contract before knowing her endowment. In our environment, the agent may exit after she learns her endowments and would surely leave if required to make large payments without future compensation.

One paper which also endogenizes the default option is Lustig (2001). His model extends the environment by Alvarez and Jermann (2000), in which a small number of agents with income risk enter long-term contracts with endogenous borrowing constraints, to incorporate the trading of a risky Lucas tree. Lustig's innovation is to assume that agents lose their share of the tree upon defaulting on their long-term contract, but are allowed to re-enter a long-term contractual relationship in the next period rather than being forced into autarky. He thereby provides a model in which all asset trades are fully collateralized, and examines the asset pricing consequences. Our assumption of allowing agents to re-enter contractual relationships is similar to Lustig's. His work, however, focuses on asset pricing consequences in the presence of aggregate uncertainty, when the number of participating agents is small, while our paper studies the allocational consequences of long-term contracts with the option of re-contracting elsewhere, assuming a large number of agents and no aggregate uncertainty. Second, while a durable asset is a necessary ingredient in his model, the stark implications in our paper derive from its absence.

Another literature that studies consumption insurance with long-term contracts derives incomplete risk-sharing from the presence of private information and moral hazard. In this literature it is usually assumed that both agents and competitive principals can commit to the long-term contract. Competition of principals for agents takes place only at the first period, with no re-contracting allowed at future dates. Green (1987) offers a partial equilibrium treatment of such an economy, while Atkeson and Lucas (1992, 1995) extend the analysis to general equilibrium, Atkeson (1991) applies such a model to sovereign lending, Phelan (1994) incorporates aggregate shocks and Malcomson and Spinnewyn (1988) study the importance of commitment to long-term contracts in achieving efficient allocations in a dynamic moral hazard environment. Whether a sequence of short-term contracts is able to attain outcomes as good as long-term contracts under private information is also the central point of investigation in the work of Fudenberg et al. (1990) and Rey and Salanie (1990, 1996).

One area of applications in which the assumption of one- or two sided limited commitment is particularly natural are dynamic employer-worker relationships. Consequently there exists a rich literature that characterizes (optimal) wage contracts between employers and workers. Important examples include Harris and Holmstrom (1982) and Thomas and Worrall (1988) and Beaudry and DiNardo (1991); a comprehensive review of this literature in provided by Malcomson (1999). Our work is related to this literature since our optimal risk-sharing contracts derived below will share some qualitative features with wage contracts studied in this literature, in particular Harris and Holmstrom (1982).

The structure of the paper is as follows. Section 2 describes the model and defines equilibrium. Section 3 provides the analysis. After proving existence of equilibrium, we argue in subsection 3.3 that, depending on the relationship between the discount factor of agents and the interest rate, either no, full or partial risk sharing is possible. The following subsections then analyze these cases in turn: subsection 3.4 provides the upper bound for the interest rate which allows for the autarky result. In section 3.5 we provide a complete characterization of the contract in the case of iid income and $\beta R = 1$, resulting in full risk sharing and constant consumption above average income in the limit. For the iid two-income case, we also provide a complete characterization of the partial insurance equilibrium for $\beta R < 1$ in subsection 3.6. Finally, in subsection 3.7 we show that there is a general duality between the long-term contracts economy considered in this paper and a consumption-savings problem with state-contingent one-period Arrow securities and borrowing constraints. The case without moving costs corresponds to strict short sales constraints, yielding the Bulow-Rogoff (1989) result as a special case. We also show that one can reinterpret competition with other principals as a requirement of renegotiation-proof contracts. Section 4 concludes. A sequential formulation of the game between agents and principals can be found in appendix A. A separate appendix contains details of some of the longer proofs in the main text.[1]

2 The Model

2.1 The Environment

The economy consists of a continuum of principals $j \in [0, 1]$, each initially associated with a measure $\mu_j \geq 0$ of atomless agents. The total population of atomless "agents" is $\sum_j \mu_j = 1$. We denote a generic agent by i. Each individual i has stochastic endowment process $\{y_{t,i}\}_{t=0}^{\infty}$ of the single consumption good with finite support $Y = \{y_1, \ldots y_m\}$ and transition matrix π. We assume that the set Y is ordered in that $y_{j-1} < y_j$ for all $j = 2, \ldots, m$. Endowment realizations are publicly observable. Let Π denote the stationary measure associated with π, assumed to be unique. Also assume that $\sum_y y\Pi(y) = 1$ and that initial distribution over endowments at each principal at date 0 is given by Π. Agent value consumption according to utility function

$$U((c_{t,i})_{t=0}^{\infty}) = (1 - \beta)E_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_{t,i}) \right] \tag{1}$$

where $u : \mathbf{R}_+ \to \mathbf{D}$ is the period utility function, with range $\mathbf{D}$, and where $0 < \beta < 1$. We assume that $u(c)$ is continuously differentiable, concave and strictly increasing in c and satisfies the Inada conditions.

A principal j has no endowment of the consumption good and consumes $\gamma_{t,j}$ in period t. We explicitly allow consumption or "profits" of the principal, $\gamma_{t,j}$,

[1] This appendix is available at www.econ.upenn.edu/~dkrueger/harapp.pdf

to be negative. This also avoids ever having to worry about bankruptcy of a principal. The principal is risk neutral and values consumption according to

$$U^{(P)}\left((\gamma_{t,j})_{t=0}^{\infty}\right) = \left(1 - \frac{1}{R}\right) E_0 \left[\sum_{t=0}^{\infty} \frac{\gamma_{t,j}}{R^t}\right] \tag{2}$$

where $0 < \frac{1}{R} < 1$ is the discount factor of the principal and an exogenous parameter of the model, which can also be interpreted as the exogenous gross interest rate. We allow this interest rate to differ from the discount factor β of the agent. The normalization with the factor $(1 - 1/R)$ has the advantage, that a constant $\gamma_{t,j} \equiv \gamma$ results in $U^{(P)} = \gamma$, so that both are expressed in the same units; this simplifies some of the expressions below (a similar argument justifies the normalization of the agents' utility function by $(1 - \beta)$). One may interpret $U^{(P)}$ as the net present value of a stream of profits $\gamma_{t,j}$ discounted at some given market return R, assuming that goods can be traded across principals and possibly some further markets, although we shall not explore this interpretation further to keep matters simple.[2]

For each principal the resource constraint posits that

$$Y_{t,j} = C_{t,j} + \gamma_{t,j} \tag{3}$$

where $Y_{t,j} = \int y_{t,i}\mu_j(di)$ is total endowment of agents associated with principal j and $C_{t,j} = \int c_{t,i}\mu_j(di)$ is total consumption of these agents.

2.2 Market Structure

In this economy agents wish to obtain insurance against stochastic endowment fluctuations from risk neutral principals. We want to characterize long-term consumption insurance contracts that competitive profit-maximizing principals offer to agents that cannot commit to honor these contracts. After the realization of income $y_{t,i}$, but before consumption takes place, an agent is free to leave the principal and join a competitor. She takes the current income realization with her. We assume that moving is "painful" to the agent, inflicting a disutility $\nu(y_{t,i}) \geq 0$. For most of the paper, we will concentrate on the case $\nu(.) \equiv 0$. Agents never revisit a principal they once left. A principal has the ability to commit to long-term contracts with his agents, but has no ability to reach them once they have left for a competitor. In short, this is an environment with one-sided commitment. Section 3.7 demonstrates that the same consumption allocation as with long-term contracts arises if agents are allowed to trade one-period state-contingent savings-loan contracts, subject to judiciously specified short-sale constraints. In that section we also discuss the connection between our long-term competitive contracts and renegotiation-proof contracts in a bilateral bargaining game between a single principal and agent.

[2] For instance, one may re-interpret R as the world gross interest rate in an open economy version of this model.

We now formulate a game of competition between principals, offering consumption contracts to potential movers and to agents already with the principal. We proceed directly to the recursive formulation of each individual principal's optimization problem, and then to define a symmetric stationary recursive equilibrium. We thereby skip the step of first describing the game as unfolding sequentially; for completeness, that formulation can be found in appendix A.

2.3 Recursive Equilibrium

An agent enters the period with current state (y, w), describing her current income y and the expected discounted utility w from the contract she had been promised by the principal last period. The fact that utility promises w and the current shock y form a sufficient description of an agent's state, in the sense that the resulting policy functions of the recursive problem induce consumption and investment sequences that solve the corresponding sequential optimization problem, has been demonstrated by Atkeson and Lucas (1992) for a private information economy and adapted to the environment presented here by Krueger (1999). Both papers borrow the idea of promised utility as a state variable from Abreu, Pierce and Stacchetti (1986) and Spear and Srivastava (1987).

The objective of the principal is to maximize the contribution to his own utility ("profit") from the contract with a particular agent. He is constrained to deliver the utility promise w by giving the agent current consumption c and utility promises from next period onwards, contingent on next period's income realization, $w'(y')$. If the principal promises less utility from tomorrow onward in a particular income realization y' than a competing principal, the agent will leave the location, and the principal makes zero profits from the contract with that particular agent from then on.[3] We denote the utility promise by competing principals as $U^{Out}(y')$, which the principal takes as given (but which is determined in equilibrium). The recursive problem of a principal can be stated as

$$P(y, w) = \max_{c, \{w'(y')\}_{y' \in Y}} \left(1 - \frac{1}{R}\right)(y - c) + \tag{4}$$

$$\frac{1}{R} \sum_{y' \in Y} \pi(y'|y) \begin{cases} P(y', w'(y')) & \text{if } w'(y') \geq U^{Out}(y') - \nu(y') \\ 0 & \text{if } w'(y') < U^{Out}(y') - \nu(y') \end{cases} \tag{5}$$

$$\text{s.t. } w = (1 - \beta)u(c) + \beta \sum_{y' \in Y} \pi(y'|y) w'(y') \tag{6}$$

where $\nu(y')$ is the "pain" of moving to a competing principal.

The promise keeping constraint (6) says that the principal delivers lifetime utility w to an agent which was promised w, either by allocating current or future

[3] If the agent is indifferent, we make the tie-breaking assumption that the agent stays with the current principal. Note that an agent always finds it preferable to sign up with a competing intermediary rather than live in financial autarky (even if she could save in a risk-free technology with gross return R) because a financial intermediary offers contracts that smooth consumpion across states and not only across time.

utility to the agent. This constraint of the principal makes our assumption of *one-sided* commitment explicit: in contrast to the agents principals are assumed to be able to commit to the long-term relationship. Finally, that continuation profits split into two parts is due to the fact that, in order to retain an agent, the principal has to guarantee her at least as much continuation utility, in any contingency, as the agent would obtain from a competing principal.

Let us now consider what our assumption of competition among principals amounts to. For a principal it only makes sense to attract a new agent if the profit from this new contract is non-negative. On the other hand, suppose that an agent could be attracted with a contract generating positive profit. Then another principal could make a profit by offering a slightly better contract. Hence, perfect competition between principals implies that the profit from a new contract exactly equals zero and that the utility promised to the newcomer is the highest utility promise achievable subject to this constraint. We require the result of this argument as an equilibrium condition by imposing that $U^{Out}(y)$ equals the highest lifetime utility w satisfying $P(y, w) = 0$.

Given this condition we can simplify the dynamic programming problem above, as the principal is always indifferent between letting an agent go by offering $w'(y') < U^{Out}(y') - \nu(y')$ or letting him stay by offering him exactly $w'(y') = U^{Out}(y') - \nu(y')$ and making zero expected profits from tomorrow onwards. We restrict attention to the latter case.[4] The dynamic programming problem (4) can then be restated as a cost minimization problem

$$V(y, w) = \min_{c, \{w'(y')\}_{y' \in Y}} \left(1 - \frac{1}{R}\right) c + \frac{1}{R} \sum_{y' \in Y} \pi(y'|y) V(y', w'(y')) \quad (7)$$

$$\text{s.t. } w = (1 - \beta) u(c) + \beta \sum_{y' \in Y} \pi(y'|y) w'(y') \quad (8)$$

$$w'(y') \geq U^{Out}(y') - \nu(y') \text{ for all } y' \in Y \quad (9)$$

where (9) now capture the constraints that competition impose on the principal and the argument above that it is never strictly beneficial for a principal to lose an agent to a competing principal.

With this recursive formulation of the principal's problem we can now restate the zero-profit condition. Let $a(y)$ be the (normalized) present discounted value of the endowment stream discounted at interest rate R and given current endowment y. Hence $a(y)$ is defined recursively as

$$a(y) = \left(1 - \frac{1}{R}\right) y + \frac{1}{R} \sum_{y'} \pi(y'|y) a(y) \quad (10)$$

One can read $a(y)$ as the human wealth of an agent with current income y, as evaluated by the principal. Perfect competition implies that the normalized

[4] If $\nu(y) = 0$ for all y this restriction is without loss of generality, since an agent starts the next period with promise $U^{Out}(y')$, independent of whether she moved or not.

expected net present value of consumption spent on this agent exactly equals her human wealth $a(y)$, i.e.

$$V(y, U^{Out}(y)) = a(y) \text{ for all } y \in Y \tag{11}$$

and that the utility $U^{Out}(y)$ promised to a newcomer is the highest utility promise achievable subject to the principal breaking even.

In order to define equilibrium we have to precisely fix the domain of admissible utility promises. Let $\mathcal{W} = [\underline{w}, \bar{w}]$ be this domain, with $\underline{w}$ being its lower and $\bar{w}$ its upper bound and let $Z = Y \times [\underline{w}, \bar{w}]$. For the results to follow it is useful to provide explicit bounds $[\underline{w}, \bar{w}]$. To do so define $\bar{a} = \max_j a(y_j)$ and

$$\begin{aligned} \bar{w} &= \max_{(c_t)_{t=0}^{\infty}} (1-\beta) \sum_{t=0}^{\infty} \beta^t u(c_t) \\ \text{s.t. } \left(1 - \frac{1}{R}\right) \sum_{t=0}^{\infty} \frac{1}{R^t} c_t &\leq \bar{a} \end{aligned}$$

That is, $\bar{w}$ is the lifetime utility an agent with highest lifetime income $\bar{a}$ could maximally receive from a principal who does not worry about the agent leaving the contract at some future point and who wishes to avoid a loss.[5] Furthermore, pick some $0 < \underline{y} < y_1$ and define $\underline{w} = u(\underline{y})$. Note that $\underline{w} < \min_j w_{aut}(y_j)$ (where $w_{aut}(y_j)$ is the utility from consuming its income forever, given current income y_j) That is, $\underline{w}$ is the lifetime utility from consuming a constant endowment $\underline{y}$ smaller than the lowest income realization y_1. In order to assure that the dynamic programming problem of the principal is always well-defined we impose the following

Condition 1 *The bounds* $[\underline{w}, \bar{w}]$ *satisfy*

$$\underline{w} > (1-\beta)\inf(\mathbf{D}) + \beta\bar{w}$$

where D *is the range of the period utility function.*

Note that this condition, purely in terms of fundamentals of the economy, is always satisfied for utility functions that are unbounded below (e.g. CRRA functions with $\sigma \geq 1$).[6] For other period utility functions, for $\bar{w}$ as defined above and a given $\underline{w} = u(\underline{y})$ there always exists a $\beta \in (0, 1)$ low enough such that condition 1 is satisfied. We are now ready to define a symmetric stationary recursive competitive equilibrium.

[5] For example, if $u(c) = (c^{1-\sigma} - 1)/(1-\sigma)$, a tedious but simple calculation shows that $\bar{w} = u(\bar{c})$, where

$$\bar{c} = \frac{1 - R^{-1}(\beta R)^{1/\sigma}}{1 - R^{-1}} \left(\frac{1-\beta}{1 - \beta(\beta R)^{((1/\sigma)-1)}} \right)^{\frac{1}{1-\sigma}} \bar{a} \leq \bar{a} \text{ for } \beta R \leq 1$$

with the inequality strict for $\beta R < 1$ and $\sigma < \infty$.

[6] Consequenctly our analysis also goes through for utility functions with monotonic transformations which are unbounded below.

Definition 2 *A symmetric stationary recursive equilibrium is functions* $V : Z \to \mathbf{R}$, $c : Z \to \mathbf{R}_+$, $w' : Z \times Y \to [\underline{w}, \bar{w}]$, $U^{Out} : Y \to [\underline{w}, \bar{w}]$, *principal consumption* $\gamma \in \mathbf{R}$ *and a positive measure* Φ *on the Borel sets of* Z *such that*

1. *(Solution of Bellman equation): V solves the functional equation above and c, w' are the associated policies, given $U^{Out}(y')$ for all $y' \in Y$*

2. *(Feasibility)*

$$\gamma + \int (c(y,w) - y)\, d\Phi = 0 \tag{12}$$

3. *(Outside Option): for all $y \in Y$*

$$U^{Out}(y) \in \arg\max_w \{w | V(y,w) = a(y)\} \tag{13}$$

4. *(Stationary Distribution)*

$$\Phi = H(\Phi) \tag{14}$$

where H is the law of motion for the measure over (y,w) induced by the income transition matrix π and the optimal policy function w'.

The law of motion H is given as follows. The exogenous Markov chain π for income together with the policy function w' define a Markov transition function on the measurable space $(Z, \mathcal{B}(Z))$ where $\mathcal{B}(Z)$ denotes the Borel sigma algebra on Z. Define the transition function $Q : Z \times \mathcal{B}(Z) \to [0,1]$ by

$$Q((y,w),A) = \sum_{y' \in Y} \begin{cases} \pi(y'|y) & \text{if } (y', w'(y,w;y')) \in A \\ 0 & \text{else} \end{cases} \tag{15}$$

for all $A \in \mathcal{B}(Z)$. Then the law of motion is defined as

$$H(\Phi)(A) = \int Q(z,A)\Phi(dz) \text{ for all } A \in \mathcal{B}(Z) \tag{16}$$

Two comments are in order. First, the range for w defined by $[\underline{w}, \bar{w}]$ is meant to precisely fix the domain of the relevant functions rather than act as another restriction. Second, agents arrive with a "blank" history at a new principal, i.e. the principal does not make particular use of the information that new arrivals must be agents who have previously defaulted. This assumption rules out cooperation by principals in punishing defaulting agents. In economic terms, this renders credit rating agencies irrelevant for allocations. While it might be interesting to study an extension allowing for such institutions, the assumption of perfect competition among principals is not different from the usual assumption maintained in Walrasian economies.[7]

[7] In the context of the sovereign debt literature Kletzer and Wright (2000) study an economy

3 Analysis

The analysis of our model contains several parts. In subsection 3.1 we establish basic properties of the principals' dynamic program problem and in subsection 3.2. we prove existence of equilibrium. Subsections 3.3 to 3.6. contain characterizations of the equilibrium risk sharing contract under different assumptions about the relative magnitude of the time discount factor of agents, β, and principal, $\frac{1}{R}$. Finally, subsection 3.7. argues that the consumption allocations characterized in the previous subsection would also arise as a solution to a simple consumption-savings problem or as an outcome of a renegotiation-proof bilateral contract between a single principle and agent.

3.1 Properties of the Bellman Equation

Let us first state properties of solutions to the dynamic programming problem of the principal. Define the cost function $C : \mathbf{D} \rightarrow \mathbf{R}_+$ as the inverse of the period utility function u. That is, $C(u)$ is the consumption needed to deliver current utility u. From the properties of the utility function it follows that $C(.)$ is strictly convex, differentiable, strictly increasing, and $\inf_{u \in D} C(u) = 0$ and $\sup_{u \in D} C(u) = \infty$. Rather than current consumption c we let the principal choose current utility $h = u(c)$ with associated cost $C(h)$. The Bellman equation then reads as

$$V(y, w) = \min_{h \in D, \{w'(y') \in [\underline{w}, \bar{w}]\}_{y' \in Y}} \left(1 - \frac{1}{R}\right) C(h) + \frac{1}{R} \sum_{y' \in Y} \pi(y'|y) V(y', w'(y')) \tag{17}$$

$$\text{s.t. } w = (1 - \beta)h + \beta \sum_{y' \in Y} \pi(y'|y) w'(y') \tag{18}$$

and subject to (9). We define an optimal contract, given outside options, as solution to the dynamic programming problem of the principal, or formally

Definition 3 *Given $[\underline{w}, \bar{w}]$ and $U^{Out}(y)_{y \in Y}$ in $[\underline{w}, \bar{w}]$, an optimal contract for $\left((U^{Out}(y))_{y \in Y}, \underline{w}, \bar{w}\right)$ is a solution $V(y, w)$ to the Bellman equation on the domain Z together with associated decision rules $h = h(y, w)$, $w'(y') = w'(y, w; y')$.*

We are now ready to establish basic properties of the optimal contract (dynamic program) of the principle.

Proposition 4 *Let outside options $(U^{Out}(y))_{y \in Y} \in [\underline{w}, \bar{w}]$ and $\beta < 1 < R$ by given. Further suppose that condition 1 is satisfied. Then, an optimal contract for $\left((U^{Out}(y))_{y \in Y}, \underline{w}, \bar{w}\right)$ exists and has the following properties.*

with one borrower countries and multiple lenders. They allow lenders to act strategically and construct renegotiation-proof trigger strategies of lenders that call for punishments of lenders who offer contracts inducing agents to leave the original lender. In this paper we take the view that perfect competition is an interesting benchmark to analyze, and that this analysis provides a complement to theirs. Perfect competition has the additional appeal that the informational requirements for the principals are substantially lower than with strategic interactions among principals.

1. *$V(y,w)$ is strictly convex, strictly increasing, continuous and differentiable in w.*

2. *The decision rules are unique and continuous.*

3. *The decision rules and the value function satisfy the first order conditions and the envelope condition*

$$
\begin{aligned}
(1-\beta)\lambda &= \left(1-\frac{1}{R}\right)C'(h) &\quad (19)\\
\lambda\beta &= \frac{1}{R}\frac{\partial V}{\partial w}(y', w'(y,w;y')) - \mu(y') &\quad (20)\\
\lambda &= \frac{\partial V}{\partial w}(y,w) &\quad (21)\\
\lambda &\geq 0 &\quad (22)\\
\mu(y') &\geq 0, \text{ for all } y' \in Y &\quad (23)
\end{aligned}
$$

   where λ and $\mu(y')$ are the Lagrange multipliers on the first and second constraints.

4. *The decision rule $h(y,w)$ is strictly increasing in w. The decision rule $w'(y,w;y')$ is weakly increasing in w, and strictly so, if the continuing participation constraint $w'(y,w;y') \geq U^{Out}(y') - \nu(y')$ is not binding.*

5. *If the income process is iid, then $V(y,w)$ depends on w alone, $V(y,w) \equiv V(w)$. If additionally $U^{Out}(y') - \nu(y')$ is weakly increasing in y', then $w'(y,w;y')$ is weakly increasing in y'.*

Proof. All arguments are similar to those in Krueger (1999) and fairly standard, apart possibly from the strict convexity of the value function. We will give a sketch of the argument here and defer details to the technical appendix.

1. Assumption 1 assures that the constraint set is non-empty. A standard contraction mapping argument then assures existence, strict monotonicity and convexity of V. Strict convexity follows from the equivalence of the sequential and recursive formulation of the problem where the strict convexity of the value function of the sequential problem follows from strict convexity of the cost function C.

2. Differentiability and uniqueness of the decision rules follow from strict convexity of V.

3. Standard first order and envelope conditions.

4. From first order conditions and strict convexity of C and V (in w).

5. Current income y appears in the Bellman equation only in the conditional probabilities $\pi(y'|y)$, independent of y in the iid case. The properties for $w'(w,y;y')$ follow from the first order conditions and strict convexity of V.

■

3.2 Existence and Properties of Equilibrium

If condition 1 is satisfied, existence of an equilibrium can now be guaranteed.

Proposition 5 *Let condition 1 be satisfied. Then an optimal contract and outside options* $\left\{U^{Out}(y)\right\}_{y\in Y}$ *satisfying* (13) *exist. If the income process is iid (or* $w'(y,w;y')$ *associated with* $\left\{U^{Out}(y)\right\}_{y\in Y}$ *is weakly increasing in* y*), then an equilibrium exists.*

Proof. Again we defer details to the technical appendix. There we first prove that there exist outside options $U^{Out} = (U^{Out}(y_1),\dots,U^{Out}(y_m))$ and associated value and policy functions $V_{U^{Out}}, h_{U^{Out}}, w'_{U^{Out}}(y')$ of the principals solving $V_{U^{Out}}(y,U^{Out}(y)) = a(y)$ for all y. Then we prove that the Markov transition function induced by π and $w'_{U^{Out}}(y')$ has a stationary distribution.

For the first part define the function $f : [\underline{w},\bar{w}]^m \to [\underline{w},\bar{w}]^m$ by

$$f_j\left[U^{Out}\right] = \min\{\tilde{w}\in[\underline{w},\bar{w}] : V_{U^{Out}}(y_j,\tilde{w}) \geq a(y_j)\} \text{ for all } j=1,\dots,m$$

We need to show three things: 1) The function f is well defined on all of $[\underline{w},\bar{w}]^m$, 2) The function f is continuous, 3) Any fixed point w^* of f satisfies $V_{w^*}(y_j,w_j^*) = a(y_j)$ for all $j=1,\dots,m$. Part 1 is straightforward.[8] Part 2) requires to show that the cost function $V_{U^{Out}}(.,.)$ is uniformly continuous in the outside options U^{Out} (which is involved, but not conceptually difficult, and for which assumption 1 again needed assure that the cost function is well defined for all possible outside options U^{Out}). Finally, part 3) has to rule out that at the fixed point satisfies $V_{w^*}(y_j,w^*) > a(y_j)$, which is done by constructing an allocation that attains lifetime utility w^* at costs lower than V_{w^*} (which is nontrivial and again requires condition 1).

For the second part we establish that π and $w'_{U^{Out}}(y')$ indeed induce a well-defined Markov transition function which satisfies the conditions of Corollary 4 in Hopenhayn and Prescott (1992). Then their result guarantees the existence of stationary measure Φ (although not its uniqueness). ∎

A useful property of the equilibrium for our further analysis is that the outside option of an agent is an increasing function of his income.

Proposition 6 *Suppose the income process is iid. Then in any equilibrium* $U^{Out}(y)$ *is increasing in* y*.*

Proof. $U^{Out}(y)$ solves $V(U^{Out}(y)) = a(y)$. The result follows since $a(y)$ is increasing in y and since $V(w)$ is increasing in w. ∎

---

[8] Note, however, that the more natural definition of f as

$$V_{U^{Out}}(y_j, f_j\left[U^{Out}\right]) = a(y_j)\} \text{ for all } j=1,\dots,m$$

would have made it impossible to show that f is well-defined on all of $[\underline{w},\bar{w}]$, unless very restrictive assumptions on $[\underline{w},\bar{w}]$ are made.

3.3 Three Risk Sharing Regimes

In the following subsections we will explicitly characterize the optimal risk-sharing contract between the principal and the agent under different assumptions about the relative patience of both parties. We will show that, loosely speaking, the more patient the agent is relative to the principle, (that is, the higher are interest rates) the more risk sharing she can obtain in an optimal contract. For these next subsections we assume that there is no cost of moving between principles, $\nu(y) \equiv 0$.

Before discussing our results in detail we give an overview over our findings, using three figures that plot the optimal utility promises tomorrow, $w'(y')$ against utility promises today. For these figures, it has been assumed that there are two income states, $y_1 < y_2$, and that income is iid. Also plotted is the expected discounted future utility promise $\beta \sum_{y'} \pi(y')'(w')$, since the vertical distance of this line and the 45^0-line amounts to current utility $(1-\beta)h = (1-\beta)u(c)$. Figure 1 pertains to an impatient agent (relative to the interest rate), figure 2 shows the case where the agent is patient and figure 3 exhibits an intermediate case.

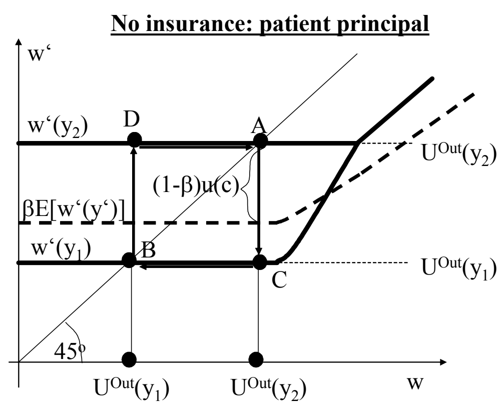

Figure 1:

In Figure 1 the agent is impatient and the principal is patient. For high $\frac{1}{R}$ it is optimal for the principal to give high current utility $(1-\beta)u(c)$ and low continuation utilities, subject to the constraints $w'(y') \geq U^{Out}(y')$. In the figure, for all current promises $w \in [U^{Out}(y_1), U^{Out}(y_2)]$ the continuation promises are always at the constraint: $w'(y') = U^{Out}(y')$. An agent starting with current promises $w = U^{Out}(y_2)$ (point A), upon receiving one bad shock moves to

$w' = U^{Out}(y_1)$ (point B via C), and an agent with one good shock moves from point B to A (via D). Note that agents, at no point in the contract, have continuation utility higher than their outside option (i.e. the principal does not share risk with the agent). In fact, we will show in the next section that for a sufficiently low R (and nonnegative consumption of the king) the stationary equilibrium is autarky: the equilibrium outside options equal the utility obtained consuming the endowment in each period, the allocation equals the autarkic allocation, and the stationary promise distribution has only positive mass $\Pi(y)$ at $U^{Out}(y)$.

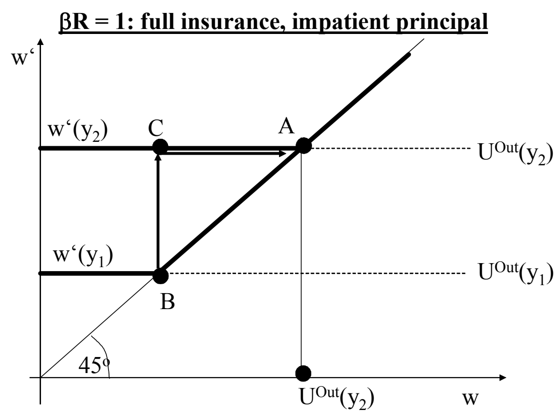

Figure 2:

Figure 2 depicts the other extreme, for a patient agent and impatient principal, with discount factors satisfying $\beta R = 1$. Now it is beneficial for the principal to economize on current utility and give high utility promises from tomorrow onwards. For *iid* income shocks future promises coincide with the 45^0-line whenever $w \geq U^{Out}(y')$ and are constrained by $U^{Out}(y')$ below these points. It is easy to see (and we will formalize this in section 3.5) that, as the agent experiences good income shocks, continuation utility and future consumption move up (to $U^{Out}(y_{\max})$ and the corresponding consumption level) and stay there forever: eventually an agents' consumption is perfectly smooth as he obtains complete consumption insurance.

Finally, in Figure 3 we depict an intermediate case in which partial insurance obtains. Consider an agent with current utility promise $w = U^{Out}(y_2)$ (point A). If this agent experiences a bad income shock $y' = y_1$ her future utility promise $w'(y')$ is lower than today's promise (and the same is true for consumption). However, the drop in promises and consumption is not as drastic as in Figure 1:

now it takes two bad income shocks to hit $U^{Out}(y_1)$ (from point A via D to B). Thus, the agent is partially insured against income risk. However, in contrast to Figure 2 insurance is not perfect: utility promises and consumption drop with a low income realization, even for an agent that previously had worked herself up to point A. For the *iid* case with two income shocks, section 3.6 below will provide a complete characterization of the consumption dynamics, including the optimal number and size of downward consumption steps shown in Figure 3.

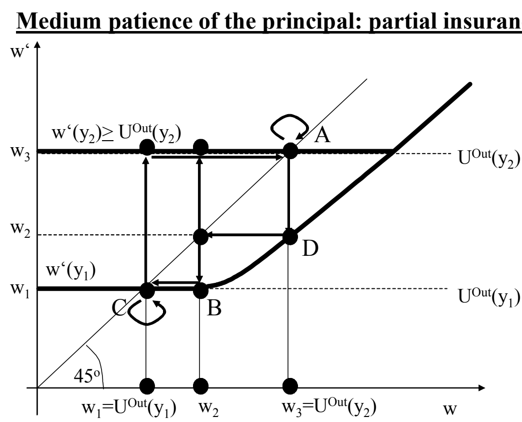

Figure 3:

3.4 No Risk Sharing: Autarky

As a starting point of our characterization of equilibrium, we shall analyze conditions, under which the principals do not lose resources in steady state, i.e. we seek equilibria which also deliver the following condition:[9]

Definition 7 (Nonnegative steady state profit condition:)

$$\gamma = \int (y - V(y,w))d\Phi \geq 0 \tag{24}$$

Leaving a location has no consequences for an agents' ability to engage in future risk sharing arrangements and generates no cost other than giving up current promises w. One may interpret the promise w as relationship capital:

[9] Note that, given our normalization of current costs by $\left(1 - \frac{1}{R}\right)$ total current period profits and the present discounted value of all future profits are identical in a stationary equilibrium.

the principal guarantees a particular level of happiness to the agent as a consequence of past events. Our first proposition shows that the threat of losing this goodwill is not enough to support risk sharing, if the principals need to enjoy nonnegative profits in steady state.

Proposition 8 *Assume* $\nu(y) \equiv 0, y \in Y$.

1. *If an equilibrium satisfies the nonnegative steady state profit condition, then it has to implement the autarkic allocation almost everywhere* (*a.e*)*:*

$$c(y,w) = y \qquad \Phi - a.e. \tag{25}$$

2. *Conversely, if an equilibrium implements the autarkic allocation, then the nonnegative steady state profit condition is satisfied.*

Proof. Remember that in any equilibrium $V(y,w)$ has to be weakly increasing in w. For $(y,w) \in$ supp Φ, we must have $w \geq U^{Out}(y)$ and thus

$$V(y,w) \geq V(y, U^{Out}(y)) = a(y) \tag{26}$$

On the other hand, the assumed non-negativity of γ together with

$$\sum_{y \in Y} a(y)\Pi(y) = \sum_{y \in Y} y\Pi(y) = E[y] \tag{27}$$

implies

$$\int V(y,w)d\Phi \leq E[y] = \int a(y)d\Phi = E[y] \tag{28}$$

Together,

$$V(y,w) = a(y) \;\; \Phi - a.e. \tag{29}$$

Now, comparing the two equations

$$V(y,w) \;\; = \;\; \left(1 - \frac{1}{R}\right) c(y,w) + \frac{1}{R}\sum_{y'} \pi(y'|y) V(y', w'(y,w;y')) \tag{30}$$

$$a(y) \;\; = \;\; \left(1 - \frac{1}{R}\right) y + \frac{1}{R}\sum_{y'} \pi(y'|y) a(y') \tag{31}$$

shows that $c(y,w) = y$ almost everywhere. The second part of the proposition follows trivially from the definitions ■

The equilibrium distribution Φ in proposition 8 is easy to calculate. Since agents consume their endowment, it follows that their remaining lifetime utility is given by the continuation utility from consuming the stochastic income stream in each period, starting with current income y. This utility from "autarky" $w_{aut}(y)$ is recursively defined as

$$w_{aut}(y) = (1-\beta)u(y) + \beta \sum_{y' \in Y} \pi(y'|y) w_{aut}(y') \tag{32}$$

The distribution Φ therefore assigns weight $\Pi(y)$ to the atoms $(y, w_{aut}(y))$ and zero to everything else. Proposition 8 says that any equilibrium must necessarily have $c(y, w_{aut}(y)) = y$ for all $y \in Y$. Similarly, the promised utility at these points is obviously $w'(y, w_{aut}(y); y') = w_{aut}(y')$. Comparing the result above to the definition of an equilibrium, we see that the proposition does not yet deliver the full specification required for a stationary equilibrium. Such a specification requires the consumption function as well as all other functions listed in the equilibrium definition to be defined on the set $Z = Y \times [\underline{w}, \bar{w}]$ rather than just the points $(y, w_{aut}(y))$. If we restricted the domain Z to just include the latter list of points, proposition 8 would essentially establish that we always have an equilibrium implementing autarky. With that assumption, the principals would have no choice but to implement the autarky solution! They would not be allowed to deviate from the autarky utility promises, even if they preferred to do so. Or, assume instead that $Z = \{(y, \bar{w}) \mid y \in Y\}$ where $\bar{w} = u(E[y])$ is the utility promise from complete risk sharing. In that case, principals would have no choice but to always implement the complete risk sharing solution. Thus choices of the domain Z of this type would completely predetermine the outcome.

Now we aim at constructing the equilibrium contract on the entire set $Z = Y \times [\underline{w}, \bar{w}]$. This requirement of the equilibrium turns out to have bite in that it rules out the existence of equilibria satisfying the nonnegative steady state profit condition for some R altogether and leads to the explicit and unique construction of the equilibrium functions outside the support of Φ otherwise. Indeed, in part 3.5, where we dispense with the non-negativity of profits condition, we shall obtain complete risk sharing if $R\beta = 1$ and income is iid. This is obviously squarely at odds with the autarky result above. The next proposition shows, that no equilibrium *satisfying the nonnegative profit* in steady state condition exists for $R\beta = 1$, thus resolving this conflict.

More generally, for the autarky result to hold we need to rule out that a principal would find it profitable to deviate from an equilibrium in which all other principals offer the autarky contract: this can be achieved under the assumption that the principal is sufficiently patient (interest rates are low enough). Intuitively, deviating from the autarky solution involves offering a agent with a high income now a better contract by taking some of his endowment now for the promise of additional consumption goods in future periods, when his endowment is low. A sufficiently patient principal is deterred by the future costs of sticking to such a contract. Based on this argument, we expect there to be an upper bound on the preference parameter R of the principal for the autarky result to emerge. The results below will show that this is indeed the case.

Define

$$h(y,w) = \frac{w - \beta \sum_{y' \in Y} \pi(y'|y) w_{aut}(y')}{1-\beta} \tag{33}$$

Note that $h(y, w_{aut}(y)) = u(y)$, see equation (32), and that $\partial h / \partial w = (1-\beta)^{-1}$. Furthermore, since $h(y,w)$ is strictly increasing in w, we can define

$$\bar{h} = \max_{\underline{w} \le w \le \bar{w}, y \in Y} h(y,w) = \max_{y \in Y} h(y, \bar{w}) \tag{34}$$

$$\underline{h} = \min_{\underline{w} \le w \le \bar{w}, y \in Y} h(y,w) = \min_{y \in Y} h(y, \underline{w}) \tag{35}$$

Proposition 9 *Assume that condition 1 holds.*

1. *An equilibrium satisfying the nonnegative steady state profit condition exists, if and only if*

$$R \in (1, \bar{R}] \tag{36}$$

   where

$$\bar{R} = \frac{C'(\underline{h})}{\beta C'(\bar{h})} \le \frac{1}{\beta} \tag{37}$$

2. *If it exists, the equilibrium has the following form*

$$V(y,w) = \left(1 - \frac{1}{R}\right)(C(h(y,w)) - y) + a(y) \tag{38}$$

$$c(y,w) = C(h(y,w)) \tag{39}$$

$$w'(y,w;y') = w_{aut}(y') \tag{40}$$

$$U^{Out}(y) = w_{aut}(y) \tag{41}$$

   profits $\gamma = 0$ and a positive measure Φ as constructed above.

3. *No equilibrium satisfying the nonnegative steady state profit condition exists for any R, if $\bar{R} \le 1$.*

4. *Suppose that $\beta R = 1$. Then, no equilibrium satisfying the nonnegative steady state profit condition exists.*

5. *Suppose that the endowment process is iid. Then*

$$\bar{R} = \frac{1}{\beta} \frac{C'(u(y_1))}{C'(u(y_m))} = \frac{1}{\beta} \frac{u'(y_m)}{u'(y_1)} \tag{42}$$

Proof. For (tedious, but straightforward) details of the proof of the first part see the technical appendix. For any equilibrium, proposition 8 implies that $U^{Out}(y) = w_{aut}(y)$ and that $w'(y') = w'(y,w;y') = w_{aut}(y')$. From promise-keeping $h = h(y,w)$ must be as in equation (33). The consumption function

c follows from the definition of $C(.)$; once this function is known, the value function implied by $c(y, w)$ can be easily calculated. We have thus constructed an equilibrium if and only if the functions above satisfies the (necessary and sufficient) first order optimality conditions of the principal

$$C'(h(y, w)) \geq R\beta C'(h(y', w'(y, w; y'))$$

on Z. But this condition is true on all of Z if and only if (37) is satisfied. A violation of (37) makes cost minimization of principals and the requirement of $\gamma \geq 0$ on the stationary equilibrium mutually exclusive, leading to the nonexistence result. The other results stated in the proposition follow immediately from its first part. ∎

Note that condition (42) is essentially the same as the one stated in Krueger and Perri (1999): in their environment with endogenous gross interest rate R, risk sharing can only be obtained if indeed $\frac{1}{\beta}\frac{u'(y_m)}{u'(y_1)} < 1$, whereas autarky obtains if $\frac{1}{\beta}\frac{u'(y_m)}{u'(y_1)} \geq 1$.

3.5 Perfect Risk Sharing: Full Insurance in the Limit

Now consider the case $\beta R = 1$ in figure 2. If the agent is already at point A with high current promises $w = U^{Out}(y_2)$, she will stay there, no matter which income she receives in the future. If the agent is at point B, she will stay there as long as income is low, $y_t = y_1$. But the first time the agent receives high income y_2, she jumps to point C on the $w'(y_2)$-branch and then stays at point A forever, with constant utility promises and consumption. The same is true for an agent with initial utility promise $w \in (U^{Out}(y_1), U^{Out}(y_2))$. We call this full insurance in the limit: full insurance is not obtained upon entering the contract, but eventually. The stationary distribution is a unit point mass at $U^{Out}(y_2)$, reflecting the fact that eventually all agents are fully insured against income fluctuations. We shall now prove these claims formally. First, we demonstrate in the next proposition, that full insurance obtains if and only if $\beta R = 1$.

Proposition 10 *Suppose income is iid. Denote the probability of income y_i as $\pi(y_i) > 0$.*

1. *Suppose that $\beta R = 1$. Then any optimal contract implies full insurance in the limit, i.e. constant consumption from the first time that the highest income level y_m is realized.*

   (a) *The utility promises take the form $w'(w; y') = max(w, U^{Out}(y'))$.*

   (b) *The decision rule for current utility $h = h(w)$ is defined by (6). The cost function $V(w)$ satisfies $V'(w) = C'(h(w))$ and $V(w) = C(w)$ for $w \geq max_y U^{Out}(y)$*

   (c) *The consumption level c_i of an agent who, so far in his life, had maximal income y_i is given by*

$$c_i = a(y_i) - \frac{1}{R-1}\sum_{j>i}\pi(y_j)\left[a(y_j) - a(y_i)\right] \tag{43}$$

The equilibrium outside options $U^{Out}(y_i)$ satisfy the recursion

$$U^{Out}(y_i) = \frac{(1-\beta)u(c_i) + \beta \sum_{j>i} \pi(y_j) U^{Out}(y_j)}{(1-\beta) + \beta \sum_{j>i} \pi(y_j)} \tag{44}$$

with $U^{Out}(y_m) = u(c_m)$. Current utility is given by

$$h_i = (1-\beta)u(c_i) \tag{45}$$

(d) If $n = 2$, then

$$c_2 = (1 - \frac{1}{R}) y_2 + \frac{1}{R} E[y] \tag{46}$$

and

$$c_1 = y_1 \tag{47}$$

2. *Conversely, suppose there is full insurance in the limit and $\beta R \leq 1$. Then, $\beta R = 1$.*

Proof.

1. The first part follows directly from proposition 4, the fact that income is iid and the assumption that $\beta R = 1$. The other parts follow from the first part which implies that the contract is like a "ratchet": once some level w_i of utility promises is reached, the promise will not fall, and will rise to w_j, if $y = y_j$ and $j > i$. Define $V_i = V(U^{Out}(y_i))$ and c_i to be the consumption level associated with $w_i = U^{Out}(y_i)$. These satisfy

$$V_i = (1 - \frac{1}{R}) c_i + \frac{1}{R} \left(\sum_{j>i} \pi(y_j) V_j + V_i * (1 - \sum_{j>i} \pi(y_j)) \right)$$

   Substituting $V_i = a(y_i)$ and solving for c_i delivers (43). Equations (44)-(47) follow from simple calculations.

2. Full insurance in the limit implies, that for some $\tilde{w} \in [\underline{w}, \bar{w}]$, one has $w'(\tilde{w}; y') \equiv \tilde{w}$ for all y'. Suppose $\beta R < 1$. The first order conditions imply that $w'(w; y') < w$ whenever the constraint $w'(w; y') \geq U^{Out}(y')$ does not bind. But $U^{Out}(y_1) < U^{Out}(y_m)$, so it cannot be the case that all constraints bind and $w'(\tilde{w}; y_1) = w'(\tilde{w}; y_m)$; a contradiction.

■

The proposition shows that on the equilibrium path of the contract, an agent receives consumption c_i, where y_i is the maximal income since starting the contract. The agent "ratches" herself up a ladder of permanent consumption claims. At the highest level of income, the agent receives consumption equal to the permanent income at that point,

$$c_m = a(y_m) = (1 - \frac{1}{R})y_m + \frac{1}{R}E[y]$$

Note that $E[y] < c_m < y_m$. The principal receives an "up front" payment $y_m - c_m$ for which he provides the permanent consumption level c_m at an expected steady state loss $E[y] - c_m$. Since this is also the absorbing state for any contract starting with $w \leq U^{Out}(y_m)$, consumption of the principal in a stationary equilibrium equals $\gamma = E(y) - c_m$. For all $i < m$ we have that $c_i < a(y_i)$, i.e. the agent, in the current period, consumes even less than his expected income, where $\frac{1}{R-1}\sum_{j>i}\pi(y_j)\left[a(y_j) - a(y_i)\right]$ is the insurance premium for having consumption never drop below c_i again in the future, regardless of future income realizations. The qualitative features of our optimal contract (racheting-up of consumption, perfect insurance in the limit) are similar to those in Harris and Holmstrom's (1982) study of optimal wage contracts; in addition we provide, in an arguably simpler environment, a full characterization of the optimal risk-sharing contract.

As an interpretation of the full insurance contract, consider observed health insurance or car insurance contracts. At the start of the contract, it often provides agents with pre-existing diseases or drivers who just had an accident with no insurance at all. Only good risks (healthy people, good drivers) are given an insurance contract, pay a premium, and can then be assured of continuing coverage in the future. After the point of payment, it is no longer sensible for the agent to switch insurance agencies and pay anew.

As a by-product of the previous proposition we can also fully characterize the stationary distribution Φ associated with complete risk sharing in the limit.

Proposition 11 *Any stationary distribution Φ on $Y \times [\underline{w}, \bar{w}]$ is given by the cross product of the stationary distribution on incomes Y times a distribution on $[\underline{w}, \bar{w}]$ given by a point mass on $w = U^{Out}(y_m)$ and an arbitrary distribution Ψ on the interval $[U^{Out}(y_m), \bar{w}]$.*

Proof. Follows immediately from the properties of $w'(w; y)$. ■

With any starting utility promise $w \leq U^{Out}(y_m)$, agents reach the absorbing utility promise level $U^{Out}(y_m)$. Agents starting at a utility promise $w > U^{Out}(y_m)$ will stay at that promise forever. The part Ψ comes about from agents who have been given exceedingly generous utility promises from the start. If all agents in the stationary distribution started from signing up with competitive principals, then the unique stationary distribution is given by $\Phi(y, U^{Out}(y_m)) = \pi(y)$, and zero elsewhere.

Finally, note that the same qualitative results as above can be proved under the assumption of $\beta R > 1$. Now, however, the absorbing state is the upper bound on utility promises $\bar{w}$, consumption is never declining and finally increasing to $c_{\bar{w}}$ with probability 1. However, now $\bar{w}$ acts as a real constraint and in its absence the optimal contract (if it exists) has ever-increasing utility promises and consumption. In general, a stationary equilibrium will not exist for $\beta R > 1$, which led us to focus on the case $\beta R = 1$ for perfect insurance in the limit.

3.6 Partial Risk Sharing

Finally, partial insurance will obtain[10], if

$$\bar{R} < R < \frac{1}{\beta}.$$

A typical situation is shown figure 3. Suppose, the agent starts from point A, i.e. high income. If income remains high, then the agent will remain at point A and utility promise $w_3 = U^{Out}(y_2)$. But if income is low, the agent follows the promise D and point B on the $w'(y_1)$-branch, reaching promise w_2 next period. If income is low again, the agent finally arrives at point C, i.e., the agent will land a level at the utility level $w_1 = U^{Out}(y_1)$. With high income at either point B or at point C, the agent will move to the $w'(y_2)$-branch, and therefore to point A and utility level w_3. The stationary distribution is given by point masses on the points $\{w_1, w_2, w_3\}$, with probabilities resulting from the dynamics described above and the income probabilities. Depending on the parameters, there may be more points like D and B, i.e., the dynamics may need a number of bad income draws to reach the lowest promise level $U^{Out}(y_1)$. The ratcheting-up part of the contract is similar to the full insurance case; now however, because of the wedge between β and R consumption has a downward drift (unlike in Harris and Holmstrom (1982)) if not constrained by the outside options. For the *iid* case with two income shocks we can provide a complete characterization of the consumption dynamics and the stationary consumption distributions, with the intuition provided by Figure 3.

Proposition 12 *Suppose, income can take two values $y_1 < y_2$ and is iid, with $\pi = \pi(y_1)$, and $\beta R < 1$. Then any equilibrium is characterized by a natural number $n \geq 2$, promise levels $w_1 = U^{Out}(y_1) < w_2 < \ldots < w_{n-1} < w_n = U^{Out}(y_2)$, costs $V(w_i) = V_i$, $V_1 < V_2 < \ldots < V_n$ and consumption levels $c_1 < c_2 < \ldots < c_n$ satisfying the following equations.*

$$\begin{aligned}
V_1 &= a(y_1) \\
V_n &= a(y_2) \\
V_j &= \left(1 - \frac{1}{R}\right) c_j + \frac{\pi}{R} V_{\max\{j-1,1\}} + \frac{(1-\pi)}{R} V_n \text{ for all } j = 1, \ldots, n \\
u'(c_j^{(n)}) &= \beta R u'(c_{j-1}^{(n)}) \text{ for } j = 2, \ldots, n \\
c_j &= c_j^{(n)} \text{ for } j = 2, \ldots, n \\
c_1 &= y_1 \\
c_2 &\geq y_1 > c_1^{(n)}
\end{aligned}$$

[10] Note that we have not ruled out that partial insurance can also happen for $R \leq \bar{R}$; what we showed above is that it can't happen in a stationary equilibrium with nonnegative profits for the principal.

Proof. All equations except $c_1 = y_1$ follow directly from the first order conditions. The result $c_1 = y_1$ follows from combining the first three equations into

$$a(y_1) = \left(1 - \frac{1}{R}\right) c_1 + \frac{1}{R}(\pi a(y_1) + (1-\pi)a(y_2))$$

and solving for c_1. ■

These equations can be solved reasonably easily. Iterating the third equation and combining with the first two yields, after some algebra:

$$\sum_{j=0}^{n-2} \left(\frac{\pi}{R}\right)^j (c_{n-j} - a(y_2)) = \left(\frac{\pi}{R}\right)^{n-1} (y_2 - y_1) \tag{48}$$

where $c_j, j \geq 2$ are found from recursively solving the Euler equations

$$u'(c_{n-j}) = (R\beta)^{-j} u'(c_n) \tag{49}$$

and therefore

$$c_{n-j} = (u')^{-1} \left[(R\beta)^{-j} u'(c_n)\right] \tag{50}$$

$$= c_{n-j}(c_n) \tag{51}$$

for $j = 0, \ldots, n-2$. Here $(u')^{-1}$ is the inverse of the marginal utility function, a strictly decreasing function which maps $\mathbf{R}_{++}$ into itself. Evidently

$$c_n > c_{n-1} > \ldots > c_2.$$

and the functions $c_{n-j}(c_n)$ are strictly increasing and continuous in c_n. In the next proposition we characterize the consumption allocation for a fixed number of steps n.

Proposition 13 *For any given $n \geq 2$ a unique solution to* (48) *exists. It satisfies $c_n = y_2$ for $n = 2$ and $c_n \in (a(y_2), y_2)$ for $n > 2$. Furthermore c_n is decreasing in n, strictly if $c_2(c_n) > y_1$.*

Proof. The existence of a solution follows from the intermediate value theorem, since the left hand side of (48) is continuous in c_n, weakly smaller than 0 for $c_n = a(y_2)$ and increasing without bound as c_n increases, whereas the right hand side is positive and constant in c_n. This argument also shows that $c_n > a(y_2)$. For $n = 2$ (autarky) equation (48) immediately implies $c_n = c_2 = y_2$. For $n > 2$ (partial risk sharing) we have $c_n < y_2$ since otherwise the principal can never break even, since agents with bad shock y_1 get to consume more than y_1 with positive probability and agents with high shock y_2 consume c_n.

Now we show that c_n is weakly decreasing in n. For fixed n define $\{c_{n-j}^{(n)}\}_{j=1}^{n}$ as the sequence of consumption levels in the previous proposition. We want to show that for $n \geq 0$ we have $c_n^{(n)} \leq c_{n-1}^{(n-1)}$, with strict inequality if $c_2^{(n)} > y_1$. Suppose not, then $c_n^{(n)} > c_{n-1}^{(n-1)}$, and from (50)

$$c_{n-j}^{(n)} > c_{n-j-1}^{(n-1)} \text{ for all } j = 0, \ldots, n-3 \tag{52}$$

Denoting the left hand side of (48) by $\Gamma(n)$ we find, using (52), that

$$\Gamma(n-1) < \Gamma(n) - \left(\frac{\pi}{R}\right)^{n-2} [c_2^{(n)} - a(y_2)]$$

and thus, using (48) again

$$\left(\frac{\pi}{R}\right)^{n-2} [y_2 - y_1] < \left(\frac{\pi}{R}\right)^{n-1} [y_2 - y_1] - \left(\frac{\pi}{R}\right)^{n-2} [c_2^{(n)} - a(y_2)]$$

which implies

$$c_2^{(n)} < y_1,$$

a contradiction to the definition of $c_2^{(n)}$. If $c_2^{(n)} > y_1$ we can repeat the argument above to show that $c_n^{(n)} < c_{n-1}^{(n-1)}$. ■

The previous proposition shows the partial nature of insurance. An agent with high income y_2 consumes less than his endowment, yet more than the present discounted value of his future income. On the other hand, with bad income shocks his consumption declines only slowly, in $n-1$ steps, towards y_1. In addition, the results show that in order to have consumption decline in many steps (a lot of insurance against bad shocks, high n), the principal, in order to break even, delivers less consumption with the good income shock: $c_n^{(n)}$ is decreasing in n.

Finally one can characterize the optimal number of steps, n^*. From proposition 12 we know that $c_1^{(n^*)} = y_1$ and that $c_2^{(n^*)}$ must satisfy

$$u'(y_1) \geq u'\left(c_2^{(n^*)}\right) > \beta R u'(y_1) \tag{53}$$

Existence of an equilibrium step number n^* is guaranteed through the general existence proof, under condition 1. It is also evident that $n^* = 2$ (with associated $c_2^{(2)} = y_2$) is the unique equilibrium step number if and only if $R < \frac{u'(y_2)}{\beta u'(y_1)}$, which confirms our results in section 3.4. Finally, the next proposition shows that n^* is always unique.

Proposition 14 *Suppose $R \geq \frac{u'(y_2)}{\beta u'(y_1)}$. Then the optimal number of steps $n^* \geq 3$ satisfying* (53) *is unique.*

Proof. The assumption in the proposition rules out autarky $n^* = 2$ as optimal. Now we prove uniqueness. Let $n^* \geq 3$ satisfy (53). Take arbitrary

$\tilde{n} \neq n^*$. First suppose $\tilde{n} < n^*$, so that $\tilde{n} = n^* - k$ for some $k \geq 1$. But then, using the fact that $c_{n^*}^{(n^*)} \leq c_{\tilde{n}}^{(\tilde{n})}$

$$\begin{aligned} u'\left(c_2^{(\tilde{n})}\right) &= (\beta R)^{-(\tilde{n}-2)} u'\left(c_{\tilde{n}}^{(\tilde{n})}\right) \leq (\beta R)^{-(n^*-2)+k} u'\left(c_{n^*}^{(n^*)}\right) \\ &= (\beta R)^k u'\left(c_2^{(n^*)}\right) \leq (\beta R)^k u'(y_1) \leq (\beta R) u'(y_1) \end{aligned}$$

which violates the second inequality of (53) for $n = \tilde{n}$. If one supposes that $\tilde{n} > n^*$, then a similar argument shows that the first inequality of (53) is violated for $n = \tilde{n}$. ■

Conditional on an optimal step number n it is also straightforward to calculate the stationary distribution.

Proposition 15 *Assume $\nu(y) \equiv 0, y \in Y$. Suppose, income can take two values and is iid, with $\pi = P(y = y_1)$. The stationary distribution is given by atoms at w_i with weights λ_i given by*

$$\begin{aligned} \lambda_1 &= \pi^{n-1} \\ \lambda_j &= \pi^{n-j}(1-\pi) \text{ for } j = 2, \ldots, n \end{aligned}$$

where n is the optimal number of steps analyzed above.

Proof. This follows from noting the following. Given any current promise w_j, the probability for reaching w_n in the next period is $1 - \pi$. Thus,

$$\lambda_n = (1-\pi) \sum_{j=1}^{n} \lambda_j = 1 - \pi$$

Next, for $1 < j < n$, w_j can be reached only from w_{j+1} and income y_1. Thus $\lambda_j = \pi \lambda_{j+1}$. Finally, for $j = 1$, w_1 can be reached from both w_1 or from w_2, provided income is y_1. Thus $\lambda_1 = \pi(\lambda_1 + \lambda_2)$. Solving these equations for $\lambda_j, j = 1, \ldots, n-1$ gives the result. ■

This completes the characterization of the optimal contract and the resulting invariant consumption and lifetime utility distribution.

3.7 Equivalence Results

3.7.1 A Consumption-Savings Reformulation

The competitive equilibrium with risk sharing contracts between principals and agents we studied can also be implemented by letting the agent trade in state-contingent one-period Arrow securities, subject to carefully chosen short-sale constraints.

More precisely, consider the consumption-savings problem

$$\begin{aligned} W(y,b) &= \max_{c,(b(y'))_{y'\in Y}} \left\{ (1-\beta)u(c) + \beta \sum_{y'} \pi(y'|y) W(y', b(y')) \right\} \\ c + \frac{1}{R} \sum_{y'\in Y} \pi(y'|y) b(y') &= y + b \\ b(y') &\geq \underline{b}(y'), \text{ for all } y' \in Y \end{aligned}$$

where $\underline{b}(y')$ is a collection of state-contingent borrowing constraints. If $\underline{b}(y') = 0$ for all $y' \in Y$, then the agent is prevented from borrowing altogether.

The price of an Arrow security paying one unit of consumption tomorrow, conditional on income realization y' is given by $q(y'|y) = \frac{\pi(y'|y)}{R} b(y')$. We now want to relate the solution of this consumption-savings problem to the competitive risk sharing contracts equilibrium studied above. Note that the consumption-savings problem treats the interest rate R and the borrowing constraints $\underline{b}(y')$ as exogenous.

Proposition 16 *Any contract equilibrium $V(y,w), w'(y,w;y'), c(y,w)$ and $\left\{U^{Out}(y)\right\}_{y\in Y}$ can be implemented as a solution $W(y,b), b'(y,b;y'), C(y,b)$ to the consumption-savings problem above with borrowing constraint given by*

$$\underline{b}(y) = \frac{R}{R-1} \left(V(y, U^{Out}(y) - \nu(y)) - a(y) \right) \tag{54}$$

Conversely, for given borrowing constraints $\underline{b}(y) \leq 0$ and associated solution to the consumption-saving problem $W(y,b), b'(y,b;y'), C(y,b)$ there exist moving costs

$$\nu(y) = W(y,0) - W(y, \underline{b}(y)) \geq 0$$

such that the solution to the consumption-savings problem can be implemented as a contract equilibrium $V(y,w), w'(y,w;y'), c(y,w)$ and $\left\{U^{Out}(y)\right\}_{y\in Y}$. The moving costs satisfy

$$U^{Out}(y) - \nu(y) = W(y, \underline{b}(y)) \text{ for all } y \in Y \tag{55}$$

Proof. The details of the proof are again relegated to the technical appendix; the main logic follows standard duality theory. In the contract economy the state variables of a contract are (y,w), in the consumption-savings problem they are (y,b). Define the mapping between state variables as

$$b(y,w) = \frac{R}{R-1} \left(V(y,w) - a(y) \right) \tag{56}$$

$$w(y,b) = W(y,b) \tag{57}$$

where both functions are strictly increasing in their second arguments and thus invertible. With this mapping it is easy to see that the objective function of the contract problem implies the budget constraint of the consumption problem and the promise keeping constraint implies that the Bellman equation of consumption problem is satisfied. Reversely, the objective of the consumption problem implies the promise keeping constraint and the resource constraint implies the Bellman equation in the contract problem.

The nontrivial parts of the proof shown in the appendix are that any feasible contract satisfies the borrowing constraint in the consumption problem and that any feasible consumption allocation satisfies the utility constraints in the contract problem. It is finally shown (by tedious but straightforward construction) that the contract allocation is in fact optimal (and not just feasible) in the consumption problem. If there existed a superior consumption allocation it is feasible in the contract problem and yields lower costs, a contradiction. The reverse logic shows that the consumption allocation solves the cost minimization problem in the contract minimization problem. ∎

The proposition above shows that the contracting problem and the consumption problem are dual to each other. Furthermore, the proposition connects two strands of the previous literature. Via our equivalence result, all our findings from the contract economy carry over immediately to Bewley (1986)-type economies with trade in state-contingent Arrow securities subject to short-sale constraints. The borrowing constraints $\underline{b}(y)$ implied by our optimal risk sharing contracts are reminiscent of Alvarez and Jermann's (2000) borrowing constraints that are "not too tight". The proposition shows that, as for their borrowing constraints, agents can be allowed to borrow up to the point at which they are, state by state, indifferent between repaying their debt or defaulting. Instead of suffering financial autarky, as in Alvarez and Jermann (2000), the consequences of default in our model amount to being expelled from the relationship with the current financial intermediary and having to hook up with a competitor (and bear the utility cost $v(y)$, if any, from doing so). Thus one may interpret a solution to our consumption-savings problem as an equilibrium with solvency constraints in the spirit of Alvarez and Jermann.[11] We have as immediate

Corollary 17 *A contract equilibrium with zero moving costs, $\nu(y) \equiv 0$, can be implemented as a solution to the consumption problem with a short-sale constraint, $\underline{b}(y) = 0$. Reversely, a solution to the consumption-savings problem prohibiting borrowing can be implemented as a contract equilibrium with $\nu(y) \equiv 0$.*

This corollary provides a link to Bulow and Rogoff (1989): if there are no moving costs for agents between intermediaries (such as direct trade sanctions), then competition among intermediaries rules out international debt. The equivalence result also shows that in the contract economy without moving costs at no point in the contract does the "balance" of the agent with the financial intermediary become negative: after the initial payment of the agent the principal owes the agent more future consumption (in an expected discounted value sense) than he receives in expected discounted income from that agent, at each

[11] Always with the proviso that we assume a fixed gross interest rate R, whereas in their formulation the interest rate is an endogenously determined equilibrium object.

point and contingency in the contract. This, again, highlights the necessary pre-payment feature of the optimal risk sharing contracts derived in this paper.

3.7.2 Renegotiation-Proof Contracts

A further interpretation of our environment is one with a single principal and agent in which contracts signed between the principal and the agent have to be renegotiation-proof. The agent is assumed to have all bargaining power in the renegotiation, can demand renegotiation at any point of time, but experiences a disutility $\nu(y)$ "up front" from opening the renegotiation. This is equivalent to resetting the promise level w at the beginning of any period. With his bargaining power the agent presses the principal to the point of indifference, i.e. given current income y, she will demand a level of utility satisfying

$$V(y,w) = a(y)$$

i.e., $w = U^{Out}(y)$. The renegotiation-proof contract is thus constrained by $w'(y,w;y') \geq U^{Out}(y) - \nu(y)$. The equivalence between renegotiation-proof contracts with all the bargaining power held by the agent, and our environment then follows.

4 Conclusion

In this paper we have constructed a model of long-term relationships between risk averse agents with random income shocks and risk-neutral profit-maximizing principals. We have assumed one-sided commitment in that only the principal can commit a priori to the long-term contract. The outside option for the agent is given by contracts offered by competing principals. The paper analyzes the benchmark case of perfect competition and zero costs of moving: agents neither lose resources during the period of moving, nor are they required to stick to the current contract for at least one period.

We have shown that nonnegative steady state profits for the principal necessarily imply that the equilibrium implements the autarky solution, i.e., that there cannot be any risk sharing in that case. Furthermore, we have shown that these autarky equilibria can only arise if the principal is sufficiently patient. Otherwise, with sufficiently impatient principals, risk sharing will be observed: this includes in particular the benchmark case of equal discount rates for agents and principals. Agents signing up with the principal will initially pay some "contract fee" in the high income state. The principal in turn promises to provide costly insurance later on in the life of the contract. The agent thus turns into a liability to the principal in expected income terms. Competition therefore only affects the size of the initial up-front payment but not the unfolding of the contract later on.

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A The Sequential Formulation of the Game Between Locations

To formulate the game sequentially, we need some more notation. For dates q and $t \geq q$, let $y_i^{t,q} = (y_{q,i}, y_{q+1,i}, \dots, y_{t,i})$ denote the endowment history from q to t for agent i. A contract for agents newly arriving at principal j at date q specifies mappings $c_{t;q,j}(y_i^{t,q}), t = q, q+1, \dots$, defining consumption given the location-specific endowment history from date q to date t. We assume that agents never return to a location they left, so that there is no issue of "resurrecting" old records. Agents originally present at principal j are assumed to draw their initial income $y_{0,i}$ from some initial distribution. For the recursive formulation, we shall assume that this initial distribution is the stationary distribution.

Principals behave competitively and agents arrive with a "blank" history, i.e. a new principal does not make particular use of the fact that new arrivals must be agents who have defaulted on their previous principal. As mentioned above, to make this assumption more appealing, one could assume that a fraction ϵ of agents is forced to move every period anyhow: in equilibrium, these are the only movers, so it is reasonable for the principal in the new location not to attach any particular significance to the fact that an agent has switched locations. The game unfolds as follows:

1. A new date t begins and each agent draws his new endowment $y_{t,i}$.

2. Each principal $j \in [0;1]$ issues a new contract $c_{s;q,j}(y_i^{s,t}), s = t, t+1, \dots$ for agents willing sign with him this period.

3. Each agent decides whether to move or not, choosing the new principal j according to his or her preferences. He (or she) keeps the endowment process y including the current endowment $y_{t,i}$.

4. Given the current principal j, arrival date $q \leq t$ in that location and current endowment history $y^{t,q}$, agents provide their income to the principal and receive consumption goods $c_{t;q,j}(y_i^{t,q}), t = q, q+1, \dots$.

5. Let $\mu_{j,t}$ denote the measure of agents at principal j at date t (after the moving decision). The principal receives the total resource surplus

$$S_{j,t} = \int (y_{t,i} - c_{t,i})\, \mu_{j,t}(di) \tag{58}$$

   We allow $S_{j,t}$ to be negative. The objective of the principals is to maximize

$$U_j^{(P)} = \left(1 - \frac{1}{R}\right) E_0 \left[\sum_{t=0}^{\infty} R^{-t} S_{j,t}\right]$$

   We focus on stationary symmetric sub-game perfect equilibria, giving rise to the recursive formulation in the main body of the paper.

Competitive Risk Sharing Contracts with One-Sided Commitment: Technical Appendix

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Abstract

This appendix provides details of some of the proofs in the main paper

Proposition 1 (Proposition 4 in Main Text) *Let outside options* $(U^{Out}(y))_{y\in Y} \in [\underline{w}, \bar{w}]$ *and* $\beta < 1 < R$ *by given. Further suppose that condition 1 in the main text is satisfied. Then, an optimal contract for* $\left((U^{Out}(y))_{y\in Y}, \underline{w}, \bar{w}\right)$ *exists and has the following properties.*

1. $V(y,w)$ *is strictly convex, strictly increasing, continuous and differentiable in* w.

2. *The decision rules are unique and continuous.*

3. *The decision rules and the solution to the dynamic programming problem satisfy the first order conditions and the envelope condition*

$$(1-\beta)\lambda = \left(1 - \frac{1}{R}\right) C'(h) \tag{1}$$

$$\lambda\beta = \frac{1}{R}\frac{\partial V}{\partial w}(y', w'(y,w;y')) - \mu(y') \tag{2}$$

$$\lambda = \frac{\partial V}{\partial w}(y,w) \tag{3}$$

$$\lambda \geq 0 \tag{4}$$

$$\mu(y') \geq 0, \text{ for all } y' \in Y \tag{5}$$

where λ *and* $\mu(y')$ *are the Lagrange multipliers on the first and second constraints.*

4. *The decision rule* $h(y,w)$ *is strictly increasing in* w. *The decision rule* $w'(y,w;y')$ *is weakly increasing in* w, *and strictly so, if the continuing participation constraint* $w'(y,w;y') \geq U^{Out}(y') - \nu(y')$ *is not binding.*

5. *If the income process is iid, then $V(y,w)$ depends on w alone, $V(y,w) \equiv V(w)$. If additionally $U^{Out}(y') - \nu(y')$ is weakly increasing in y', then $w'(y,w;y')$ is weakly increasing in y'.*

Proof. While the proof draws on fairly standard techniques as in Stokey, Lucas with Prescott (1989), some points require particular attention.

1. Condition 1 assures that the equation

$$w = (1-\beta)h + \beta \sum_{y' \in Y} \pi(y'|y) w'(y')$$

subject to $w'(y') \geq U^{\text{Out}}(y') - \nu(y')$ always has a solution. Existence and convexity of V follows from a standard contraction argument. The contraction argument also shows that $V(y,\cdot)$ is strictly increasing.

To prove that $V(y,\cdot)$ is strictly convex we make use of the fact that under our assumptions the value function from the sequential and the recursive problem of the principal coincide. Now consider some y and $w_1 \neq w_2$ and, for each w_i, consider a stochastic sequence of optimal choices $(y_{t,i}, w_{t,i}, h_{t,i})_{t=0}^{\infty}$ from iterating the solution to the dynamic programming problem forward, starting with $y_{0,i} = y$ and $w_{0,i} = w_i$. Note that we do not require at this point that there is a unique solution. Note that $\underline{w} - \beta\bar{w} \leq h_t \leq \bar{w} - \beta\underline{w}$. This implies convergence of discounted sums of the $h_{t,i}$ and, per iteration,

$$w_i = (1-\beta) E\left[\sum_{j=0}^{\infty} \beta^t h_{t,i}\right]$$

A similar iteration for $V(y,w_i)$ and observing the upper bound $V(y,w) \leq V(y,\bar{w})) \leq C(\bar{w}) < \infty$ yields

$$V(y,w_i) = E\left[\sum_{j=0}^{\infty} \frac{1}{R^t} C(h_{t,i})\right]$$

Consider now the convex combination $w_\lambda = \lambda w_1 + (1-\lambda)w_2$. A feasible plan is given by the convex combination of $(w_{t,i}, h_{t,i})_{t=0}^{\infty}$, with costs given by

$$V_\lambda = E\left[\sum_{j=0}^{\infty} \frac{1}{R^t} C(h_{t,\lambda})\right] < \lambda V(y,w_1) + (1-\lambda)V(y,w_2)$$

where $h_{t,\lambda} = \lambda h_{t,1} + (1-\lambda)h_{t,2}$ and the inequality follows from the strict convexity of the cost function C. Obviously $V(y,w_\lambda) \leq V_\lambda < \lambda V(y,w_1) + (1-\lambda)V(y,w_2)$.

2. With strict convexity, differentiability and uniqueness of the decision rules now follow from standard arguments.

3. Likewise, deriving the first-order conditions are standard.

4. The monotonicity properties in w follow from the first order conditions, the fact, that $V(y,w)$ is strictly convex in w and the strict convexity of the cost function $c = C(h)$.

5. The fact that V does not depend on y in the iid case follows from the fact that y appears in the Bellman equation within the conditional probabilities $\pi(y'|y)$, which are independent of y in the iid case. The properties for $w'(w,y;y')$ follow from the first order conditions, the fact that $U^{\text{Out}}(y) - \nu(y)$ is increasing in y and strict convexity of V.

■

Proposition 2 (Proposition 5 in Main Text) *Let condition 1 be satisfied. Then an optimal contract and outside options* $\left\{U^{Out}(y)\right\}_{y\in Y}$ *satisfying* (13) *in the main text exist. If the income process is iid (or* $w'(y,w;y')$ *associated with* $\left\{U^{Out}(y)\right\}_{y\in Y}$ *is weakly increasing in* y*), then an equilibrium exists.*

Proof. We will prove this proposition for $\nu(y) \equiv 0$; equivalent arguments prove existence for the general case. We first prove that there exists outside options $U^{Out} = (U^{Out}(y_1), \ldots, U^{Out}(y_m))$ and associated value and policy functions $V_{U^{Out}}, h_{U^{Out}}, w'_{U^{Out}}(y')$ of the principals solving $V_{U^{Out}}(y, U^{Out}(y)) = a(y)$ for all y. Then we prove that the Markov transition function induced by π and $w'_{U^{Out}}(y')$ has a stationary distribution.

For the first part define the function $f : [\underline{w}, \bar{w}]^m \to [\underline{w}, \bar{w}]^m$ by

$$f_j\left[U^{Out}\right] = \min\{\tilde{w} \in [\underline{w}, \bar{w}] : V_{U^{Out}}(y_j, \tilde{w}) \geq a(y_j)\} \text{ for all } j = 1, \ldots, m$$

We need to show three things: 1) The function f is well defined on all of $[\underline{w}, \bar{w}]^m$, 2) The function f is continuous, 3) Any fixed point w^* of f satisfies $V_{w^*}(y_j, w_j^*) = a(y_j)$ for all $j = 1, \ldots, m$. We discuss each point in turn

1. If the set $\{\tilde{w} \in [\underline{w}, \bar{w}] : V_{U^{Out}}(y_j, \tilde{w}) = a(y_j)\}$ is non-empty for all j, then the function f is well-defined since the minimization over $\tilde{w}$ is a minimization of a continuous function over a compact set. To show that the set is non-empty for all j and all $U^{Out} \in [\underline{w}, \bar{w}]^m$ it suffices to show that $V_{\underline{w}}(y_j, \bar{w}) \geq a(y_j)$ for all j, since $V_{\underline{w}}(y_j, w)$ is strictly increasing in w and $V_{\underline{w}}(y_j, w) \leq V_{U^{Out}}(y_j, w)$ for all $U^{Out} \in [\underline{w}, \bar{w}]^m$ and all $w \in U^{Out} \in [\underline{w}, \bar{w}]$. Let $\hat{V}$ denote the cost function of a principal that does not face the competition constraints. But then

$$V_{\underline{w}}(y_j, \bar{w}) \geq \hat{V}(y_j, \bar{w}) = \bar{a} \equiv \max_i a(y_i) \geq a(y_j)$$

where the first equality follows from the definition of $\bar{w}$ in the main text. This proves the first point in our list.

2. In order to prove that f is continuous in $U^{Out} \in [\underline{w}, \bar{w}]^m$ it is useful to rewrite f as

$$f_j\left[U^{Out}\right]\begin{cases} = \underline{w} & \text{if } V_{U^{Out}}(y_j, \underline{w}) > a(y_j) \\ \text{solves } V_{U^{Out}}(y_j, f_j\left[U^{Out}\right]) = a(y_j) & \text{if } V_{U^{Out}}(y_j, \underline{w}) \leq a(y_j) \end{cases}$$

In the first case $f_j\left[U^{Out}\right]$ is independent of U^{Out}, in the second case it moves continuously in U^{Out} as long as $V_{U^{Out}}(.,.)$ is uniformly continuous in U^{Out}. Furthermore the switching point between the two cases moves continuously with U^{Out} if $V_{U^{Out}}(.,.)$ is uniformly continuous in U^{Out}. Thus it suffices to show uniform continuity of $V_{U^{Out}}(.,.)$ in U^{Out}. Take a sequence $\{U_n^{Out}\}_{n=0}^{\infty}$ in $[\underline{w}, \bar{w}]^m$ converging to U^{Out}. Let $\|.\|$ denote the sup-norm and $T_{U_n^{Out}}$ the operator associated with the Bellman equation (7) for outside options U_n^{Out}. Let $T_{U_n^{Out}}^k$ denote the $T_{U_n^{Out}}$ operator being applied k times. Note that by the triangle inequality

$$\left\|V_{U_n^{Out}} - V_{U^{Out}}\right\| \leq \left\|V_{U_n^{Out}} - T_{U_n^{Out}}^n V_{U^{Out}}\right\| + \left\|T_{U_n^{Out}}^n V_{U^{Out}} - V_{U^{Out}}\right\|$$

Since $T_{U_n^{Out}}$ is a contraction mapping for all U_n^{Out}, $\left\|V_{U_n^{Out}} - T_{U_n^{Out}}^n V_{U^{Out}}\right\|$ converges to 0 as $n \to \infty$. Again by the triangle inequality and the fact that $T_{U_n^{Out}}$ is a contraction mapping with modulus $\frac{1}{R}$ we have

$$\begin{aligned}\left\|T_{U_n^{Out}}^n V_{U^{Out}} - V_{U^{Out}}\right\| &\leq \sum_{k=1}^{n}\left\|T_{U_n^{Out}}^k V_{U^{Out}} - T_{U_n^{Out}}^{k-1} V_{U^{Out}}\right\| \\ &\leq \sum_{k=0}^{n-1} R^{-k}\left\|T_{U_n^{Out}} V_{U^{Out}} - V_{U^{Out}}\right\| \\ &= \sum_{k=0}^{n-1} R^{-k}\left\|T_{U_n^{Out}} V_{U^{Out}} - T_{U^{Out}} V_{U^{Out}}\right\|\end{aligned}$$

where the last equality follows from the fact that $V_{U^{Out}}$ is a fixed point of $T_{U^{Out}}$. Since $\sum_{k=0}^{n-1} R^{-k}$ converges, if we can show that $\left\|T_{U_n^{Out}} V_{U^{Out}} - T_{U^{Out}} V_{U^{Out}}\right\|$ converges to 0 in n we have demonstrated that $\left\|V_{U_n^{Out}} - V_{U^{Out}}\right\|$ converges to 0 in n, that is, $V_{U_n^{Out}}$ converges to $V_{U^{Out}}$ uniformly. Consider the function

$$\psi(y, w, U^{Out}) = \min_{h, \{w'(y')\} \in \Gamma(y, wU^{Out})} \left(1 - \frac{1}{R}\right) C(h) \frac{1}{R} + \sum_{y' \in Y} \pi(y'|y) V_{U^{Out}}(y', w'(y'))$$

with constraint set

$$\Gamma(y, w, U^{Out}) = \{h, w'(y') | h \in D, w'(y') \in [\underline{w}, \bar{w}], (8) \text{ and } (9) \text{ of main text}\}.$$

on $Y \times [\underline{w}, \bar{w}]^{m+1}$. Since for all $(y, w, U^{Out}) \in Y \times [\underline{w}, \bar{w}]^{m+1}$ the objective function is continuous and the constraint set is continuous, non-empty and compact-valued, by the theorem of the maximum ψ is continuous, and thus continuous in particular with respect to U^{Out}. Uniformity follows from the compactness of the space $Y \times [\underline{w}, \bar{w}]^{m+1}$, so that $V_{U_n^{Out}}$ converges to $V_{U^{Out}}$ uniformly. From points 1. and 2. it follows that the function f defined above is well-defined and continuous on the compact space $[\underline{w}, \bar{w}]^m$. Brouwer's fixed point theorem now guaranties existence of a $w^* \in [\underline{w}, \bar{w}]^m$ such that $w^* = f(w^*)$. In point 3. we now argue that this fixed point has the desired property that $V_{w^*}(y_j, w_j^*) = a(y_j)$.

3. Suppose not. Then there is some non-empty index set $\mathcal{J}$, so that $V_{w^*}(y_j, w_j^*) = a(y_j)$ for all $j \notin \mathcal{J}$ and $V_{w^*}(y_j, w_j^*) > a(y_j)$ as well as $w_j^* = \underline{w}$ for $j \in \mathcal{J}$. We will now show that V_{w^*} cannot be the solution to its Bellman equation. Let $\Delta = \max_j (V_{w^*}(y_j, w_j^*) - a(y_j))$: by assumption in this proof by contradiction, $\Delta > 0$. Let T be large enough so that

$$\frac{1 - R^{-T}}{1 - R^{-1}} (\underline{y} - \min_j y_j) + \frac{1}{R^T} \Delta < 0$$

Such a T exists, since we have assumed in our definition of the lower bound $\underline{w}$ that $\underline{y} - \min_j y_j < 0$. We will now construct sequential allocations that attain utility promises $w^*(y_0)$ at lower cost than $V_{w^*}(y_0, w^*(y_0))$, contradicting the fact that $V_{w^*}(.,.)$ is the cost function associated with outside options w^*. Let $V_{seq}(y, w)$ denote the cost function from the sequential allocation to be constructed now. Let $\sigma_t = (w_0, j_0, \ldots, j_t)$ be the history of income states until date t, including the initial promise $w_0 \in [\underline{w}, \bar{w}]$. A sequential decision rule must specify choices $\left(h(\sigma_t), (w'(y_j'; \sigma_t))_{j=1}^m\right)$ for any t and any history σ_t. Consider the following allocation: for a given t and σ_t :

- If $t < T$ and if $j_s \in \mathcal{J}$ for all $s \leq t$ and if $w_0 = w^*(y_0)$, provide utility $h_t = \underline{h}$ (where $\underline{h}$ is specified below). Furthermore, set the promises $(w'(y_j'; \sigma_t))_{j=1}^m$ equal to the fixed point w^*.
- If $t \geq T$ or if $t < T$ and if $j_s \notin \mathcal{J}$ for some $s \leq t$ or if $w_0 \neq w^*(y_0)$, provide utility h_{j_t} and promises $(w'(y_j'; \sigma_t))_{j=1}^m$ according to the decision rule of the proposed solution $V_{w^*}(y, w)$, where $y = y_t$ and $w = w'(y_t; \sigma_{t-1})$, if $t > 0$, and $w = w_0$, if $t = 0$.

We need to check that a) the continuation promise is always at least as large as the outside option w^*, b) the realized utility $w(\sigma_t)$ to the agent is always equal to the promise $w = w'(y_t; \sigma_{t-1})$, if $t > 0$, and $w = w_0$ if $t = 0$. and c) the resulting cost function V_{seq} is nowhere higher than the candidate solution V_{w^*}, i.e. $V_{seq}(y_0, w_0) \leq V_{w^*}(y_0, w_0)$. Furthermore, the costs are not larger than $a(y_j)$ for all j and all initial states $(y_0 = y_j, w^*(y_0))$. Then, since the costs exceed $a(y_j)$ for some j in the proposed solution V_{w^*} to the Bellman equation, this then renders a contradiction that V_{w^*} is a solution to the Bellman equation. Let us check each of the items a) - c) above.

a. **Continuation promise is always is at least as large as the outside options** w^*. True by construction, since it is true for the decision rules of V_{w^*} and since it is also true at the changed decisions under the first bullet point.

b. **Realized utility** $w(\sigma_t)$ **of agent equals the promise** $w = w'(y_t;\sigma_{t-1})$**, if** $t > 0$**, and** $w = w_0$ **if** $t = 0$**.** Since this must be true along histories simply following the decision rules given by the candidate solution V_{w^*}, we only need to check this for the case that $t < T$, $j_s \in \mathcal{J}$ for all $s \le t$ and $w_0 = w^*(y_0)$. But along these paths, $w_0 = \underline{w}$ for $t = 0$ and $w'(y_t;\sigma_{t-1}) = w^*(y')$ for $t > 0$. The claim follows with

$$\underline{w} = (1-\beta)\underline{h} + \beta \sum_{y' \in Y} \pi(y'|y_t) w^*(y')$$

Note that condition 1 in the main text assures that the $\underline{h}$ required by this equation lies in $\mathbf{D}$ and that consequently there exists a $\underline{c} \in (0, \underline{y}]$ such that $\underline{h} = u(\underline{c})$. The fact that $\underline{c} \le \underline{y}$ follows from the fact that $w^*(\underline{y'}) \ge \underline{w}$ and the definition of $\underline{y}$ in the main text.

c. **Resulting cost function is nowhere higher than the candidate solution** V_{w^*}**, i.e.** $V_{seq}(y_0, w_0) \le V_{w^*}(y_0, w_0)$**. Furthermore, the costs are not larger than** $a(y_j)$ **for all** j **and all initial states** $(y_0 = y_j, w^*(y_0))$**.** The claim is true by construction if the sequential decisions just follows along the decision rules of the candidate solution V_{w^*}, since in particular, $V_{w^*}(y_j, w_j^*) = a(y_j)$ for all $j \notin \mathcal{J}$. Consider now $j \in \mathcal{J}$, the initial state $y_0 = y_j$ and the initial promise $w^*(y_0) = \underline{w}$. Consider all possible histories σ_T until date T and define the stopping time $\tau(\sigma_T)$ to be the earliest date $t = 1, \ldots, T$, at which $j_t \notin \mathcal{J}$, and set $\tau(\sigma_T) = T$, if all $j_t, t = 1, \ldots, T$ belong to the set $\mathcal{J}$. Let $\pi(\sigma_T, t)$ the probability of the history, truncated at date t. The costs can now be calculated directly as

$$\begin{aligned} V_{seq}(y_0, \underline{w}) &= \left(1 - \frac{1}{R}\right) \sum_{\sigma_T} \left\{ \left(\sum_{t=0}^{\tau(\sigma_T)-1} \frac{1}{R^t} \pi(\sigma_T, t)\underline{c} \right) + \frac{1}{R^{\tau(\sigma_T)}} \pi(\sigma_T, \tau(\sigma_T)) V(y_{j_t}, w^*(y_{j_t})) \right\} \\ &\le \left(1 - \frac{1}{R}\right) \sum_{\sigma_T} \left\{ \left(\sum_{t=0}^{\tau(\sigma_T)-1} \frac{1}{R^t} \pi(\sigma_T, t)\underline{y} \right) + \frac{1}{R^{\tau(\sigma_T)}} \pi(\sigma_T, \tau(\sigma_T)) V(y_{j_t}, w^*(y_{j_t})) \right\} \end{aligned}$$

where the inequality follows from the fact that $\underline{c} \le \underline{y}$. Now note the following. For any history σ_T, for which $j_t \notin \mathcal{J}$ for some $t \le T$, we have $V(y_{j_t}, w^*(y_{j_t})) = a(y_{j_t})$ at $t = \tau(\sigma_T))$ and we have costs no larger than $\underline{y} < \min y_j$ for $t < \tau(\sigma_T))$, so that costs conditional on these paths are no higher than the net present value $a(y_0, \sigma_T)$ of the income, conditional on these paths, where

$$a(y_0, \sigma_T) = \left(1 - \frac{1}{R}\right)\left(\left(\sum_{t=0}^{T-1} \pi(\sigma_T, t)\frac{1}{R^t} y_{j_t}\right) + \pi(\sigma_T, T)\frac{1}{R^T} a(y_{j_T})\right)$$

Finally consider the histories σ_T, for which $j_t \in \mathcal{J}$ for all $t = 0, \ldots, T$. For these paths, the contribution $\kappa(\sigma_T)$ to the costs are

$$\begin{aligned}
\kappa(\sigma_T) &= \left(1 - \frac{1}{R}\right)\left(\left(\sum_{t=0}^{T-1} \frac{1}{R^t}\pi(\sigma_T, t)\underline{c}\right) + \frac{1}{R^T}\pi(\sigma_T, T)V(y_{j_t}, w^*(y_{j_t}))\right) \\
&\leq \left(1 - \frac{1}{R}\right)\left(\left(\sum_{t=0}^{T-1} \frac{1}{R^t}\pi(\sigma_T, t)\underline{y}\right) + \frac{1}{R^T}\pi(\sigma_T, T)V(y_{j_t}, w^*(y_{j_t}))\right) \\
&\leq a(y_0, \sigma_T) + \left(1 - \frac{1}{R}\right)\left(\left(\sum_{t=0}^{T-1} \frac{1}{R^t}\pi(\sigma_T, t)(\underline{y} - \min_j y_j)\right) + \frac{1}{R^T}\pi(\sigma_T, T)\Delta\right) \\
&\leq a(y_0, \sigma_T) + \left(1 - \frac{1}{R}\right)\pi(\sigma_T, T)\left(\frac{1 - R^{-T}}{1 - R^{-1}}(\underline{y} - \min_j y_j) + \frac{1}{R^T}\Delta\right) \\
&\leq a(y_0, \sigma_T)
\end{aligned}$$

by our assumption about T. This finishes the check of point c), thus part 3. and therefore the entire proof of the existence of equilibrium outside options $U^{Out} = w^*$. It remains to be shown that a stationary distribution associated with the optimal decision rules for equilibrium outside options w^* exist.

From now on let $(h, w'(y'))$ denote the optimal policies of the principal associated with the equilibrium outside options $U^{Out} = w^*$, whose existence was established above. We know that $\{w'(y, w; y')\}_{y' \in Y}$ are continuous functions on Z. Let $Q : Z \times \mathcal{B}(Z) \to [0, 1]$ denote the Markov transition function as defined in the main text. Since the policy functions are continuous and hence measurable, by theorem 9.13 of Stokey et al. (1989) Q is indeed a well-defined transition function. Furthermore, by their theorem 8.2. the operator $T^* : \Lambda(Z \times \mathcal{B}(Z)) \to \Lambda(Z \times \mathcal{B}(Z))$ mapping the space of probability measures on $(Z \times \mathcal{B}(Z))$ into itself and defined as

$$T^*\Phi(A) = \int Q(z, A)\Phi(dz)$$

is well-defined. Showing that there exists an invariant probability measure Φ associated with Q amounts to showing the existence of a fixed point of the operator T^*. If $w'(y, w; y')$ is increasing in both y and in w then this result (but not uniqueness) follows from Corollary 4 of Hopenhayn and Prescott (1992), the assumptions of which are easily verified under this condition. We established above monotonicity in w; with respect to monotonicity in y we note that if income is *iid*, this is trivially true, but any other assumption rendering this true (in particular, assumptions on the transition matrix π) does work as well. ■

Proposition 3 (Proposition 9 in Main Text) *Assume that condition 1 holds.*

1. *An equilibrium satisfying the nonnegative steady state profit condition exists, if and only if*

$$R \in (1, \bar{R}] \tag{6}$$

where

$$\bar{R} = \frac{C'(\underline{h})}{\beta C'(\bar{h})} \leq \frac{1}{\beta} \tag{7}$$

2. *If it exists, the equilibrium has the following form*

$$V(y, w) = \left(1 - \frac{1}{R}\right)(C(h(y, w)) - y) + a(y) \tag{8}$$

$$c(y, w) = C(h(y, w)) \tag{9}$$

$$w'(y, w; y') = w_{aut}(y') \tag{10}$$

$$U^{Out}(y) = w_{aut}(y) \tag{11}$$

profits $\gamma = 0$ and a positive measure Φ as constructed above.

3. *No equilibria satisfying the nonnegative steady state profit condition exist for any R, if $\bar{R} \leq 1$.*

4. *Suppose that $\beta R = 1$. Then, no equilibrium satisfying the nonnegative steady state profit condition exists.*

5. *Suppose that the endowment process is iid. Then*

$$\bar{R} = \frac{1}{\beta} \frac{C'(u(y_1))}{C'(u(y_m))} = \frac{1}{\beta} \frac{u'(y_m)}{u'(y_1))} \tag{12}$$

Proof. Suppose there is an equilibrium. We shall show that this either leads to a contradiction, or that it has to be of the form given above. Consider the Bellman equation (13)

$$V(y, w) = \min_{h \in D, \{w'(y') \in [\underline{w}, \bar{w}]\}_{y' \in Y}} \left(1 - \frac{1}{R}\right) C(h) \frac{1}{R} + \sum_{y' \in Y} \pi(y'|y) V(y', w'(y')) \tag{13}$$

$$\text{s.t. } w = (1 - \beta)h + \beta \sum_{y' \in Y} \pi(y'|y) w'(y') \tag{14}$$

with the outside option given by the autarky level $U^{Out}(y) = w_{aut}(y)$ for all $y \in Y$ and with the domain $Z = Y \times [\underline{w}, \bar{w}]$. As discussed above, proposition 8 implies that $w'(y') = w'(y, w; y') = w_{aut}(y')$. It also implies that $V(y, w_{aut}(y)) = a(y)$.

The first fact and the first constraint in the Bellman equation together imply that $h = h(y, w)$, defined as in equation (15):

$$h(y, w) = \frac{w - \beta \sum_{y' \in Y} \pi(y'|y) w_{aut}(y')}{1 - \beta} \tag{15}$$

All these facts and a short calculation (alternatively one may guess the functional form of the value function and determine its coefficients) in turn imply that $V(y, w)$ as well as $c(y, w)$ are of the form stated in the proposition.

It remains to check that V indeed solves the Bellman equation: if it does, we have constructed an equilibrium. If not, there cannot be an equilibrium. Note that $V(y, w)$ is strictly convex and differentiable in w: checking the constraints and the first-order conditions is therefore necessary as well as sufficient. The constraints are satisfied by construction: what remains are the first-order conditions, given in proposition (1).

Taking the functional form for V in the proposition and differentiating shows equations (1), (3) and (4) to be satisfied. Equation (2) can be rewritten as an equation defining $\mu(y')$: it therefore only remains to check the inequality constraints (5). Define

$$\nu(y, w; y') = \frac{(1 - \beta)R}{1 - \frac{1}{R}} \mu(y') \tag{16}$$

Rewriting (5) with the help of the other equations to see that

$$\nu(y, w; y') = C'(u(y')) - R\beta C'(h(y, w)) \tag{17}$$

Therefore, we have obtained a solution if and only if $\nu(y, w; y') \geq 0$ for all $y, y' \in Y$ and $\underline{w} \leq w \leq \bar{w}$. Minimizing over this set finally shows, that it suffices to check the worst case scenario $y' = \min_{y \in Y} y$ and $h(y, w) = \bar{h}$. Hence we have shown that an equilibrium exists if and only if (7) is satisfied. Note that these inequalities have to be satisfied in order to be able to implement the autarky solution, as demanded by proposition 7 in the main text. The other results trivially follow from the first part of the proposition. ■

Proposition 4 (Proposition 16 in Main Text) *Any contract equilibrium* $V(y, w)$, $w'(y, w; y')$, $c(y, w)$ *and* $\left\{U^{Out}(y)\right\}_{y \in Y}$ *can be implemented as a solution* $W(y, b)$, $b'(y, b; y')$, $C(y, b)$ *to the consumption-savings problem above with borrowing constraint given by*

$$\underline{b}(y) = \frac{R}{R - 1} \left(V(y, U^{Out}(y) - \nu(y)) - a(y)\right) \tag{18}$$

Conversely, for given borrowing constraints $\underline{b}(y) \leq 0$ *and associated solution to the consumption-saving problem* $W(y, b), b'(y, b; y'), C(y, b)$ *there exist moving costs*

$$\nu(y) = W(y, 0) - W(y, \underline{b}(y)) \geq 0$$

such that the solution to the consumption-savings problem can be implemented as a contract equilibrium $V(y,w), w'(y,w;y'), c(y,w)$ *and* $\left\{U^{Out}(y)\right\}_{y\in Y}$. *The moving costs satisfy*

$$U^{Out}(y) - \nu(y) = W(y, \underline{b}(y)) \text{ for all } y \in Y \tag{19}$$

Proof. In the contract economy the state variables of a contract are (y,w), in the consumption-savings problem they are (y,b). Define the mapping between state variables as

$$\begin{aligned} b(y,w) &= \frac{R}{R-1}\left(V(y,w) - a(y)\right) && (20) \\ w(y,b) &= W(y,b) && (21) \end{aligned}$$

where both functions are strictly increasing in their second arguments and thus invertible. We denote the inverse of $b(y,w)$ by $w = b^{-1}(y,b)$: it is that lifetime utility level w which requires initial bond holdings b to realize that level in the bond economy. Let $b = w^{-1}(y,w)$ be similarly defined. Furthermore define $\underline{b} = w^{-1}(y, \underline{w})$ and $\bar{b} = w^{-1}(y, \bar{w})$.[1] Finally define the map between policies and value functions in the two problems as

$$\begin{aligned} c(y,b) &= C(h(y,w(y,b))) && (22) \\ b'(y,b;y') &= b(y', w'(y,w(y,b);y')) \\ W(y,b) &= b^{-1}(y,b) \end{aligned}$$

and

$$\begin{aligned} h(y,w) &= u(c(y,b(y,w))) && (23) \\ w'(y,w;y') &= w(y', b'(y,b(y,w);y')) \\ V(y,w) &= w^{-1}(y,w) \end{aligned}$$

With policies so defined it is straightforward to verify that (W,c,b') constructed from the contract problem as in (22) satisfy the Bellman equation of the consumption problem (since the underlying (V,h,w') satisfy the promise keeping constraint in the contract problem) and the budget constraint (since (V,h,w') satisfy the Bellman equation of the contract problem). Reversely, one can equally easily verify that (V,h,w') constructed from the consumption problem as in (23) satisfy the Bellman equation of the contract problem (since the underlying (W,c,b') satisfy the budget constraint in the contract problem) and the promise keeping constraint (since (W,c,b') satisfy the Bellman equation of the consumption problem).

[1] When mapping the bond economy into the contract economy bounds on bond holdings $[\underline{b}, \bar{b}]$ map into utility bounds $[\underline{w}, \bar{w}]$ in a similar fashion. A lower bound on bond holdings is always guaranteed via the borrowing constraints; when we start with the bond economy we assume the existence of an upper bound on bond holdings. Under fairly general conditions this upper bound can be chosen without imposing additional binding restrictions on the problem although this is not crucial for the equivalence result).

Now we show that any feasible contract satisfies the borrowing constraint in the savings problem and that any consumption allocation satisfies the utility constraints in the contract problem. First we show that the bond holdings

$$b'(y, b(w, y); y') = \frac{R}{R-1} \left(V(y', w'(y, b^{-1}(y, b); y')) - a(y') \right) \tag{24}$$

derived from the contract problem satisfy the short-sale constraints

$$\underline{b}(y') = \frac{R}{R-1} \left(V(y', U^{Out}(y') - \nu(y')) - a(y') \right)$$

in the consumption problem. Since V is strictly increasing in its second argument and the function $w'(y, w; y')$ satisfies

$$w'(y, w; y') \geq U^{Out}(y') - \nu(y')$$

we have

$$\begin{aligned} b'(y, b(w, y); y') &= \frac{R}{R-1} \left(V(y', w'(y, b^{-1}(y, b); y')) - a(y') \right) \\ &\geq \frac{R}{R-1} \left(V(y, U^{Out}(y') - v(y')) - a(y') \right) \\ &= \underline{b}(y) \end{aligned}$$

Second we show that the future utility promises

$$w'(y, w; y') = W(y', b'(y, b; y')) \tag{25}$$

derived from the savings problem satisfy the contract enforcement constraints. We note, since

$$b'(y, b; y') \geq \underline{b}(y')$$

that

$$w'(y, w; y') = W(y', b'(y, b; y')) \geq W(y', \underline{b}(y')) = W(y', 0) - \nu(y') \tag{26}$$

using that the function W is strictly increasing in its second argument and the definition of $v(y')$. Now we note that in the consumption problem lifetime utility $w = W(y, b)$ can be delivered at lifetime cost $b + \frac{Ra(y)}{R-1}$. Thus $V(y, w) = V\left[y, W(y, b)\right] = a(y) + \frac{R-1}{R} b$ is the cost in per-period terms. From this it follows that

$$V\left[y', W(y', 0)\right] = a(y') = V(y', U^{Out}(y'))$$

and thus

$$W(y', 0) = U^{Out}(y') \tag{27}$$

Combining (25)-(27) yields

$$w'(y,w;y') \geq U^{Out}(y') - \nu(y)$$

that is, the utility promises derived from the consumption problem satisfy the enforcement constraint. From $W(y',0) = U^{Out}(y')$ also (19) immediately follows.

These results imply that consumption-bond policies and value function (W, c, b') derived from the contract policies are feasible for the consumption problem and the value function satisfies the Bellman recursion. It remains to be shown that for the given value function W on the right hand side of Bellman's equation for the consumption problem, the policies (c, b') are indeed optimal. Reversely, it remains to be shown that for the given value function V on the right hand side of the contract minimization problem (h, w') are the optimal policies. Since both directions follow exactly the same logic we shall only prove the direction that (c, b') are indeed optimal.

We will do so by arguing that if (c, b') weren't optimal one can construct a contract policy from the superior consumption policy that yields lower costs than the optimal contract allocation, a contradiction. So suppose that

$$\begin{aligned}
\left(\tilde{c}, \tilde{b}'(y')\right) &\in \arg\max_{c,(b(y'))_{y'\in Y}} \left\{(1-\beta)u(c) + \beta \sum_{y'} \pi(y'|y) W(y', b(y'))\right\} \\
c + \frac{1}{R}\sum_{y'\in Y} \pi(y'|y) b(y') &= y + b \\
b(y') &\geq \underline{b}(y'), \text{ for all } y' \in Y
\end{aligned}$$

for a given (y, b) and that at least for some $(y^*, b^*) \in Y \times [\underline{b}, \bar{b}]$ the bond-consumption policy (c, b') derived from the contract policies (h, w') yields strictly worse utility. Let $(\tilde{h}, \tilde{w}')$ denote the contract policy associated with $(\tilde{c}, \tilde{b}')$. Determine the associated utility promise $w^*(y^*, b^*)$ from (21). Since the original contract allocation associated with (h, w') satisfies the promise keeping constraint and $(\tilde{c}, \tilde{b}')$ yields weakly higher lifetime utility than (c, b'), strictly so for (y^*, w^*) one can define a new contract policy $(\hat{h}, \hat{w}')$ with

$$\begin{aligned}
\hat{h} &= \tilde{h} \quad \text{for } (w,y) \neq (w^*, y^*) \\
\hat{h} &= \tilde{h} - \varepsilon \quad \text{for } (w,y) \neq (w^*, y^*) \\
\hat{w}' &= \tilde{w}'
\end{aligned}$$

that also satisfies the promise-keeping constraint for small enough $\varepsilon > 0$ (and the outside option constraints, since $(\tilde{h}, \tilde{w}')$ does, as it is derived from which satisfy the short-sale constraint). By definition

$$\begin{aligned}
V(y^*,w^*) &= \left(1-\frac{1}{R}\right)C(h(y^*,w^*))+\frac{1}{R}\sum_{y'}\pi(y'|y)V(y',w'(y^*,w^*;y')) \\
&= \left(1-\frac{1}{R}\right)C(\tilde{h}(y^*,w^*))+\frac{1}{R}\sum_{y'}\pi(y'|y)V(y',\tilde{w}'(y^*,w^*;y')) \\
&> \left(1-\frac{1}{R}\right)C(\hat{h}(y^*,w^*))+\frac{1}{R}\sum_{y'}\pi(y'|y)V(y',\hat{w}'(y^*,w^*;y'))
\end{aligned}$$

and thus $(\tilde{h},\tilde{w}')$ has lower costs that (h,w'), a contradiction to the fact that (h,w') is cost-minimal in the contract problem. Note that the second equality comes from the fact that both (c,b') and $(\tilde{c},\tilde{b}')$ satisfy the budget constraint in the consumption problem. Also note that a similar proof, based on the sequence form of consumption maximization and contract minimization problems immediately gives the same result. ■

References

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