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# Orderings, excess functions, and the nucleolus

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**Abstract**

The nucleolus of a cooperative game can be described with the aid of the leximin ordering but also on the basis of two other orderings. In this note the relation between these orderings is studied in a more general framework. The results are applied to the nucleolus corresponding to so-called normal excess functions. Also the Kohlberg criterion is extended to this more general case.


*Keywords:* Cooperative games; Nucleolus; Orderings; Excess functions

## 1. Introduction

The nucleolus of a cooperative game with transferable utility, introduced in Schmeidler (1969), lexicographically minimizes the nonincreasingly ordered excesses of the coalitions over the imputation set of the game. More generally, let $M$ be a finite set of agents (e.g. coalitions) and let $\Pi \subseteq \mathbb{R}^M$ be some set of feasible vectors (e.g. the excess vectors corresponding to a collection of payoff vectors in a game, see Section 3 below). Consider the ordering $\succeq_d$ ($d$ from 'desirable') defined by

$$a \succeq_d b :\text{. there is a } t' \in \mathbb{R} \text{ with } \begin{cases} |B_t(a)| = |B_t(b)| & \text{for all } t > t' \\ |B_{t'}(a)| < |B_{t'}(b)| \end{cases}$$

$$\text{or } |B_t(a)| = |B_t(b) \text{ for all } t \in R$$

for all $a$, $b \in \Pi$, where $B_t(_a) := \{j \in M : a_j \geq t\}$ denotes the set of agents for which the corresponding coordinates in $a$ are at least $t$, and $|B_t(a)|$ denotes the cardinality of this set. It is straightforward to verify that

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$$a \succeq_d b \Leftrightarr a^* \succeq_{\text{lexmin}} b^* ,$$

where $a^*$ arises from $a$ by rearranging the coordinates in nonincreasing order and $\succeq_{\text{lexmin}}$ denotes lexicographical minimization; that is, $a^* \succeq_{\text{lexmin}} b^*$ if, and only if, $a_j^* < b_j^*$ for the smallest j with $a_j^* \neq b_j^*$. Thus, the nucleolus may alternatively be defined using $\succeq_d$. This ordering has the advantage that it is closer in formulation to the two orderings to be defined next. For a comprehensive survey on the merits of the nucleolus and related solutions see Maschler (1992).

In Justmann (1977), Justman considers the ordering

$$\begin{aligned} a \succeq_J b : \Leftrightarrow \quad & a = b, \text{ or there is a } j \in M \text{ such that:} \\ & a_j < b_j \text{ and for all } i \in M: a_i > b_i \text{ implies } a_i \leq a_j . \end{aligned}$$

He shows that under certain conditions an iterative process based on this ordering converges to the nucleolus, when applied to a game. Also Osborne and Rubinstein propose an alternative ordering in their definition of the nucleolus (see Osborne and Rubinstein, 1994). This is the following ordering:

$$\begin{aligned} a \succeq_{OR} : \Leftrightarrow \quad & a = b, \text{ or there is a } j \in M \text{ such that:} \\ & a_j < b_j \text{ and for all } i \in M: a_i > b_i \text{ implies } a_i \leq b_j . \end{aligned}$$

The advantage of both these orderings over the desirability relation $\succeq_d$ or the lexmin ordering $\succeq_{\text{lexmin}}$ is that they admit a more transparent interpretation in terms of objections and counterobjections (cf. Osborne and Rubinstein, 1994, p. 286). To see this, first realize that in agreement with their interpretation as excesses, coordinates should be seen as disutilities; so lower coordinates are better. Concerning the ordering $\succeq_J$ one can imagine some agent $i$ objecting against some proposal $a$ by referring to the alternative $b$; then agent $j$ may counterobject by stating, not only that $a$ is better for him, $j$, than $b$, but also that $a$ is in fact better for agent $i$ than for agent $j$ himself; thus, by insisting on $a$ agent $j$ actually accepts that he will end up less satisfied than agent $i$. Observe that such an interpretation assumes that coordinates of different agents can be meaningfully compared. The relation $\succeq_{OR}$ can be given an interpretation in the same spirit, with the difference that this time agent $j$, in counterobjecting against $b$, refers to the fact that $b$ is worse for him than $a$ is for agent $i$.

The first objective of this note is to clarify the relations between the three orderings defined above. This is done in Section 2. In Section 3 the result is applied to so called normal excess vectors in cooperative games with transferable utility, which leads to a generalization of existing results. A corresponding generalization of the Kohlberg criterion involving balanced collections completes the paper.

### *1.1. Notation*

$\subseteq$ denotes set inclusion, $\subset$ denotes strict set inclusion.

## 2. Comparison of $\succeq_J$, $\succeq_d$, and $\succeq_{OR}$

Let $\Pi \subseteq \mathbb{R}^M$, as in the Introduction. The following lemma is immediate from the definitions.

**Lemma 1.** *Let* $a, b \in \Pi$. *Then:*

$$(i)\ a \succeq_J b \Rightarrow a \succeq_{OR} b$$

$$(ii)\ a \succeq_J b, a \neq b \Rightarrow \textit{not } b \succeq_J a, \textit{not } b \succeq_{OR} a\ .$$

As to the relation with $\succeq_d$ we have the following result.

**Theorem 1.** *Let* $a, b \in \Pi$. *Then:* $a \succeq_J b \Rightarrow a \succeq_d \Rightarrow a \succeq_{OR} b$.

**Proof.** Assume $a \succeq_J b$. If $a = b$ then $|B_t(a)| = |B_t(b)|$ for all $t \in \mathbb{R}$, so $a \succeq_J b$.

Now assume $a \neq b$, and let $j \in M$ as in the definition of $a \succeq_J b$. Define $t' := \max\{b_k : b_k \neq a_k\}$. Take $t \geq t'$ and suppose that $b_i < t \leq a_i$ for some $i \in M$. Then $a_i \leq a_j < b_j \leq t'$, a contradiction. This shows that $B_t(a) \subseteq B_t(b)$ and hence $|B_t(a)| \leq |B_t(b)|$ for all $t \leq t'$.

For $t = t'$ this inequality is strict: for $k \in M$ with $b_k = t'$, $a_k > b_k$ would imply $b_k < a_k \leq a_j < b_j$ contradicting the definition of $t'$, hence $a_k < b_k = t'$. For $t > t'$ the inequality is in fact an equality, which can be seen as follows. Suppose, to the contrary, that there is an $i \in M$ with $i \in B_t(b) \backslash B_t(a)$. Then $b_i \geq t > a_i$, so $b_i \leq t' < t$ by definition of $t'$, a contradiction.

It follows that also in this case $a \succeq_d b$.

Next, assume $a \succeq_d b$. If $a = b$ then $a \neq_{OR} b$. Now assume $a \neq b$. Let $t'$ be such that $B_t(a) = B_t(b)$ for all $t > t'$, and $B_{t'}(a) \neq B_{t'}(b)$.

Suppose $B_{t'}(b) \subset B_{t'}(a)$, then obviously $b \succeq_d a$. Together with $a \succeq_d b$ this implies $|B_t(a)| = |B_t(b)|$ for all $t$, and in particular for $t = t'$, a contradiction. It follows that $B_{t'}(b) \backslash B_{t'}(a) \neq \emptyset$.

So, take $j \in B_{t'}(b) \backslash B_{t'}(a)$. Then $b_j > a_j$. For $i$ with $a_i > b_i$ one must have $a_i \leq t'$ since otherwise $i \in B_t(a) = B_t(b)$ for $t = a_i$, and thus, $b_i \leq a_i$, a contradiction. Therefore, $a_i \leq t \leq b_j$, hence $a \succeq_{OR} b$. $\square$

**Remark.** *In the proof of Theorem 1 the following has in fact been shown as well.*

$$a \succeq_J b \quad \Rightarrow \quad \text{there is a } t' \in R \text{ with } \begin{cases} B_t(a) = B_t(b) & \text{for all } t > t' \\ B_{t'}(a) \subset B_{t'}(b) \end{cases}$$
$$\text{or } a = b\ ,$$

*and*

$$a \succeq_{OR} b \quad \Leftarrow \quad \text{there is a } t' \in R \text{ with } \begin{cases} B_t(a) = B_t(b) & \text{for all } t > t' \\ B_{t'}(b)/B_{t'}(a) \neq \emptyset \end{cases}$$
$$\text{or } a = b\ ,$$

It can be shown that the implication in the $\succeq_j$-case is strict, and in the $\succeq_{OR}$-case it is an equivalence.

In general – that is, without specific conditions on the feasible set $\Pi$ – the implications in Theorem 1 cannot be reversed. For instance, let $M = \{1, 2\}$, $a = (2, 0)$, and $b = (1, 2)$. Then neither $a \succeq_J b$ nor $b \succeq_J a$; $a \succeq_d b$ but not $b \succeq_d a$; and $a \succeq_{OR} b$ as well as $b \succeq_{OR} a$.

Under the following condition the reversal of the implications in Theorem 1, at least for maximal elements of the three relations, will hold. Call a set $\Pi \subseteq \mathbb{R}^M$ *weakly convex* if for all a, $b \in \Pi$ and every $\epsilon > 0$ there is a $c \in \Pi$ with (Euclidean) distance to $a$ smaller than $\epsilon$ and with for all $i \in M$:

$$a_i > b_i \Rightarrow a_i > c_i > b_i$$

$$a_i < b_i \Rightarrow a_i < c_i < b_i$$

$$a_i = b_i \Rightarrow a_i = c_i = b_i \tag{1}$$

Thus, the condition of weak convexity of a set means that between any two points of the set there exists another point of the set as close to one of the two points as desired.

Obviously, convex sets are weakly convex. Further, one easily shows that a closed set is weakly convex whenever for each $a,b \in \Pi$ there exists a $c \in \Pi$ with property (1).

**Theorem 2.** *Let $\Pi \subseteq \mathbb{R}^M$ be weakly convex, and let $\succeq$, $\succeq'$ be any of the three orderings $\succeq_J$, $\succeq_d$, $\succeq_{OR}$. Let $a \in \Pi$. If $a \succeq b$ for all $b \in \Pi$, then $a \succeq' b$ for all $b \in \Pi$.*

**Proof.** In view of Theorem 1 it is sufficient to prove the implication for $\succeq = \succeq_{OR}$ and $\succeq' = \succeq_J$. So let $a \succeq_{OR} b$ for all $b \in \Pi$. If $a = b$ then by definition $a \succeq_J b$. Now suppose $a \neq b$, then in particular $a \geq b$. Choose a point $c$ satisfying (1) and so close to $a$ that $c_j < a_i$ for all $i$, $j \in M$ with $a_i > a_j$. Because $a \succeq_{OR} c$ there is a coordinate $j$ with $a_j < c_j$ and $a_i > c_i \Rightarrow a_i \leq c_j$ for all $i \in M$. By (1), $a_j < b_j$. Take any $i \in M$ with $a_i > b_i$. Then, by (1), $a_i > c_i$, hence $a_i \leq c_j$. By the choice of $c$ this implies $a_i \leq a_j$. It follows that $a_j \succeq b$. □

Theorem 2 applies in particular to a convex set $\Pi$ (the usual case for the nucleolus of a cooperative game). Convex sets are connected but, clearly, connectedness of $\Pi$ is not a necessary condition for the conclusion of the theorem to hold. It is also not a sufficient condition, as the next example shows.

**Example 1**. Let $M = \{1, 2\}$, let

$$\Pi = \{(x, 1) \in \mathbb{R}^{\mathbb{M}} : 0 \leq x \leq 1\} \cup \{(1, x) \in \mathbb{R}^M : 0 \leq x \leq 1\}$$

and let $a = (0, 1)$. Then it is straightforward to verify that $a \succeq_{OR} b$ for all $b \in \Pi$, but not (e.g.) $a \succeq_J (1, 0)$. Observe that the set $\Pi$ is connected, but clearly does not satisfy (1).

The proof of Theorem 2 uses the assumption that $c$ may be chosen sufficiently close to $a$. The following example shows that this assumption cannot be dropped.

**Example 2.** Let $M = \{1, 2\}$, and

$$\Pi = \{(0, 1)\} \cup \{(x, 1 - x/2) \in \mathbb{R}^{\mathbb{M}} : 1 < x \leq 2\},$$

and further, let $a = (0,1)$. Again, it is straightforward to verify that $a \succeq_{OR} b$ for all $b$, but not (e.g.) $a \succeq_J (2, 0)$. $\Pi$ satisfies the property that for each $a$, $b$ there exists a $c \in \Pi$ with property (1), but for $a=(0, 1)$ we cannot choose $c$ arbitrarily close to $a$. Observe that $\Pi$ is not closed.

## 3. Normal excess functions and the nucleolus

A *(cooperative) game (with transferable utility)* is a pair $(N, v)$, or briefly $v$, where $N:=\{1,...,n\}$ is the set of *players* and the *characteristic function* $v:2^N \to \mathbb{R}$ assigns to each *coalition* $S \in 2^N$ the worth $v(S)$, with the convention that $v(\emptyset)=0$. *Feasible allocations* are vectors in a given set $F(v) \not\subseteq \mathbb{R}^N$, to be interpreted as possible payoff vectors for the players in the game $v$. One way to evaluate feasible payoff vectors is to consider *excess* or complaint functions $e=(e_S)_{S \in 2^N \setminus \{\emptyset\}}$, where the excess $e_S(v,x)$ measures the dissatisfaction of coalition $S$ in case the allocation $x \in F(v)$ is chosen as the outcome of the game $v$. (The best known example is $e_S(v, x):=v(S)-\Sigma_{i\in S}x_i$ for every coalition $S$.) Thus, for a given collection of excess functions, any feasible allocation $x$ gives rise to a corresponding vector of excesses $e(x)=e(v, x)=(e_S(v, x))_{S\neq\emptyset}$ in $\mathbb{R}^M$, where the indices in $M$ correspond to the nonempty coalitions.

For a given collection of excess functions $e$ the *nucleolus* of a game $v$ with feasible allocation set $F(v)$ consists of those feasible allocations of which the corresponding excess vectors are most desirable:

$$\nu(v, F(v), e)=\{x \in F(v)\colon e(x) \succeq_d e(y) \text{ for all } y \in F(v)\}\,.$$

Most nucleoli considered in the literature (Schmeidler, 1969; Kohlberg, 1971; Grotte, 1971; Sobolev, 1975; Owen, 1977; Wallmeier, 1980, 1983; Potters and Tijs, 1992) are defined with respect to excess functions that share the following property. A collection of excess functions $e$ is called normal if for every game $v$ with feasible allocation set $F(v)$ every $e_S$ is continuous on $F(v)$ and satisfies:

$$e_S(x)<e_S(y) \Leftrightarrow x(S)>y(S) \text{ for all } x, y \in F(v)\,,$$

where $x(S):=\Sigma_{i\in S}\, x_i$. Thus, normality implies that the excess functions depend only on the sums of the individual coordinates.

It is not hard to prove that if $F(v)$ is convex and $e$ is a normal collection of excess functions, then the image $e(F(v))$ is a weakly convex set. This observation implies the following immediate corollary of Theorem 2.

**Theorem 3**. *Let $v$ be a game and let $F(v)$ be a convex set. Let $e$ be a normal collection of excess functions. Then*

$$\nu(v, F(v), e)=\{x \in F(v)\colon e(x) \succeq_J e(y) \text{ for all } y \in F(v)\}$$

$$=\{x \in F(v)\colon e(x) \succeq_{OR} e(y) \text{ for all } y \in F(v)\}$$

The rest of this section and of the paper deals with the Kohlberg criterion. As is well known, the nucleolus can be characterized in terms of so called balanced collections of coalitions (Kohlberg, 1971). This Kohlberg criterion will be extended now to normal excess functions.

A collection $C \subseteq 2^N \setminus \{\emptyset\}$ of coalitions is *balanced* if it is empty or there exist positive weights $\hat{\lambda}_S > 0$, $S \in C$, such that for each player $i \in N$ the sum $\Sigma_{S \ni i, S \in C}\ \lambda_S$ equals 1.

In the following lemma balancedness is characterized in terms of sidepayments (a $y \in \mathbb{R}^N$ is a *side-payment* if $y \neq 0$ and $y(N) = 0$). The lemma states that a non-empty collection of coalitions is balanced if and only if it cannot perform a reallocation, beneficial for at least one coalition in the given collection without hurting others (see also Zumsteg (1995) for an application in the context of the computation of the nucleolus).

**Lemma 2.** *A non-empty collection $C \subseteq 2^N$ of coalitions is balanced if and only if for each side-payment $y \in \mathbb{R}^N$ either $y(S) = 0$ for all $S \in C$, or there are two coalitions $S,T \in C$ with $y(S) > 0$ and $y(T) < 0$.*

Instead of this lemma a slightly different version will be formulated and proved, using Farkas' Lemma. Let $\mathcal{T}$ be a collection of coalitions. A collection $C \subseteq 2^N \setminus \{\emptyset\}$ is called $\mathcal{T}$-balanced if there is a subset $\mathcal{T}'$ of $\mathcal{T}$ such that $C \cup \mathcal{T}'$ is balanced. So, the standard notion of balancedness is incorporated in this definition by taking $\mathcal{T}$ equal to the empty set. Further, observe that each collection is $\{\{i\}: i \in N\}$-balanced.

**Lemma 3.** *Let $T \subseteq 2^N \setminus \{\emptyset\}$. A non-empty collection $C \not\subseteq 2^N \setminus \{\emptyset\}$ of coalitions is $\mathcal{T}$-balanced if and only if for each side-payment $y \in \mathbb{R}^N$ with $y(S) \geq 0$ for all $S \in C \cup \mathcal{T}$ there is no coalition $S \in C$ with $y(S) > 0$.*

**Proof.** Let $e^S$ denote the indicator vector of coalition $S$, i.e. $e^S \in \mathbb{R}^N$ and $e_i^S = 1$ if $i \in S$, $e_i^S = 0$ if $i \notin S$.

*Proof of 'if':* The inequalities $y \cdot e^N \geq 0$, $y \cdot -e^N \geq 0$, $y \cdot e^S \geq 0$, for $S \in C \cup \mathcal{T}$, imply $y \cdot -e^S \geq 0$, for $S \in C$; according to Farkas' Lemma this implies that for each $S \in C$, $-e^S$ has to be a nonnegative weighted sum of the indicator vectors in $C \cup \mathcal{T}$, and the vectors $e^N$ and $-e^N$. Therefore,

$$-e^S = \lambda_N^S e^N - \mu_N^S e^N + \sum_{T \in C \cup \mathcal{T}} \lambda_T^S e^T\ ,\ S \in C\ ,$$

with all weights nonnegative. Observe that $\lambda_N^S < \mu_N^S$ for each $S \in C$ and, thus, there exist nonnegative weights $\gamma_T^S$, $T \in C \cup \mathcal{T}$, with $\gamma_S^S > 0$, such that

$$e^N = \sum_{T \in C \cup \mathcal{T}} \gamma_T^S e^T\ .$$

Now define the nonnegative weights $\lambda_T = (\Sigma_{S \in C}\ \gamma_T^S)/(|C|)$, $T \in C \cup \mathcal{T}$. Define $\mathcal{T}' := \{T \in \mathcal{T}:\ \lambda_T > 0\}$. Observe that $\lambda_T > 0$ for every $T \in C$. Then the collection $C \cup \mathcal{T}'$ is balanced, as follows from

$$e^N = \frac{\sum_{S \in C} \sum_{T \in C \cup \mathcal{T}} \gamma_T^S e^T}{|C|} = \sum_{T \in C \cup \mathcal{T}} \lambda_T e^T = \sum_{T \in C \cup \mathcal{T}'} \lambda_T e^T \ .$$

Hence, $C$ is $\mathcal{T}$-balanced.

*Proof of 'only if':* Let $C \cup \mathcal{T}'$ be balanced for a subset $\mathcal{T}'$ of $\mathcal{T}$, and let $y$ be a side-payment with $y(S) \geq 0$ for all $S \in C \cup \mathcal{T}$. Then, with $\lambda_S > 0$ for $S \in C \cup \mathcal{T}'$ such that $e^N = \sum_{S \in C \cup \mathcal{T}'} \lambda_S e^S$,

$$0 = y \cdot e^N = y \cdot \Big( \sum_{S \in C \cup \mathcal{T}'} \lambda_S e^S \Big) = \sum_{S \in C \cup \mathcal{T}'} \lambda_S y(S) \ .$$

This is only possible if $y(S) = 0$ for all $S \in C \cup \mathcal{T}'$. □

The following theorem gives a characterization of the nucleolus in terms of balanced collections for a specific case of normal excess functions. It is an extension of a result in Potters and Tijs (1992).

**Theorem 4.** *Let $\mathcal{T} \subseteq 2^N$, let $v$ be a game with $F(v) = \{x \in \mathbb{R}^N$: $x(S) \geq v(S)$ for all $S \in \mathcal{T}$, $x(N) = v(N)$. Let $e$ be a normal collection of excess functions, and let $x \in F(v)$. Then the following two assertions are equivalent:*

1. *for each $t \in \mathbb{R}$ there is a subset $\mathcal{T}'$ of $\{S \in \mathcal{T}$: $x(S) = v(S)\}$ such that $B_t(e(v, x)) \cup \mathcal{T}'$ is balanced;*
2. *$x$ belongs to the nucleolus.*

**Proof.** In this proof the notation $B_t(x)$ is used instead of $B_t(e(v, x))$.

*Proof of '(i)⇒(ii)':* Let $z \in F(v)$, $z \neq x$ be arbitrary. Take $t'$ such that $B_t(x) = B_t(z)$ for all $t > t'$, and $B_t'(x) \neq B_{t'}(z)$. Such a $t'$ exists because otherwise $x(S) = z(S)$ for all coalitions, implying $x = z$.

*Claim:* $x(S) \leq z(S)$ for all $S$ with $e_S(x) \geq t'$.To prove this claim, let $S$ be a coalition with $s := e_S(x) \geq t'$. Consider the following three cases:

- $s > t'$: Then $S \in B_s(x) = B_s(z)$ and $S \notin B_t(x) = B_t(z)$ for all $t > s$, implying $s = e_S(z)$.
- $s = t'$ and $S \in B_{t'}(z)$: If $e_S(z) = t > s$ then $S \in B_t(x)$ implying $s \geq t > s$. Therefore, $s = e_S(z)$.
- $s = t'$ and $S \notin B_{t'}(z)$: Obviously, $e_S(z) < s$.

Hence, in all cases, $e_S(x) \geq e_S(z)$ or, equivalently, $x(S) \geq z(S)$. This proves the Claim.

Define $y = z - x$, then $y(S) \geq 0$ for all $S \in B_{t'}(x)$. Obviously, $y(S) \geq 0$ for all $S \in \mathcal{T}$ with $x(S) = v(S)$. By Lemma 3 it follows that $y(S) = 0$ for all $S \in B_{t'}(x)$. Hence, $x(S) = z(S)$, and therefore $e_S(x) = e_S(z)$ for all $S \in B_{t'}(x)$, implying $B_{t'}(x) \subseteq B_{t'}(z)$. Because $B_{t'}(x) \neq B_t'(z)$ there is a coalition $T$ with $e_T(z) = t' > e_T(x)$. Also, $e_S(x) > e_S(z)$ implies $e_S(x) \leq t' = e_T(z)$. So $e(x) \succeq_{OR} e(z)$. By Theorem 3, $x$ belongs to the nucleolus.

*Proof of '(ii)⇒(i)':* Let $x \in F(v)$ and suppose there is a $t \in \mathbb{R}$ such that $B_t(x) \cup \mathcal{T}'$ is not balanced for each subset $\mathcal{T}'$ of $\{S \in T\colon x(S)=v(S)\}$. According to Lemma 3 there is a side-payment $y$ and coalition $S' \in B_t(x)$ such that $y(S')>0$ and $y(S) \geq 0$ for each coalition in $B_t(x) \cup \{S \in \mathcal{T}\colon x(S)=v(S)\}$. Obviously, $y$ may be chosen to have Euclidean length of 1. Choose $\delta>0$ such that $B_t(x)=B_{t-\delta}(x)$. This is possible since by taking the largest excess $t'$ of a coalition outside $B_t(x)$ one has $t'<t$, and any $0<\delta<t-t'$ can be taken.

By continuity of the excess functions there is an $\epsilon>0$ so that for each $z \in \Pi$ within Euclidean distance from $x$ one has $e_S(z) \geq t-\delta$ for $S \in B_t(x)$, and $e_S(z)<t-\delta$ for $S \notin B_t(x)$, i.e., $B_{t-\delta}(z)=B_{t-\delta}(x)$.

For $z:=x+\epsilon y$ it holds that $z(S) \geq x(S)$ for all $S \in B_{t-\delta}(z)$, with strict inequality for $S=S'$. This implies $e_S(z) \leq e_S(x)$ for all $S \in B_{t-\delta}(z)$ and $e_{S'}(z)<e_{S'}(x)$. In particular, $e_S(z)>e_S(x)$ implies $S \notin B_{t-\delta}(z)$, so $e_S(z)<t-\delta \leq e_{S'}(z)$. Consequently, $e(z) \succeq_J e(x)$ which yields, by Lemma 1, that $e(x) \succeq_{OR} e(z)$ does not hold. By Theorem 3, $x$ is not in the nucleolus. □

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