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Strategy-proof division with single-peaked preferences and individual endowments

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Abstract. We consider the problem of (re)allocating the total endowment of an infinitely divisible commodity among agents with single-peaked preferences and individual endowments. We propose an extension of the so-called uniform rule and show that it is the unique rule satisfying Pareto optimality, strategy-proofness, reversibility, and an equal-treatment condition. The resulting rule turns out to be peaks-only and individually rational: the allocation assigned by the rule depends only on the peaks of the preferences, and no agent is worse off than at his individual endowment.

1. Introduction

Consider the problem of allocating teaching hours among the members of a university department. It is reasonable to assume that preferences for teaching are single-peaked: each individual has an optimally preferred number of teaching hours, below which and above which preference is decreasing. The existing distribution of teaching hours may be unsatisfactory, for instance because preferences have changed over time. Then the question arises how to reallocate teaching hours.

The special instance of this problem in which only the total endowment plays a role (and, consequently, individual endowments are not modeled), has been studied extensively in economic literature. The allocation rule featuring pre-eminently in this literature is the so-called "uniform rule". This rule was already described as a strategy-proof rationing scheme by Benassy

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[3]. Sprumont [9] showed that it is the unique Pareto optimal, anonymous, and strategy-proof rule. As usual, strategy-proofness means that no agent can gain by misrepresenting his preferences. Anonymity implies that only the preferences and not the names of the agents matter. Ching [5] weakens anonymity to a condition called equal treatment of equals. Other characterizations of the uniform rule were obtained by Thomson ([10], [11], [12]), using monotonicity and consistency properties, Otten et al. [7], applying conditions from bargaining theory, and Angeles de Frutos and Massó [1] using a condition called Lorenz maximality.

In this paper we consider the more general setting where individual endowments do play a role (not just through the total endowment). There are several ways in which individual endowments may influence the allocation. An agent who, at the reallocation assigned by the rule to be used, is worse off than at his individual endowment, might refuse to participate in the reallocation operation if he has the chance; indeed, applying the uniform rule to a problem with individual endowments may lead to non-individually rational allocations. However, the procedure which underlies the uniform rule can be extended to an individually rational reallocation mechanism – the uniform reallocation rule.

As in the original allocation situation we are interested in properties like Pareto optimality, strategy-proofness, equal treatment of equals and anonymity (see Ching [5], Sprumont [9]). Whereas Pareto optimality, strategy-proofness and anonymity can easily be "translated" to the reallocation setting, the notion of equality has to be adjusted in order to formulate an equal-treatment property for reallocation rules. This is done – loosely speaking – by comparing the agents' net demands, i.e., the difference between the reported preference peaks and the individual endowments. If equal-treatment is imposed, then every agent is indifferent between his net trade and the net trade of an equal agent.

In the reallocation model that we consider in this study individual endowments and allotments are bounded from below by zero. This causes an asymmetry between excess demand and excess supply which does not exist in allocation problems. So, in order to "translate" results from allocation to reallocation situations it is not sufficient to only adjust the properties. Therefore, we impose a property of reversibility which describes symmetry between problems in excess demand and problems in excess supply. Suppose a problem is "reversed" in the sense that each agent's peak and individual endowment are interchanged, and each agent's preference relation is reflected in its peak and translated to its individual endowment. Then reversibility requires that only the signs of the net allotment change. If in addition, we impose Pareto optimality, equal-treatment and strategy-proofness with respect to the reported preferences – the individual endowments are assumed to be publicly known – we obtain a characterization of the uniform reallocation rule. In Klaus et al. [6] some variations of the reallocation model (e.g. allowing for debts) and their impact on the result are studied.

Besides in situations as the one at the beginning of this introduction, the uniform reallocation rule can be used in exchange economies with two goods and fixed prices, where rationing of one of the goods entails an allocation of the other good, in view of fixed prices and budgets. For more than two goods, one needs a multi-dimensional rule.

Another situation where uniform reallocation can be applied is studied in Barberà et al. [2]. There, situations are considered where agents have natural claims or are treated with different priorities, for example in investment situations. Proceeding from this situation they deviate from uniform division and introduce a class of Pareto optimal and strategy-proof allotment rules: the class of sequential allotment rules. Allowing for asymmetric treatment of the agents, the stepwise procedure of sequential allotment incorporates guaranteed levels of the agents' shares. Applying uniform division in the sequential allotment procedure, the uniform reallocation rule as defined in the allocation context with individual endowments is obtained.

The uniform reallocation rule can be seen as an extension of the uniform rule for problems without individual endowments, by starting from equal division of the amount to be divided in such problems. The proof of the characterization of the uniform reallocation rule is structured in a similar way as Ching's [5] elegant proof for the uniform rule.

The organization of the paper is as follows. In Section 2 we formulate the model and the uniform reallocation rule with its main properties. In Section 3 we state and prove the characterization of the uniform reallocation rule. Section 4 is devoted to some variations of the preference domain, and to showing independence of the axiom systems.

2. The uniform reallocation rule

Let $N = \{1, 2, \ldots, n\}$ denote the set of agents. Each agent $i \in N$ has a *single-peaked preference* on $\mathbb{R}$, i.e. a complete and transitive binary relation R_i on $\mathbb{R}$ for which there exists a point $p(R_i) \in \mathbb{R}_+$ with the following property: for all $\alpha, \beta \in \mathbb{R}$ with $\beta < \alpha \leq p(R_i)$ or $\beta > \alpha \geq p(R_i)$ we have $\alpha P_i \beta$, where P_i is the asymmetric part of R_i. As usual, $\alpha R_i \beta$ is interpreted as "α is weakly preferred to β", and $\alpha P_i \beta$ as "α is strictly preferred to β". The symmetric part of R_i is denoted by I_i: $\alpha I_i \beta$ means that individual i is indifferent between α and β. The point $p(R_i)$ is called the *peak* of R_i and will also be denoted by p_i. By $\mathscr{R}$ we denote the class of all single-peaked preferences. An element $R = (R_1, \ldots, R_n)$ of $\mathscr{R}^N$ is called a *preference profile*. Furthermore, each agent i has an *individual endowment* $e_i \in \mathbb{R}_+$. A *reallocation problem* (or briefly: *problem*) is a pair (e, R) where $e = (e_1, \ldots, e_n)$ is a vector of individual endowments and $R \in \mathscr{R}^N$ is a preference profile. By $E := \sum_{i \in N} e_i$ we denote the *total endowment* of a problem (e, R).

We say that the problem (e, R) is in *excess demand* if $\sum_{i=1}^n p(R_i) > E$. If $\sum_{i=1}^n p(R_i) = E$ the problem (e, R) is *balanced*. If $\sum_{i=1}^n p(R_i) < E$ the problem (e, R) is in *excess supply*.

In what follows it is useful to distinguish between suppliers and demanders in a reallocation problem (e, R). The set of *demanders* is defined as $D(e, R) := \{i \in N | p(R_i) > e_i\}$ and the set of *suppliers* as $S(e, R) := \{i \in N | p(R_i) \leq e_i\}$.[1]

For a problem (e, R), a (*feasible*) *allocation* or an *allotment* is a vector $x \in \mathbb{R}_+^N$ with $\sum_{i=1}^n x_i = E$. A *rule* φ is a map assigning to every problem (e, R) a feasible allocation $\varphi(e, R)$. For $i \in N$, $\varphi_i(e, R)$ denotes the *share* of agent i and $\Delta\varphi_i(e, R) := \varphi_i(e, R) - e_i$ denotes the *net allotment change* or *net trade* for agent i.

Let φ be a rule. We are interested in the following possible properties of φ.

A standard property which needs no further explanation is the following Pareto optimality condition.

Pareto optimality. For every problem (e, R) there is no (feasible) allocation x with $x_i R_i \varphi_i(e, R)$ for all $i \in N$ and $x_i P_i \varphi_i(e, R)$ for at least one $i \in N$.

Strategy-proofness ensures that no agent benefits from (strategically) misrepresenting his preference. So, if preferences are private information, in the game where each agent reports his preference it is a (weakly) dominant strategy to reveal one's true preference.

Strategy-proofness. For all $j \in N$ and all problems $(e, R), (e', R')$ with $e = e'$ and $R_i = R'_i$ for all $i \neq j$, we have $\varphi_j(e, R) R_j \varphi_j(e', R')$.

Equal-treatment requires that, if the individual endowments and preferences of two agents are equal up to a translation, then each agent should be indifferent between his own share and the translated share of the other agent.[2]

For $i \in N$, a preference $R_i \in \mathscr{R}$ and a number $\tau \in \mathbb{R}$ such that $p(R_i) + \tau \geq 0$, we define the *translated preference* $R_i + \tau$ by: for all $\alpha, \beta \in \mathbb{R}$

$$\alpha(R_i + \tau)\beta \quad \text{if} \quad (\alpha - \tau)R_i(\beta - \tau).$$

Equal-treatment. For all $i, j \in N$, every $\tau \in \mathbb{R}$, and every problem (e, R) with $R_j = R_i + \tau$ and $e_j = e_i + \tau$, we have $\varphi_j(e, R) I_j (\varphi_i(e, R) + \tau)$.

The equal-treatment property as described above is a natural extension of Ching's equal treatment of equals for division problems with total endowments.[3] Because net trades are compared it can also be seen as a weaker form of fairness as introduced in Schmeidler and Vind [8].

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[1] The inclusion of non-traders, i.e. agents with peaks equal to their individual endowments, among the suppliers is arbitrary, but convenient for what follows.

[2] The uniform reallocation rule defined below actually satisfies the stronger version where we would have equality instead of indifference.

[3] Equal-treatment applied on reallocation problems with equal individual endowments implies equal treatment of equals for the division of the total endowment.

With reversibility we introduce a notion of symmetry in the model. Suppose a problem of excess demand (respectively, supply) can be obtained by reversing – in a sense to be specified below – a problem of excess supply (respectively, demand). Then reversibility requires that also the net allotment changes be reversed. To formalize this condition we need the following notations.

Let $R_i \in \mathscr{R}$. Then $R_i^r \in \mathscr{R}$ is the *reflection* of R_i (in the peak $p(R_i)$), if for all $\alpha, \beta \in IR$

$$\alpha R_i^r \beta \quad \text{if} \quad (2p(R_i) - \alpha) R_i (2p(R_i) - \beta).$$

By $R^r := (R_1^r, \ldots, R_n^r)$ we denote the reflection of $R \in \mathscr{R}^N$.

Let (e, R) be a problem. The *reversed problem* of (e, R) with endowment vector $p(R)$ and preference profile $(R^r - (p(R) - e))$ is denoted by $\overline{(e, R)}$. So, in the reversed problem (a "reflection" of the original problem) the role of endowments and peaks is interchanged whereby each preference is reflected in the peak and translated from the peak to the endowment. By this all agents demand (supply) at (e, R) as much as they supply (demand) in the reversed problem. With the following condition we link the outcome of a problem to the outcome of the corresponding reversed problem.

Reversibility. For all $i \in N$ and every problem (e, R) and its reversed problem $\overline{(e, R)}$ we have $\Delta\varphi_i(e, R) = -\Delta_{\varphi_i}\overline{(e, R)}$.

Because all extensions of well-known division rules we consider here – the uniform, the proportional and the hierarchical reallocation rule – satisfy reversibility, the property does not seem too demanding. Nevertheless, it turns out that it is crucial when extending Ching's result to the reallocation setting.

A well-known rule, satisfying strategy-proofness and Pareto optimality (see Sprumont [9] or Ching [5]), is the *uniform rule* U defined[4] by

$$U_j(e, R) := \begin{cases} \min\{p(R_j), \lambda\} & \text{if} \quad \sum_{i=1}^n p(R_i) \geq E \\ \max\{p(R_j), \lambda\} & \text{if} \quad \sum_{i=1}^n p(R_i) \leq E \end{cases}$$

for every $j \in N$, where λ solves $\sum_{i=1}^n U_i(e, R) = E$. So, all agents either receive their optimal share $p(R_i)$ or a maximal (minimal) equal share λ in case of excess demand (supply). The uniform rule does not take the individual initial distribution of the total endowment into account; for instance, it does not satisfy equal-treatment. As an alternative, we propose the *uniform reallocation rule* U^r defined by

[4] This definition is adapted to our context. In the original literature only the total endowment is specified and no individual endowments.

$$U_j^r(e,R) := \begin{cases} \min\{p(R_j), e_j + \lambda\} & \text{if} \quad \sum_{i=1}^n p(R_i) \geq E \\ \max\{p(R_j), e_j - \lambda\} & \text{if} \quad \sum_{i=1}^n p(R_i) \leq E \end{cases}$$

for every $j \in N$, where $\lambda \geq 0$ and λ solves $\sum_{i=1}^n U_i^r(e,R) = E$. So, in case of excess demand the uniform reallocation rule works as follows. All suppliers receive their peaks. Next, the total resulting supply is distributed uniformly among the demanders (who already possess their individual endowments). In case of excess supply the uniform reallocation rule is dual to the excess demand case: Demanders are satiated and the total amount of the good they absorb is subtracted uniformly from the individual endowments of the suppliers. Observe that the uniform reallocation rule is well defined: it assigns a feasible allocation to every reallocation problem.

It is easy to see that the uniform reallocation rule is an extension of the uniform rule in a sense specified by the following lemma.

Lemma 1. *For every problem* (e, R) *applying the uniform rule gives the same result as applying the uniform reallocation rule from equal individual endowments, i.e.,*

$$U(e,R) = U^r(\tilde{e},R)$$

where $\tilde{e} = \left(\frac{E}{n}, \ldots, \frac{E}{n}\right)$.

Proof. Let (e,R) be a reallocation problem. Assume that $\sum_{i=1}^n p_i \geq E$, the other case is similar. For agents $i \in N$ such that $p_i \leq \frac{E}{n}$ we have $U_i^r(\tilde{e},R) = p_i = U_i(e,R)$. If $i \in N$ with $p_i > \frac{E}{n}$, then $U_i^r(\tilde{e},R) = \min\{p_i, \frac{E}{n} + \lambda\}$, where λ solves $\sum_{i=1}^n U_i^r(\tilde{e},R) = E$. Let $\lambda' := \frac{E}{n} + \lambda$. Then, $U_i^r(\tilde{e},R) = \min\{p_i, \lambda'\}$, where λ' solves $\sum_{i=1}^n U_i^r(\tilde{e},R) = E = \sum_{i=1}^n U_i(e,R)$. Hence, $U_i^r(\tilde{e},R) = U_i(e,R)$. ■

Let φ be a rule with, for every problem (e,R), either $\varphi_i(e,R) \leq p(R_i)$ for all $i \in N$ or $\varphi_i(e,R) \geq p(R_i)$ for all $i \in N$. We call such a rule **same-sided**. By single-peakedness of the preferences, it is easy to show that a same-sided rule is Pareto optimal, and that, conversely, every Pareto optimal rule must be same-sided. (Sprumont [9] actually uses same-sidedness as definition of Pareto optimality.) For later reference, we state this observation as a lemma.

Lemma 2. *A rule* φ *is Pareto optimal if and only if it is same-sided.*

3. The characterization result

This section is entirely devoted to the following characterization of the uniform reallocation rule.

Theorem 1. *The uniform reallocation rule is the unique rule satisfying Pareto optimality, strategy-proofness, reversibility, and equal-treatment.*

We start by showing that the uniform reallocation rule has all the properties mentioned in the theorem.

Proposition 1. *The uniform reallocation rule is Pareto optimal, strategy-proof, reversible, and satisfies equal-treatment.*

Proof. Pareto optimality of U^r follows immediately from same-sidedness, see Lemma 2. In order to show strategy-proofness of U^r, let $j \in N$ and let (e, R) and (e, R') be reallocation problems with $R_i = R_i'$ for all $i \neq j$. We have to prove that

$$U_j^r(e, R) R_j U_j^r(e, R'). \tag{1}$$

We assume that $\sum_{i=1}^n p_i > E$, the other case is similar. Then, $U_j^r(e, R) = \min\{p_j, e_j + \lambda\}$, where $\lambda \geq 0$ solves $\sum_{i=1}^n U_i^r(e, R) = E$. If $U_j^r(e, R) = p_j$, then (1) holds because p_j is the peak of R_j. Otherwise, $U_j^r(e, R) = e_j + \lambda$, i.e., agent j is a demander. We distinguish two cases.

Case 1. $p_j' > e_j + \lambda$

Observe that in the profile R' agent j is still a demander. Consequently, $D(e, R') = D(e, R)$ and by feasibility and same-sidedness,

$$\begin{aligned}
\sum_{i \in D(e,R')} (p_i' - e_i) = (p_j' - e_j) + \sum_{i \in D(e,R),\, i \neq j} (p_i - e_i) \\
> \lambda + \sum_{i \in D(e,R),\, i \neq j} (U_i^r(e, R) - e_i) \\
= \sum_{i \in D(e,R)} (U_i^r(e, R) - e_i) \\
= \sum_{i \in S(e,R)} (e_i - p_i) \\
= \sum_{i \in S(e,R')} (e_i - p_i').
\end{aligned}$$

Hence, $\sum_{i=1}^n p_i' > \sum_{i=1}^n e_i = E$. Therefore, $U_j^r(e, R') = \min\{p_j', e_j + \lambda'\}$, where $\lambda' \geq 0$ solves $\sum_{i=1}^n U_i^r(e, R') = E$. Because $p_j' > e_j + \lambda$ and $p_i' = p_i$ for $i \neq j$ it follows that $\lambda' = \lambda$. Hence, $U_j^r(e, R') = e_j + \lambda = U_j^r(e, R)$, and (1) follows.

Case 2. $p_j' \leq e_j + \lambda$

If $\sum_{i=1}^n p_i' > E$, then $U_j^r(e, R') = \min\{p_j', e_j + \lambda'\} \leq e_j + \lambda = U_j^r(e, R) \leq p_j$. If $\sum_{i=1}^n p_i' \leq E$, then $U_j^r(e, R') = \max\{p_j', e_j - \lambda'\} \leq e_j + \lambda = U_j^r(e, R) \leq p_j$. So in both cases (1) holds.

Next, we show the reversibility of U^r. Let (e, R) be a problem. We assume that $\sum_{i=1}^n p_i \geq E$, the other case is similar. Then, the reversed problem $\overline{(e, R)} = (p, R^r - (p - e))$ has the endowment vector $\bar{e} = p$, the peak vector $\bar{p} = e$ and it holds that $\sum_{i=1}^n \bar{p}_i = E \leq \sum_{i=1}^n p_i = \sum_{i=1}^n \bar{e}_i =: \bar{E}$. Let $i \in N$. By the definition of the uniform reallocation rule, we have

$$
\begin{aligned}
\Delta U_i^r\overline{(e,R)} &= U_i^r\overline{(e,R)} - \bar{e}_i \\
&= \max\{\bar{p}_i, \bar{e}_i - \lambda\} - \bar{e}_i,\ \lambda \geq 0 \text{ solves } \sum_{i=1}^{n} \max\{\bar{p}_i, \bar{e}_i - \lambda\} = \bar{E} \\
&= \max\{e_i, p_i - \lambda\} - p_i,\ \lambda \geq 0 \text{ solves } \sum_{i=1}^{n} \max\{e_i, p_i - \lambda\} = \sum_{i=1}^{n} p_i \\
&= \max\{e_i - p_i, -\lambda\},\ \lambda \geq 0 \text{ solves } \sum_{i=1}^{n} \max\{e_i - p_i, -\lambda\} = 0 \\
&= -\min\{p_i - e_i, \lambda\},\ \lambda \geq 0 \text{ solves } \sum_{i=1}^{n} \min\{p_i - e_i, \lambda\} = 0 \\
&= -(\min\{p_i, e_i + \lambda\} - e_i),\ \lambda \geq 0 \text{ solves } \sum_{i=1}^{n} \min\{p_i, e_i + \lambda\} = E \\
&= -\Delta U_i^r(e,R).
\end{aligned}
$$

Hence, U^r is reversible.
The proof of equal-treatment of U^r is straightforward and left to the reader. ■

The proof of the converse direction of Theorem 1 is structured in a similar way as Ching's [5] elegant proof for the uniform rule. We also show that Pareto optimality and strategy-proofness imply own-peak monotonicity and uncompromisingness.

The first property means that increasing the peak of an agent while leaving the remaining problem unchanged, does not decrease the amount allocated to that agent. The second condition implies that if an agent's peak differs from the share assigned by the rule, then his share does not change if his peak remains at the same side of the allocation. Ching's result and ours are, however, logically independent, and – except for the global structure – the proofs of the characterization results proceed rather differently.

Own-peak monotonicity. For every $j \in N$ and all problems (e,R) and (e',R') with $e = e', R_i = R'_i$ for all $i \neq j$ and $p(R'_j) \leq p(R_j)$:

$$\varphi_j(e',R') \leq \varphi_j(e,R).$$

Uncompromisingness. For every $j \in N$ and all problems (e,R) and (e',R') with $e = e', R_i = R'_i$ for all $i \neq j$: if $p(R_j) < \varphi_j(e,R)$ and $p(R'_j) \leq \varphi_j(e,R)$ or if $p(R_j) > \varphi_j(e,R)$ and $p(R'_j) \geq \varphi_j(e,R)$, then $\varphi_j(e,R) = \varphi_j(e',R')$.

Own-peak monotonicity was introduced by Ching [5]. Uncompromisingness is a well-known property in connection with strategy-proofness (see for instance Border and Jordan [4] in the context of public goods). Both properties are convenient to work with because they only use information concerning the peaks of the preferences. In the following two lemmas it is shown that both properties are implied by Pareto optimality and strategy-proofness.

Lemma 3. *Let φ be a Pareto optimal and strategy-proof rule. Then φ is own-peak monotonic.*

Proof. Let $j \in N$ and let (e, R) and (e, R') be reallocation problems with $R_i = R'_i$ for all $i \neq j$, and $p'_j \leq p_j$. We wish to show that $\varphi_j(e, R') \leq \varphi_j(e, R)$. Suppose that,

$$\varphi_j(e, R') > \varphi_j(e, R). \tag{2}$$

We derive a contradiction which completes the proof. By strategy-proofness it follows that

$$\varphi_j(e, R) R_j \varphi_j(e, R') \tag{3}$$

and

$$\varphi_j(e, R') R'_j \varphi_j(e, R). \tag{4}$$

This yields $p_j < \varphi_j(e, R')$ ((2) and (3)) and $p'_j > \varphi_j(e, R)$ ((2) and (4)). Thus,

$$\varphi_j(e, R) < p'_j \leq p_j < \varphi_j(e, R').$$

Now, because $p_j > \varphi_j(e, R)$, by same-sidedness it follows that

$$p_i \geq \varphi_i(e, R) \quad \text{for all} \quad i \in N.$$

Similarly, by $p'_j < \varphi_j(e, R')$ and same-sidedness

$$p'_i \leq \varphi_i(e, R') \quad \text{for all} \quad i \in N.$$

Because $p_i = p'_i$ for all $i \neq j$ we have

$$\varphi_i(e, R) \leq \varphi_i(e, R') \quad \text{for all} \quad i \neq j.$$

Furthermore, by assumption,

$$\varphi_j(e, R) < \varphi_j(e, R').$$

Hence,

$$E = \sum_{i=1}^{n} \varphi_i(e, R) < \sum_{i=1}^{n} \varphi_i(e, R') = E,$$

which is the desired contradiction. ■

An immediate but important consequence of own-peak monotonicity of a rule φ is that unilateral preference changes of an agent j with the same peak do not change j's share.

Individual peak-onliness. For every $j \in N, \varphi_j(e, R) = \varphi_j(e', R')$ whenever $e = e', R_i = R'_i$ for all $i \neq j$, and $p(R_j) = p(R'_j)$.

We proceed with the result concerning uncompromisingness.

Lemma 4. *Let φ be a Pareto optimal and strategy-proof rule. Then φ is uncompromising.*

Proof. Let $j \in N$ and let $(e, R), (e, R')$ be reallocation problems with $R_i = R'_i$ for all $i \neq j, p_j > \varphi_j(e, R)$, and $p'_j \geq \varphi_j(e, R)$ (the other case is similar). We wish to prove that $\varphi_j(e, R) = \varphi_j(e, R')$.

If $\varphi_j(e, R') < \varphi_j(e, R)$, then $\varphi_j(e, R) P'_j \varphi_j(e, R')$, violating strategy-proofness. Therefore, $\varphi_j(e, R') \geq \varphi_j(e, R)$.

Assume that R_j is a preference which is symmetric around its peak: because φ is own-peak monotonic by Lemma 3 and therefore individually peak-only, this is without loss of generality. If $\varphi_j(e, R') > \varphi_j(e, R)$, then $\varphi_j(e, R') \geq 2p_j - \varphi_j(e, R)$ because otherwise $\varphi_j(e, R') P_j \varphi_j(e, R)$, violating strategy-proofness. By Pareto optimality this implies $p'_j \geq \varphi_j(e, R') \geq 2p_j - \varphi_j(e, R)$. Hence, as long as $p'_j < 2p_j - \varphi_j(e, R)$ we have $\varphi_j(e, R') = \varphi_j(e, R)$. By repeating this argument, each time we double the range of peaks p'_j with $\varphi_j(e, R) = \varphi_j(e, R')$. This implies $\varphi_j(e, R) = \varphi_j(e, R')$ for all $p_j > \varphi_j(e, R)$. ■

Remark 1. The repetition argument in the proof of Lemma 4 can be avoided by taking a sufficiently asymmetric preference R_j instead (see Ching [5] Lemma 2). We deliberately used the above argument to be able to conclude later (see Remark 4) that Theorem 1 remains valid if only symmetric preferences are allowed.

Our next task is to prove the converse of Proposition 1.

Proposition 2. *Let φ be a rule satisfying Pareto optimality*, *strategy-proofness*, *reversibility*, *and equal-treatment. Then φ is the uniform reallocation rule U^r*.

Proof. Let (e, R) be an arbitrarily chosen reallocation problem. We have to show that $\varphi(e, R) = U^r(e, R)$.

Case 1. Assume that $\sum_{i=1}^{n} p_i \geq E$.

We assume that $\varphi(e, R) \neq U^r(e, R)$ and derive a contradiction. By Lemmas 3 and 4 both φ and U^r are own-peak monotonic and uncompromising.

Let $m \in \arg\max \{p_i - e_i | i \in D(e, R)\}$, and let $M(e, R) := \{i \in D(e, R) | R_i = R_m + \tau_i, \}$, where $\tau_i := e_i - e_m$. In words, agent m is an arbitrary but fixed agent with maximal demand, and $M(e,\ R)$ is the set of maximal demanders that have the same preferences as agent m up to a translation; so $M(e,\ R)$ contains at least agent m.

We say that $\Gamma(e, R)$ holds if the following two conditions are satisfied:

(i) For all $i \in S(e, R)$: $\varphi_i(e, R) = p_i$.
(ii) $D(e, R) = M(e, R)$.

Suppose $\Gamma(e, R)$ holds. Then, by definition, for all $i \in S(e, R)$: $U^r_i(e, R) = p_i$. With (i) and feasibility this implies

$$\sum_{i \in D(e,R)} U^r_i(e, R) = \sum_{i \in D(e,R)} \varphi_i(e, R).$$

By (ii) and equal-treatment therefore $U_i^r(e,R) = \varphi_i(e,R)$ for all $i \in D(e,R)$. Hence, $U^r(e,R) = \varphi(e,R)$, violating the assumption $\varphi(e,R) \neq U^r(e,R)$. This contradiction completes the proof for the case that $\Gamma(e,R)$ holds. Otherwise, we have the following claim:

Claim. If $\Gamma(e,R)$ does not hold, then there is a problem (e,R') satisfying the following two conditions:

(iii) $M(e,R') \supset M(e,R), M(e,R') \neq M(e,R)$.
(iv) $\varphi(e,R') \neq U^r(e,R')$.

We will prove this claim below. First observe that for (e,R') as in the claim $\Gamma(e,R')$ cannot be true, because otherwise (iv) would be violated. Hence, by repeated application of the claim we can find an infinite sequence of reallocation problems satisfying (iii) and (iv) but not both (i) and (ii). By (iii), however, the number of maximal demanders with the same preferences as m up to a translation increases at every step, an obvious impossibility since N is finite. So we have a contradiction.

We are left to prove the Claim. Suppose that $\Gamma(e,R)$ does not hold. We distinguish two cases.

Case 1.1. There exists a $k \in S(e,R)$ with $\varphi_k(e,R) \neq p_k$. Then, by same-sidedness and feasibility, $\varphi_k(e,R) < p_k$. Define $R'_i := R_i$ for all $i \neq k$ and $R'_k := R_m + \tau_k$. In other words, we turn agent k into an agent in $M(e,R')$, so that the number of maximal demanders that up to a translation have the same preferences as m, is increased. By uncompromisingness, $\varphi_k(e,R') = \varphi_k(e,R)$, hence $\varphi_k(e,R') < e_k \leq U_k^r(e,R')$. So also $\varphi(e,R') \neq U^r(e,R')$.

Case 1.2. For all $k \in S(e,R)$: $\varphi_k(e,R) = p_k$. Because $\Gamma(e,R)$ does not hold, we have $M(e,R) \neq D(e,R)$. Note that $A := \{i \in D(e,R) | \varphi_i(e,R) > U_i^r(e,R)\} \neq \emptyset$. We distinguish two subcases.

In the first subcase, there is a $j \in A$ with $j \notin M(e,R)$, i.e., $R_j \neq R_m + \tau_j$. Define R' by $R'_i := R_i$ for all $i \neq j$ and $R'_j := R_m + \tau_j$. Then $p'_j \geq p_j$, so that by own-peak monotonicity $\varphi_j(e,R') \geq \varphi_j(e,R) > U_j^r(e,R)$. By uncompromisingness, $U_j^r(e,R') = U_j^r(e,R)$. Hence $\varphi_j(e,R') > U_j^r(e,R')$, and in particular, $\varphi(e,R') \neq U^r(e,R')$.

In the second subcase, $A \subset M(e,R)$, i.e., there is no $j \in A$ with $R_j \neq R_m + \tau_j$. By feasibility:

$$\sum_{i \in D(e,R)} U_i^r(e,R) = E - \sum_{i \in S(e,R)} p_i = \sum_{i \in D(e,R)} \varphi_i(e,R).$$

Hence, there exists a $j \in D(e,R) \backslash M(e,R)$ such that $p_j \geq U_j^r(e,R) > \varphi_j(e,R)$. Again, define R' by $R'_i := R_i$ for all $i \neq j$ and $R'_j := R_m + \tau_j$. Uncompromisingness of φ implies $\varphi_j(e,R') = \varphi_j(e,R)$. Own-peak monotonicity of U^r implies $U_j^r(e,R) \leq U_j^r(e,R')$. Thus, $U_j^r(e,R') > \varphi_j(e,R')$, and in particular,

$\varphi(e,R') \neq U^r(e,R')$. In both subcases, (e,R') satisfies conditions (iii) and (iv). This completes the proof of the claim. Hence, $\varphi(e,R) = U^r(e,R)$ if $\sum_{i=1}^n p_i > E$.

Case 2. Assume that $\sum_{i=1}^n p_i \leq E$.

Let $\overline{(e,R)}$ be the reversed problem of (e,R). Denote the endowment vector of $\overline{(e,R)}$ by $\bar{e} = p$ and the peak vector by $\bar{p} = e$. Hence, $\sum_{i=1}^n \bar{p}_i = \sum_{i=1}^n e_i = E \geq \sum_{i=1}^n p_i = \sum_{i=1}^n \bar{e}_i =: \bar{E}$. Let $i \in N$. Applying Case 1 yields

$$\Delta\varphi_i\overline{(e,R)} = \Delta U_i^r\overline{(e,R)}.$$

By reversibility of φ and U^r it follows that $\Delta\varphi_i\overline{(e,R)} = -\Delta\varphi_i(e,R)$ and $\Delta U_i^r\overline{(e,R)} = -\Delta U_i^r(e,R)$. Hence, $\Delta\varphi_i(e,R) = \Delta U_i^r(e,R)$. So, $\varphi = U^r$. ■

Remark 2. The excess demand part of the proof of Proposition 2 does not use reversibility. This condition, however, is necessary (see Example 4 in Section 4) to prove the proposition for excess supply. Because the peaks of the agents have zero as lower bound, transforming a problem with excess supply into an excess supply problem with exclusively maximal suppliers is not always possible (Example 4). Consequently, for the excess supply case the proof technique used for the excess demand case above, cannot be applied. Reversibility compensates this asymmetry of the model between the excess demand and the excess supply case.

Proof of Theorem 1. Theorem 1 follows from Propositions 1 and 2. ■

Remark 3. Theorem 1 shows that Pareto optimality, strategy-proofness, reversibility and equal-treatment together imply **peak-onliness**, i.e., $\varphi(e,R) = \varphi(e',R')$ whenever $e = e'$ and $p(R_i) = p(R_i')$ for all $i \in N$.

4. Some remarks, and independence of the axioms

In this section we first make some observations concerning the domain of preferences and the domain of the reallocation problems. Second, we briefly discuss the connections between allocation and reallocation problems when translation invariance is imposed. Finally, we show the logical independence of the axioms in Theorem 1.

Remark 4. By going over the proofs – see also Remark 1–the reader may verify that our results, in particular Theorem 1, remain valid if the domain of single-peaked preferences is replaced by the much smaller domain of all single-peaked preferences which are symmetric around their peaks or that linearly depend only on the distance to the peaks. It is interesting to note that there is a trade off between the domain restriction and strategy-proofness: On the one hand the restriction of the preference domain weakens strategy-proofness (less preference profiles can be used in unilateral deviations). On the other hand by the domain restriction extra information about rules is implied, e.g., when preferences are symmetrical they are completely described by their peaks. Consequently, on this domain a rule is peak-only by definition.

Remark 5. If we extend the domain of the reallocation problems by allowing negative endowments and peaks, all results remain true with little changes in the proofs. In the characterization of the uniform reallocation rule in Theorem 1 we can then omit reversibility. For further model variations (e.g., a restriction of the preferences to the nonnegative real line) see Klaus et al. [6].

Remark 6.[5] Comparing our characterization of the uniform reallocation rule with Ching's characterization of the uniform allocation rule (Ching [5], Theorem 1), we observe that the equal-treatment condition we impose is not logically equivalent with Ching's equal treatment of equals. In addition, we impose a reversibility property which guarantees a symmetrical treatment of excess demand and excess supply problems. Now, imposing translation invariance – i.e., translating a problem along the nonnegative real line has no impact on net allotment changes – would compensate the asymmetry between excess demand and excess supply in the reallocation setting as well. Then, we can translate any reallocation problem to a problem where all agents have identical individual endowments. In this case, Pareto optimality, strategy-proofness and equal-treatment are logically equivalent to the corresponding properties for allocation problems and the uniform allocation equals the uniform reallocation (Lemma 1). As a consequence, Ching's result can be straightforwardly applied and by translation invariance, we obtain the uniform reallocation for the initial reallocation problem. With a similar argumentation, Sprumont's characterizations of the uniform rule by Pareto optimality, strategy-proofness and anonymity or envy-freeness respectively directly imply corresponding characterizations of the uniform reallocation rule.

Finally, we discuss logical independence of the axiom systems in Theorem 1. In this discussion we include consideration of the following property of a rule φ.

Individual rationality. For every problem (e, R) and every $i \in N$

$$\varphi_i(e, R) R_i e_i.$$

In words, no individual should be worse off than at his individual endowment. All examples below will be individually rational; this indicates that, in general, individual rationality cannot replace any of the axioms used in Theorems 1.

[5] This remark is due to an anonymous referee.

Example 1. The *endowment rule* φ^e satisfies individual rationality, strategy-proofness, reversibility, and equal-treatment, but not Pareto optimality. It is defined as follows. For every reallocation problem (e, R)

$$\varphi^e(e, R) = e.$$

Example 2. The *proportional rule* φ^p is Pareto optimal, individually rational, reversible and satisfies equal-treatment, but is not strategy-proof. The proportional rule satiates in case of excess demand (supply) all suppliers (demanders) and the demanders (suppliers) proportional to their net demand (supply). It is defined as follows. For a reallocation problem (e, R) with vector of peaks p:

- if $\sum_{i=1}^n p_i \geq E$, then

$$\varphi_i(e, R) := p_i \text{ if } i \in S(e, R);$$

allocate $S := \sum_{i \in S(e,R)} e_i - p_i$ among the agents in $D(e, R)$ by giving each agent $i \in D(e, R)$ the amount $e_i + \tau(p_i - e_i)$, where $0 \leq \tau \leq 1$ is determined by feasibility.

- if $\sum_{i=1}^n p_i \leq E$, $\varphi^p(e, R)$ is defined similarly.

Example 3. The *hierarchical rule* φ^h is Pareto optimal, individually rational, reversible and strategy-proof, but does not satisfy equal-treatment. The hierarchical rule satiates in case of excess demand (supply) all suppliers (demanders) and the demanders (suppliers) according to their number. It is defined as follows. For a reallocation problem (e, R) with vector of peaks p:

- if $\sum_{i=1}^n p_i \geq E$, then

$$\varphi_i(e, R) := p_i \text{ if } i \in S(e, R);$$

allocate $S := \sum_{i \in S(e,R)} e_i - p_i$ among the agents in $D(e, R)$ as follows: first serve the demander with the lowest number as well as possible; if there is something left, serve the agent in $D(e, R)$ with the second lowest number, etc.

- if $\sum_{i=1}^n p_i \leq E$, $\varphi^h(e, R)$ is defined similarly.

Example 4. The following rule $\hat{\varphi}$, defined for 3-person reallocation problems with set of agents $N = \{1, 2, 3\}$, is Pareto optimal, individually rational, strategy-proof and equally treating, but not reversible.

- Let (e, R) be in excess supply with: $D(e, R) = \{1\}, e_2 = \frac{p_1 - e_1}{2} =: M$. Then, $e_3 > e_2$ and $e_3 - p_3 > e_2 - p_2$.

For such problems we define $\hat{\varphi}$ as follows: First agent 3 may give as much of his endowment to agent 1 as he wishes,

$$\hat{\varphi}_3(e,R) = \max\{p_3, e_3 - 2M\}.$$

After this, agent 2 may hand in the remaining amount of the good to satiate agent 1. So,

$$\hat{\varphi}_2(e,R) = e_2 - (2M - (e_3 - \hat{\varphi}_3(e,R))),$$
$$\hat{\varphi}_1(e,R) = p_1.$$

- For all other reallocation problems, $\hat{\varphi}(e,R) = U^r(e,R)$.

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