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Note

A note on a consistency property for permutations

Jean Derks[a,*], Hans Peters[b]

[a] *Department of Mathematics, University of Maastricht, P.O. Box 616, 6200 MD Maastricht, The Netherlands*

[b] *Department of Economics, University of Maastricht, P.O. Box 616, 6200 MD Maastricht, The Netherlands*

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Abstract

In cooperative game theory allocation of earnings to players may take place on the basis of selectors or—more restrictively—consistent selectors, or on the basis of a permutation representing the queueing of the players. This note gives a graph theoretic characterization of those situations in which the latter allocation method results in allocation with consistent selectors.

Keywords: Selector; Permutation; Consistency; Cycle

1. Introduction

For a finite set $N=\{1,\dots,n\}$ we consider functions, the so-called selectors, assigning to each non-empty subset of N an element of that subset; formally, a *selector* is a function $\alpha: 2^N\backslash\{\emptyset\}\to N$ with $\alpha(S)\in S$ for every non-empty subset S of N.

In [2] the notion of selector is introduced in a game theoretic context, where the elements of N are the players, and a non-empty subset S of N is a coalition. In this context, a selector assigns a representative to each coalition. We will adopt the notation and terminology from the game theoretic literature since the problem that we tackle in this note is inspired by an application in this field.

A selector α is called *consistent* if the representative of any coalition S is also the representative of the subsets of S containing that representative; i.e., if S, $T\subseteq N$, and $T\subseteq S$ so that $\alpha(S)\in T$, then $\alpha(T)=\alpha(S)$.

* Corresponding author.

E-mail addresses: jean.derks@math.unimaas.nl (J. Derks), h.peters@ke.unimaas.nl (H. Peters).

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A selector is mainly used in allocation methods of the total earnings among the players. Assuming that these earnings (e.g., the so-called dividends of a cooperative game) are split up over the coalitions, allocation is simply performed by giving the earnings of each coalition to its representative. If the average is taken over all selectors or over all consistent selectors, this allocation method results in the well-known Shapley Value (cf. [3,1]).

A different approach towards allocation is the assignment of the earnings according to a specified queueing $\pi(1), \pi(2), \ldots, \pi(n)$ of the players, as follows: allocate to player $\pi(j)$ all positive earnings of those coalitions S with no member to the left of $\pi(j)$ (so, $\pi(j)$ is the first of the players in S), and all negative earnings of those coalitions S with only players to the left of $\pi(j)$ (i.e., $\pi(j)$ is the last player in S). In other words, if we let the players take their turn from right to left then each player, when it is his turn, may grab all positive earnings of coalitions containing him and not already taken, and he may pass on the negative earnings of coalitions as long as there are still players waiting in that coalition; if not he has to accept also these earnings. Allocations based on this method are called *greedy* for obvious reasons, and considered first in [1]; there it is shown that taking the average of all greedy allocations (one for each permutation π) again results in the Shapley value.

It is not hard to see that the greedy allocations correspond to allocations based on specific selectors. Moreover, in [1] it is shown that the convex hull of all greedy allocations is equal to the convex hull of all allocations based on arbitrary selectors, the so-called selectope. Also, the question is raised to characterize those situations where all greedy allocations correspond to consistent selectors. Because allocations based on consistent selectors are exactly the marginal values (cf. [4]) this question concerns the coincidence of the Weber set (i.e., the convex hull of the marginal values) and the selectope. The purpose of the present note is to give a (hyper)graph theoretic characterization of this coincidence.

2. The characterization result

For a permutation π denote the player in coalition S who has no predecessors in S by $\min_\pi(S)$ and denote the player in S for whom all other players in S are predecessors by $\max_\pi(S)$. Let $\mathscr{R}$ and $\mathscr{L}$ denote two disjoint subsets of coalitions. In the game theoretic context the elements of $\mathscr{R}$ are those coalitions with negative earnings, and $\mathscr{L}$ those with positive earnings.

We say that a selector α is an $(\mathscr{L},\mathscr{R})$-match for permutation π if

$$\alpha(S) = \max_\pi(S) \quad \text{for all } S \text{ with } S\in\mathscr{R}, \tag{1}$$

$$\alpha(S) = \min_\pi(S) \quad \text{for all } S \text{ with } S\in\mathscr{L}. \tag{2}$$

Observe that the representatives of the coalitions in $\mathscr{R}$ according to α are the Right-most players according to π, and the representatives of the coalitions in $\mathscr{L}$ according to α are the Left-most players according to π. Further, observe that an allocation on the basis of an $(\mathscr{L},\mathscr{R})$-match yields the same outcome as the greedy allocation method based on queueing according to π.

A reformulation of the question at the end of the Introduction is: For which pairs $(\mathscr{L},\mathscr{R})$ do all permutations have consistent $(\mathscr{L},\mathscr{R})$-matches?

Let $\mathscr{R}$ and $\mathscr{L}$ be disjoint sets of coalitions. A sequence $S_1,S_2,\dots,S_k$ of k different coalitions from $\mathscr{R}\cup\mathscr{L}$ is called an $(\mathscr{L},\mathscr{R})$-*cycle* if there are k different players $i_1,\dots,i_k$ such that

(i) for each $r=1,\dots,k$, it holds that $i_r \in S_r \cap S_{r+1}$ (where S_{k+1} denotes the coalition S_1);
(ii) at least one coalition is taken from $\mathscr{R}$ and at least one from $\mathscr{L}$.

$(\mathscr{L},\mathscr{R})$-cycle free collections are exactly those pairs that we are looking for:

Theorem. *Let $\mathscr{R}$ and $\mathscr{L}$ be disjoint collections of coalitions. The following two conditions are equivalent*:

(a) *There exists no $(\mathscr{L},\mathscr{R})$-cycle.*
(b) *For each permutation π of N there exists a consistent $(\mathscr{L},\mathscr{R})$-match.*

Proof. For the implication (b) $\Rightarrow$ (a), suppose there exists an $(\mathscr{L},\mathscr{R})$-cycle $S_1,S_2,\dots,S_k$. Since each $(\mathscr{L},\mathscr{R})$-cycle contains two consecutive coalitions, one in $\mathscr{L}$ and one in $\mathscr{R}$, we may assume without loss of generality that $S_1\in\mathscr{L}$ and $S_k\in\mathscr{R}$. Choose different players i_r in the intersection of S_r and S_{r+1}, for every $r=1,\dots,k$. One may also assume $i_r\notin S_\ell$ for all $r=1,\dots,k$ and $\ell\neq r,r+1$ because otherwise at least one of the two shorter cycles $S_\ell,S_{\ell+1},\dots,S_r$ and $S_\ell,S_r,S_{r+1},\dots,S_{\ell-1}$ is again an $(\mathscr{L},\mathscr{R})$-cycle, so that the proof can be restarted with the appropriate shorter cycle.

With these players $i_1,\dots,i_k$ define the following mapping π:

$$\pi(i_r)=\begin{cases} |\{j\colon\ j\geqslant r \text{ and } S_j\in\mathscr{L}\}| & \text{if } S_r\in\mathscr{L},\\ n-|\{j\colon\ j>r \text{ and } S_j\in\mathscr{R}\}| & \text{if } S_r\in\mathscr{R}.\end{cases}$$

Observe first that $\pi(i_r)\in N$ for all i_r because an $(\mathscr{L},\mathscr{R})$-cycle contains at most n coalitions. Let L denote $|\{j\colon\ j\in\mathscr{L}\}|$, and R denote $|\{j\colon\ j\in\mathscr{R}\}|$. With these notations we have $\pi(i_r)\leqslant L$ whenever $S_r\in\mathscr{L}$, and $\pi(i_r)>n-R$ whenever $S_r\in\mathscr{R}$. Further, $\pi(i_r)\neq\pi(i_l)$ whenever S_r, $S_l\in\mathscr{L}$ or S_r, $S_l\in\mathscr{R}$, so that π is injective. By extending π to N and by taking the positions of the other players arbitrarily among the remaining positions $L+1,\dots,n-R$, we obtain a permutation.

We will now show that the permutation π has the property $i_r=\min_\pi(S_r)$ for all $S_r\in\mathscr{L}$. For $S_r\in\mathscr{L}$ there are only two players among $i_1,\dots,i_k$ who are member of S_r, namely i_{r-1} and i_r (where the index 0 is taken to mean k). If $S_{r-1}\in\mathscr{L}$ then $\pi(i_{r-1})=\pi(i_r)+1$, and if $S_{r-1}\in\mathscr{R}$ then $\pi(i_{r-1})>n-R\geqslant L\geqslant\pi(i_r)$. Therefore, $\pi(i_{r-1})>\pi(i_r)$. The other players of S_r obtain positions between $L+1$ and $n-R$, so that we must have $i_r=\min_\pi(S_r)$.

The property $i_r=\max_\pi(S_r)$ for all $S_r\in\mathscr{R}$ can be shown similarly.

Now suppose that there exists a consistent selector α satisfying (1) and (2). So, by construction of π we have $\alpha(S_r)=i_r$ for each $r=1,\dots,k$. Consider the coalition

$S = \bigcup_{r=1}^{k} S_r$. Then $\alpha(S)$ is a member of one of the coalitions $S_1, \ldots, S_k$, say S_r. Consistency thus implies that $\alpha(S) = i_r$; therefore, $\alpha(S) \in S_{r+1}$, so that, again by consistency, $\alpha(S) = i_{r+1}$, a contradiction. This proves the implication (b) $\Rightarrow$ (a).

In order to prove the implication (a) $\Rightarrow$ (b), suppose there are no $(\mathscr{L}, \mathscr{R})$-cycles. Let π be an arbitrary permutation of the player set. Consider the directed graph (N, E) on N with edge set E equal to the union of the sets $\{(\max_\pi(S), j)\colon\ j \in S,\ S \in \mathscr{R}\}$, and $\{(\min_\pi(S), j)\colon\ j \in S,\ S \in \mathscr{L}\}$.

Suppose there exists a directed cycle in this graph, say $i_1, \ldots, i_k$. Then consider for each edge $(i_r,\ i_{r+1})$ in this cycle a coalition $S_r \in \mathscr{R} \cup \mathscr{L}$ such that $i_r,\ i_{r+1} \in S_r$, and $i_r = \max_\pi(S_r)$ if $S_r \in \mathscr{R}$ or $i_r = \min_\pi(S_r)$ if $S_r \in \mathscr{L}$. It is impossible that all these coalitions can be chosen from $\mathscr{R}$, because otherwise $\pi(i_1) > \pi(i_2) > \cdots > \pi(i_k) > \pi(i_1)$. Similarly, not all these coalitions can be chosen from $\mathscr{L}$. This implies that $S_1, \ldots, S_k$ is an $(\mathscr{L}, \mathscr{R})$-cycle, a contradiction. Hence, there are no directed cycles in (N, E). Therefore, there is a permutation $j_1, \ldots, j_n$, with the property that each edge (j_l, j_r) in E satisfies $l > r$. Consider the following selector α which chooses for each coalition S the right-most player:

$$\alpha(S) = j_r \quad \text{with } r = \max\{l\colon\ j_l \in S\},\ \ S \subseteq N.$$

This α is consistent. Now let $S \in \mathscr{R}$, and suppose that $\alpha(S) \neq \max_\pi(S)$. This implies $(j_l, j_r) \in E$ for $\max_\pi(S) = j_l$ and $\alpha(S) = j_r$, and, therefore, $l > r$; but this contradicts the definition of α. Therefore, α satisfies (1). Similarly one shows that α satisfies (2).

This concludes the proof of the Theorem. □

Clearly, no $(\mathscr{L}, \mathscr{R})$-cycles exist if the carriers of both collections have at most one player in common, that is, $|(\bigcup\{S\colon\ S \in \mathscr{R}\}) \cap (\bigcup\{S\colon\ S \in \mathscr{L}\})| \leqslant 1$.

There is no clear intuition of the existence or non-existence of a $(\mathscr{L}, \mathscr{R})$-cycle in the game theoretic context although the very appealing implication of the non-existence is that the set of selector-based allocations coincides with its subset of allocations coming from consistent selectors. For a detailed discussion the reader is referred to [1].

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