A globally and universally stable price adjustment process

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Abstract

A price adjustment process for an exchange economy is given that converges generically to a Walrasian equilibrium. The assumptions made with respect to consumptions sets, preferences and initial endowments are standard. No restrictions are made with respect to the starting price system. The well-known fact that the number of Walrasian equilibria is generically odd follows as a special case of the main theorem. In the case of gross substitutability of demand functions convergence always takes place even without making differentiability assumptions. In this special case, the prices of commodities in excess demand (supply) are strictly increasing (decreasing), and therefore the qualitative behaviour of the process resembles the Walrasian tatonnement process. Moreover, on every market the absolute value of the total excess demand is monotonically decreasing.

1. Introduction

Since Walras (1874), economists have been interested in the problem of finding an adjustment process that generates, for a given economy and an arbitrarily specified starting price system, a path of price systems that converge to a price system at which the total excess demand is equal to zero. The classical Walrasian tatonnement process may fail to converge if some rather restrictive assumptions on the economy are not satisfied. Examples of economies where this process does not

converge are given in Scarf (1960). It is not difficult to construct other examples since, by the work of Sonnenschein (1973), Mantel (1974), and Debreu (1974), every continuous function that satisfies Walras' law and is defined for elements of the unit simplex with all components greater than or equal to some arbitrarily small positive number, is the total excess demand function of some pure exchange economy. Therefore it is interesting to look for adjustment processes that converge for every total excess demand function, i.e. convergence should hold universally, a requirement not met by the Walrasian tatonnement process.

A universally convergent process to find a fixed point of a function has been presented in Kellog et al. (1976), and a universally convergent process to find a zero point of a total excess demand function is given in Smale (1976). In Varian (1977) it is shown that the boundary conditions on the total excess demand function used by Smale can be relaxed if the adjustment process is extended in a particular way outside the original domain. These processes converge, for a generic economy, to a Walrasian equilibrium price system for almost every starting price system belonging to the boundary of the domain. However, it seems likely that an actual adjustment process may start with an arbitrarily chosen price system in the interior of the domain. In Keenan (1981) it is shown that Smale's process is not globally convergent, i.e. there may exist an open set of starting price systems for which the process does not converge to some equilibrium.

A globally and universally convergent process is presented in Kamiya (1990). Under rather weak conditions on the total excess demand function, among which the boundary condition that the excess demand of a commodity is positive if its price is zero, so that the excess demand function is also assumed to be defined on the boundary of the unit simplex, convergence is guaranteed for almost every starting price system in the interior of the unit simplex. It might be possible to weaken this boundary condition in a similar way as Varian (1977) did for Smale's process. However, from an economic point of view such a solution is not completely satisfactory since outside the original domain the adjustment process is artificially defined and, for example, does not depend on the excess demand at the price system reached, but instead on the excess demand at another price system.

In this paper an alternative globally and universally convergent price adjustment process is considered, proposed in van der Laan and Talman (1987), which has a nice economic interpretation. Van der Laan and Talman (1987) claim that, under certain regularity assumptions on the total excess demand function, their process is globally and universally convergent. However, it is not clear how strong these regularity assumptions are. In this paper it will be shown that indeed for every starting price system in the domain, their process converges generically in the initial endowments to a Walrasian equilibrium price system using only standard conditions on utility functions and consumption sets. Under these conditions the total excess demand function is only well defined on the interior of the unit simplex. It is not excluded that the excess demand of a commodity is not defined or becomes negative if its price goes to zero.

In Section 2 the price adjustment process is described and a definition of convergence of the process is given for which continuity, instead of differentiability, of the total excess demand function is sufficient. Defining the process also for continuous total excess demand functions will be very useful in Section 4, where the special case of total excess demand functions that satisfy gross substitutability in the finite increment form is considered, the same case as the one for which Arrow et al. (1959) showed convergence of the Walrasian tatonnement process as formulated by Samuelson (1941). In this case it is sufficient to assume continuity of the total excess demand function in order to prove convergence of the process considered in this paper.

The process is illustrated using the first example of Scarf (1960). For the economy given in this example, the price adjustment process converges for every starting price system in the unit simplex. In Section 2 we present the main result that holds for an arbitrary exchange economy that satisfies the standard assumptions. Corollaries of this result are the generic convergence of the price adjustment process, and the well-known result (see Dierker, 1972) that generically the number of Walrasian equilibria is odd. In Section 3 the proof of the main result is given. In Section 4 the adjustment process is analyzed for the special case where a continuous total excess demand function satisfies the gross substitutability condition. In this special case convergence not only holds generically, as in the results of Smale (1976) and Kamiya (1990), but also always occurs. In this case it can be shown that the prices of commodities in excess demand (supply) are strictly increasing (decreasing) during the adjustment process. Therefore the process has some features that are qualitatively the same as for the Walrasian tatonnement process. In the gross substitutability case it is also shown that if a market reaches an equilibrium situation during the process, then it stays in equilibrium for the rest of the process. An even stronger result will be proved if on every market the absolute value of the total excess demand is monotonically decreasing.

2. The price adjustment process

In what follows, for $k \in \mathbb{N}$, I_k denotes the set of integers $\{1, \ldots, k\}$, $\mathbb{R}^k_+$ denotes the non-negative orthant of the k-dimensional Euclidean space $\mathbb{R}^k$, and $\mathbb{R}^k_{++}$ denotes the set $\{x \in \mathbb{R}^k \mid \forall j \in I_k, x_j > 0\}$. Moreover, 0^k (1^k) denotes a k-dimensional vector of zeros (ones), and $0^{k \times l}$ ($1^{k \times l}$) denotes a $k \times l$ matrix of zeros (ones), for $k, l \in \mathbb{N}$. In this section the price adjustment process is described for an exchange economy $\mathscr{E} = (\{X^i, u^i, \omega^i\}_{i=1}^m)$ and a given starting price system v. There are m consumers, indexed $i = 1, \ldots, m$, and $n + 1$ commodities, indexed $j = 1, \ldots, n + 1$. Each consumer is defined by a consumption set X^i, a utility function $u^i : X^i \to \mathbb{R}$, and a vector of initial endowments ω^i. The vector $(\omega^{1^\mathrm{T}}, \ldots, \omega^{m^\mathrm{T}})^\mathrm{T}$ will be denoted by ω. The excess demand correspondence of this economy is given by a (possibly empty-valued) correspondence $z : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$,

which associates with every price system $p \in \mathbb{R}^{n+1}$ the set $z(p)$ of total excess demands compatible with the selection by every consumer of an optimal consumption bundle in his budget set. A vector p^* is a Walrasian equilibrium price system if $0^{n+1} \in z(p^*)$. Sufficient conditions on consumption sets, initial endowments, and preferences can be given such that if $p^* \in \mathbb{R}^{n+1}$ is a Walrasian equilibrium price system, then $p^* \in \mathbb{R}^{n+1}_{++}$. Moreover, z is a function that is continuous, homogeneous of degree zero on $\mathbb{R}^{n+1}_{++}$, and satisfies Walras' law, $p \cdot z(p) = 0$, $\forall p \in \mathbb{R}^{n+1}_{++}$; see, for example, Hildenbrand and Kirman (1988). Let S^n denote the relative interior of the n-dimensional unit simplex, so $S^n = \{p \in \mathbb{R}^{n+1}_{++} \mid \sum_{j=1}^{n+1} p_j = 1\}$. Let $\bar{S}^n$ denote the closure of S^n. By the homogeneity of degree zero there is no loss of generality in normalizing the price systems such that they belong to S^n.

From now on let z be a continuous function defined on S^n that satisfies Walras' law. Moreover, let the starting price system v be an arbitrary element of S^n. The vector $s \in \mathbb{R}^{n+1}$ is called a feasible sign vector if, for every $j \in I_{n+1}$, $s_j \in \{-1, 0, +1\}$, for some $k \in I_{n+1}$, $s_k = -1$, while for another $k \in I_{n+1}$, $s_k = +1$. Let $\mathcal{S}$ denote the set of feasible sign vectors in $\mathbb{R}^{n+1}$. Given a sign vector $s \in \mathcal{S}$ we define the sets $I^-(s) = \{j \in I_{n+1} \mid s_j = -1\}$, $I^0(s) = \{j \in I_{n+1} \mid s_j = 0\}$, and $I^+(s) = \{j \in I_{n+1} \mid s_j = +1\}$. Moreover, let $k^-(s)$, $k^0(s)$, and $k^+(s)$ denote the number of elements in the sets $I^-(s)$, $I^0(s)$, and $I^+(s)$, respectively. Note that for a feasible sign vector s it holds that $k^0(s) \leq n - 1$. To describe the adjustment process, for every sign vector $s \in \mathcal{S}$ the sets $A(s)$, $B(s)$, and $C(s)$ of price systems are defined by

$$A(s) = \left\{ p \in S^n \mid \forall k \in I_{n+1}, \frac{p_k}{v_k} = \min_{j \in I_{n+1}} \frac{p_j}{v_j} \text{ if } s_k = -1, \right.$$

$$\left. \text{and } \frac{p_k}{v_k} = \max_{j \in I_{n+1}} \frac{p_j}{v_j} \text{ if } s_k = +1 \right\},$$

$$B(s) = \left\{ p \in S^n \mid \forall j \in I_{n+1}, z_j(p) \leq 0 \text{ if } s_j = -1, z_j(p) = 0 \text{ if } s_j = 0, \right.$$

$$\left. \text{and } z_j(p) \geq 0 \text{ if } s_j = +1 \right\},$$

$$C(s) = A(s) \cap B(s).$$

Hence when $p \in C(s)$, then $s_j = -1 (s_j = +1)$ implies that there is excess supply (demand) on market j and the price of commodity j is relatively, i.e. with respect to the starting price v_j, minimal (maximal), and $s_j = 0$ implies that market j is in equilibrium. So the sign vector $s \in \mathcal{S}$ characterizes the state of every market. The set $\bigcup_{s \in \mathcal{S}} C(s)$ will be denoted by C. Clearly, there is a sign vector $\bar{s} \in \mathcal{S}$ such that for every $j \in I_{n+1}$, $z_j(v) > 0$ implies $\bar{s}_j = +1$ and $z_j(v) < 0$ implies $\bar{s}_j = -1$, where Walras' law guarantees that indeed $\bar{s}$ can be chosen in $\mathcal{S}$. Then it holds that $v \in B(\bar{s})$, obviously $v \in A(\bar{s})$, hence $v \in C(\bar{s})$, and therefore $v \in C$. Let us consider a Walrasian equilibrium price system $p^* \in S^n$. Clearly there is a sign vector $\bar{s} \in \mathcal{S}$ such that for every $k \in I_{n+1}$, $\bar{s}_k = -1$ implies $p_k^*/v_k =$

$\min_{j \in I_{n+1}} p_j^* / v_j$ and $\bar{s}_k = +1$ implies $p_k^* / v_k = \max_{j \in I_{n+1}} p_j^* / v_j$. Then $p^* \in A(\bar{s})$, clearly $p^* \in B(\bar{s})$, and therefore $p^* \in C(\bar{s}) \subset C$. Hence, the set C contains both the starting price system v and all Walrasian equilibrium price systems.

A subset of a topological space X is connected if it is not the union of two non-empty, disjoint sets, which are open in the induced topology. The component of a point x in a topological space X is the union of all connected subsets of X containing x. It is not difficult to show that each component is connected and therefore the component of an element x is the largest connected subset of X containing x. Intuitively, a set is connected if it is of one piece.

Definition 2.1. Let an economy $\mathcal{E} = (\{X^i, u^i, \omega^i\}_{i=1}^m)$ be given with continuous total excess demand function $z: S^n \to \mathbb{R}^{n+1}$, and let $v \in S^n$ be a starting price system. Then the price adjustment process is given by the component of the set C that contains the starting price system v.

Since in the definition of the price adjustment process under consideration no differentiability assumptions are used, we should also give a definition of convergence without using such assumptions. A subset T of $\mathbb{R}^k$ is called an arc if it is homeomorphic to the unit interval $[0,1]$. A subset T of $\mathbb{R}^k$ is called a loop if it is homeomorphic to the unit circle, i.e. the set $\{x \in \mathbb{R}^2 \,|\, (x_1)^2 + (x_2)^2 = 1\}$.

Definition 2.2. Let an economy $\mathcal{E} = (\{X^i, u^i, \omega^i\}_{i=1}^m)$ be given with continuous total excess demand function $z: S^n \to \mathbb{R}^{n+1}$ and let $v \in S^n$ be a starting price system. If $z(v) \neq 0^{n+1}$ then the price adjustment process is convergent if the component of the set C that contains v is an arc having v and a Walrasian equilibrium price system p^* of the economy $\mathcal{E}$ as its boundary points, whereas the arc does not contain another Walrasian equilibrium price system.

In the next section it is proved that generically the price adjustment process is convergent. If the price adjustment process is convergent, then there exists a continuous function $\pi: [0, 1] \to C$ which is one-to-one and satisfies that $\pi(0) = v$ and $\pi(1)$ is a Walrasian equilibrium price system, so $z(\pi(1)) = 0^{n+1}$. Moreover, $\pi([0, 1])$ is the component of C that contains v. So there exists a unique, continuous path of price systems leading from the starting price system v to a Walrasian equilibrium price system. An element of the set $[0, 1]$ could be considered to be a normalized time parameter. Although the arc $\pi([0, 1])$ is uniquely determined, the function π is clearly not unique, and different functions correspond to different speeds of adjustment. The adjustment process is therefore described by considering explicitly the path of price systems followed. In the case when an adjustment process is implicitly defined by a system of differential equations, this path corresponds to its trajectory. Note that it is only required that the arc contains some Walrasian equilibrium price system, which means that even if the starting price system is 'sufficiently close' to an equilibrium price system,

then the adjustment process may converge to another equilibrium. So in the terminology of Saari and Simon (1978) or Saari (1985), Definition 2.2 corresponds to an effective or globally convergent mechanism, but not to a locally effective or locally convergent mechanism.

The adjustment process can be followed numerically using the $(2^{n+1} - 2)$-ray algorithm described in Doup et al. (1987). This algorithm generates a piecewise linear path of points that correspond to the adjustment process for a piecewise linear approximation of the total excess demand function. The information needed at some price system p reached by the algorithm is given by a finite number (at most $n + 1$) of price systems already generated by the algorithm, the excess demands at these price systems, and the starting price system v. This means that the amount of information needed is roughly the same as the amount indicated by Saari and Simon (1978).

In the case of differentiability, the approach taken above is related to those of Kellog et al. (1976), Smale (1976, 1981), and Kamiya (1990). In Smale (1976) commodity $n + 1$ is considered to be a numeraire commodity and a process is defined which follows price systems in the set

$$\left\{ p \in \mathbb{R}_+^n \mid \exists \lambda \in \mathbb{R}_+, \forall k \in I_n, \ z_k\!\left(\left(p^{\mathsf{T}}, 1\right)^{\mathsf{T}}\right) = \lambda z_k\!\left(\left(v^{\mathsf{T}}, 1\right)^{\mathsf{T}}\right) \right\},$$

with v the initial values of the prices of the first n commodities. It is easily verified that taking $\lambda = 1$ yields that $p = v$ is an element of the set, and taking $\lambda = 0$ yields that p^* is an element of the set if $(p^{*\mathsf{T}}, 1)^{\mathsf{T}}$ is a Walrasian equilibrium price system. We define $T^n = \{ p \in \mathbb{R}_+^n \mid \sum_{j=1}^n (p_j)^2 \le 1 \}$. In Kamiya (1990) an adjustment process is defined that generates prices in the set

$$\left\{ p \in T^n \mid \exists \lambda \in [0, 1], \forall k \in I_n, (1 - \lambda) z_k\!\left(\left(p^{\mathsf{T}}, \left(1 - \sum_{j=1}^n (p_j)^2\right)^{1/2}\right)^{\mathsf{T}}\right) \right.$$

$$= \lambda (p_k - v_k) \Bigg\},$$

with v the initial values of the prices of the first n commodities. It is easily verified that $\lambda = 1$ yields $p = v$ as a unique solution. By considering $\lambda = 0$ it follows that p^* is in the set if $(p^{*\mathsf{T}}, \sqrt{1 - \sum_{j=1}^n p_j^{*2}})^{\mathsf{T}}$ is an equilibrium price system. By making suitable differentiability, regularity, and boundary conditions it can be shown that for the adjustment processes of Smale (1976) and Kamiya (1990) the components that contain the starting price system v of the sets defined above are arcs which can be described by continuously differentiable functions. Such a continuously differentiable arc can be described by a system of differential equations (see, for example, Garcia and Zangwill, 1981), which corresponds to the system of differential equations given in Smale (1976) and Kamiya (1990).

The price adjustment proces considered in this paper has a nice economic interpretation and can be described as follows. First the sign of the excess demand is evaluated at the starting price system v. We consider the case where, for every $j \in I_{n+1}$, $z_j(v) \neq 0$. In Section 3 this will be shown to be the generic case. The prices of commodities $j \in I_{n+1}$ with $z_j(v) < 0$ will be decreased relatively, while the prices of commodities $j \in I_{n+1}$ with $z_j(v) > 0$ will be increased relatively. We define the sign vector $s^0 \in \mathscr{S}$ by $z_j(v) > 0$, and $s_j^0 = -1$ if $z_j(v) < 0$. So the process starts by leaving v along the ray $A(s^0)$ of price systems. The ratio of prices of commodities in excess demand is kept constant among those in excess demand, and similarly for the ratio of prices of commodities in excess supply. Prices are adjusted in this way until one of the markets, say market k, attains an equilibrium situation. Let us assume that there is a single market which attains an equilibrium. This will be shown to be the generic case. Then the process continues by keeping market k in equilibrium, while the price p_k is increased (decreased) relatively in the case when there was a negative (positive) excess demand on market k before attaining equilibrium. Other prices are kept relatively minimal in the case of excess supply and relatively maximal in the case of excess demand. Hence a path in $C(s^1)$ is followed, where $s_k^1 = 0$ and $s_j^1 = s_j^0$, $\forall j \in I_{n+1} \setminus \{k\}$. It is shown in this section that for every $s \in \mathscr{S}$ the set $C(s)$ is compact and in Section 4 that generically it is a finite collection of arcs and loops. Two situations now can occur at the other end-point of the path in $C(s^1)$. Either another market, say market k', attains an equilibrium situation. In this case prices are adjusted in such a way that markets k and k' are kept in equilibrium, while the price in market k' is increased (decreased) relatively in the case when there was a negative (positive) excess demand on market k' before attaining equilibrium. Again, other prices are kept either relatively minimal or relatively maximal. Hence a path of price systems in $C(s^2)$ is followed, where $s_{k'}^2 = 0$ and $s_j^2 = s_j^1$, $\forall j \in I_{n+1} \setminus \{k'\}$. Or the price on market k becomes relatively minimal or maximal. In this case market k is no longer kept in equilibrium but is allowed to become in excess supply or excess demand, while p_k is kept relatively minimal or relatively maximal, respectively. So then a path of prices in $C(s^2)$ is followed, where $s_k^2 = -1$ or $+1$ and $s_j^2 = s_j^1$, $\forall j \in I_{n+1} \setminus \{k\}$.

The general case is as follows. Suppose the process follows a path of prices in $C(s^l)$ for some $l \in \mathbb{N}$. Then at the end-point either market $k \in I^-(s^l) \cup I^+(s^l)$ attains an equilibrium situation, in which case a path of price systems in $C(s^{l+1})$ is followed, where $s_k^{l+1} = 0$ and $s_j^{l+1} = s_j^l$, $\forall j \in I_{n+1} \setminus \{k\}$, or the price of some commodity $k \in I^0(s^l)$ becomes relatively minimal (maximal) in which case a path of prices in $C(s^{l+1})$ is followed, where $s_k^{l+1} = -1$ ($s_k^{l+1} = +1$) and $s_j^{l+1} = s_j^l$, $\forall j \in I_{n+1} \setminus \{k\}$. It will be shown that the process described above generically converges to a Walrasian equilibrium price system.

In the Walrasian tatonnement process, as formulated in Samuelson (1941), i.e. $p(0) = v$ and $dp(t)/dt = z(p(t))$, it is possible that after some time the adjustment process reaches a price system which is such that the price of a commodity is

higher than the starting price, while there is an excess supply on the market of this commodity. Similarly, it can happen that the Walrasian tatonnement process reaches a price system at which there is excess demand in the market of a commodity, while the price of this commodity is lower than the starting price of this commodity. This is a remarkable phenomenon since initially the Walrasian tatonnement process changes prices in such a way that the prices of commodities in excess supply are lowereu and the prices of commodities in excess demand are raised with respect to the starting price system. Any price system on the path generated by the adjustment process in this paper has the natural property that the price of a commodity in excess demand is higher than the corresponding starting price, while the price of a commodity in excess supply is lower than the corresponding starting price. More details concerning the economic interpretation of the price adjustment process can be found in van der Laan and Talman (1987) and van den Elzen (1993).

The price adjustment process can be illustrated using the first example in Scarf (1960) concerning an exchange economy with three commodities. For this example it is well known that the Walrasian tatonnement process is unstable for every starting price system except for the unique Walrasian equilibrium price system. In Scarf's example, initial endowments ω^i and utility functions u^i are specified for three consumers, yielding a total excess demand function of the economy, z: $S^2 \to \mathbb{R}^3$, which is defined by

$$z_1(p) = \frac{-p_2}{p_1 + p_2} + \frac{p_3}{p_1 + p_3}, \quad \forall p \in S^2,$$

$$z_2(p) = \frac{-p_3}{p_2 + p_3} + \frac{p_1}{p_1 + p_2}, \quad \forall p \in S^2,$$

$$z_3(p) = \frac{-p_1}{p_1 + p_3} + \frac{p_2}{p_2 + p_3}, \quad \forall p \in S^2.$$

The unique Walrasian equilibrium price system is given by $p^* = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})^T$. It is easily verified that $z_1(p) = 0$ iff $p_2 = p_3$, $z_2(p) = 0$ iff $p_1 = p_3$, and $z_3(p) = 0$ iff $p_1 = p_2$. Let us consider the starting price system $v = (\frac{11}{18}, \frac{2}{18}, \frac{5}{18})^T$. In Fig. 1 the sets $A(s)$ and $B(s)$ are drawn for every $s \in \mathcal{S}$. In Fig. 2 the set C is depicted. In Scarf's example there is an excess demand in the markets of the first two commodities at $v = (\frac{11}{18}, \frac{2}{18}, \frac{5}{18})^T$. The process therefore starts by following a path in $C((+1, +1, -1)^T)$, having v as a boundary point. So the prices of the first two commodities are relatively increased. At $p = (\frac{11}{15}, \frac{2}{15}, \frac{2}{15})^T$ the market of the first commodity attains an equilibrium situation. So this market is kept in equilibrium, the relative price of the second commodity is kept maximal, and the relative price of the third commodity minimal, so a path in $C((0, +1, -1)^T)$ is followed. At $p = (\frac{11}{21}, \frac{5}{21}, \frac{5}{21})^T$ the price of the first commodity becomes relatively minimal and equal to the relative price of commodity 3. Hence the process continues by
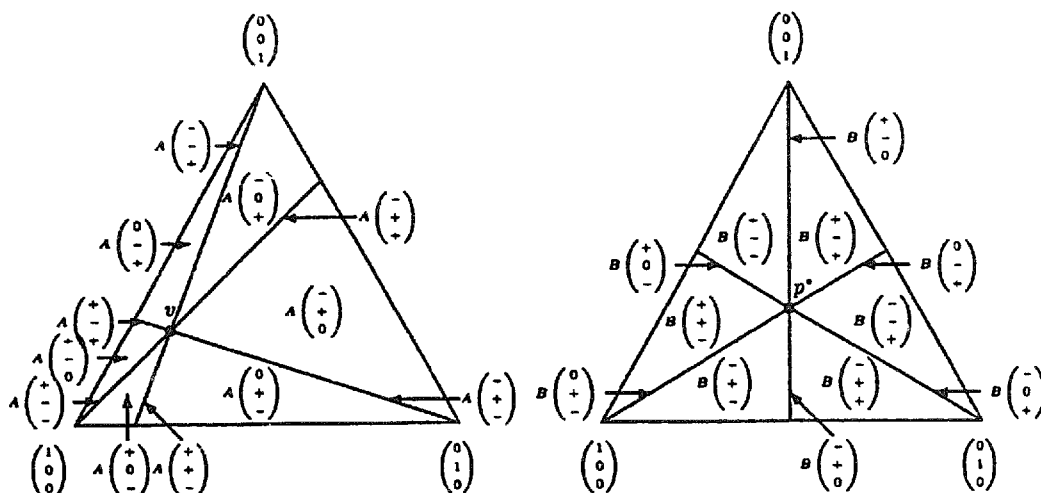
Fig. 1. The sets $A(s)$ and $B(s)$, $s \in \mathcal{S}$, in the first example of Scarf (1960).

following a path in $C((-1,+1,-1)^{\mathrm{T}})$, where the prices of commodities 1 and 3 are relatively decreased and the price of commodity 2 is relatively increased. The market of commodity 1 is no longer in equilibrium. At $p = (\frac{11}{27}, \frac{11}{27}, \frac{5}{27})^{\mathrm{T}}$ market 3 attains an equilibrium situation and so a path in $C((-1,+1,0)^{\mathrm{T}})$ is followed. At $p^* = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})^{\mathrm{T}}$ the process reaches a Walrasian equilibrium price system. Clearly the price adjustment process is convergent in the sense of Definition 2.2. It can be shown that in Scarf's example the price adjustment process converges for every starting price system $v \in S'$.

To show that convergence is a generic property of the adjustment process, the following standard assumptions on consumption sets and preferences have to be made.

Assumption 1. For every $i \in I_m$ the consumption set X^i is equal to $\mathbb{R}^{n+1}_{++}$.

Assumption 2. For every $i \in I_m$ the utility function $u^i : X^i \to \mathbb{R}$ is strictly increasing, strictly quasi-concave, three times continuously differentiable, the indifference surfaces of u^i have non-zero Gaussian curvature at every $x^i \in X^i$, and the closure of the indifference surfaces in $\mathbb{R}^{n+1}$ is a subset of $\mathbb{R}^{n+1}_{++}$.

If the economy $\mathscr{E}$ satisfies Assumptions 1 and 2, and for every consumer $i \in I_m$ it holds that $\omega^i \in X^i$, then the total excess demand function $z : S^n \to \mathbb{R}^{n+1}$ is twice continuously differentiable on S^n. Let m consumption sets and utility functions, $(\{X^i, u^i\}_{i=1}^m)$, and a starting price system, $v \in S^n$, be given. We denote $\Omega = \prod_{i=1}^m X^i$. We define the set of regular initial endowments, denoted by Ω^*, as the set of initial endowments $\omega \in \Omega$ for which the components of the set C for the economy $\mathscr{E} = (\{X^i, u^i, \omega^i\}_{i=1}^m)$ with starting price system v are given by: (1) a unique arc containing v and one Walrasian equilibrium price system which are
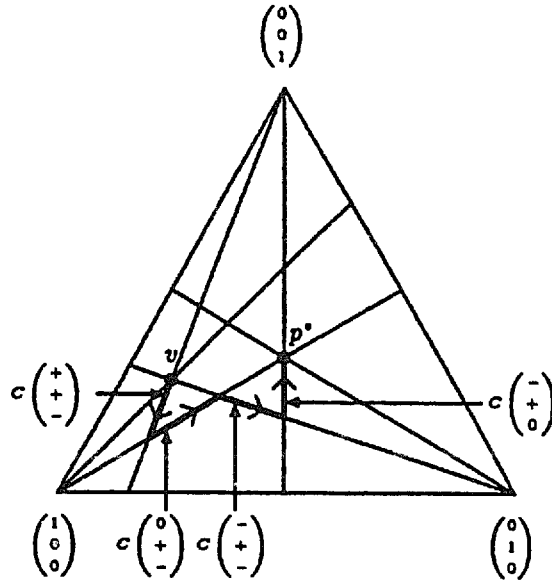
Fig. The set C in the first example of Scarf (1960).

boundary points of the arc; (2) a finite number of arcs containing two Walrasian equilibrium price systems both being boundary points; and (3) a finite number of loops containing no Walrasian equilibrium price systems.

Theorem 2.3. Let $(\{X^i, u^i\}_{i=1}^m)$ *that satisfy Assumptions* 1 *and* 2 *be given, and let* $v \in S^n$ *be a starting price system. Then the set of non-regular initial endowments* $\Omega \setminus \Omega^*$ *has a closure in* Ω *with Lebesgue measure zero.*

Theorem 2.3 will be proved in Section 3. In fact, the proof of Theorem 2.3 yields that the path of the prices followed by the price adjustment process is a one-dimensional piecewise twice continuously differentiable manifold, i.e. a one-dimensional continuous manifold which is a finite union of twice continuously differentiable manifolds, some possibly of lower dimensions. Moreover, the other components of the set C are either loops or arcs, both being one-dimensional piecewise twice continuously differentiable manifolds. Since $\omega \in \Omega^*$ implies that the price adjustment process converges, Theorem 2.3 immediately implies the next result.

Corollary 2.4. Let $(\{X^i, u^i\}_{i=1}^m)$ *that satisfy Assumptions* 1 *and* 2 *be given, and let* $v \in S^n$ *be a starting price system. Then the price adjustment process for the economy* $\mathscr{E} = (\{X^i, u^i, \omega^i\}_{i=1}^m)$ *with starting price system* v *converges, except for a set of initial endowments in* Ω *having a closure in* Ω *with Lebesgue measure zero.*

Since every Walrasian equilibrium price system is an element of C, Theorem 2.3 confirms the well-known result of Dierker (1972) that generically there is an odd number of Walrasian equilibria in an economy.

Corollary 2.5. Let $(\{X^i, u^i\}_{i=1}^m)$ *that satisfy Assumptions* 1 *and* 2 *be given. Then the number of Walrasian equilibria of the economy* $\mathscr{E} = (\{X^i, u^i, \omega^i\}_{i=1}^m)$ *is a finite, odd number, except for a set of initial endowments in* Ω *having a closure in* Ω *with Lebesgue measure zero.*

Let $(\{X^i, u^i\}_{i=1}^m)$ and a starting price system $v \in S^n$ be given. For every $\omega \in \Omega$ we define the set $P(\omega)$ as the component of the set C containing v; we define for every $s \in \mathscr{S}$ the set $Q_s(\omega)$ as the set $C(s)$; and we define the set $Q(\omega)$ as the set C for the economy $\mathscr{E} = (\{X^i, u^i, \omega^i\}_{i=1}^m)$ with starting price system v. In this way we obtain the price adjustment process correspondence $P : \Omega \to S^n$ and a correspondence $Q : \Omega \to S^n$. Note that the price adjustment correspondence P and the correspondence Q are non-empty valued since for every $\omega \in \Omega$ the starting price system v is contained in $P(\omega) \subset Q(\omega)$. To make clear the dependence of the total excess demand on the initial endowments, some additional notation is needed. Let $\delta^i(p, w^i)$ denote the demand of consumer $i \in I_m$ at price system $p \in \mathbb{R}_{++}^{n+1}$ and wealth $w^i \in \mathbb{R}_{++}$. The total excess demand function $\zeta : \mathbb{R}_{++}^{n+1} \times \Omega \to \mathbb{R}^{n+1}$ is defined by $\zeta(p, \omega) = \sum_{i=1}^m \delta^i(p, p \cdot \omega^i) - \sum_{i=1}^m \omega^i$, $\forall (p, \omega) \in \mathbb{R}_{++}^{n+1} \times \Omega$. For a non-empty compact set $T \subset \mathbb{R}^k$ we define the function $d_T : \mathbb{R}^k \to \mathbb{R}$ by $d_T(t') = \min\{\|t' - t\|_\infty \mid t \in T\}$, $\forall t' \in \mathbb{R}^k$. It is not difficult to show that d_T is a continuous function. For two non-empty compact subsets T^1 and T^2 of $\mathbb{R}^k$, we define $e(T^1, T^2) = \min\{\|t^1 - t^2\|_\infty \mid t^1 \in T^1, t^2 \in T^2\}$. If T^1 and T^2 are disjoint, then obviously $e(T^1, T^2) > 0$.

Theorem 2.6. Let $(\{X^i, u^i\}_{i=1}^m)$ *that satisfy Assumptions* 1 *and* 2 *be given, and let* $v \in S^n$ *be a starting price system. Then the correspondences* P *and* Q *are compact-valued and upper semi-continuous.*

Proof. First the correspondence Q is shown to be upper semi-continuous and compact-valued. Let $(\omega^q)_{q \in \mathbb{N}}$ be a sequence in Ω converging to $\bar{\omega} \in \Omega$ and let $(p^q)_{q \in \mathbb{N}}$ be a sequence in S^n such that $p^q \in Q(\omega^q)$. It will be shown that $(p^q)_{q \in \mathbb{N}}$ has a subsequence that converges to a point $\bar{p} \in Q(\bar{\omega})$. Since $\bar{S}^n$ is compact, $(p^q)_{q \in \mathbb{N}}$ has a subsequence $(p^{q^r})_{r \in \mathbb{N}}$ that converges to a point $\bar{p} \in \bar{S}^n$. Moreover, since the set of sign vectors $\mathscr{S}$ is finite the subsequence can be taken such that $\exists s \in \mathscr{S}$, $\forall r \in \mathbb{N}$, and $p^{q^r} \in Q_s(\omega^{q^r})$. Clearly, if $j \in I^+(s)$, then $p_j^{q^r} \geq v_j$. Note that $\forall r \in \mathbb{N}$, if $j \in I^-(s)$, then $\zeta_j(p^{q^r}, \omega^{q^r}) \leq 0$, if $j \in I^0(s)$, then $\zeta_j(p^{q^r}, \omega^{q^r}) = 0$, and if $j \in I^+(s)$, then $\zeta_j(p^{q^r}, \omega^{q^r}) \geq 0$. Consequently, it holds for every $r \in \mathbb{N}$ that

$$\|\zeta(p^{q^r}, \omega^{q^r})\|_\infty = \max\left\{ \max_{j \in I^-(s)} - \zeta_j(p^{q^r}, \omega^{q^r}), \max_{j \in I^+(s)} \zeta_j(p^{q^r}, \omega^{q^r}) \right\}$$

$$\leq \max\left\{ \sup_{r \in \mathbb{N}} \left\| \sum_{i=1}^m \omega^{i q^r} \right\|_\infty, \frac{\sup_{r \in \mathbb{N}} \left\| \sum_{i=1}^m \omega^{i q^r} \right\|_\infty}{\min_{j \in I^+(s)} v_j} \right\}.$$

Note that the right-hand side of the inequality above is finite. Suppose $\bar{p} \in \bar{S}^n \setminus S^n$. Then Assumptions 1 and 2 imply that $\| \zeta(p^{q'}, \omega^{q'}) \|_\infty$ goes to infinity if $p^{q'} \to \bar{p} \in \bar{S}^n \setminus S^n$ and $\omega^{q'} \to \bar{\omega} \in \Omega$, and a contradiction is obtained. Using $p^{q'} \to \bar{p} \in S^n$, $\omega^{q'} \to \bar{\omega} \in \Omega$, and the continuity of ζ, it follows that $\bar{p} \in Q_s(\bar{\omega})$. So Q is upper semi-continuous and compact-valued.

Now let $(\omega^q)_{q \in \mathbb{N}}$ be a sequence in Ω converging to $\bar{\omega} \in \Omega$ and let $(p^q)_{q \in \mathbb{N}}$ be a sequence in S^n such that $p^q \in P(\omega^q)$. It will be shown that $(p^q)_{q \in \mathbb{N}}$ has a subsequence that converges to a point $\bar{p} \in P(\bar{\omega})$. Without loss of generality it can be assumed that, using the previous paragraph, $p^q \to \bar{p}$ with $\bar{p} \in Q(\bar{\omega})$. Since $P(\bar{\omega})$ is the largest connected subset of $Q(\bar{\omega})$ that contains v, the closure of a connected set is connected, and since $Q(\bar{\omega})$ is compact, it follows that $P(\bar{\omega})$ is compact.

Exercise 4c of section 5.1 in Munkres (1975, p. 235) states that for a compact Hausdorff space X and an element $x \in X$ the component of X that contains x is equal to the intersection of all sets that contain x which are both open and closed in X. Suppose $\bar{p} \notin P(\bar{\omega})$. Using the result mentioned above and the compactness of $P(\bar{\omega})$ it follows that there exist compact disjoint sets T^1 and T^2 such that $v \in T^1$, $\bar{p} \in T^2$, and $T^1 \cup T^2 = Q(\bar{\omega})$. Hence there exists $\epsilon > 0$ such that $e(T^1, T^2) > \epsilon$. By the upper semi-continuity of the correspondence Q there exists an $N \in \mathbb{N}$ such that $\forall q \geq N$, $\forall p \in P(\omega^q)$, $d_{Q(\bar{\omega})}(p) < \frac{1}{2}\epsilon$. We consider some $q \geq N$ such that $\| p^q - \bar{p} \|_\infty < \frac{1}{2}\epsilon$. We define $U^1 = \{ p \in P(\omega^q) \,|\, d_{T^1}(p) < \frac{1}{2}\epsilon \}$ and $U^2 = \{ p \in P(\omega^q) \,|\, d_{T^2}(p) < \frac{1}{2}\epsilon \}$. By the continuity of d_{T^1} and d_{T^2} the sets U^1 and U^2 are open in $P(\omega^q)$. Clearly, U^1 and U^2 are disjoint, $U^1 \cup U^2 = P(\omega^q)$, and U^1 and U^2 are non-empty since $v \in U^1$ and $p^q \in U^2$. So $P(\omega^q)$ is not connected, which is a contradiction.   □ Q.E.D.

The correspondences P and Q are compact-valued and upper-semicontinuous, and the image set S^n is totally bounded when given the Euclidean metric, i.e. for every $\epsilon > 0$, $S^n \subset \mathbb{R}^{n+1}_{++}$ can be covered by a finite number of sets of diameter less than ϵ. Therefore it follows immediately, in the same way as in Dierker (1974, p. 85), that the correspondences P and Q are continuous on a residual subset of Ω, i.e. on a countable intersection of sets open and dense in Ω. Therefore, from an economic point of view, Theorem 2.6 is interesting since it means that the adjustment process itself is in some sense stable against perturbations in the initial endowments. The upper semi-continuity and the compact-valuedness of Q will be used in the proof of Theorem 2.3.

3. Generic convergence of the process

In this section consumption sets and utility functions $(\{X^i, u^i\}_{i=1}^m)$ that satisfy Assumptions 1 and 2, and a starting price system $v \in S^n$, are given. Then for every $s \in \mathscr{S}$ and $\omega \in \Omega$ the sets $B(s)$, $C(s)$, and C for the economy $\mathscr{E} =$

$(\{X^i, u^i, \omega^i\}_{i=1}^m)$ with starting price system v can be derived. To make clear the dependence on the initial endowments, the notation is changed in this section to $B_\omega(s)$, $C_\omega(s)$, and C_ω, respectively.

For some $r \geq 1$ a subset M of $\mathbb{R}^k$ is called a C^r l-dimensional manifold with generalized boundary (MGB), if for every $\bar{x} \in M$ there exists a local C^r coordinate system of $\mathbb{R}^k$ around $\bar{x}$, i.e. a C^r diffeomorphism $\varphi : U \to V$, where U is an open subset of $\mathbb{R}^k$ containing $\bar{x}$ and V is open in $\mathbb{R}^k$, and some $b(\bar{x}) \geq 0$ such that $\varphi(\bar{x}) = 0^k$ and $\varphi(U \cap M)$ equals $\{(y_1, \ldots, y_{k-l}, y_{k-l+1}, \ldots, y_{k-l+1+b(\bar{x})}, \ldots, y_k)^T \in V \mid y_1 = \cdots = y_{k-l} = 0, y_{k-l+1} \geq 0, \ldots, y_{k-l+b(\bar{x})} \geq 0\}$. If, for every element $\bar{x}$ of an MGB M, $b(\bar{x}) \leq 1$, then M is called a manifold with boundary and it is easily shown that the set of elements $\bar{x}$ for which $b(\bar{x}) = 1$ is an $(l-1)$-dimensional manifold, called the boundary of M. Let J^1 and J^2 be two finite index sets and let $g_j, \forall j \in J^1$, and $h_j, \forall j \in J^2$, be C^r functions defined on some open subset X of $\mathbb{R}^k$. We define

$$M[g, h] = \{x \in X \mid g_j(x) = 0, \forall j \in J^1, h_j(x) \geq 0, \forall j \in J^2\}.$$

For $x \in X$ we define $J^0(x) = \{j \in J^2 \mid h_j(x) = 0\}$. If for every $\bar{x} \in M[g, h]$ it holds that $\{\partial g_j(\bar{x}), \forall j \in J^1, \partial h_j(\bar{x}), \forall j \in J^0(\bar{x})\}$ is a set of independent vectors, then $M[g, h]$ is called a C^r regular constraint set (RCS). In Jongen et al. (1983, lemma 3.1.2, example 3.1.3) it is shown that every C^r RCS is a $(k - |J^1|)$-dimensional C^r MGB with, for every $\bar{x} \in M[g, h]$, $b(\bar{x}) = |J^0(\bar{x})|$.

Let some sign vector $s \in \mathscr{S}$ be given. Without loss of generality it can be assumed that $I^0(s) = I_{k^0(s)}$, $I^-(s) = I_{k^0(s)+k^-(s)} \setminus I_{k^0(s)}$, and $I^+(s) = I_{n+1} \setminus I_{k^0(s)+k^-(s)}$. Let some $j^- \in I^-(s)$ and $j^+ \in I^+(s)$ be given. The price system p is an element of the set $C_\omega(s)$ if and only if the element $(p, \omega) \in \mathbb{R}_{++}^{n+1} \times \Omega$ satisfies

$$\zeta_j(p, \omega) = 0, \quad \forall j \in I^0(s), \tag{1}$$

$$p_j v_{j+1} - p_{j+1} v_j = 0, \forall j \in I_{k^0(s)+k^-(s)-1} \setminus I_{k^0(s)}, \tag{2}$$

$$p_j v_{j+1} - p_{j+1} v_j = 0, \quad \forall j \in I_n \setminus I_{k^0(s)+k^-(s)}, \tag{3}$$

$$\sum_{j=1}^{n+1} p_j - 1 = 0, \tag{4}$$

$$-\zeta_j(p, \omega) \geq 0, \quad \forall j \in I^-(s), \tag{5}$$

$$\zeta_j(p, \omega) \geq 0, \quad \forall j \in I^+(s), \quad \text{if } k^0(s) \leq n - 2, \tag{6}$$

$$p_j v_{j^-} - p_{j^-} v_j \geq 0, \quad \forall j \in I^0(s), \tag{7}$$

$$p_{j^+} v_j - p_j v_{j^+} \geq 0, \quad \forall j \in I^0(s), \tag{8}$$

$$p_{j^+} v_{j^-} - p_{j^-} v_{j^+} \geq 0. \tag{9}$$

Note that if $k^-(s) = 1$, then no constraints are specified in (2). The same holds with respect to (3) if $k^+(s) = 1$. Since $k^-(s)$ and $k^+(s)$ are both greater than or equal to one, there are all together n equations in (1)–(4). If $k^0(s) > n - 2$, or equivalently $k^0(s) = n - 1$, then $k^-(s) = k^+(s) = 1$. In this case the inequality in (6) follows by Walras' law from inequality (5) and therefore inequality (6) is not specified in this case. It will be shown in what follows that for a given generic $\omega \in \Omega$, (1)–(9) constitute a one-dimensional C^2 RCS.

To show Theorem 2.3 it is useful to define for every $s \in \mathscr{S}$ and $\omega \in \Omega$ a set $D_\omega(s)$ as follows:

$$D_\omega(s) = \left\{ p \in S^n \,\middle|\, \frac{p_j}{v_j} = \frac{p_{j'}}{v_{j'}} \text{ if } j,j' \in I^-(s), \ \zeta_j(p, \omega) = 0 \text{ if } j \in I^0(s), \right.$$

$$\left. \text{and } \frac{p_j}{v_j} = \frac{p_{j'}}{v_{j'}} \text{ if } j,j' \in I^+(s) \right\}.$$

Clearly, $C_\omega(s) \subset D_\omega(s)$. The difference between these two sets is that no inequality constraints are taken into account in the specification of $D_\omega(s)$. In Lemma 3.1 we show that except for a set of initial endowments of Lebesgue measure zero, the set $D_\omega(s)$ is a C^2 one-dimensional manifold. Hence it consists of a number of disjoint sets that are diffeomorphic to either a unit circle or an open unit interval. The function $\psi_s : \mathbb{R}^{n+1}_{++} \times \Omega \to \mathbb{R}^n$ is defined such that $\psi_s(p, \omega)$ is the left-hand side of (1)–(4). We define $\psi_{s,\omega} : \mathbb{R}^{n+1}_{++} \to \mathbb{R}^n$ by $\psi_{s,\omega}(p) = \psi_s(p, \omega)$, $\forall p \in \mathbb{R}^{n+1}_{++}$. Note that $D_\omega(s) = \psi_{s,\omega}^{-1}(\{0^n\})$.

Lemma 3.1. *Let $(\{X^i, u^i\}_{i=1}^m)$ that satisfy Assumptions 1 and 2 be given, and let $v \in S^n$ be a starting price system. Moreover, let a sign vector $s \in \mathscr{S}$ be given. Then $\psi_{s,\omega} \pitchfork \{0^n\}$ and $D_\omega(s)$ is a C^2 one-dimensional manifold, except for a set of initial endowments $\omega \in \Omega$ with Lebesgue measure zero.*

Proof. The matrix of partial derivatives of ψ_s evaluated at a point $(\bar{p}, \bar{\omega})$ that satisfies $\psi_s(\bar{p}, \bar{\omega}) = 0^n$ is denoted by M and is given in Table 1. Moreover, in Table 1 two submatrices, M^1 and M^2, of M are defined. We will show that the matrix M has rank n. First it is proved that for every $i \in I_m$, $\partial_{\omega^i}\zeta(\bar{p}, \bar{\omega})$ has rank n. Note that $\bar{p}^{\mathrm{T}}\partial_{\omega^i}\zeta(\bar{p}, \bar{\omega}) = 0^{n+1^{\mathrm{T}}}$ and $\partial_{\omega^i}\zeta(\bar{p}, \bar{\omega}) = \partial_{\omega^i}\delta^i(\bar{p}, \bar{p} \cdot \bar{\omega}^i)\bar{p}^{\mathrm{T}} - I^{n+1}$, where I^{n+1} denotes the $(n+1) \times (n+1)$ identity matrix. For $j \in I_{n+1}$ let e^j denote the $(n+1)$-dimensional unit vector with $e^j_j = 1$. Then $\partial_{\omega^i}\zeta(\bar{p}, \bar{\omega})(\bar{p}_{j'}e^j - \bar{p}_je^{j'}) = \bar{p}_je^j - \bar{p}_{j'}e^{j'}$, $\forall j,j' \in I_{n+1}$, and so the rank of $\partial_{\omega^i}\zeta(\bar{p}, \bar{\omega})$ is equal to n. We consider the first $k^0(s)$ rows of $\partial_{\omega^i}\zeta(\bar{p}, \bar{\omega})$. These rows have to be independent. Suppose not, then $k^0(s) \leq n - 1$ implies the existence of $y \in \mathbb{R}^{n+1}$ such that $y_n = y_{n+1} = 0$ and $y^{\mathrm{T}}\partial_{\omega^i}\zeta(\bar{p}, \bar{\omega}) = 0^{n+1^{\mathrm{T}}}$. Since $\bar{p}^{\mathrm{T}}\partial_{\omega^i}\zeta(\bar{p}, \bar{\omega}) = 0^{n+1^{\mathrm{T}}}$, this implies that the rank of $\partial_{\omega^i}\zeta(\bar{p}, \bar{\omega})$ is less than or equal to $n - 1$, which is a contradiction.

Table 1
The matrix M

$$M = \begin{array}{|c|c|} \hline \begin{matrix} \partial_p \zeta_1(\tilde{p}, \tilde{\omega}) \\ \vdots \\ \partial_p \zeta_{k^0(s)}(\tilde{p}, \tilde{\omega}) \end{matrix} & \begin{matrix} \partial_\omega \zeta_1(\tilde{p}, \tilde{\omega}) \\ \vdots \\ \partial_\omega \zeta_{k^0(s)}(\tilde{p}, \tilde{\omega}) \end{matrix} \\ \hline \end{array} \quad k^0(s)$$

Now let $y \in \mathbb{R}^n$ be such that $y^T M = 0^{(m+1)(n+1)^T}$. By the previous paragraph $y^T \partial_\omega \psi_s(\tilde{p}, \tilde{\omega}) = 0^{n+1^T}$ implies $y_j = 0, \forall j \in I_{k^0(s)}$. Suppose $y_n \neq 0$. Without loss of generality it can be assumed that $y_n < 0$. If $k^0(s) \geq 1$ or if $k^-(s) = 1$, then a contradiction is obtained with $y^T \partial_{p_1} \psi_s(\tilde{p}, \tilde{\omega}) = 0$. If $k^0(s) = 0$ and $k^-(s) \geq 2$, then $y_n < 0$ and $y^T \partial_{p_1} \psi_s(\tilde{p}, \tilde{\omega}) = 0$ implies $y_1 > 0$. It is easily seen that $y_j > 0$ and $y^T \partial_{p_{j+1}} \psi_s(\tilde{p}, \tilde{\omega}) = 0$ implies $y_{j+1} > 0, \forall j \in I_{k^-(s)-2}$. Hence $y_{k^-(s)-1} > 0$, which implies that $y^T \partial_{p_{k^-(s)}} \psi_s(\tilde{p}, \tilde{\omega}) < 0$, a contradiction. Consequently, $y_n = 0$.

The independence of the rows of M^1 and M^2 yields $y_{k^0(s)+1} = \cdots = y_{n-1} = 0$. So M has rank n and consequently ψ_s is transverse to the origin: $\psi_s \pitchfork \{0^n\}$. By the transversality theorem (see, for example, theorem I.2.2 of Mas-Colell, 1985) and since ψ_s is a twice continuously differentiable function, it follows that the complement of the set $\{\omega \in \Omega \mid \psi_{s,\omega} \pitchfork \{0^n\}\}$ has Lebesgue measure zero. Since $\psi_{s,\omega}$ maps from a manifold with dimension $n+1$ into a manifold with dimension n and $\psi_{s,\omega}$ is a twice continuously differentiable function, $\psi_{s,\omega} \pitchfork \{0^n\}$ implies that $\psi_{s,\omega}^{-1}(\{0^n\})$ and hence $D_\omega(s)$ is a C^2 one-dimensional manifold. □ Q.E.D.

For some given $s \in \mathcal{S}$ and $\omega \in \Omega$ we consider the set of price systems p in $D_\omega(s)$ that satisfy $\zeta_k(p, \omega) = 0$ for some $k \in I^-(s) \cup I^+(s)$. Hence one of the inequalities in (5) or (6) is satisfied with equality. If for $\bar{s}$ defined by $\bar{s}_k = 0$ and

$\tilde{s}_j = s_j$, $\forall j \in I_{n+1} \setminus \{k\}$, it holds that $\tilde{s} \in \mathscr{S}$, then these price systems are in the intersection of the sets $D_\omega(s)$ and $D_\omega(\tilde{s})$. If we consider the system of equations that defines $C_\omega(\tilde{s})$, then it follows that one of the inequalities in (7) or (8) is satisfied with equality. For every $\omega \in \Omega$, $s \in \mathscr{S}$, and $k \in I^-(s) \cup I^+(s)$, we define the set $D_\omega(s, k)$ as follows:

$$
D_\omega(s, k)
$$

$$
= \left\{ p \in S^n \left| \frac{p_j}{v_j} = \frac{p_{j'}}{v_{j'}} \text{ if } j,j' \in I^-(s), \ \zeta_j(p, \omega) = 0 \text{ if } j \in I^0(s) \cup \{k\}, \right. \right.
$$

$$
\left. \text{and } \frac{p_j}{v_j} = \frac{p_{j'}}{v_{j'}} \text{ if } j,j' \in I^+(s) \right\}.
$$

In Lemma 3.2 we show that except for a set of initial endowments of Lebesgue measure zero the set $D_\omega(s, k)$ is a zero-dimensional manifold and hence a discrete set of points. Given a sign vector $s \in \mathscr{S}$ the commodities can be relabelled such that $I^0(s) = I_{k^0(s)}$, $I^-(s) = I_{k^0(s)+k^-(s)} \setminus I_{k^0(s)}$, and $I^+(s) = I_{n+1} \setminus I_{k^0(s)+k^-(s)}$. It is easily verified that the price system p is an element of the set $D_\omega(s, k)$ if and only if the element $(p, \omega) \in \mathbb{R}^{n+1}_{++} \times \Omega$ satisfies Eqs. (1)–(4), and

$$
\zeta_k(p, \omega) = 0. \tag{10}
$$

Now a function $\psi_{s,k} : \mathbb{R}^{n+1}_{++} \times \Omega \to \mathbb{R}^{n+1}$ is defined such that $\psi_{s,k}(p, \omega)$ is the left-hand side of (1)–(4) and (10). We define $\psi_{s,k,\omega} : \mathbb{R}^{n+1}_{++} \to \mathbb{R}^{n+1}$ by $\psi_{s,k,\omega}(p) = \psi_{s,k}(p, \omega)$, $\forall p \in \mathbb{R}^{n+1}_{++}$.

Lemma 3.2. Let $(\{X^i, u^i\}_{i=1}^m)$ that satisfy Assumptions 1 and 2 be given, and let $v \in S^n$ be a starting price system. Moreover, let a sign vector $s \in \mathscr{S}$ and a commodity $k \in I^-(s) \cup I^+(s)$ be given. Then $\psi_{s,k,\omega} \pitchfork \{0^{n+1}\}$ and $D_\omega(s, k)$ is a zero-dimensional manifold, except for a set of initial endowments $\omega \in \Omega$ with Lebesgue measure zero.

Proof. The matrix of partial derivatives of $\psi_{s,k}$ evaluated at a point $(\bar{p}, \bar{\omega})$ that satisfies $\psi_{s,k}(\bar{p}, \bar{\omega}) = 0^{n+1}$ is denoted by $\tilde{M}$. It is shown that the matrix $\tilde{M}$ has rank $n + 1$. Let $y \in \mathbb{R}^{n+1}$ satisfy $y^T \tilde{M} = 0^{(m+1)(n+1)^T}$. As in the proof of Lemma 3.1, it can be shown that the rows $1, \ldots, k^0(s)$, k of $\partial_\omega \zeta(\bar{p}, \bar{\omega})$ are independent for every $i \in I_m$, since $k^0(s) \le n - 1$ and $k \notin I^0(s)$. So $y^T \partial_\omega \psi_{s,k}(\bar{p}, \bar{\omega}) = 0^{n+1^T}$ implies $y_1 = \cdots = y_{k^0(s)} = y_{n+1} = 0$. The proof that $y_{k^0(s)+1} = \cdots = y_n = 0$ is now identical to the corresponding part of the proof of Lemma 3.1. Hence $\tilde{M}$ has rank $n + 1$ and $\psi_{s,k} \pitchfork \{0^{n+1}\}$. By the transversality theorem it follows that the complement of the set $\{\omega \in \Omega \mid \psi_{s,k,\omega} \pitchfork \{0^{n+1}\}\}$ has Lebesgue measure zero. Since $\psi_{s,k,\omega}$ maps from a manifold with dimension $n + 1$ into a manifold with dimension $n + 1$ and $\psi_{s,k,\omega}$ is a twice continuously differentiable function,

$\psi_{s,k,\omega} \pitchfork \{0^{n+1}\}$ implies that $\psi_{s,k,\omega}^{-1}(\{0^{n+1}\})$ and hence $D_{\omega}(s, k)$ is a zero-dimensional manifold. ☐ Q.E.D.

For some sign vector $s \in \mathcal{S}$ with $k^0(s) \le n - 2$ and some $\omega \in \Omega$ we consider the set of price systems $p \in D_{\omega}(s)$ that satisfies $z_{k^1}(p) = z_{k^2}(p) = 0$ for some $k^1, k^2 \in I^-(s) \cup I^+(s)$, with $k^1 \ne k^2$. This set is denoted by $D_{\omega}(s, k^1, k^2)$. The next lemma shows that the set $D_{\omega}(s, k^1, k^2)$ is empty, except for a set of initial endowments of Lebesgue measure zero. Note that the condition $k^0(s) \le n - 2$ is crucial, since for a sign vector s with $k^0(s) = n - 1$ a corresponding set $D_{\omega}(s, k^1, k^2)$ is equal to the set of Walrasian equilibrium price systems in $A(s)$ of the economy $\mathcal{E} = (\{X^i, u^i, \omega^i\}_{i=1}^m)$.

The price system p is an element of the set $D_{\omega}(s, k^1, k^2)$ if and only if the element $(p, \omega) \in \mathbb{R}_{++}^{n+1} \times \Omega$ satisfies Eqs. (1)–(4) and

$$\zeta_{k^1}(p, \omega) = 0, \tag{11}$$

$$\zeta_{k^2}(p, \omega) = 0. \tag{12}$$

The function $\psi_{s,k^1,k^2} : \mathbb{R}_{++}^{n+1} \times \Omega \to \mathbb{R}^{n+2}$ is defined such that $\psi_{s,k^1,k^2}(p, \omega)$ is the left-hand side of (1)–(4), (11), and (12). The function $\psi_{s,k^1,k^2,\omega} : \mathbb{R}_{++}^{n+1} \to \mathbb{R}^{n+2}$ is defined by $\psi_{s,k^1,k^2,\omega}(p) = \psi_{s,k^1,k^2}(p, \omega)$, $\forall p \in \mathbb{R}_{++}^{n+1}$.

Lemma 3.3. Let $(\{X^i, u^i\}_{i=1}^m)$ that satisfy Assumptions 1 and 2 be given. Moreover, let a sign vector $s \in \mathcal{S}$ that satisfies $k^0(s) \le n - 2$, two different commodities $k^1, k^2 \in I^-(s) \cup I^+(s)$, and a starting price system $v \in S^n$ be given. Then $\psi_{s,k^1,k^2,\omega} \pitchfork \{0^{n+2}\}$ and $D_{\omega}(s, k^1, k^2)$ is empty, except for a set of initial endowments $\omega \in \Omega$ with Lebesgue measure zero.

Proof. We note that $k^0(s) \le n - 2$ and $k^1, k^2 \notin I^0(s)$ imply that the rows $1, \ldots, k^0(s), k^1$, and k^2 of $\partial_{\omega^i} \zeta(\bar{p}, \bar{\omega})$ are independent for every $i \in I_m$. Similar to the proof of Lemma 3.1 and Lemma 3.2 it can be shown that $\psi_{s,k^1,k^2} \pitchfork \{0^{n+2}\}$ and therefore the complement of the set $\{\omega \in \Omega \mid \psi_{s,k^1,k^2,\omega} \pitchfork \{0^{n+2}\}\}$ has Lebesgue measure zero. Since $\psi_{s,k^1,k^2,\omega}$ maps from an $(n + 1)$-dimensional manifold into an $(n + 2)$-dimensional manifold, $\psi_{s,k^1,k^2,\omega} \pitchfork \{0^{n+2}\}$ implies that $\psi_{s,k^1,k^2,\omega}^{-1}(\{0^{n+2}\})$ and hence $D_{\omega}(s, k^1, k^2)$ is an empty set by the definition of transversality. ☐ Q.E.D.

It holds that $v \in C_{\omega}(s)$ for a unique $s \in \mathcal{S}$ if and only if $\zeta_j(v, \omega) \ne 0$ for every $j \in I_{n+1}$. Therefore it is shown in Lemma 3.4 that, except for a set of initial endowments of Lebesgue measure zero, all components of the excess demand at price system v are unequal to zero. It is sufficient to show that the set of initial endowments for which the excess demand of one of the commodities is equal to zero at v has Lebesgue measure zero. Hence for given $j \in I_{n+1}$ we define the function $\psi_j : \{v\} \times \Omega \to \mathbb{R}$ by $\psi_j(v, \omega) = \zeta_j(v, \omega)$, $\forall \omega \in \Omega$. We define $\psi_{j,\omega} : \{v\} \to \mathbb{R}$ by $\psi_{j,\omega}(v) = \psi_j(v, \omega)$.

Lemma 3.4. Let $(\{X^i, u^i\}_{i=1}^m)$ *that satisfy Assumptions* 1 *and* 2 *be given, and let* $v \in S^n$ *be a starting price system. Then for every* $j \in I_{n+1}$, $\psi_{j,\omega} \pitchfork \{0\}$ *and* $\zeta_j(v, \omega) \neq 0$, *except for a set of initial endowments* $\omega \in \Omega$ *with Lebesgue measure zero.*

Proof. Clearly it holds that $\psi_j \pitchfork \{0\}$. So the complement of the set $\{\omega \in \Omega \mid \psi_{j,\omega} \pitchfork \{0\}\}$ has Lebesgue measure zero. Since $\psi_{j,\omega}$ maps from a zero-dimensional manifold into a one-dimensional manifold, $\psi_{j,\omega}(v) \pitchfork \{0\}$ implies that $\psi_{j,\omega}^{-1}(\{0\})$ is an empty set by the definition of transversality. $\square$ Q.E.D.

All the preliminary work has now been done to provide a proof of Theorem 2.3. The proof consists of three parts. In the first part it is shown that for almost every $\omega \in \Omega$ the set $C_\omega(s)$ is a compact C^2 one-dimensional manifold with a boundary for every $s \in \mathscr{S}$. In the second part the sets $C_\omega(s)$ are linked and it is shown that for almost every $\omega \in \Omega$ the set C_ω consists of a finite number of arcs and loops. There is a unique arc that has the starting price system v and a unique Walrasian equilibrium price system as boundary points. The other arcs have two Walrasian equilibrium price systems as boundary points. In part three of the proof it is shown that the closure of the set of initial endowments for which the result of the second part does not hold has Lebesgue measure zero.

Proof of Theorem 2.3. Let $\omega \in \Omega$ be given which satisfies, for every $j \in I_{n+1}$, $\psi_{j,\omega} \pitchfork \{0\}$, for every $s \in \mathscr{S}$, $\psi_{s,\omega} \pitchfork \{0^n\}$, $\forall k \in I^-(s) \cup I^+(s)$, $\psi_{s,k,\omega} \pitchfork \{0^{n+1}\}$, and for every $s \in \mathscr{S}$ with $k^0(s) \leq n - 2$, $\forall k^1, k^2 \in I^-(s) \cup I^+(s)$ with $k^1 \neq k^2$, $\psi_{s,k^1,k^2,\omega} \pitchfork \{0^{n+2}\}$. By Lemmas 3.1, 3.2, 3.3, and 3.4, almost every element of Ω satisfies this finite number of requirements.

Part 1. $C_\omega(s)$ *is a compact* C^2 *one-dimensional manifold with boundary* $\forall s \in \mathscr{S}$. It is shown that when the left-hand sides of Eqs. (1)–(9) are considered as functions of p from the open set $\mathbb{R}_{++}^{n+1}$ into $\mathbb{R}$ they yield a one-dimensional C^2 RCS. Let $\bar{s} \in \mathscr{S}$ be given and let $\bar{p} \in C_\omega(\bar{s})$.

If $\bar{p} = v$, then since for every $j \in I_{n+1}$, $\psi_{j,\omega} \pitchfork \{0\}$, it holds that $I^0(\bar{s}) = \emptyset$ and the inequalities (5) and (6) are not binding. Hence $J^0(\bar{p})$ consists of a unique element corresponding with Eq. (9). It is easily verified that the derivatives with respect to p of (2)–(4) and (9) at $\bar{p}$ constitute an independent set of vectors.

We consider the case with $\bar{p} \neq v$. Then (9) holds with inequality. Suppose that two (or more) equations in (5)–(8) hold with equality. Since $\bar{p} \neq v$, (7) and (8) cannot be binding for the same commodity in $I^0(\bar{s})$ and therefore the two equations that hold with equality correspond to different commodities $k^1, k^2 \in I_{n+1}$. We define $\tilde{s}$ by $\tilde{s}_j = \bar{s}_j$, $\forall j \in I_{n+1} \setminus \{k^1, k^2\}$ and for $i \in I_2$, $\tilde{s}_{k^i} = -1$ if k^i corresponds to an equation in (5) or (7), and $\tilde{s}_{k^i} = +1$ if k^i corresponds to an equation in (6) or (8). If, for some $i \in I_2$, k^i corresponds to (7) or (8), or if

$k^0(\bar{s}) \leq n - 2$, then $k^0(\tilde{s}) \leq n - 2$. Moreover, $\bar{p} \in D_\omega(\bar{s}, k^1, k^2)$, which is a contradiction since $\psi_{\bar{s}, k^1, k^2, \omega} \pitchfork \{0^{n+2}\}$. So $k^0(\bar{s}) = n - 1$ and k^1, k^2 correspond to two different equations in (5) and (6), again leading to a contradiction, since there is only one equation specified in (5) and (6) in this case. Consequently at most one of the inequalities in (5)–(9) is satisfied with equality.

If none of the inequalities in (5)–(9) is satisfied with equality, then it follows that the derivatives with respect to p of Eqs. (1)–(4) at $\bar{p}$ constitute a set of independent vectors since $\psi_{\bar{s}, \omega} \pitchfork \{0^n\}$. Moreover, $b(\bar{p}) = 0$. If one of the inequalities in (5)–(9) is satisfied with equality, then, since the case $\bar{p} \neq v$ is considered, one of the inequalities in (5)–(8) is satisfied with equality, say the one corresponding to commodity $k \in I_{n+1}$. We define $\tilde{s}$ by $\tilde{s}_j = \bar{s}_j$, $\forall j \in I_{n+1} \setminus \{k\}$, $\tilde{s}_k = -1$ if k corresponds to (5) or (7), and $\tilde{s}_k = +1$ if k corresponds to (6) or (8). Then $\psi_{\tilde{s}, k} \pitchfork \{0^{n+1}\}$ implies that the derivatives with respect to p of the binding inequality and (1)–(4) at $\bar{p}$ constitute a set of independent vectors. Moreover, $b(\bar{p}) = 1$. Since (1)–(4) form n functions defined on $\mathbb{R}^{n+1}_{++}$ and (1)–(9) are C^2 functions, a one-dimensional C^2 RCS is obtained. Since $\forall \bar{p} \in C_\omega(\bar{s})$, $b(\bar{p}) \leq 1$, it follows that $C_\omega(\bar{s})$ is a one-dimensional C^2 manifold with boundary, where the boundary is given by the set of points $\bar{p} \in C_\omega(s)$ with $b(\bar{p}) = 1$, a zero-dimensional manifold.

The compactness of $C_\omega(\bar{s})$ follows immediately from the proof of Theorem 2.6. Consequently, $C_\omega(\bar{s})$ is a compact C^2 one-dimensional manifold with boundary and therefore a finite union of disjoint sets, being diffeomorphic to either the unit circle or the closed unit interval [0, 1]. We denote these sets by $C^1_\omega(\bar{s}), \ldots, C^{j(\bar{s})}_\omega(\bar{s})$. Note that for every $j \in I_{j(\bar{s})}$, $\bar{p} \in C^j_\omega(\bar{s})$ is a point of the boundary of $C^j_\omega(\bar{s})$ if and only if $b(\bar{p}) = 1$.

Part 2. C_ω is a finite union of arcs and loops for almost every $\omega \in \Omega$. Let $p^0 \in C_\omega$ be given. So for some $s^0 \in \mathscr{S}$ and for some $j^0 \in I_{j(s^0)}$, $p^0 \in C^{j^0}_\omega(s^0)$. Either $C^{j^0}_\omega(s^0)$ is a component of C_ω, being diffeomorphic to the unit circle and has no boundary, or $C^{j^0}_\omega(s^0)$ is a subset of C_ω, being diffeomorphic to the unit interval and having two boundary points, p^1 and p^{-1}. We consider p^1. Either $p^1 = v$, or exactly one of the inequalities in (5)–(8) is binding. Four cases have to be considered.

2.1. If $p^1 = v$, then since for every $j \in I_{n+1}$, $\psi_{j, \omega} \pitchfork \{0\}$, $\nexists s \in \mathscr{S} \setminus \{s^0\}$ with $p^1 \in C_\omega(s)$.

2.2. If $k^0(s^0) = n - 1$ and the inequality in (5) is binding, then by Walras' law p^1 is a Walrasian equilibrium price system. Suppose for some $s \in \mathscr{S} \setminus \{s^0\}$, $p^1 \in C_\omega(s)$. Using $p^1 \neq v$ it follows that $I^0(s^0) \neq I^0(s)$ and that $(I^-(s^0) \cup I^-(s)) \cap (I^+(s^0) \cup I^+(s)) = \emptyset$. Let $\bar{s}$ be the sign vector defined by $\bar{s}_j = 0$, $\forall j \in I^0(s^0) \cap I^0(s)$, $\bar{s}_j = -1$, $\forall j \in I^-(s^0) \cup I^-(s)$, $\bar{s}_j = +1$, $\forall j \in I^+(s^0) \cup I^+(s)$. Let k^1 and k^2 be two different elements of $I^-(\bar{s}) \cup I^+(\bar{s})$. Then, since $k^0(\bar{s}) \leq n - 2$ and $p^1 \in D_\omega(\bar{s}, k^1, k^2)$, a contradiction with $\psi_{\bar{s}, k^1, k^2} \pitchfork \{0^{n+2}\}$ is obtained. Consequently, $\nexists s \in \mathscr{S} \setminus \{s^0\}$ such that $p^1 \in C_\omega(s)$.

2.3. If $k^0(s^0) = n - 1$ and an inequality in (7) or (8) is binding corresponding

with some commodity $k \in I^0(s)$, then we define s^1 by $s_j^1 = s_j^0$, $\forall j \in I_{n+1} \setminus \{k\}$, $s_k^1 = -1$ if an inequality in (7) is binding, and $s_k^1 = +1$ if an inequality in (8) is binding. Clearly p^1 is a boundary point of $C_\omega^{j^1}(s^1)$ for some $j^1 \in I_{j(s^1)}$. Moreover, $\nexists s \in \mathscr{P} \setminus \{s^0, s^1\}$ such that $p^1 \in C_\omega(s)$ since otherwise again a contradiction is obtained as before.

2.4. If $k^0(s^0) \le n - 2$, then it can be shown in a similar way as in Case 2.3 that there is a unique $s^1 \in \mathscr{P}$ such that p^1 is a boundary point of $C_\omega^{j^1}(s^1)$ for some $j^1 \in I_{j(s^1)}$.

The set $C_\omega^{j^1}(s^1)$ obtained in Cases 2.3 and 2.4 has two boundary points, p^1 and, say, p^2. Using the same arguments as above, either $p^2 = v$ or p^2 is a Walrasian equilibrium price system or $p^2 \in C_\omega^{j^2}(s^2)$ for some unique $s^2 \in \mathscr{P} \setminus \{s^1\}$ and $j^2 \in I_{j(s^2)}$. Repeating these arguments a number of sets $C^0 = C_\omega^{j^0}(s^0)$, $C^1 = C_\omega^{j^1}(s^1)$, $C^2 = C_\omega^{j^2}(s^2)$, ... is obtained such that each set is a component of $C_\omega(s)$ for some $s \in \mathscr{P}$ being diffeomorphic to the unit interval, $C^j \cap C^{j+1}$ is a common boundary point and $C^j \ne C^{j+1}$. Therefore, after a finite number of k steps either a set C^k is obtained having v or a Walrasian equilibrium price system as a boundary point while $C^0, \ldots, C^k$ are all different, or $C^j = C^k$ for some $j \in \mathbb{N}$, $j < k$, and $C^0, \ldots, C^{k-1}$ are all different.

In the second case it will be shown that $j = 0$. Then it is easily verified that $C^0 \cup \ldots \cup C^{k-1}$ is a component of C_ω containing p^0, being homeomorphic to the unit circle. Suppose $j > 1$, then $C^j \cap C^{k-1}$ is a boundary point of either C^{j-1} or C^{j+1}. Clearly $j + 1 \le k - 1$. Suppose $j + 1 = k - 1$, then C^{j+1} has one boundary point in common with C^j and the other boundary point in common with $C^{j+2} = C^k = C^j$. The sets C^{j-1}, C^j, and C^{j+1} are different and share a common boundary point, which gives a contradiction. Consequently $j + 1 < k - 1$. The three sets C^{j-1}, C^j, and C^{k-1} are different and the three sets C^j, C^{j+1}, and C^{k-1} are different, while the three sets in one of these two collections of sets have a common boundary point, which gives a contradiction.

In the first case, we consider the other boundary point of $C_\omega^{j^0}(s^0)$, denoted p^{-1}. Again, a number of sets C^0, C^{-1}, ... is obtained such that after a finite number of k' steps, either a set $C^{-k'}$ is obtained having v or a Walrasian equilibrium price system as a boundary point, the sets $C^{-k'}, \ldots, C^k$ are all different, and it is easily shown that the set $\bigcup_{j \in \{-k', \ldots, k\}} C^j$ is the component of C_ω that contains p^0 which is homeomorphic to the unit interval, or there is $j > -k'$ such that $C^{-k'} = C^j$, with the sets $C^{-k'+1}, \ldots, C^k$ are all different. Suppose $j = k$, then since C^k has v or a Walrasian equilibrium price system as the boundary point it holds that $C^{-k'+1} = C^{k-1}$, which gives a contradiction unless $-k' + 1 = k - 1$. In the final case C^{k-1} has one boundary point in common with $C^{k-2} = C^{-k'} = C^k$ and the other boundary point in common with C^k. This implies that C^{k-1} and C^k have v or the same Walrasian equilibrium price system as a boundary point, which is a contradiction. Consequently, $j < k$. Clearly $-k' + 1 \le j - 1$. Let us suppose that $-k' + 1 = j - 1$, then the three different sets $C^{-k'+1}$, C^j, and C^{j+1} have a common boundary point, which gives a contradiction. Consequently, $-k' + 1 < j$

$- 1$. Hence $C^{-k'+1} \cap C^j$ is a boundary point of either C^{j-1} or C^{j+1} and therefore either $C^{-k'+1}$, C^{j-1}, and C^j are three different sets with a common boundary point, or $C^{-k'+1}$, C^j, and C^{j+1} are three different sets with a common boundary point, which gives a contradiction.

Consequently, C_ω has a finite number of components, being either arcs or loops. The boundary of C_ω is given by the collection consisting of the starting price system v and the Walrasian equilibrium price systems. Therefore the component that contains v is an arc with a Walrasian equilibrium price system as the other boundary point. If there exists another Walrasian equilibrium price system, say p^*, then the component that contains p^* is an arc having p^* and a third Walrasian equilibrium price system as boundary points.

Part 3. The closure of $\Omega \setminus \Omega^*$ in Ω has Lebesgue measure zero. It has already been shown that $\Omega \setminus \Omega^*$ has Lebesgue measure zero. If $\omega \in \Omega \setminus \Omega^*$, then by Parts 1 and 2 of the proof there exists $p \in S^n$ such that (p, ω) belongs to the set Σ, defined by

$$\Sigma = \{(p, \omega) \in S^n \times \Omega \mid \exists s \in \mathcal{S} \text{ such that } p \in C_\omega(s)$$

and rank $\partial \psi_{s,\omega}(p) \leq n - 1$, or

$$\exists s \in \mathcal{S},$$

$$\exists k \in I^-(s) \cup I^+(s) \text{ such that } p \in C_\omega(s), \ \zeta_k(p, w) = 0,$$

and rank $\partial \psi_{s,k,\omega}(p) \leq n$, or

$$\exists s \in \mathcal{S}, \ k^0(s) \leq n - 2, \ \exists k^1,$$

$$k^2 \in I^-(s) \cup I^+(s), \ k^1 \neq k^2 \text{ such that } p \in C_\omega(s)$$

and $\zeta_{k^1}(p, \omega) = \zeta_{k^2}(p, \omega) = 0$, or

$$p = v \text{ and } \exists j \in I_{n+1} \text{ such that } \zeta_j(p, \omega) = 0\}.$$

It is easily shown that Σ is closed relative to $S^n \times \Omega$ since Σ can be obtained by finite unions and intersections of sets being closed in $S^n \times \Omega$, owing to the continuity of the functions ζ, $\partial \psi_s$, and $\partial \psi_{s,k}$ and the continuity in p of $\min_{j \in I_{n+1}} p_j/v_j$ and $\max_{j \in I_{n+1}} p_j/v_j$. We define the projection $\pi: \Sigma \to \Omega$ by $\pi(p, \omega) = \omega$, $\forall(p, \omega) \in \Sigma$. Then $\Omega \setminus \Omega^* \subset \pi(\Sigma)$ and $\pi(\Sigma)$ is a subset of a measure zero set by Lemmas 3.1, 3.2, 3.3, and 3.4. It will be shown that $\pi(\Sigma)$ is closed in Ω. Since the image by a continuous proper mapping of a closed set is closed, it is sufficient to show that π is proper. Let T be a compact subset of Ω. It has to be shown that $\pi^{-1}(T)$ is compact. Clearly $\pi^{-1}(T)$ is a closed set in Σ and therefore it is closed in $S^n \times \Omega$. Moreover, it is a subset of the set $\{(p, \omega) \in S^n \times T \mid p \in Q(\omega)\}$, which is compact by the compact-valuedness and upper semi-continuity of Q (Theorem 2.6). Consequently, $\pi^{-1}(T)$ is compact. □ Q.E.D.

4. The gross substitution case

In this section a starting price system $v \in S^n$ and a total excess demand function $z : \mathbb{R}_{++}^{n+1} \to \mathbb{R}^{n+1}$ are given, with z satisfying the following assumptions.

Assumption 3. The function $z : \mathbb{R}_{++}^{n+1} \to \mathbb{R}^{n+1}$ is continuous.

Assumption 4. For every $\lambda \in \mathbb{R}_{++}$, $\forall p \in \mathbb{R}_{++}^{n+1}$, $z(\lambda p) = z(p)$.

Assumption 5. For every $p \in \mathbb{R}_{++}^{n+1}$, $p \cdot z(p) = 0$.

Assumption 6. The function z is bounded from below, and if $(p^r)_{r \in \mathbb{N}}$ is a sequence in $\mathbb{R}_{++}^{n+1}$ converging to an element $p \in \mathbb{R}_{+}^{n+1} \backslash \{0^{n+1}\}$ with $p_k = 0$ for some $k \in I_{n+1}$, then $\sum_{j=1}^{n+1} | z_j(p^r) |$ goes to infinity.

Assumption 7 (gross substitutability, finite increment form). If $\tilde{p}$, $\bar{p} \in \mathbb{R}_{++}^{n+1}$ are such that for some $k \in I_{n+1}$, $\tilde{p}_k < \bar{p}_k$, and $\forall j \in I_{n+1} \backslash \{k\}$, $\tilde{p}_j = \bar{p}_j$, then $\forall j \in I_{n+1} \backslash \{k\}$, $z_j(\tilde{p}) < z_j(\bar{p})$.

Owing to the homogeneity of degree zero (Assumption 4) it is possible to normalize the set of prices to the set S^n on which the adjustment process is defined. For $\tilde{p}$, $\bar{p} \in S^n$ we define the sets $J_{\max}(\tilde{p}, \bar{p}) = \{k \in I_{n+1} \mid \bar{p}_k/\tilde{p}_k = \max_{j \in I_{n+1}} \bar{p}_j/\tilde{p}_j\}$ and $J_{\min}(\tilde{p}, \bar{p}) = \{k \in I_{n+1} \mid \bar{p}_k/\tilde{p}_k = \min_{j \in I_{n+1}} \bar{p}_j/\tilde{p}_j\}$. Clearly, $J_{\min}(\tilde{p}, \bar{p}) \neq \emptyset$ and $J_{\max}(\tilde{p}, \bar{p}) \neq \emptyset$. Moreover, if $\tilde{p} \neq \bar{p}$, then $k \in J_{\max}(\tilde{p}, \bar{p})$ implies $\bar{p}_k/\tilde{p}_k > 1$, and $k \in J_{\min}(\tilde{p}, \bar{p})$ implies $\bar{p}_k/\tilde{p}_k < 1$. The following lemma will appear to be very useful.

Lemma 4.1. Let a total excess demand function z that satisfies Assumptions 4 and 7 and $\tilde{p}$, $\bar{p} \in S^n$ with $\tilde{p} \neq \bar{p}$ be given. Then $k \in J_{\max}(\tilde{p}, \bar{p})$ implies $z_k(\tilde{p}) > z_k(\bar{p})$ and $k \in J_{\min}(\tilde{p}, \bar{p})$ implies $z_k(\tilde{p}) < z_k(\bar{p})$.

Proof. Let $k \in J_{\max}(\tilde{p}, \bar{p})$ and define $\hat{p} \in \mathbb{R}_{++}^{n+1}$ by $\hat{p} = (\bar{p}_k/\tilde{p}_k)\tilde{p}$. By Assumption 4, $z(\hat{p}) = z(\tilde{p})$. Clearly, $\hat{p}_k = \bar{p}_k$, $\forall j \in I_{n+1}$, $\hat{p}_j \geq \bar{p}_j$, and $\exists j \in I_{n+1}$, $\hat{p}_j > \bar{p}_j$. Given $\hat{p}$, we decrease the prices for commodities $j \in I_{n+1} \backslash \{k\}$ until $\bar{p}$ is reached. Using Assumption 7 repeatedly yields $z_k(\bar{p}) < z_k(\hat{p}) = z_k(\tilde{p})$. The case with $k \in J_{\min}(\tilde{p}, \bar{p})$ can be treated similarly.   □ Q.E.D.

Using Lemma 4.1 it is trivial to show that in the gross substitution case a Walrasian equilibrium, if it exists, is unique.

We define for every $\lambda \in (0, 1]$ the set S_λ^n by $S_\lambda^n = \{p \in S^n \mid \min_{j \in I_{n+1}} p_j/v_j = \lambda\}$. Clearly, $S_1^n = \{v\}$. In the case $n = 2$ and $\lambda \in (0, 1)$ the set S_λ^n consists of the sides of a triangle. For arbitrary $n \in \mathbb{N}$ it holds that for λ^1, $\lambda^2 \in (0, 1]$, with $\lambda^1 \neq \lambda^2$, the sets $S_{\lambda^1}^n$ and $S_{\lambda^2}^n$ are disjoint, and that $\bigcup_{\lambda \in (0, 1]} S_\lambda^n = S^n$. The first step

in proving that under Assumptions 3–7 the price adjustment process converges is to show that if the adjustment process has reached the set S_λ^n and did not find a Walrasian equilibrium price system, then the adjustment process intersects the set $S_{\lambda-\delta}^n$ for every δ small enough.

Lemma 4.2. *Let a total excess demand function z that satisfies Assumptions 3–5 and 7 be given, and let $v \in S^n$ be a starting price system. If for some $\lambda \in (0, 1]$ and for some $s \in \mathscr{S}$, $\bar{p} \in C(s) \cap S_\lambda^n$ and $z(\bar{p}) \neq 0^{n+1}$, then $\exists \bar{s} \in \mathscr{S}$, $\exists \epsilon > 0$, such that $\forall \delta \in [0, \epsilon]$, $C(\bar{s}) \cap S_{\lambda-\delta}^n \neq \emptyset$. If $\forall j \in I^+(s)$, $z_j(\bar{p}) > 0$, and $\forall j \in I^-(s)$, $z_j(\bar{p}) < 0$, then $\bar{s}$ can be taken equal to s.*

Proof. Let $\bar{s}$ be such that $\bar{s}_j = +1$ if $z_j(\bar{p}) > 0$, $\bar{s}_j = 0$ if $z_j(\bar{p}) = 0$, and $\bar{s}_j = -1$ if $z_j(\bar{p}) < 0$. Note that $\bar{s} = s$ if the requirements in the last part of Lemma 4.2 are satisfied. Since $z(\bar{p}) \neq 0^{n+1}$, it holds by Walras' law that $\bar{s} \in \mathscr{S}$. Clearly, $\bar{p} \in C(\bar{s}) \cap S_\lambda^n$. If, for every $j \in I_{n+1}$, $z_j(\bar{p}) \neq 0$, then Lemma 4.2 is clearly true by the continuity of z. So we consider the case where $I^0(\bar{s}) \neq \emptyset$. For $\delta \in [0, \lambda)$ we define the set

$$E(\bar{p}, \bar{s}, \delta) = \Big\{ p \in S^n \,|\, \forall k \in I^-(\bar{s}), \; p_k/\bar{p}_k = 1 - \delta/\lambda, \; \forall k \in I^0(\bar{s}),$$

$$1 - \delta/\lambda \leq p_k/\bar{p}_k \leq \max_{j \in I_{n+1}} p_j/\bar{p}_j, \; \forall k \in I^+(\bar{s}),$$

$$p_k/\bar{p}_k = \max_{j \in I_{n+1}} p_j/\bar{p}_j \Big\}.$$

It is easily verified that $E(\bar{p}, \bar{s}, \delta)$ is a compact subset of $A(\bar{s}) \cap S_{\lambda-\delta}^n$. By continuity of the total excess demand function z there exists $\epsilon \in (0, \lambda)$ such that if $0 \leq \delta \leq \epsilon$, then for every $p \in E(\bar{p}, \bar{s}, \delta)$, $z_j(p) > 0$ if $j \in I^+(\bar{s})$ and $z_j(p) < 0$ if $j \in I^-(\bar{s})$. For $\delta = 0$, Lemma 4.2 is obviously true. We consider an arbitrary $\delta \in (0, \epsilon]$, and consider $p^* \in \arg\min_{p \in E(\bar{p}, \bar{s}, \delta)} (\max_{j \in I^0(\bar{s})} |z_j(p)|)$.

Suppose $\max_{j \in I^0(\bar{s})} |z_j(p^*)| > 0$. We define the sets $I_-^0 = \{k \in I^0(\bar{s}) \,|\, z_k(p^*) = -\max_{j \in I^0(\bar{s})} |z_j(p^*)| \}$ and $I_+^0 = \{k \in I^0(\bar{s}) \,|\, z_k(p^*) = \max_{j \in I^0(\bar{s})} |z_j(p^*)| \}$. Suppose $k \in I_-^0$ and $p_k^*/\bar{p}_k = 1 - \delta/\lambda$. By Lemma 4.1 and since $p^* \in E(\bar{p}, \bar{s}, \delta)$, this implies $0 = z_k(\bar{p}) < z_k(p^*)$, a contradiction since $k \in I_-^0$. Hence, $k \in I_-^0$ implies $p_k^*/\bar{p}_k > 1 - \delta/\lambda$. Similarly, it can be shown that $k \in I_+^0$ implies $p_k^*/\bar{p}_k < \max_{j \in I_{n+1}} p_j^*/\bar{p}_j$. Next, three possible cases will be considered, each leading to a contradiction with the supposition that $\max_{j \in I^0(\bar{s})} |z_j(p^*)| > 0$. Therefore $\max_{j \in I^0(\bar{s})} |z_j(p^*)| = 0$, and this result, together with the choice of ϵ, implies $p^* \in B(\bar{s})$. Moreover, $p^* \in E(\bar{p}, \bar{s}, \delta) \subset A(\bar{s}) \cap S_{\lambda-\delta}^n$, and therefore $p^* \in C(\bar{s}) \cap S_{\lambda-\delta}^n$.

Case 1. If $I_-^0 \neq \emptyset$ and $I_+^0 = \emptyset$, then, for $\alpha > 0$, we define p^α by $p_k^\alpha = p_k^*$, $\forall k \in I^-(\bar{s}) \cup (I^0(\bar{s}) \setminus I_-^0)$, $p_k^\alpha = (1 - \alpha)p_k^*$, $\forall k \in I_-^0$,

$$p_k^\alpha = \left(1 + \frac{\alpha \sum_{j \in I_-^0} p_j^*}{\sum_{j \in I^+(\bar{s})} p_j^*} \right) p_k^*, \quad \forall k \in I^+(\bar{s}).$$

Since $k \in I_-^0$ implies $p_k^*/\bar{p}_k > 1 - \delta/\lambda$, α can be chosen small enough to guarantee that $p^\alpha \in E(\bar{p}, \bar{s}, \delta)$, $|z_k(p^\alpha)| < \max_{j \in I^0(\bar{s})} |z_j(p^*)|$, $\forall k \in I^0(\bar{s})$ $\setminus I_-^0$, and $z_k(p^\alpha) < 0$, $\forall k \in I_-^0$. By Lemma 4.1 it now holds that $\max_{j \in I^0(\bar{s})} |z_j(p^\alpha)| < \max_{j \in I^0(\bar{s})} |z_j(p^*)|$, which contradicts the definition of p^*.

Case 2. If $I_-^0 = \emptyset$ and $I_+^0 \neq \emptyset$, then we define the, possibly empty, set $K = \{k \in I^0(\bar{s}) \mid p_k^*/\bar{p}_k = \max_{j \in I_{n+1}} p_j^*/\bar{p}_j\}$. Moreover, for $\alpha > 0$, we define p^α by $p_k^\alpha = p_k^*$, $\forall k \in I^-(\bar{s}) \cup (I^0(\bar{s}) \setminus (I_+^0 \cup K))$, $p_k^\alpha = (1 + \alpha)p_k^*$, $\forall k \in I_+^0$,

$$p_k^\alpha = \left(1 - \frac{\alpha \sum_{j \in I_+^0} p_j^*}{\sum_{j \in I^+(\bar{s}) \cup K} p_j^*}\right) p_k^*, \quad \forall k \in I^+(\bar{s}) \cup K.$$

Since $k \in I_+^0$ implies $p_k^*/\bar{p}_k < \max_{j \in I_{n+1}} p_j^*/\bar{p}_j$, α can be chosen small enough to guarantee that $p^\alpha \in E(\bar{p}, \bar{s}, \delta)$, $|z_k(p^\alpha)| < \max_{j \in I^0(\bar{s})} |z_j(p^*)|$, $\forall k \in I^0(\bar{s})$ $\setminus I_+^0$, and $z_k(p^\alpha) > 0$, $\forall k \in I_+^0$. By Lemma 4.1 and the construction of p^α, $\max_{j \in I^0(\bar{s})} |z_j(p^\alpha)| < \max_{j \in I^0(\bar{s})} |z_j(p^*)|$, which contradicts the definition of p^*.

Case 3. If $I_-^0 \neq \emptyset$ and $I_+^0 \neq \emptyset$, then, for $\alpha > 0$, we define p^α by $p_k^\alpha = p_k^*$, $\forall k \in I_{n+1} \setminus (I_-^0 \cup I_+^0)$, $p_k^\alpha = (1 - \alpha)p_k^*$, $\forall k \in I_-^0$,

$$p_k^\alpha = \left(1 + \frac{\alpha \sum_{j \in I_-^0} p_j^*}{\sum_{j \in I_+^0} p_j^*}\right) p_k^*, \quad \forall k \in I_+^0.$$

Clearly, α can be chosen small enough to guarantee that $p^\alpha \in E(\bar{p}, \bar{s}, \delta)$ and $|z_k(p^\alpha)| < \max_{j \in I^0(\bar{s})} |z_j(p^*)|$, $\forall k \in I^0(\bar{s}) \setminus (I_-^0 \cup I_+^0)$, $z_k(p^\alpha) < 0$, $\forall k \in I_-^0$, and $z_k(p^\alpha) > 0$, $\forall k \in I_+^0$. By Lemma 4.1, a contradiction is obtained as before. $\square$ Q.E.D.

The next step is to show that if $\bar{s}$ and $\bar{s}$ in $\mathcal{S}$ are such that $k^0(\bar{s}) = k^0(\bar{s})$ and $\bar{p} \in C(\bar{s})$, then there is no $\bar{p} \in C(\bar{s}) \setminus C(\bar{s})$. This is the result of Lemma 4.4. So, if during the price adjustment process the region $A(\bar{s})$ is reached, and therefore $k^0(\bar{s})$ markets are in equilibrium, then every price system $\bar{p}$ ever generated by the process with $k^0(\bar{s})$ markets in equilibrium satisfies $\bar{p} \in A(\bar{s})$. Moreover, it is shown in Lemma 4.4 that if two price systems $\bar{p}$ and $\bar{p}$ are reached by the adjustment process with the same number of markets in equilibrium and with the minimal price ratio (with respect to the starting price system v) of $\bar{p}$ greater than that of $\bar{p}$, then $\bar{p}$, $\bar{p} \in C(s)$ for a uniquely determined sign vector s. Moreover, $I^-(s) = J_{\min}(\bar{p}, \bar{p})$ and $I^+(s) = J_{\max}(\bar{p}, \bar{p})$. So the prices of commodities in excess supply (demand) have been decreased (increased) maximally. To show Lemma 4.4, the technical Lemma 4.3 has to be shown first.

Lemma 4.3. Let a starting price system $v \in S^n$ be given. Moreover, let $\bar{s}$, $\bar{s} \in \mathcal{S}$ with $\bar{s} \neq \bar{s}$, $k^0(\bar{s}) = k^0(\bar{s})$, $\bar{p} \in A(\bar{s})$, and $\bar{p} \in A(\bar{s})$, with $\bar{p} \neq \bar{p}$ be given. Then

$$J_{\max}(\bar{p}, \bar{p}) \cap (I^-(\bar{s}) \cup I^0(\bar{s})) \cap (I^0(\bar{s}) \cup I^+(\bar{s})) \neq \emptyset$$

or

$$J_{\min}(\tilde{p}, \bar{p}) \cap \left(I^0(\tilde{s}) \cup I^+(\tilde{s}) \right) \cap \left(I^-(\bar{s}) \cup I^0(\bar{s}) \right) \neq \emptyset.$$

Proof. Suppose, on the contrary, that

$$J_{\max}(\tilde{p}, \bar{p}) \subset I^+(\tilde{s}) \cup I^-(\bar{s}) \tag{13}$$

and

$$J_{\min}(\tilde{p}, \bar{p}) \subset I^-(\tilde{s}) \cup I^+(\bar{s}). \tag{14}$$

It will be shown that if (13) and (14) hold, then

$$\cdot J_{\max}(\tilde{p}, \bar{p}) \cap I^+(\tilde{s}) = \emptyset \quad \text{or} \quad J_{\min}(\tilde{p}, \bar{p}) \cap I^+(\bar{s}) = \emptyset, \tag{15}$$

$$J_{\max}(\tilde{p}, \bar{p}) \cap I^+(\tilde{s}) = \emptyset \quad \text{or} \quad J_{\min}(\tilde{p}, \bar{p}) \cap I^-(\bar{s}) = \emptyset, \tag{16}$$

$$J_{\max}(\tilde{p}, \bar{p}) \cap I^-(\tilde{s}) = \emptyset \quad \text{or} \quad J_{\min}(\tilde{p}, \bar{p}) \cap I^-(\bar{s}) = \emptyset, \tag{17}$$

$$J_{\max}(\tilde{p}, \bar{p}) \cap I^-(\tilde{s}) = \emptyset \quad \text{or} \quad J_{\min}(\tilde{p}, \bar{p}) \cap I^+(\bar{s}) = \emptyset. \tag{18}$$

From (14)–(16) it follows that $J_{\max}(\tilde{p}, \bar{p}) \cap I^+(\tilde{s}) = \emptyset$, and from (14), (17), and (18) it follows that $J_{\max}(\tilde{p}, \bar{p}) \cap I^-(\tilde{s}) = \emptyset$. Together with (13) this yields $J_{\max}(\tilde{p}, \bar{p}) = \emptyset$, which is a contradiction, and this proves the lemma. It remains to be shown that (13) and (14) imply (15)–(18). Let $j^1 \in J_{\max}(\tilde{p}, \bar{p})$ and $j^2 \in J_{\min}(\tilde{p}, \bar{p})$.

Suppose $j^1 \in I^+(\tilde{s})$. If $j^2 \in I^+(\bar{s})$, then $1 > \bar{p}_{j^2}/\tilde{p}_{j^2} = (\bar{p}_{j^2}/v_{j^2})(v_{j^2}/\tilde{p}_{j^2}) \geq (\bar{p}_{j^1}/v_{j^1})(v_{j^1}/\tilde{p}_{j^1}) = \bar{p}_{j^1}/\tilde{p}_{j^1} > 1$, which is a contradiction. Hence (15) is true. If $j^2 \in I^-(\bar{s})$, then for every $k \in I^-(\bar{s})$ it holds that $\bar{p}_k/\tilde{p}_k \leq \bar{p}_{j^2}/\tilde{p}_{j^2}$, so $k \in J_{\min}(\tilde{p}, \bar{p})$, and by (14), $k \in I^-(\tilde{s})$. For every $k \in I^+(\tilde{s})$ it holds that $\bar{p}_k/\tilde{p}_k \geq \bar{p}_{j^1}/\tilde{p}_{j^1}$, so $k \in J_{\max}(\tilde{p}, \bar{p})$, and by (13), $k \in I^+(\tilde{s})$. Consequently, $I^+(\tilde{s}) \subset I^+(\bar{s})$ and $I^-(\bar{s}) \subset I^-(\tilde{s})$. Since $\bar{s} \neq \tilde{s}$ and $k^0(\tilde{s}) = k^0(\bar{s})$, a contradiction is obtained. Hence (16) is true.

Suppose $j^1 \in I^-(\tilde{s})$. If $j^2 \in I^-(\bar{s})$, then $1 > \bar{p}_{j^2}/\tilde{p}_{j^2} \geq \bar{p}_{j^1}/\tilde{p}_{j^1} > 1$, which is a contradiction. Hence (17) is true. If $j^2 \in I^+(\bar{s})$, then for every $k \in I^+(\bar{s})$ it holds that $\bar{p}_k/\tilde{p}_k \leq \bar{p}_{j^2}/\tilde{p}_{j^2}$, so $k \in J_{\min}(\tilde{p}, \bar{p})$, and by (14), $k \in I^+(\tilde{s})$. For every $k \in I^-(\tilde{s})$ it holds that $\bar{p}_k/\tilde{p}_k \geq \bar{p}_{j^1}/\tilde{p}_{j^1}$, so $k \in J_{\max}(\tilde{p}, \bar{p})$, and by (13), $k \in I^-(\tilde{s})$. Since $\bar{s} \neq \tilde{s}$ and $k^0(\tilde{s}) = k^0(\bar{s})$, a contradiction is obtained. So (18) is true. □Q.E.D.

Lemma 4.4. Let a total excess demand function z that satisfies Assumptions 4 and 7, a starting price system $v \in S^n$, and sign vectors $\tilde{s}, \bar{s} \in \mathcal{S}$ with $k^0(\tilde{s}) = k^0(\bar{s})$ be given. If $C(\tilde{s}) \neq \emptyset$, then $C(\bar{s}) \setminus C(\tilde{s}) = \emptyset$. Moreover, if there are price systems $\tilde{p} \in C(\tilde{s})$ and $\bar{p} \in C(\tilde{s})$ with $\min_{j \in I_{n+1}} \tilde{p}_j/v_j > \min_{j \in I_{n+1}} \bar{p}_j/v_j$, then $\tilde{s} = \bar{s}$, $J_{\min}(\tilde{p}, \bar{p}) = I^-(\tilde{s})$, and $J_{\max}(\tilde{p}, \bar{p}) = I^+(\tilde{s})$.

Proof. Suppose, on the contrary, that there exist $\tilde{p} \in C(\tilde{s})$ and $\bar{p} \in C(\bar{s}) \setminus C(\tilde{s})$. Clearly, $\tilde{s} \neq \bar{s}$. Moreover, $\tilde{p} \in A(\tilde{s})$ and $\bar{p} \in A(\bar{s})$ and therefore by Lemma 4.3

there exists a $k \in J_{max}(\tilde{p}, \bar{p}) \cap (I^-(\tilde{s}) \cup I^0(\tilde{s})) \cap (I^0(\tilde{s}) \cup I^+(\tilde{s}))$ or there exists a $k \in J_{min}(\tilde{p}, \bar{p}) \cap (I^0(\tilde{s}) \cup I^+(\tilde{s})) \cap (I^-(\tilde{s}) \cup I^0(\tilde{s}))$. In the first case, by Lemma 4.1 $z_k(\tilde{p}) > z_k(\bar{p})$. However, $z_k(\tilde{p}) \leq 0$ and $z_k(\bar{p}) \geq 0$, which is a contradiction. In the second case, by Lemma 4.1 $z_k(\tilde{p}) < z_k(\bar{p})$. However, $z_k(\tilde{p}) \geq 0$ and $z_k(\bar{p}) \leq 0$, which is a contradiction. This proves the first part of Lemma 4.4.

If $\tilde{p} \in C(\tilde{s})$ and $\bar{p} \in C(\bar{s})$, then by the first part of the lemma it holds that $C(\tilde{s}) = C(\bar{s})$. If $j \in I^-(\tilde{s})$, then $\bar{p}_j/\tilde{p}_j = (\bar{p}_j/v_j)(v_j/\tilde{p}_j) < (\tilde{p}_j/v_j)(v_j/\tilde{p}_j) = 1$ and therefore $J_{max}(\tilde{p}, \bar{p}) \subset I^0(\tilde{s}) \cup I^+(\tilde{s})$. Suppose $j \in J_{max}(\tilde{p}, \bar{p}) \cap I^0(\tilde{s})$. Then by Lemma 4.1 and since $\tilde{p}, \bar{p} \in C(\tilde{s})$, $0 = z_j(\tilde{p}) > z_j(\bar{p}) = 0$, which is a contradiction. Consequently, $J_{max}(\tilde{p}, \bar{p}) \subset I^+(\tilde{s})$. If $j, j' \in I^+(\tilde{s})$, then $\bar{p}_j/\tilde{p}_j = \bar{p}_{j'}/\tilde{p}_{j'}$. Hence $J_{max}(\tilde{p}, \bar{p}) = I^+(\tilde{s})$ and $J_{min}(\tilde{p}, \bar{p}) \subset I^-(\tilde{s}) \cup I^0(\tilde{s})$. Suppose $j \in J_{min}(\tilde{p}, \bar{p}) \cap I^0(\tilde{s})$. Then $0 = z_j(\tilde{p}) < z_j(\bar{p}) = 0$, which is a contradiction. It follows that $J_{min}(\tilde{p}, \bar{p}) = I^-(\tilde{s})$. In a similar way it can be shown that $J_{min}(\tilde{p}, \bar{p}) = I^-(\bar{s})$ and $J_{max}(\tilde{p}, \bar{p}) = I^+(\bar{s})$, hence $I^-(\tilde{s}) = I^-(\bar{s})$, $I^+(\tilde{s}) = I^+(\bar{s})$, and therefore $\tilde{s} = \bar{s}$.   □ Q.E.D.

The next step in proving the convergence of the price adjustment process is to show that the adjustment process intersects each set S_λ^n at most once. First it is shown that, given $s \in \mathscr{S}$, the intersection of $C(s)$ and S_λ^n contains at most one element.

Lemma 4.5. *Let a total excess demand function z that satisfies Assumptions 4 and 7 be given, and let $v \in S^n$ be a starting price system. Then for every $\lambda \in (0, 1]$ and for every $s \in \mathscr{S}$ the set $C(s) \cap S_\lambda^n$ contains at most one element.*

Proof. Suppose $\tilde{p}, \bar{p} \in C(s) \cap S_\lambda^n$ with $\tilde{p} \neq \bar{p}$. Then $\forall j \in j^-(s)$, $\bar{p}_j/\tilde{p}_j = \lambda v_j/\lambda v_j = 1$. So $\tilde{p} \neq \bar{p}$ implies that there exists a $j^1 \in J_{max}(\tilde{p}, \bar{p}) \cap (I^0(s) \cup I^+(s))$ and there exists a $j^2 \in J_{min}(\tilde{p}, \bar{p}) \cap (I^0(s) \cup I^+(s))$. By Lemma 4.1, $z_{j^1}(\tilde{p}) > z_{j^1}(\bar{p})$ and $z_{j^2}(\tilde{p}) < z_{j^2}(\bar{p})$. So $j^1, j^2 \notin I^0(s)$. If $j^1, j^2 \in I^+(s)$, then $1 < \bar{p}_{j^1}/\tilde{p}_{j^1} = \bar{p}_{j^2}/\tilde{p}_{j^2} < 1$, which is a contradiction. Consequently, $C(s) \cap S_\lambda^n$ contains at most one element.   □ Q.E.D.

After these preliminary lemmas it is possible to show the convergence of the price adjustment process. First, it will be shown that the price adjustment process intersects each set S_λ^n at most once for every $\lambda \in (0, 1]$. Secondly, the continuity of the price adjustment process will be shown.

Theorem 4.6. *Let a total excess demand function z that satisfies Assumptions 3–7 be given and let $v \in S^n$ be a starting price system. If v is not a Walrasian equilibrium price system, then the set C is an arc that contains v and a Walrasian equilibrium price system as boundary points.*

Proof. We define the (possibly empty valued) correspondence $\Pi : (0, 1] \to S^n$ by $\Pi(\lambda) = C \cap S_\lambda^n$. First it is shown that there exists a $\lambda^* \in (0, 1]$ such that Π is a

function on a compact set $[\lambda^*, 1]$ and is empty valued on the set $(0, \lambda^*)$. In a similar way as in the proof of Theorem 2.6 it can be shown that for every $s \in \mathscr{S}$ the set $C(s)$ is compact and therefore that the set $C = \bigcup_{s \in \mathscr{S}} C(s)$ is compact. We define $\lambda^* = \min_{p \in C}(\min_{j \in I_{n+1}} p_j/v_j)$. Obviously it holds that $\Pi(\lambda) = \emptyset$ if $\lambda < \lambda^*$. Moreover, by Lemma 4.2, $\Pi(\lambda^*)$ is a Walrasian equilibrium price system. By the compactness of C, Lemma 4.2, the uniqueness of the Walrasian equilibrium price system, and since $v \in \Pi(1)$, it follows that $\Pi(\lambda) \neq \emptyset, \forall \lambda \in [\lambda^*, 1]$.

Now let some $\lambda \in [\lambda^*, 1]$ be given and suppose $\bar{p} \in C(\bar{s})$, $\bar{p} \in C(\bar{s})$, $\bar{p} \neq \bar{p}$, and $\bar{p}, \bar{p} \in S_\lambda^n$. By Lemmas 4.4 and 4.5, $k^0(\bar{s}) \neq k^0(\bar{s})$. We assume without loss of generality that $k^0(\bar{s}) < k^0(\bar{s})$. Since $C(\bar{s})$ is compact there is a price system p^1 such that $p^1 \in \arg\min_{p \in C(\bar{s})}(\min_{j \in I_{n+1}} p_j/v_j)$. By Lemma 4.2 it follows that for some $k \in I^-(\bar{s}) \cup I^+(\bar{s})$, $z_k(p^1) = 0$. Hence $p^1 \in C(s^1)$, where s^1 is defined by $s_k^1 = 0$, and $s_j^1 = \bar{s}_j, \forall j \in I_{n+1} \setminus \{k\}$. Since $k^0(\bar{s}) < k^0(\bar{s}) \leq n - 1$ and by Walras' law, k can be chosen such that $s^1 \in \mathscr{S}$. Repeating this argument a finite number of times, a price system $\hat{p} \in C(\hat{s})$ is found, where $\hat{s} \in \mathscr{S}$, $k^0(\hat{s}) = k^0(\bar{s})$, and for some $k \in I^0(\hat{s})$ it holds that $\hat{p}_k/v_k = \min_{j \in I_{n+1}} \hat{p}_j/v_j$ or $\hat{p}_k/v_k = \max_{j \in I_{n+1}} \hat{p}_j/v_j$. Suppose that $\min_{j \in I_{n+1}} \hat{p}_j/v_j = \min_{j \in I_{n+1}} \bar{p}_j/v_j$, then $\bar{p}, \bar{p} \in S_\lambda^n$ implies $\min_{j \in I_{n+1}} \hat{p}_j/v_j = \min_{j \in I_{n+1}} \bar{p}_j/v_j$. Using Lemma 4.5 yields that the minimizing argument equals $\bar{p}$ in every step, and hence $\hat{p} = \bar{p}$. By Lemma 4.5, $\bar{p} \notin C(\bar{s})$, and since $\bar{p} \in C(\hat{s})$ a contradiction with Lemma 4.4 is obtained. Consequently, $\min_{j \in I_{n+1}} \hat{p}_j/v_j < \min_{j \in I_{n+1}} \bar{p}_j/v_j$. By Lemma 4.4, $\bar{s} = \hat{s}$, $J_{\min}(\bar{p}, \hat{p}) = I^-(\hat{s})$, and $J_{\max}(\bar{p}, \hat{p}) = I^+(\hat{s})$. We consider the case where $\hat{p}_k/v_k = \min_{j \in I_{n+1}} \hat{p}_j/v_j$. Let $k' \in J_{\min}(\bar{p}, \hat{p})$. Since $\hat{p}_k/v_k = \min_{j \in I_{n+1}} \hat{p}_j/v_j$ and $J_{\min}(\bar{p}, \hat{p}) = I^-(\hat{s}) = I^-(\bar{s})$, it holds that $\hat{p}_k/\bar{p}_k \leq \hat{p}_{k'}/\bar{p}_{k'}$. So $k \in J_{\min}(\bar{p}, \hat{p})$, which contradicts $k \in I^0(\hat{s})$. Similarly a contradiction is obtained if $\hat{p}_k/v_k = \max_{j \in I_{n+1}} \hat{p}_j/v_j$. This shows that for $\lambda \in [\lambda^*, 1]$, $\Pi(\lambda)$ is single-valued.

Either $\lambda^* = 1$ and $C = \{v\}$, or $\lambda^* < 1$. In the latter case we define the function $\pi : [0, 1] \to C$ by $\{\pi(t)\} = \Pi((\lambda^* - 1)t + 1), \forall t \in [0, 1]$. The function π is one-to-one and onto. It remains to be shown that π is continuous. The continuity of π^{-1} then follows immediately using the compactness of $[0, 1]$. Let $(t^r)_{r \in \mathbb{N}}$ be a sequence in $[0\ 1]$ with limit $\bar{t}$. We consider the sequence $(\pi(t^r))_{r \in \mathbb{N}}$. If π is not continuous, then by the compactness of C there is no loss of generality in assuming that $\pi(t^r)$ converges to a limit $\bar{p} \in C$ and $\bar{p} \neq \pi(\bar{t})$. Since $\pi(t^r) \in S_{(\lambda^* - 1)t^r + 1}^n$ it holds that $\min_{j \in I_{n+1}} \bar{p}_j/v_j = \lim_{r \to \infty} \min_{j \in I_{n+1}} \pi_j(t^r)/v_j = (\lambda^* - 1)\bar{t} + 1$. Hence $\{\bar{p}, \pi(\bar{t})\} \subset C \cap S_{(\lambda^* - 1)\bar{t} + 1}^n = \{\pi(\bar{t})\}$, which is a contradiction. □Q.E.D.

In the gross substitution case the adjustment process has very interesting economic properties as will be made clear in the three final theorems. In Theorem 4.7 it is shown that during the adjustment process the number of markets in equilibrium is increasing. More precisely, if a market attains an equilibrium situation, it remains in equilibrium during the rest of the adjustment process. In

Theorem 4.8 this result is even strengthened and it is shown that on every market the absolute value of the excess demand is monotonically decreasing. In Theorem 4.9 it is shown that during the entire process, prices of commodities in excess demand are strictly increasing, while prices of commodities in excess supply are strictly decreasing. Theorem 4.9 makes clear that the prices on markets out of equilibrium are adjusted in a way that is qualitatively the same as Walrasian tatonnement, while Theorem 4.7 states an important difference: markets in equilibrium remain in equilibrium. Let p^* be the unique Walrasian equilibrium for an economy with a demand function that satisfies gross substitutability. We define $\lambda^* = \min_{j \in I_{n+1}} p_j^* / v_j$ and define the function $\pi : [0, 1] \to S^n$ by $\{\pi(t)\} = C \cap S^n_{(\lambda^* - 1)t + 1}$. In the proof of Theorem 4.7 we show that π is a homeomorphism between C and $[0, 1]$ if $\lambda^* < 1$. Moreover, $\pi(0) = v$ and $\pi(1) = p^*$. If $\lambda^* = 1$, then this function is still well defined and is a constant function, assigning to every $t \in [0, 1]$ the Walrasian equilibrium price system v.

Theorem 4.7. *Let a total excess demand function z that satisfies Assumptions 3–7 be given and let $v \in S^n$ be a starting price system. We take t^1, $t^2 \in [0, 1]$, with $t^1 < t^2$. If s^1, $s^2 \in \mathscr{S}$ are such that $\pi(t^1) \in C(s^1)$ and $\pi(t^2) \in C(s^2)$ and if $\lambda^* \neq 1$, then $I^-(s^1) \supset I^-(s^2)$, $I^0(s^1) \subset I^0(s^2)$, and $I^+(s^1) \supset I^+(s^2)$.*

Proof. Let $p^1 = \pi(t^1)$ and $p^2 = \pi(t^2)$. Note that $\min_{j \in I_{n+1}} p_j^1 / v_j > \min_{j \in I_{n+1}} p_j^2 / v_j$, since $\lambda^* \neq 1$. Suppose $k \in (I^0(s^1) \setminus I^0(s^2)) \cup (I^-(s^2) \setminus I^-(s^1)) \cup (I^+(s^2) \setminus I^+(s^1))$. If $k^0(s^1) = k^0(s^2)$, then by Lemma 4.4 it holds that $s^1 = s^2$, which contradicts the choice of k. If $k^0(s^1) > k^0(s^2)$, then by the same arguments as in the proof of Theorem 4.6, starting with $C(s^2)$, there exists an $\hat{s} \in \mathscr{S}$ and a $\hat{p} \in C(\hat{s})$, such that $k^0(\hat{s}) = k^0(s^1)$, $\min_{j \in I_{n+1}} \hat{p}_j / v_j \leq \min_{j \in I_{n+1}} p_j^2 / v_j$, $\exists k' \in I^0(\hat{s})$, $\hat{p}_{k'}/v_{k'} = \min_{j \in I_{n+1}} \hat{p}_j / v_j$ or $\hat{p}_{k'}/v_{k'} = \max_{j \in I_{n+1}} \hat{p}_j / v_j$. Since $\min_{j \in I_{n+1}} \overline{p}_j / v_j < \min_{j \in I_{n+1}} p_j^1 / v_j$, it holds by Lemma 4.4 that $s^1 = \hat{s}$, $J_{\min}(p^1, \hat{p}) = I^-(\hat{s})$, and $J_{\max}(p^1, \hat{p}) = I^+(\hat{s})$. We consider the case where $\hat{p}_{k'}/v_{k'} = \min_{j \in I_{n+1}} \hat{p}_j / v_j$. Let $j' \in J_{\min}(p^1, \hat{p})$. Since $p^1 \in C(\hat{s})$ and $J_{\min}(p^1, \hat{p}) = I^-(\hat{s})$, $\hat{p}_{k'}/p_{k'}^1 \leq \hat{p}_j / p_j^1$. So $k' \in J_{\min}(p^1, \hat{p})$, which contradicts $k' \in I^0(\hat{s})$. Similarly, a contradiction is obtained if $\hat{p}_{k'}/v_{k'} = \max_{j \in I_{n+1}} \hat{p}_j / v_j$.

If $k^0(s^1) < k^0(s^2)$, then again the construction of the proof of Theorem 4.6 can be used, starting with $C(s^1)$. There exists an $\hat{s} \in \mathscr{S}$ and a $\hat{p} \in C(\hat{s})$ such that $k^0(\hat{s}) = k^0(s^2)$, $\min_{j \in I_{n+1}} \hat{p}_j / v_j \leq \min_{j \in I_{n+1}} p_j^1 / v_j$, $\exists k' \in I^0(\hat{s})$, $\hat{p}_{k'}/v_{k'} = \min_{j \in I_{n+1}} \hat{p}_j / v_j$ or $\hat{p}_{k'}/v_{k'} = \max_{j \in I_{n+1}} \hat{p}_j / v_j$. Moreover, $I^-(s^1) \supset I^-(\hat{s})$, $I^0(s^1) \subset I^0(\hat{s})$, and $I^+(s^1) \supset I^+(\hat{s})$. If $\min_{j \in I_{n+1}} \hat{p}_j / v_j \neq \min_{j \in I_{n+1}} p_j^2 / v_j$, then, using Lemma 4.4, $\hat{s} = s^2$, which contradicts the existence of k. So we consider the case where $\min_{j \in I_{n+1}} \hat{p}_j / v_j = \min_{j \in I_{n+1}} p_j^2 / v_j$. Now $\hat{p} = p^2$ since $C \cap S^n_{(\lambda^* - 1)t^2 + 1}$ contains a unique element. If $j \in I^-(\hat{s})$, then $\hat{p}_j / p_j^1 < 1$. Hence $J_{\max}(p^1, \hat{p}) \subset I^0(\hat{s}) \cup I^+(\hat{s})$. If $j \in J_{\max}(p^1, \hat{p}) \cap I^0(\hat{s})$, then by Lemma 4.1, $z_j(p^1) > z_j(\hat{p}) = 0$, therefore $j \in I^+(s^1)$, and hence $\hat{p}_j / p_j^1 \leq \hat{p}_j / p_j^1$, with $j' \in I^+(\hat{s})$. Since $I^+(\hat{s}) \subset I^+(s^1)$ it follows that $\hat{p}_j / p_j^1 = \hat{p}_j / p_j^1$ if j, $j' \in I^+(\hat{s})$. Consequently, $I^+(\hat{s}) \subset$

$J_{\max}(p^1, \hat{p})$. It follows in a similar way that $I^-(\hat{s}) \subset J_{\min}(p^1, \hat{p})$. Consider k as defined in the beginning of the proof. If $k \in I^-(s^2) \setminus I^-(s^1)$, then $z_k(\hat{p}) \leq 0$ since $\hat{p} = p^2$. Let $j \in I^-(\hat{s})$ then $j \in I^-(s^1)$, and using $k \in I^-(s^2)$ and $\hat{p} = p^2$ it holds that $\hat{p}_k/p_k^1 \leq \hat{p}_j/p_j^1$. Consequently, $k \in J_{\min}(p^1, \hat{p})$. By Lemma 4.1 it holds that $0 \leq z_k(p^1) < z_k(\hat{p}) \leq 0$, which is a contradiction. The case where $k \in I^+(s^2) \setminus I^+(s^1)$ yields a contradiction in a similar way. Finally, we consider the case where $k \in I^0(s^1) \setminus I^0(s^2)$. Hence $k \in I^-(s^2) \cup I^+(s^2)$ and it can be shown that $k \in J_{\min}(p^1, \hat{p}) \cup J_{\max}(p^1, \hat{p})$. Since $k \in I^0(s^1) \subset I^0(\hat{s})$, $0 = z_k(p^1) = z_k(\hat{p})$. By Lemma 4.1 it holds that $z_k(\hat{p}) \neq 0$, which is a contradiction. □ Q.E.D.

Theorem 4.8. *Let a total excess demand function z that satisfies Assumptions 3–7 be given and let $v \in S^n$ be a starting price system. Take t^1, $t^2 \in [0, 1]$ with $t^1 < t^2$ and take $k \in I_{n+1}$. If $z_k(\pi(t^1)) < 0$, then $z_k(\pi(t^1)) < z_k(\pi(t^2)) \leq 0$, if $z_k(\pi(t^1)) = 0$, then $z_k(\pi(t^2)) = 0$, and if $z_k(\pi(t^1)) > 0$, then $z_k(\pi(t^1)) > z_k(\pi(t^2)) \geq 0$.*

Proof. If $\lambda^* = 1$, then the proof of Theorem 4.8 is trivial, so we consider the case $\lambda^* < 1$. Let $s^1, s^2 \in \mathscr{S}$ be such that $\pi(t^1) \in C(s^1)$ and $\pi(t^2) \in C(s^2)$. Let $j_- \in I^-(s^2)$, then by Theorem 4.7, $j_- \in I^-(s^1)$ and so $\pi_{j_-}(t^2)/\pi_{j_-}(t^1) = ((\lambda^* - 1)t^2 + 1)/((\lambda^* - 1)t^1 + 1) < 1$. Let $j_+, j_+ \in I^+(s^2)$, then using Theorem 4.7, $\pi_{j_+}(t^2)/\pi_{j_+}(t^1) = \pi_{j_+}(t^2)/\pi_{j_+}(t^1)$. If $j_0 \in I^0(s^2) \cap I^-(s^1)$, then $\pi_{j_0}(t^2)/\pi_{j_0}(t^1) \geq \pi_{j_-}(t^2)/\pi_{j_-}(t^1)$. Moreover $j_0 \notin J_{\max}(\pi(t^1), \pi(t^2))$ since otherwise by Lemma 4.1, $z_{j_0}(\pi(t^1)) > z_{j_0}(\pi(t^2)) = 0$, and a contradiction would be obtained. Similarly, $j_0 \in I^0(s^2) \cap I^0(s^1)$ implies $j_0 \notin J_{\min}(\pi(t^1), \pi(t^2)) \cup J_{\max}(\pi(t^1), \pi(t^2))$, and $j_0 \in I^0(s^2) \cap I^+(s^1)$ implies $\pi_{j_0}(t^2)/\pi_{j_0}(t^1) \leq \pi_{j_+}(t^2)/\pi_{j_+}(t^1)$ and $j_0 \notin J_{\min}(\pi(t^1), \pi(t^2))$. Consequently, $I^-(s^2) \subset J_{\min}(\pi(t^1), \pi(t^2))$ and $I^+(s^2) \subset J_{\max}(\pi(t^1), \pi(t^2))$. Using this result and Lemma 4.1, $z_k(\pi(t^2)) < 0$ implies $z_k(\pi(t^1)) < z_k(\pi(t^2))$ and $z_k(\pi(t^2)) > 0$ implies $z_k(\pi(t^1)) > z_k(\pi(t^2))$. By Theorem 4.7, $z_k(\pi(t^2)) \leq 0$ if $z_k(\pi(t^1)) < 0$, $z_k(\pi(t^2)) = 0$ if $z_k(\pi(t^1)) = 0$, and $z_k(\pi(t^2)) \geq 0$ if $z_k(\pi(t^1)) > 0$. □ Q.E.D.

Theorem 4.9. *Let a total excess demand function z that satisfies Assumptions 3–7 be given and let $v \in S^n$ be a starting price system. Let $\bar{t} \in [0, 1]$ be given. Then there exists $\varepsilon > 0$ such that $\forall j \in I_{n+1}$:*

$\forall t \in (\bar{t} - \varepsilon, \bar{t}) \cap [0, 1]$, $\pi_j(t) < \pi_j(\bar{t})$ *if* $z_j(\pi(\bar{t})) > 0$ *and* $\pi_j(t) > \pi_j(\bar{t})$ *if* $z_j(\pi(\bar{t})) < 0$,

$\forall t \in (\bar{t}, \bar{t} + \varepsilon) \cap [0, 1]$, $\pi_j(t) > \pi_j(\bar{t})$ *if* $z_j(\pi(\bar{t})) > 0$ *and* $\pi_j(t) < \pi_j(\bar{t})$ *if* $z_j(\pi(\bar{t})) < 0$.

Proof. For $\lambda^* = 1$ the proof of Theorem 4.9 is trivial, so we consider the case $\lambda^* < 1$. By continuity of the functions z and π it is possible to choose $\varepsilon > 0$ such that $\forall j \in I_{n+1}$, $\forall t \in (\bar{t} - \varepsilon, \bar{t} + \varepsilon) \cap [0, 1]$, $z_j(\pi(t)) > 0$ if $z_j(\pi(\bar{t})) > 0$ and $z_j(\pi(t)) < 0$ if $z_j(\pi(\bar{t})) < 0$.

Let $t \in (\bar{t} - \varepsilon, \bar{t} + \varepsilon) \cap [0, 1]$ and $z_k(\pi(\bar{t})) < 0$. Then $\pi_k(t) = ((\lambda^* - 1)t + 1)v_k$. Hence if $t \neq \bar{t}$, then $(t - \bar{t})(\pi_k(t) - \pi_k(\bar{t})) = (\lambda^* - 1)(t - \bar{t})^2 v_k < 0$.

Let us consider the case where $k \in I_{n+1}$ is such that $z_k(\pi(\bar{t})) > 0$ and $t \in (\bar{t}, \bar{t} + \varepsilon) \cap [0, 1]$. Suppose $\pi_k(t) < \pi_k(\bar{t})$, so $\pi_k(\bar{t})/\pi_k(t) > 1$. Then $\pi_j(\bar{t})/\pi_j(t) > 1$, $\forall j \in I_{n+1}$ satisfying $z_j(\pi(t)) > 0$. Also $\pi_j(\bar{t})/\pi_j(t) > 1$, $\forall j \in I_{n+1}$ satisfying $z_j(\pi(t)) < 0$. Hence for some $j \in I_{n+1}$, $z_j(\pi(t)) = 0$, and $j \in J_{\min}(\pi(t), \pi(\bar{t}))$. By Lemma 4.1 it holds that $z_j(\pi(\bar{t})) > 0$, which contradicts the choice of ε. The case where $z_k(\pi(\bar{t})) > 0$ and $t \in (\bar{t} - \varepsilon, \bar{t}) \cap [0, 1]$ can be treated similarly.   □ Q.E.D.

Acknowledgements

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