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Short communication

# The coordinate-wise core for multiple-type housing markets is second-best incentive compatible☆

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**Abstract**

We consider the generalization of Shapley and Scarf's (1974) [Shapley, L., Scarf's, H., 1974. On cores and indivisibility. Journal of Mathematical Economics 1, 23–37.] model of trading indivisible objects (houses) to so-called multiple-type housing markets. We show that the prominent solution for these markets, the coordinate-wise core rule, is second-best incentive compatible.


*JEL Classification:* D63; D70

*Keywords:* Coordinate-wise core; Indivisible goods; Second-best incentive compatibility; Strategy-proofness

## 1. Introduction

We consider the generalization of Shapley and Scarf's (1974) model of trading indivisible objects (houses) to so-called multiple-type housing markets. In Shapley and Scarf's (1974) housing markets each agent is endowed with an indivisible commodity: a house. Furthermore, each agent wishes to consume exactly one house and strictly ranks all houses in the market. Interestingly, one of the best-known solution concepts for barter economies can always be applied: the core for any housing market is non-empty (Scarf and Shapley, 1974). In addition, the core is always a singleton and it coincides with the unique competitive allocation (Roth and Postlewaite, 1977). Furthermore, the trading rule that assigns the unique core allocation for any housing market is *strategy-proof*, *i.e.*, no agent can benefit from misrepresenting his preferences (Roth, 1982). In addition, (Ma, 1994) demonstrated that the core rule is the unique trading rule satisfying *Pareto efficiency*, *strategy- proofness*, and *individual rationality*.

We consider an extension of Shapley and Scarf's (1974) housing markets – multiple-type housing markets – with several types of indivisible commodities, maybe houses and cars: each agent is endowed with an indivisible commodity of each type and wishes to consume exactly one commodity of each type.[1] Moulin (1995) introduced multiple-type housing markets, but Konishi et al. (2001) were the first to analyze the model. They demonstrate that when increasing the dimension of the model by adding other types of indivisible commodities, most of the positive results obtained for

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[1] A more realistic example of course assignments to students is described in Klaus (2006).

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the one-dimensional case disappear: even for additively separable[2] preferences the core may be empty and no *Pareto efficient*, *strategy-proof*, and *individually rational* trading rule exists. For separable preferences, Konishi et al. (2001) and Wako (2005) suggested an alternative solution to the core by first using separability to decompose a multiple-type housing market into "coordinate-wise submarkets" and second, determining the core in each submarket. Wako (2005) calls the resulting outcome the commodity-wise competitive allocation and shows that it is implementable in coalition-proof Nash equilibria. We call the rule that assigns the commodity-wise (unique) core allocation in each submarket the "coordinate-wise core rule." From its definition it follows easily that the coordinate-wise core rule satisfies *strategy-proofness* and *individual rationality*, but not *Pareto efficiency*. Miyagawa (2002) characterizes the coordinate-wise core rule by *citizen sovereignty*,[3] *strategy-proofness*, *individual rationality*, and *non-bossiness* [4]. Hence, in the absence of *Pareto efficient*, *strategy-proof*, and *individual rational* trading rules, the coordinate-wise core rule seems to be a good compromise.

In this article, we further promote the coordinate-wise core rule as a desirable solution for multiple-type housing markets. We do so by showing that the coordinate-wise core is second-best incentive compatible (Theorem 1). In other words, there exists no other *strategy-proof* trading rule that Pareto dominates the coordinate-wise core rule. Given that for multiple-type housing markets *Pareto efficiency*, *strategy-proofness*, and *individual rationality* are not compatible, we show that applying the coordinate-wise core rule is a minimal concession with respect to *Pareto efficiency* while preserving *strategy-proofness* and *individual rationality*.

## 2. Multiple-type housing markets and the coordinate-wise core

We mostly follow Miyagawa's (2002) model and notation of housing markets with multiple types. Let $N = \{1, \ldots, n\}$, $n \geq 2$, be the *set of agents*. There exist $\bar{\ell} \geq 1$ types of (distinct) indivisible objects. The *set of object types* is denoted by $L = \{1, \ldots, \bar{\ell}\}$ and each agent $i \in N$ is endowed with one object of each type $\ell \in L$, denoted by $i$. Thus, $N$ also denotes the set of objects of each type.

An allocation is a reallocation of objects among agents such that each agent again receives one object of each type. Formally, an *allocation* is a list $x = (x_i(\ell))_{i \in N, \ell \in L} \in N^{N \times L}$ such that

(i) each agent receives one object of each type, *i.e.*, for all $i \in N$ and all $\ell \in L$, $x_i(\ell) \in N$ denotes the object of type $\ell$ that agent $i$ consumes, *e.g.*, if $x_i(\ell) = j$, then agent $i$ receives agent $j$'s endowment of type $\ell$, and
(ii) no object of any type is assigned to more than one agent at allocation $x$. Thus, for all $\ell \in L$, $\cup_{i \in N}\{x_i(\ell)\} = N$.

Let $X$ denote the *set of allocations*. Given $x \in X$ and $\ell \in L$, $x(\ell) = (x_1(\ell), \ldots, x_n(\ell))$ denotes the allocation of type-$\ell$ objects. Given $x \in X$ and $i \in N$, $x_i = (x_i(1), \ldots, x_i(\bar{\ell}))$ denotes the list of objects that agent $i$ receives at allocation $x$. We call $x_i$ *agent i's (consumption) bundle*. Note that the set of bundles for each agent $i \in N$ can be denoted by $N^L$. We denote each *agent i's endowment* by $(i, \ldots, i) \in N^L$.

Each agent $i \in N$ has complete, transitive, and strict preferences $R_i$ over bundles, *i.e.*, $R_i$ is a linear order over $N^L$. Thus, for bundles $x_i, y_i \in N^L$, $x_i P_i y_i$ implies $x_i \neq y_i$ and $x_i I_i y_i$ implies $x_i = y_i$. In addition to being linear orders, we assume that preferences are separable: each agent $i \in N$ has complete, transitive, and strict *marginal preferences* $R_i(\ell)$ over the objects of each type $\ell$ and prefers consuming a bundle $x_i$ to a bundle $y_i$ if $x_i \neq y_i$ and all objects received at $x_i$ are (weakly) better than those received at $y_i$ according to the marginal preferences, *i.e.*, for all $\ell \in L$, $x_i(\ell) R_i(\ell) y_i(\ell)$. Formally, a preference relation $R_i$ over $N^L$ is *separable* if for all $\ell \in L$, there exists a linear order $R_i(\ell)$ defined over $N$, $P_i(\ell)$ being its strict part, such that for any two bundles $x_i, y_i \in N^L$, if for all $\ell \in L$, $x_i(\ell) R_i(\ell) y_i(\ell)$, and for some $\tilde{\ell}$, $x_i(\tilde{\ell}) P_i(\tilde{\ell}) y_i(\tilde{\ell})$, then $x_i P_i y_i$. By $\mathcal{R}$ we denote the *set of separable linear orders over* $N^L$. Since for all agents $i \in N$, $\mathcal{R}$ represents agent $i$'s set of preferences, by $\mathcal{R}^N = \times_{i \in N}\mathcal{R}$ we denote the *set of (preference) profiles*. Since the set of agents and their endowments remain fixed throughout, $\mathcal{R}^N$ also denotes the *set of multiple-type housing markets*. For $\bar{\ell} = 1$ our multiple-type housing market model equals the classical Shapley and Scarf (1974) housing market model.

[2] By separability, preferences between commodities of the same type do not depend on the consumption of commodities of different types. We formally introduce separable preferences in Section 2.
[3] No allocation is excluded from the range of the trading rule.
[4] No agent can influence another agent's final consumption without changing his final consumption.

A *(trading) rule* is a function $\varphi : \mathcal{R}^N \to X$ that assigns to each multiple-type housing market $R \in \mathcal{R}^N$ an allocation $\varphi(R) \in X$. By $\varphi_i(R)$ we denote the *bundle assigned by $\varphi$ to agent $i \in N$*.

Before we introduce our main rule, the coordinate-wise core rule, we need some notation. The set of all reallocations of objects among the members of coalition $S \subseteq N$ is denoted by

$$X_S = \{(x_i(\ell))_{i \in S, \ell \in L} \in N^{S \times L} : \text{for all } \ell \in L,\ \cup_{i \in S}\{x_i(\ell)\} = S\}.$$

Similarly, for $\ell \in L$ the set of all reallocations of type- $\ell$ objects among the members of coalition $S \subseteq N$ is denoted by

$$X_S(\ell) = \{(x_i(\ell))_{i \in S} \in N^S :\ \cup_{i \in S}\{x_i(\ell)\} = S\}.$$

Given $x \in X$ and $\ell \in L$, a *trading cycle* for $x(\ell)$ is a coalition $T \subseteq N$ such that

(i) agents in $T$ obtain their objects of type $\ell$ by reallocating their endowments of type $\ell$ among themselves, *i.e.*, $(x_i(\ell))_{i \in T} \in X_T(\ell)$ and
(ii) coalition $T$ is *minimal*, *i.e.*, there exists no $T' \subsetneq T$ such that $(x_i(\ell))_{i \in T'} \in X_{T'}(\ell)$.

Note that for all $x \in X$ and $\ell \in L$, there exists a partition $\{T_1, \ldots, T_m\}$ of $N$ such that for each $k \in \{1, \ldots, m\}$, $T_k$ is a trading cycle for $x(\ell)$.

An allocation is in the core if no coalition of agents can improve their welfare by reallocating their endowments among themselves. Formally, an allocation $x \in X$ is a (strict or strong) *core allocation* for the multiple-type housing market $R \in \mathcal{R}^N$ if there exist no coalition $S \subseteq N$ and no $y \in X_S$ such that for all $i \in S$, $y_i R_i x_i$, and for some $j \in S$, $y_j P_j x_j$.

For any housing market, the unique core allocation can easily be calculated by using the so-called top-trading algorithm (due to David Gale, see Shapley and Scarf (1974)). The *coordinate-wise core rule* $\varphi^{\text{cc}}$ assigns to each multiple-type housing market $R \in \mathcal{R}^N$ the unique coordinate-wise core allocation $\varphi^{\text{cc}}(R) \equiv x \in X$ that is obtained by separately calculating the core allocation $x(\ell)$ for each object type $\ell \in L$ in its associated marginal object type market, *e.g.*, by applying the top trading algorithm. Formally, for all $\ell \in L$, there exists no coalition $S \subseteq N$ and no $y(\ell) \in X_S(\ell)$, such that for all $i \in S$, $y_i(\ell) R_i(\ell) x_i(\ell)$, and for some $j \in S$, $y_j(\ell) P_j(\ell) x_j(\ell)$. For $\bar{\ell} = 1$ we call $\varphi^{\text{cc}}$ the core rule. A description of the well-known *top trading algorithm* and an illustrating example for the coordinate-wise core rule can be found in the working paper version of this note (Klaus, 2006).

## 3. Pareto efficiency, individual rationality, strategy-proofness, and second-best incentive compatibility

We now introduce and discuss some well-known properties for rules. First we consider an efficiency requirement.

**Pareto efficiency.** For all $R \in \mathcal{R}^N$ there exists no $y \in X$ such that for all $i \in N$, $y_i R_i \varphi_i(R)$, and for some $j \in N$, $y_j P_j \varphi_j(R)$.

Second, we formulate a voluntary participation condition: no agent receives a bundle that he considers worse than his endowment.

**Individual rationality.** For all $R \in \mathcal{R}^N$ and all $i \in N$, $\varphi_i(R) R_i (i, \ldots, i)$

Next, we discuss an incentive property: no agent ever benefits from misrepresenting his preference relation. In game theoretical terms, a rule is *strategy-proof* if in its associated direct revelation game form, it is a weakly dominant strategy for each agent to announce his true preference relation. Given $i \in N$, $R \in \mathcal{R}^N$, and $R'_i \in \mathcal{R}$, we denote by $(R'_i, R_{-i}) \in \mathcal{R}^N$ the new profile that is obtained from $R$ by replacing $R_i$ with $R'_i$.

**Strategy-proofness.** For all $R \in \mathcal{R}^N$, all $i \in N$, and all $R'_i \in \mathcal{R}$, $\varphi_i(R) R_i \varphi_i(R'_i, R_{-i})$.

Ma (1994) proved that for housing markets the core rule $\varphi^{\text{cc}}$ is the unique rule satisfying *Pareto efficiency*, *individual rationality*, and *strategy-proofness*. For multiple-type housing markets generally no *Pareto efficient*, *individually rational*, and *strategy-proof* rule exists (Konishi et al., 2001). Given this impossibility, Miyagawa (2002) demonstrated that by weakening *Pareto efficiency* and by strengthening *strategy-proofness* a characterization of the coordinate-wise core rule can be obtained: the coordinate-wise core rule $\varphi^{cc}$ is the unique rule satisfying *citizen sovereignty*[5], Individual rationality, and strong strategy-proofness[6] (see Miyagawa, 2002b Theorem 1). Wako (2005) considered a normal form game and showed that its unique coalition-proof equilibrium outcome equals the coordinate-wise core. Note that since in the top trading algorithm neither the names of the objects nor the names of the agents play any particular role, the coordinate-wise core also satisfies *neutrality* and *anonymity*. Thus, even though the coordinate-wise core rule $\varphi^{cc}$ is not *Pareto efficient*, it has many appealing properties. We prove another appealing property of the coordinate-wise core rule $\varphi^{cc}$: no other *strategy-proof* rule Pareto dominates $\varphi^{cc}$.

**Pareto domination of rules.** Rule $\psi$ *Pareto dominates* rule $\varphi$ if for all $R \in \mathcal{R}^N$ and all $i \in N$, $\psi_i(R) R_i \varphi_i(R)$ and for some $R' \in \mathcal{R}^N$ and $j \in N$, $\psi_j(R') P'_j \varphi_j(R')$.

**Second-best incentive compatibility.** If rule $\varphi$ is *strategy-proof* and no *strategy-proof* rule $\psi$ Pareto dominates rule $\varphi$, then $\varphi$ is *second-best incentive compatible*.

**Theorem 1.** *The coordinate-wise core rule $\varphi^{cc}$ is second-best incentive compatible.*

We use the following lemma in the proof of Theorem 1. Basically it states that if at some profile an allocation $y$ Pareto dominates the coordinate-wise core allocation $x$, then some agent who prefers $y$ to $x$ must receive in some marginal object type market an object at $y$ that, according to marginal preferences, is worse than the one received at $x$.

**Lemma 1.** *Let $R \in \mathcal{R}^N$ and $x \equiv \varphi^{cc}(R)$. Let $y \in X$ such that for all $i \in N$, $y_i R_i x_i$, and for some $j \in N$, $y_j P_j x_j$. Then, there exists $k \in N$ such that $y_k P_k x_k$* and for some $\ell \in L$, $x_k(\ell) P_k(\ell) y_k(\ell)$.

**Proof of Lemma 1.** Let $R \in \mathcal{R}^N$ and $x \equiv \varphi^{cc}(R)$. Let $y \in X$ such that for all $i \in N$, $y_i R_i x_i$, and for some $j \in N$, $y_j P_j x_j$. Since preferences are strict,

$$\text{for all } i \in N, \text{ either } y_i P_i x_I \text{ or } x_i = y_i. \tag{1}$$

Suppose, by contradiction, that no $k \in N$ exists such that $y_k P_k x_k$ and for some $\ell \in L$, $x_k(\ell) P_k(\ell) y_k(\ell)$. Hence, by (1), separability of preferences, and strictness of marginal preferences,

$$\text{for all } i \in N \text{ and all } \ell \in L, \text{ either } y_i(\ell) P_i(\ell) x_i(\ell) \text{ or } x_i(\ell) = y_i(\ell). \tag{2}$$

Since there exists $j \in N$ such that $y_j P_j x_j$, by (2) there exists a marginal object type market, *e.g.*, $\tilde{\ell} \in L$, such that

$$y_j(\tilde{\ell}) P_j(\tilde{\ell}) x_j(\tilde{\ell}) \text{ and for all } i \in N, \; y_i(\tilde{\ell}) R_i(\tilde{\ell}) x_i(\tilde{\ell}). \tag{3}$$

Thus, by (3) there exists a coalition of agents that can reallocate their endowments of type $\tilde{\ell}$ among themselves such that according to their marginal preferences for objects of type $\tilde{\ell}$, they are all weakly better off and at least one member of the coalition is strictly better off. Formally, there exists a coalition $S \subseteq N$ such that $z(\tilde{\ell}) \equiv (y_i(\tilde{\ell}))_{i \in S} \in X_S(\tilde{\ell})$ and

$$\text{for all } i \in S, \; z_i(\tilde{\ell}) R_i(\tilde{\ell}) x_i(\tilde{\ell}) \text{ and for some } k \in S, \; z_k(\tilde{\ell}) P_k(\tilde{\ell}) x_k(\tilde{\ell}). \tag{4}$$

Since $x \equiv \varphi^{cc}(R)$, (4) yields the required contradiction to the definition of the (coordinate-wise) core for the marginal object type market $\tilde{\ell}$. □

**Proof of Theorem 1.** Assume, by contradiction, that there exists a *strategy-proof* rule $\psi$ that Pareto dominates $\varphi^{cc}$. Recall that $\varphi^{cc}$ is *individually rational*. Thus, since $\psi$ Pareto dominates $\varphi^{cc}$, it is *individually rational* as well.

[5] A rule $\varphi$ satisfies citizen sovereignty if no allocation is excluded from the range of the trading rule, i.e., for all $x \in X$ there exists $R \in \mathcal{R}^N$ such that $\varphi(R) = x$.

[6] A rule $\varphi$ satisfies strong strategy-proofness if it is strategy-proof and non-bossy, i.e., for all $R \in \mathcal{R}^N$, $i \in N$, and $R'_i \in \mathcal{R}$, there exists no $S \subseteq N$ with $i \in S$ such that for all $j \in S$, $\varphi_j(R'_i, R_{-i}) R_j \varphi_j(R)$and for some $k \in S$, $\varphi_k(R'_i, R_{-i}) P_k \varphi_k(R)$.

In the following induction proof we transform agents' preferences such that agents prefer fewer and fewer object types to their respective object type endowments. For $R \in \mathcal{R}^N$ and $i \in N$, $B(R_i) \equiv \sum_{\ell \in L} |\{j \in N : j P_i(\ell) i\}|$ equals the total number of object types that agent $i$ prefers to his respective object type endowments. We denote the total number of object types that agents prefer to their respective object type endowments by $B(R) \equiv \sum_{i \in N} B(R_i)$.

**Induction Basis (Induction Step 0).** Since $\psi$ Pareto dominates $\varphi^{cc}$, for all $R \in \mathcal{R}^N$ and all $i \in N$, $\psi_i(R) R_i \varphi_i^{cc}(R)$ and for some $R^0 \in \mathcal{R}^N$ and $j \in N$, $\psi_j(R^0) P_j^0 \varphi_j^{cc}(R^0)$. To simplify notation let $x^0 \equiv \varphi^{cc}(R^0)$ and $y^0 \equiv \psi(R^0)$. By Lemma 1, there exists $j(0) \in N$ such that $y^0_{j(0)} P^0_{j(0)} x^0_{j(0)}$ and for some $\ell(0) \in L$, $x^0_{j(0)}(\ell(0)) P^0_{j(0)}(\ell(0)) y^0_{j(0)}(\ell(0))$. We now change agent $j(0)$'s preferences $R^0_{j(0)}$ to preferences $R^1_{j(0)} \in \mathcal{R}$ such that:

(i) According to $j(0)$'s new marginal preferences $R^1_{j(0)}(\ell)$ for any object type $\ell \in L$, $y^0_{j(0)}(\ell)$ is the best object of type $\ell$ and if it is different from agent $j(0)$'s endowment of type $\ell$, then (the endowment of type $\ell$) $j(0)$ is the second-best object of type $\ell$, *i.e.*, for all $\ell \in L$, $y^0_{j(0)}(\ell) R^1_{j(0)}(\ell) j(0)$ and for all $i \in N \setminus \{y^0_{j(0)}(\ell), j(0)\}$, $j(0) P^1_{j(0)}(\ell) i$.

(ii) Any commodity bundle $z_{j(0)} \in N^L$ that assigns an object of type $\ell \in L$ that does not equal $y^0_{j(0)}(\ell)$ or the endowment of type $\ell$ is worse than the endowment, *i.e.*, for all $z_{j(0)} \in N^L$ such that for some $\hat{\ell} \in L$, $z_{j(0)}(\hat{\ell}) \in N \setminus \{y^0_{j(0)}(\hat{\ell}), j(0)\}$, $(j(0), \ldots, j(0)) P^1_{j(0)} z_{j(0)}$.

Let $R^1 \equiv (R^1_{j(0)}, R^0_{-j(0)}) \in \mathcal{R}^N$. To simplify notation let $x^1 \equiv \varphi^{cc}(R^1)$ and $y^1 \equiv \psi(R^1)$. Note that at $R^1_{j(0)}$ agent $j(0)$'s best bundle equals $y^0_{j(0)}$. Thus, by *strategy-proofness* of $\psi$, $y^1_{j(0)} = y^0_{j(0)}$. By *strategy-proofness* of $\varphi^{cc}$, $x^1_{j(0)} \neq y^0_{j(0)}$. Hence, $x^1_{j(0)} \neq y^1_{j(0)}$. Then, since $\psi$ Pareto dominates $\varphi^{cc}$, $y^1_{j(0)} P^1_{j(0)} x^1_{j(0)}$ and for all $i \in N$, $y^1_i R_i x^1_i$. Thus, by Lemma 1, there exists $j(1) \in N$ such that $y^1_{j(1)} P^1_{j(1)} x^1_{j(1)}$ and for some $\ell(1) \in L$, $x^1_{j(1)}(\ell(1)) P^1_{j(1)}(\ell(1)) y^1_{j(1)}(\ell(1))$.

**Claim 1.** $j(1) \in N \setminus \{j(0)\}$

**Proof of Claim 1.** If $j(1) = j(0)$, then $x^1_{j(0)}(\ell(1)) P^1_{j(0)}(\ell(1)) y^1_{j(0)}(\ell(1))$ contradicts that according to $j(0)$'s new marginal preferences $R^1_{j(0)}(\ell(1))$, $y^1_{j(0)}(\ell(1))[= y^0_1(\ell(1))]$ is the best object of type $\ell(1)$. Hence, $j(1) \neq j(0)$. □

Define $N(1) \equiv |N \setminus \{j(0)\}|$. Note that $N(1) < |N|$.

**Induction Step k.** let $k \geq 1$ and $j(k) \in N$ such that $y^k_{j(k)} P^k_{j(k)} x^k_{j(k)}$ and for some $\ell(k) \in L$, $x^k_{j(k)}(\ell(k)) P^k_{j(k)}(\ell(k)) y^k_{j(k)}(\ell(k))$. We now change agent $j(k)$'s preferences $R^k_{j(k)}$ to preferences $R^{k+1}_{j(k)} \in \mathcal{R}$ such that:

(i) According to $j(k)$'s new marginal preferences $R^{k+1}_{j(k)}(\ell)$ for any object type $\ell \in L$, $y^k_{j(k)}(\ell)$ is the best object of type $\ell$ and if it is different from agent $j(k)$'s endowment of type $\ell$, then (the endowment of type $\ell$) $j(k)$ is the second-best object of type $\ell$, *i.e.*, for all $\ell \in L$, $y^k_{j(k)}(\ell) R^{k+1}_{j(k)}(\ell) j(k)$ and for all $i \in N \setminus \{y^k_{j(k)}(\ell), j(k)\}$, $j(k) P^{k+1}_{j(k)}(\ell) i$.

(ii) Any commodity bundle $z_{j(k)} \in N^L$ that assigns an object of type $\ell \in L$ that does not equal $y^k_{j(k)}(\ell)$ or the endowment of type $\ell$ is worse than the endowment, *i.e.*, for all $z_{j(k)} \in N^L$ such that for some $\hat{\ell} \in L$, $z_{j(k)}(\hat{\ell}) \in N \setminus \{y^k_{j(k)}(\hat{\ell}), j(k)\}$, $(j(k), \ldots, j(k)) P^{k+1}_{j(k)} z_{j(k)}$.

Let $R^{k+1} \equiv (R^{k+1}_{j(k)}, R^k_{-j(k)}) \in \mathcal{R}^N$. To simplify notation let $x^{k+1} \equiv \varphi^{cc}(R^{k+1})$ and $y^{k+1} \equiv \psi(R^{k+1})$. Note that at $R^k_{j(k)}$ agent $j(k)$'s best bundle equals $y^k_{j(k)}$. Similarly as before it follows that $y^{k+1}_{j(k)} = y^k_{j(k)}$, $x^{k+1}_{j(k)} \neq y^k_{j(k)}$, and $y^{k+1}_{j(k)} P^{k+1}_{j(k)} x^{k+1}_{j(k)}$. By Lemma 1, there exists $j(k+1) \in N$ such that $y^{k+1}_{j(k+1)} P^{k+1}_{j(k+1)} x^{k+1}_{j(k+1)}$ and for some $\ell(k+1) \in L$, $x^{k+1}_{j(k+1)}(\ell(k+1)) P^{k+1}_{j(k+1)}(\ell(k+1)) y^{k+1}_{j(k+1)}(\ell(k+1))$.

**Claim k+1.** $j(k+1) \in N \setminus \{j(0), \ldots, j(k)\}$ or $B(R^k_{j(k+1)}) > B(R^{k+1}_{j(k+1)})$

**Proof of Claim k+1.** Suppose that $j(k+1) \in \{j(0), \ldots, j(k)\}$, *i.e.*, for some $k \geq k' \geq 0$, $j(k+1) = j(k')$. Hence, $R^k_{j(k+1)}$ resulted from a previous transformation. Without loss of generality, at the end of Step $k'-1$, $R^k_{j(k+1)} = R^{k'}_{j(k+1)}$. Thus, $B(R^k_{j(k+1)}) = B(R^{k'}_{j(k+1)}) = \sum_{\ell \in L} |\{y^{k'}_{j(k+1)}(\ell) : y^{k'}_{j(k+1)}(\ell) \neq j(k+1)\}|$. By (*ii*), for all $\ell \in L$, $y^k_{j(k+1)}(\ell) \in \{y^{k'}_{j(k+1)}(\ell) : y^{k'}_{j(k+1)}(\ell) \neq j(k+1)\}$. Since for some $\ell(k+1) \in L$, $x^{k+1}_{j(k+1)}(\ell(k+1)) P^{k+1}_{j(k+1)}(\ell(k+1)) y^{k+1}_{j(k+1)}(\ell(k+1))$, $B(R^k_{j(k+1)}) > B(R^{k+1}_{j(k+1)})$. □

Define $N(k+1) \equiv |N \setminus \{j(0), \ldots, j(k)\}|$.

Note that at the end of each Induction Step $k$, $N(k) > N(k+1)$ or $B(R^k) > B(R^{k+1})$. Hence, after finitely many induction steps $\hat{k}$, $N(\hat{k}+1) = 0$ or $B(R^{\hat{k}+1}) = 0$. If $N(\hat{k}+1) = 0$, then in a contradiction to Lemma 1 no further agent $j(\hat{k}+1) \in N \setminus \{1, \ldots, j(\hat{k})\}$ exists at the end of Step $\hat{k}$. Hence, $B(R^{\hat{k}+1}) = 0$. Then, by *individual rationality*, for all $i \in N$, $x^{\hat{k}+1} = y^{\hat{k}+1}_i = (i, \ldots, i)$. However, at the end of Step $\hat{k}$, there exists $j(\hat{k}+1) \in N$ such that $x^{\hat{k}+1}_{j(\hat{k}+1)} \neq y^{\hat{k}+1}_{j(\hat{k}+1)}$; a contradiction. □

One can easily show that the coordinate-wise core is not the only second-best incentive compatible rule.

Consider the following slight variation of the coordinate-wise core. For simplicity assume that $\bar{\ell} = 1$. Fix two agents, without loss of generality, agents 1 and 2, with the specification that agent 2 can never receive agent 1's endowment when applying the top trading algorithm. Then, for any profile $R \in \mathcal{R}^N$ we calculate $\varphi^{1,2}(R)$ by applying the top trading algorithm with the extra specification that agent 2 is not allowed to point to agent 1 (for a formal definition see Klaus (2006)). Loosely speaking, the second-best incentive compatibility of $\varphi^{cc}$ and the fact that for many $R \in \mathcal{R}^N$ such that $\varphi^{1,2}(R) \neq \varphi^{cc}(R)$, agent 2's trade restriction benefits some other agent(s), imply that $\varphi^{1,2}$ is second-best incentive compatible. Note that $\varphi^{1,2}$ can easily be (coordinate-wise) extended to $\bar{\ell} > 1$. Clearly, $\varphi^{1,2}$ is *individually rational*, but neither *Pareto efficient* nor *anonymous*.

Another class of rules that are second-best incentive compatible because they are all *Pareto efficient* and *strategy-proof* (but not *individually rational*) are serial dictatorship rules: the first agent in a fixed order chooses his favorite bundle, then the second agent chooses his favorite bundle among the remaining feasible bundles, *etc.* In fact, also dictatorial rules where the choice of the next chooser may depend on previous choices, object type combinations previously chosen, identity of previous choosers, *etc.*, are *Pareto efficient* and *strategy-proof* and therefore second-best incentive compatible.

It is an open problem if, apart from the coordinate-wise core rule there are other *individually rational*, *strategy-proof*, *anonymous*, *neutral*, and second-best incentive compatible rules: the so-called top-trading rule where agents are only allowed to trade their (complete) endowments is *individually rational*, *strategy-proof*, *anonymous*, and *neutral*. We conclude with a conjecture that we could not yet verify.

**Conjecture:** The top-trading rule is second-best incentive compatible.

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