

# Implementation of the Walrasian Correspondence: The Boundary Problem\*

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## Abstract

Consider exchange economies in which preferences are continuous, convex and strongly monotonic. It is well known that the Walrasian correspondence is not Nash implementable: Maskin monotonicity (Maskin, 1999) is violated for Walrasian allocations on the boundary of the feasible set. We derive an impossibility result showing that the Walrasian correspondence is in fact not implementable in any of the solution concepts considered in the implementation literature.

Next, imposing an additional domain restriction, we construct a sequential mechanism that doubly implements the Walrasian correspondence in subgame perfect and strong subgame perfect equilibrium. The mechanism is based on price-allocation announcements, and it fits the very description of Walrasian equilibrium. We thus take care of the boundary problem that was prominent in the Nash implementation literature.

*Keywords:* Walrasian equilibrium, implementability, justified sensitivity, double implementation, subgame perfect equilibrium, strong subgame perfect equilibrium.

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# 1 Introduction

The question of the *implementation* of the *Walrasian correspondence* has received much attention over the past decades. While the Walrasian correspondence satisfies desirable requirements –e.g. efficiency, individual rationality– its strategic properties are much less appealing. Hurwicz (1972) shows that agents would not find it in their interest to honestly report their preferences or demand functions to the so-called auctioneer: the Walrasian correspondence is not *strategy-proof*. The news is not better even if agents have complete information about one another and behave according to the Nash equilibrium criterion. Hurwicz, Maskin and Postlewaite (1995) show that *Maskin monotonicity* (Maskin, 1999; first version dated 1977), a necessary condition for implementation of a *social choice correspondence* (henceforth, SCC) in *Nash equilibrium*, is violated by the Walrasian correspondence for allocations that are at the boundary of the feasible set. The Walrasian correspondence is badly behaved because it also depends on the shape of preferences outside of the feasible set. This is what generates the so-called *boundary problem*.

Two early papers in the literature on Nash implementation of the Walrasian correspondence, Hurwicz (1979) and Schmeidler (1980), go around the violation of Maskin monotonicity by constructing *mechanisms* in which allocations obtained off the equilibrium path may award unfeasible bundles with negative quantities to some agents.<sup>1</sup>

Giraud and Rochon (2001) construct an alternative mechanism that respects feasibility but rules out Walrasian allocations on the boundary. Alternatively, the literature also paid attention to the Nash implementation of the constrained Walrasian correspondence –see for instance Postlewaite and Wettstein (1989), Dutta, Sen and Vohra (1995), Tian (1992, 2000), or Sotskov (2003).<sup>2</sup> Consistent with the notion of Walrasian equilibrium, in each of these papers, strategy sets include announcements of allocations and prices. In a different literature, papers on non-cooperative bargaining such as Gale

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<sup>1</sup>In Hurwicz (1979) and Schmeidler (1980), the consumption set of agents is implicitly  $\mathbb{R}^\ell$  instead of the usual non-negative orthant. Although a feasible bundle is always preferred by agents to one that gives negative quantities of some of the goods, the unfeasibility of some bundles off-the-equilibrium path allows to go around the violation of Maskin monotonicity on the boundary of the feasible set.

<sup>2</sup>The constrained Walrasian correspondence is a supercorrespondence of the Walrasian correspondence. Thomson (1999) shows that it is the minimal monotonic extension of the Walrasian correspondence. For domains in which preferences are convex and Walrasian allocations are interior, the two correspondence coincide.

(1986, a and b), or more recently Kunimoto and Serrano (2004) provide implementation in *subgame perfect equilibrium* of the Walrasian correspondence for economies with a continuum of agents. However, Walrasian allocations on the boundary of the feasible sets are also ruled out. Finally, Yildiz (2003) considers a bargaining procedure that covers only the two-player case, and uses assumptions such as uniqueness and interiority of Walrasian allocations. To the best of our knowledge, the papers on non-cooperative bargaining are the only one in this literature that use sequential mechanisms.

We know that the class of implementable SCCs considerably expands when one considers refinements of Nash equilibrium as solution concepts. For instance, in their seminal papers, Moore and Repullo (1988) (MR in the sequel) and Abreu and Sen (1990) (henceforth AS) show that Maskin monotonicity is no longer necessary for implementation in subgame perfect equilibrium. In MR, it is claimed that the Walrasian Correspondence is implementable in subgame perfect equilibrium when preferences are continuous, convex, and monotonic. In contradiction with their claim, we first show that, without further restrictions, the boundary problem generates an impossibility. Indeed, the Walrasian correspondence defined over this class of economies is in fact not implementable in any *responsive game theoretic solution concept* –and therefore not implementable in any of the solution concepts commonly studied in the literature.<sup>3</sup>

Next, we show that, by excluding non-differentiable preferences, the Walrasian correspondence is implementable in subgame perfect equilibrium.<sup>4</sup> We then proceed to solve the boundary problem. Because MR and AS construct a *canonical mechanism* for the entire class of SCCs that are implementable in subgame perfect equilibrium, we could simply rely on their construction.<sup>5</sup> Nevertheless, once a specific SCC of interest has been identified, this mechanism is not very appealing. It involves each agent reporting preference profiles, which are infinite-dimensional objects. Also, since this mechanism

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<sup>3</sup>See section 2.2 for a formal definition of a responsive game theoretic solution concept.

<sup>4</sup>Obviously, even with differentiability, the Walrasian correspondence is not Nash implementable as long as boundary allocations are not excluded.

<sup>5</sup>Suppose we are interested in implementation of SCCs in subgame perfect equilibrium. Loosely speaking, we say that a mechanism is canonical if it is constructed for the implementation of entire class of SCCs.

is not designed for any particular SCC, it does not capture the specific characteristics of the Walrasian correspondence.

However, for Nash implementation, Saijo (1988) and McKelvey (1989) show that the size of strategy sets in general mechanisms can be reduced. In particular, McKelvey (1989) shows that announcements of preference profiles are not necessary. While such reduction remains an open question for implementation in subgame perfect equilibrium, it is possible to follow a similar approach for specific SCCs and use tailor-made strategy sets –see for instance Serrano and Vohra (2002) who, in the context of exchange economies, provide a mechanism to implement the bargaining set in subgame perfect equilibrium.

Our position is that a mechanism is appealing if (i) in equilibrium, agents always get what they asked for, (ii) announcements of preference profiles are not part of the strategy sets of agents, and (iii) the mechanism corresponds closely to the description of the correspondence studied.

Our mechanism is simple, appealing, and it doubly implements the Walrasian correspondence in subgame perfect and strong subgame perfect equilibrium. Importantly, our construction is based on the notion of allocations, prices, and moves along price hyperplanes, which are central in the story behind the Walrasian correspondence. Recall that a pair composed of a feasible allocation and a price vector is a Walrasian equilibrium if each agent gets, at that allocation, the best bundle he can obtain in his budget set: no agent wants to “move” and get a different bundle on his budget hyperplane. This fundamental property is problematic for allocations that are on the boundary of the feasible set: moves along price hyperplanes can lead to bundles that are unfeasible. The mechanism we construct allows a better understanding of the strategic issues at stake when solving the boundary problem. Interestingly, in contrast to implementation in Nash equilibrium, a change in the property of a boundary allocation when going from one preference profile to another –from Walrasian to not being Walrasian– is revealed through differences in the upper contour sets at that allocation.

The plan of the paper is as follows. In Section 2, we present the set-up. In Section 3, we prove our impossibility result. In Section 4, we present the mechanism and the implementation result. We provide some final comments in Section 5.

## 2 The set-up

### 2.1 Economic environments

There is a set  $N \equiv \{1, \dots, n\}$ ,  $n \geq 2$ , of *agents*, and a set  $L \equiv \{1, \dots, \ell\}$  of *infinitely divisible goods*. For each  $i \in N$ , let  $\mathbb{R}_+^\ell$  be agent  $i$ 's *consumption set*. For each  $i \in N$ , let  $R_i$  be a complete and transitive binary relation on  $\mathbb{R}_+^\ell$  indicating (weak) *preferences*. Let the associated *strict preference* and *indifference* relations be  $P_i$  and  $I_i$ . For each  $i \in N$ , let  $\mathcal{R}_i$  be agent  $i$ 's set of *possible preferences*. Let  $\mathcal{R} = \prod_i \mathcal{R}_i$  be the set of *preference profiles*. A typical preference profile is a list  $R = (R_i)_{i \in N} \in \mathcal{R}$ . For each  $i \in N$ , let  $\omega_i \in \mathbb{R}_+^\ell \setminus \{0\}$  be  $i$ 's *individual endowment*. The *aggregate endowment* is  $\bar{\omega} = \sum \omega_i \gg 0$ .<sup>6</sup> Each  $i \in N$  is fully characterized by  $\mathcal{R}_i$  and  $\omega_i$ . The only characteristics of agents unknown to the planner are the preferences. An *economy* is thus a preference profile.

We consider the following two domains of economies.

$\mathcal{R}^C$  (*classical domain*): For each  $i \in N$ , each  $R_i \in \mathcal{R}_i^C$  is continuous, convex and strongly monotonic.<sup>7</sup>

$\mathcal{R}^D$  (*differentiable domain*): For each  $i \in N$ , each  $R_i \in \mathcal{R}_i^D$  is convex, strongly monotonic and representable by a differentiable utility function.

A *feasible allocation* is a list of bundles  $(x_i)_{i \in N} \in \mathbb{R}_+^{\ell n}$  such that  $\sum x_i \leq \bar{\omega}$ . Given  $i \in N$ , let  $x_{il} \in \mathbb{R}_+$  be the  $l$ -th coordinate of  $x_i$ .

Let  $A \equiv \{x \in \mathbb{R}_+^{\ell n} : \sum x_i \leq \bar{\omega}\}$  be the *set of feasible allocations*.

Let  $\partial A \equiv \{x \in A : \text{for some } i \in N \text{ and some } l, m \in L, x_{il} = \bar{\omega}_l, x_{im} > 0\}$  be the (*upper*) *boundary* of  $A$ . Likewise, let  $A^\circ = A \setminus \partial A \equiv \{x \in A : \text{for each } i \in N \text{ and } l \in L, \text{ either } 0 \leq x_{il} < \bar{\omega}_l, \text{ or } x_{il} = \bar{\omega}_l \text{ implies that for each } m \neq l, x_{im} = 0\}$  be the *interior* of  $A$ .<sup>8</sup>

For each  $i \in N$ , let  $A_i$ ,  $\partial A_i$ , and  $A_i^\circ$  be the *projections* of  $A$ ,  $\partial A$  and  $A^\circ$  onto agent  $i$ 's consumption set.

<sup>6</sup>We order vectors with the usual conventions,  $\gg, >, \geq$ .

<sup>7</sup>A preference relation  $R_i$  defined over  $\mathbb{R}_+^\ell$  is *convex* if, for each  $\{x_i, y_i\} \subset \mathbb{R}_+^\ell$  such that  $x_i P_i y_i$ , we have that for each  $\lambda \in (0, 1]$ ,  $\lambda x_i + (1 - \lambda)y_i P_i y_i$ .

A preference relation  $R_i$  defined over  $\mathbb{R}_+^\ell$  is *strongly monotonic* if, for each  $\{x_i, y_i\} \subset \mathbb{R}_+^\ell$ ,  $x_i > y_i$  implies that  $x_i P_i y_i$ .

<sup>8</sup>Notice that our definition of interiority is not strictly speaking the interior of  $A$  since some corners of the Edgeworth box may be included in  $A^\circ$ . We nevertheless stick to this terminology for convenience.

For each  $x \in \partial A$ , let  $J(x) \equiv \{i \in N : x_i \in \partial A_i\}$  be the set of agents who receive a bundle  $x_i$  on the boundary of the feasible set.

Let  $F \equiv \{x \in \mathbb{R}_+^{\ell n} : \sum x_i = \bar{\omega}\}$  be the *set of balanced allocations*.

A *price vector* is  $p \in \mathbb{R}_+^\ell$  such that  $\sum p_l = 1$ . Let  $\Delta \equiv \{p \in \mathbb{R}_+^\ell : \sum p_l = 1\}$  be the *price simplex* and  $\Delta_+ \equiv \{p \in \mathbb{R}_{++}^\ell : \sum p_l = 1\}$  the set of strictly positive price vector.

For each  $i \in N$ , each  $z_i \in A_i$ , and each  $p \in \Delta$ , let  $B_i(p, z_i) \equiv \{x_i \in \mathbb{R}_+^\ell : p \cdot x_i \leq p \cdot z_i\}$  be the *budget set* of agent  $i$  at price  $p$  and bundle  $z_i$ , and  $B_i(p, z_i) \cap A_i \equiv \{x_i \in A_i : p \cdot x_i \leq p \cdot z_i\}$  his *constrained budget set* at price  $p$  and bundle  $z_i$ .

For each  $x \in \partial A$ , each  $p \in \Delta_+$  and each  $i \in J(x)$ , let  $T_i(p, x) \equiv \{p' \in \Delta_+ \setminus \{p\} : \text{there exists } y_i \in A_i \setminus \{x_i\} \text{ such that } p' \cdot y_i = p' \cdot x_i \text{ and } p \cdot y_i > p \cdot \omega_i\}$  be the set of prices at which  $i$  can attain bundles  $y_i \in B_i(p', x_i) \cap A_i$  that are not affordable in  $B_i(p, \omega_i) \cap A_i$ .<sup>9</sup>

For each  $i \in N$ , each  $R_i \in \mathcal{R}_i$ , and each  $x_i \in \mathbb{R}_+^\ell$ , let  $UC(R_i, x_i) \equiv \{y_i \in \mathbb{R}_+^\ell : y_i R_i x_i\}$  be the *upper contour set* of  $R_i$  at  $x_i$ ;  $LC(R_i, x_i) \equiv \{y_i \in \mathbb{R}_+^\ell : x_i R_i y_i\}$  be the *lower contour set* of  $R_i$  at  $x_i$ ;  $SUC(R_i, x_i) \equiv \{y_i \in \mathbb{R}_+^\ell : y_i P_i x_i\}$  be the *strict upper contour set* of  $R_i$  at  $x_i$ ; and  $IC(R_i, x_i) \equiv \{y_i \in \mathbb{R}_+^\ell : x_i I_i y_i\}$  be the *indifference curve* of  $R_i$  through  $x_i$ .

Given  $R \in \mathcal{R}$ ,  $x \in A$  is a *Walrasian allocation* for  $R$  if there is  $p \in \Delta$  such that for each  $i \in N$ ,  $x_i \in B_i(p, \omega_i)$ , and for each  $y_i \in B_i(p, \omega_i)$ ,  $x_i R_i y_i$ . For each  $R \in \mathcal{R}$ , let  $W(R)$  be the set of these allocations.

Likewise, Given  $R \in \mathcal{R}$ ,  $(x, p) \in A \times \Delta$  is a *Walrasian equilibrium* for  $R$  if for each  $i \in N$ ,  $x_i \in B_i(p, \omega_i)$ , and for each  $y_i \in B_i(p, \omega_i)$ ,  $x_i R_i y_i$ .<sup>10</sup> For each  $R \in \mathcal{R}$ , let  $WE(R)$  be the set of these allocation-price pairs.

A *social choice correspondence* is a mapping  $f : \mathcal{R} \rightarrow A$  that associates to each preference profile a non-empty subset of feasible allocations.

The *Walrasian correspondence*  $W : \mathcal{R} \rightarrow A$  associates to each economy  $R \in \mathcal{R}$  its set of Walrasian allocations  $W(R)$ .

To conclude, we recall the definition of *Maskin monotonicity* and the necessary *condition C* for subgame perfect implementation introduced in MR.

<sup>9</sup>This definition will be explained in Section 4 and Figure 3.

<sup>10</sup>Notice that, both in  $\mathcal{R}^C$  and  $\mathcal{R}^D$ , preferences are strongly montonic. Thus, for each Walrasian equilibrium  $(x, p)$ , we have  $p \in \Delta_+$ .

**Maskin monotonicity:** For each  $\{R, R'\} \subset \mathcal{R}$  and each  $a \in f(R)$ ,

$$[\text{For each } i \in N, LC(R_i, a_i) \cap A_i \subseteq LC(R'_i, a_i) \cap A_i] \implies [a \in f(R')].$$

**Condition C:** For each  $\{R, R'\} \subset \mathcal{R}$  and each  $a \in f(R) \setminus f(R')$ , there exists a finite sequence  $(a^l)_{l=1}^{k+1}$  of elements of  $A$  such that,

a) For each  $l = 0, \dots, k - 1$ , there exists  $j \in N$  for whom,

$$a_j^l R_j a_j^{l+1}.$$

b) There is  $j \in N$  for whom  $R_j \neq R'_j$  and,

$$a_j^k R_j a_j^{k+1} \text{ and } a_j^{k+1} P'_j a_j^k.$$

## 2.2 Game-form: definitions and notation

Let  $T$  be a *game tree*. Let  $S$  be the set of *nodes* of  $T$ ,  $s_0$  the *initial node*, and  $Z$  the set of *terminal nodes*. For each  $i \in N$ , let  $M_i$  be the set of (*pure*) *strategies*, and for each  $s \in S$ , let  $M_i^s$  be the set of strategies available to  $i$  at node  $s \in S$ . Let  $M \equiv \prod_i M_i$  be the set of *strategy profiles*. As is common in the implementation literature, we confine our attention to pure strategies. Let  $g$ , the *outcome function*, be a function that associates a feasible allocation with each path of play. An extensive *mechanism*  $\Gamma \equiv (N, T, g)$  –or extensive game form– is a game with possibly simultaneous moves.<sup>11</sup>

Let  $m \in M$ . Let  $g(m)$  be the allocation prescribed by the path induced by  $m$ , and  $g_i(m)$  the  $i$ -th component of  $g(m)$ . Let  $g(m, s)$  be the allocation corresponding to  $m$  starting at node  $s$ . Let  $\mathcal{G}$  be the *set of mechanisms*.

Let  $\Gamma \in \mathcal{G}$ . Given  $R \in \mathcal{R}$ ,  $\Gamma$  defines a *non-cooperative game* in extensive form  $(\Gamma, R)$ . For each  $R \in \mathcal{R}$ , a *game-theoretic solution concept*  $E$  (henceforth solution concept) describes a set of predictions on how  $\Gamma$  will be played, as a function of the agents' preferences. It is a mapping  $E : \mathcal{G} \times \mathcal{R} \rightarrow 2^M$ . For each  $R \in \mathcal{R}$ , let  $E(\Gamma, R)$  and  $E\mathcal{O}(\Gamma, R)$  be the set of  $E$ -*equilibrium* and  $E$ -*equilibrium outcomes* of  $(\Gamma, R)$ . The definition of solution concepts is indeed very broad and encompasses many concepts in which agents do not behave

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<sup>11</sup>The definition we give encompasses both static and sequential mechanisms: a static mechanism can always be represented via an extensive mechanism with simultaneous moves.

“strategically”. In order to narrow the definition, we introduce the minimal requirement on a solution concept.

**Responsiveness:**  $E$  is *responsive* with respect to  $\Gamma \in \mathcal{G}$  if for each  $\{R, R'\} \subset \mathcal{R}$  and each  $x \in EO(\Gamma, R) \setminus EO(\Gamma, R')$ , there exist  $i \in N$  and  $\{y_i, z_i\} \subset A_i$  such that,

$$y_i R_i z_i \text{ and } z_i P'_i y_i.$$

Observe that all the solution concepts commonly studied in the implementation literature –such as Nash equilibrium and any of its refinements– are responsive.

We define next the solution concepts that we consider in the paper. Let  $R \in \mathcal{R}$  and  $\Gamma \in \mathcal{G}$ .

A *subgame perfect equilibrium* of  $(\Gamma, R)$  is  $m^* \in M$  such that for each  $s \in S \setminus Z$ , each  $i \in N$ , and each  $m_i \in M_i$ ,

$$g_i(m^*, s) R_i g_i(m_i, m_{-i}^*, s).$$

Let  $SPE(\Gamma, R)$  and  $SPEO(\Gamma, R)$  be the set of subgame perfect equilibrium and subgame perfect equilibrium outcomes of  $(\Gamma, R)$ .

A *strong equilibrium* of  $(\Gamma, R)$  is  $m^* \in M$  such that for each  $H \subseteq N$ , each  $m' \neq m$  with, for each  $i \in N \setminus H$ ,  $m_i = m'_i$ , then there is  $j \in H$  for whom,

$$g_i(m^*) R_i g_i(m').$$

A *strong subgame perfect equilibrium* of  $(\Gamma, R)$  is  $m^* \in M$  such that for each proper subgame,  $m^*$  is a strong equilibrium in that subgame. Let  $SSPE(\Gamma, R)$  and  $SSPEO(\Gamma, R)$  be the set of strong subgame perfect equilibrium and strong subgame perfect equilibrium outcomes of  $(\Gamma, R)$ .

A SCC  $f$  is *implemented* by  $\Gamma$  in  $E$  if for each  $R \in \mathcal{R}$ , we have  $EO(\Gamma, R) = f(R)$ .

A SCC  $f$  is *doubly implemented* by  $\Gamma$  in subgame perfect and strong subgame perfect equilibrium if for each  $R \in \mathcal{R}$  we have,  $SSPEO(\Gamma, R) = SPEO(\Gamma, R) = f(R)$ .

We define next the more general notion of *implementability* of a SCC  $f$ .



**Implementability:** There exist  $\Gamma$  and a solution concept  $E$ , responsive with respect to  $\Gamma$ , such that  $f$  is implemented by  $\Gamma$  in  $E$ .

We need to introduce one last piece of notation. For each  $i \in N$ , let  $\epsilon_i \in \mathbb{R}_+^\ell \setminus \{0\}$  be such that  $\omega_i - \epsilon_i \in \mathbb{R}_+^\ell \setminus \{0\}$ . Such an  $\epsilon_i$  exists since we assumed that for each  $i \in N$ ,  $\omega_i > 0$ . Let  $\mathcal{P}$  be the set of one-to-one functions –*permutations*– from  $N$  into itself, and let  $\mathcal{P}^n$  be the set of *permutation profiles*. For each  $i \in N$ , let  $\pi^i \in \mathcal{P}$  be a permutation and  $\pi \equiv (\pi^i)_{i \in N}$ . For each  $\pi \in \mathcal{P}^n$ , let  $f(\pi) \equiv \pi^1(\pi^2(\dots(\pi^i \dots(\pi^n))\dots))$  be the *ordered composition of the permutations* in  $\pi$ . We call  $f(\pi)$  a *protocol*. For each  $\pi \in \mathcal{P}^n$ , let  $f_i(\pi)$  be the *i-th* agent in  $f(\pi)$  –i.e.  $f_1(\pi)$  is the first agent,  $f_n(\pi)$  the last agent in the protocol etc.

For each  $i \in N$  and each  $\pi \in \mathcal{P}^n$ , we use the notation  $(\pi^i, \pi^{-i})$  to indicate that  $i$  uses  $\pi^i$ , instead of  $\pi^i$ , while each  $j \in N \setminus \{i\}$  uses  $\pi^j$ . Since we confine our attention to pure strategies, notice that, for each  $i \in N$  and each  $\{\pi, \pi^*\} \subset \mathcal{P}^n$ , there exists  $\pi^i \neq \pi^i$  such that  $f(\pi^i, \pi^{-i}) = f(\pi^*)$ . Permutations will be used as components of strategy sets and will play a role similar to integer devices found in many of the mechanisms used in the literature. However, unlike integer devices,  $\mathcal{P}$  is a finite set which allows us to have finite strategy sets in the mechanism we construct in Section 4.<sup>12</sup>

Permutations capture here an idea of anonymity of the mechanism – equilibria should be independent of protocols– and work as a “king-maker” process because protocols will determine who will lead the game if it goes beyond Stage 1 of our mechanism. A similar device was used in Serrano and Vohra (1997). Permutations were used first as components of strategy spaces in the context of implementation, but in a different fashion, in Thomson (2005; first version dated 1995).

### 3 An impossibility Result

Given any responsive solution concept, *implementability* of a SCC requires the following property to be satisfied.

**Justified sensitivity:** For each  $\{R, R'\} \subset \mathcal{R}$  and each  $a \in f(R) \setminus f(R')$ , there exist  $i \in N$  and  $\{x_i, y_i\} \subset A_i$  such that,

$$x_i R_i y_i \text{ and } y_i P_i' x_i.$$

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<sup>12</sup>See Jackson (1992) for a criticism of integer games.

*Justified sensitivity* is a basic necessary condition for a SCC to be implemented in some responsive solution concept.<sup>13</sup> Suppose there exist  $\{R, R'\} \subset \mathcal{R}$  and  $z \in A$  such that  $z \in f(R) \setminus f(R')$ . If  $f$  satisfies *implementability*, there exist a mechanism  $\Gamma$  and a solution concept  $E$ , responsive with respect to  $\Gamma$ , such that  $f$  is implemented by  $\Gamma$  in  $E$ . Precisely, there exists  $m \in E(\Gamma, R)$  that gives  $g(m) = z$ . Assume that there does not exist  $i \in N$  and  $\{x_i, y_i\} \subset A_i$  such that  $x_i R_i y_i$  and  $y_i P'_i x_i$ . Since  $f$  is implemented by  $\Gamma$  in  $E$ , then  $z \notin E\mathcal{O}(\Gamma, R')$ . But this is in contradiction with  $E$  being responsive. In fact,  $f$  cannot be implemented by  $\Gamma$  in  $E$ : it is not possible to check information on allocations that are not feasible.<sup>14</sup>

When the domain of preference profiles is  $\mathcal{R}^C$ , the Walrasian correspondence violates *justified sensitivity*, and hence violates *implementability*.<sup>15</sup> The problem we underline here also applies to public good economies and the Lindhal correspondence. Proposition 1 below can be adapted to that case.

**Proposition 1:** *Suppose the domain is  $\mathcal{R}^C$  and  $n \geq 2$ . Then the Walrasian correspondence violates implementability.*

*Proof:* Let  $\ell = n = 2$ . Let  $\mathcal{R} \subset \mathcal{R}^C$ ,  $\mathcal{R}_1 = \{R_1, R'_1\}$ ,  $\mathcal{R}_2 = \{R_2\}$ ,  $R = (R_1, R_2)$ ,  $R' = (R'_1, R_2)$  with  $R'_2 = R_2$  and  $W : \mathcal{R} \rightarrow A$ . Let  $R_1$ ,  $R'_1$  and  $R_2$  be respectively represented by the following functions:

$$u_1(x_1, y_1) = \min\{x_1 + y_1; x_1 + \frac{1}{3}y_1 + \frac{8}{3}\}, \quad u'_1(x_1, y_1) = x_1 + y_1, \quad \text{and} \\ u_2(x_2, y_2; R_2) = 2x_2 + y_2.$$

The individual endowments are  $\omega_1 = \omega_2 = (2, 2)$ .

Let  $z^* = ((1, 4); (3, 0))$  and  $p^* = (\frac{2}{3}, \frac{1}{3})$ . Observe that  $z^* \in W(R)$  with supporting price  $p^*$ . The situation is depicted in Figure 1.

<sup>13</sup>The reader may have noticed that a preference change from  $R$  to  $R'$  could also simply imply the existence of  $i \in N$  and  $\{x_i, y_i\} \subset A_i$  such that  $x_i P_i y_i$  and  $y_i I'_i x_i$ . By continuity and strong monotonicity of preferences, this implies that there exists  $\{w_i, z_i\} \subset A_i$  such that  $w_i R_i z_i$  and  $z_i P'_i w_i$ . Therefore, whenever  $R \neq R'$ , the structure of the model and the assumptions on preferences guarantee that, if preferences change within  $A$ , there always exists a “real” preference reversal in  $A$ .

<sup>14</sup>Therefore, when a SCC  $f$  violates *justified sensitivity*, the only hope to have  $f$  implemented is to use some non-responsive solution concepts. For instance, let *Truth* be the *truthtelling solution concept* –i.e. agents always report truthfully– and  $\Gamma$  be the direct mechanism in which  $g = f$  and, for each  $i \in N$ ,  $M_i = \mathcal{R}_i$ . Trivially, for each  $R \in \mathcal{R}$ ,  $\text{Truth}\mathcal{O}(\Gamma, R) = f(R)$ .

<sup>15</sup>The violation of *justified sensitivity* implies that the Walrasian correspondence cannot be virtually implemented in any responsive solution concept. See Abreu and Sen (1991) for a definition of virtual Nash implementation.

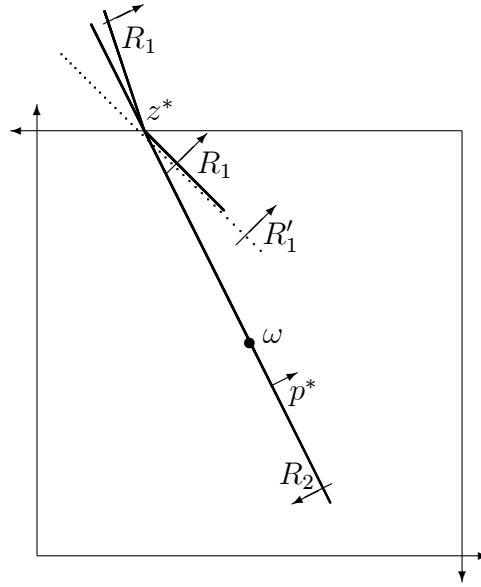


Figure 1

Since  $z^* \notin W(R')$ , *implementability* requires *justified sensitivity*. However, there does not exist  $i \in N$  and  $\{x_i, y_i\} \subset A_i$  such that  $x_i R_i y_i$  and  $y_i P'_i x_i$ . It is easy to see that the proof can be extended to any  $n$  and  $\ell$ .<sup>16</sup>  
*Q.E.D.*

By considering the domain  $\mathcal{R}^D$ , the pathologies uncovered in Proposition 1 disappear.<sup>17</sup> Differentiability of preferences guarantees that local information around  $z^*$  can be used to construct a sequence  $(a^l)_{l=1}^{k+1}$  of elements of  $A$  as required by *condition C*. To see this, consider  $\{R, R'\} \subset \mathcal{R}^D$  and  $z^* \in W(R) \setminus W(R')$  as shown in Figure 2. We can now identify  $\{z^*, x, y\} \subset A$  such that,

$$z_1^* R_1 x_1 R_1 y_1 \text{ and } y_1 P'_1 x_1.$$

When differentiability is imposed,  $IC(R_1, z_1^*) \cap A_1$  and  $IC(R'_1, z_1^*) \cap A_1$  have to differ around  $z_1^*$  if  $z^* \notin W(R')$ . Excluding non-differentiable preference relations from the domain, we do not need further restrictions to implement the Walrasian correspondence in subgame perfect equilibrium.

<sup>16</sup>It is clear that the domain  $\mathcal{R} \subset \mathcal{R}^C$  of preferences does not have to be a cartesian product for the proof to go through.

<sup>17</sup>Obviously, even with this domain restriction, the Walrasian correspondence is not Nash implementable since corner Walrasian allocations are not excluded.

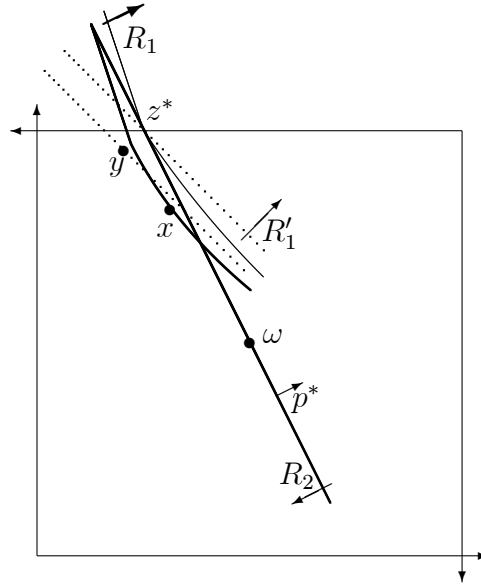


Figure 2

## 4 Taking care of the boundary problem

We now work with the differentiable domain  $\mathcal{R}^D$ . The mechanism we construct has three stages. At stage 1, agents simultaneously announce an allocation, a price and a permutation. First, for each  $R \in \mathcal{R}^D$ , and each  $(x, p) \in A^\circ \times \Delta_+$ , the information contained locally in prices is enough to determine whether  $(x, p) \in WE(R)$ . If  $(x, p) \notin WE(R)$ , there exist  $i \in N$  and  $y_i \in B_i(p, \omega_i) \cap A_i$  such that  $y_i P_i x_i$ .<sup>18</sup> The possibility of moves along budget hyperplanes is an important feature of our construction.<sup>19</sup> Accordingly, our mechanism always stops at Stage 1 if announced allocations are in  $A^\circ$ .

However when  $x \in \partial A$ , this device does not work anymore because moves along price hyperplanes can lead to infeasible bundles. To circumvent this problem, we still rely on the information contained locally in prices, but we use an alternative idea of retrading. This is where sequentiality comes into play. To see this, let us look at Figure 3. There is  $\{R, R'\} \subset \mathcal{R}^D$  and  $(z^*, p^*) \in \partial A \times \Delta_+$  such that  $(z^*, p^*) \in WE(R) \setminus WE(R')$ . Observe that (i)  $z_1^* \in \partial A_1$  and (ii)  $p'$  is such that there exists  $y_1 \in A_1$  with  $p' \cdot y_1 = p' \cdot z_1^*$

<sup>18</sup>The assumption of convexity of preferences cannot be relaxed. Thomson (1999) showed that with non-convex preferences, the Walrasian correspondence violates Maskin monotonicity even for allocations that are in  $A^\circ$ .

<sup>19</sup>This idea was already used, for instance, in Dutta, Sen and Vohra (1995) or Sotskov (2003).

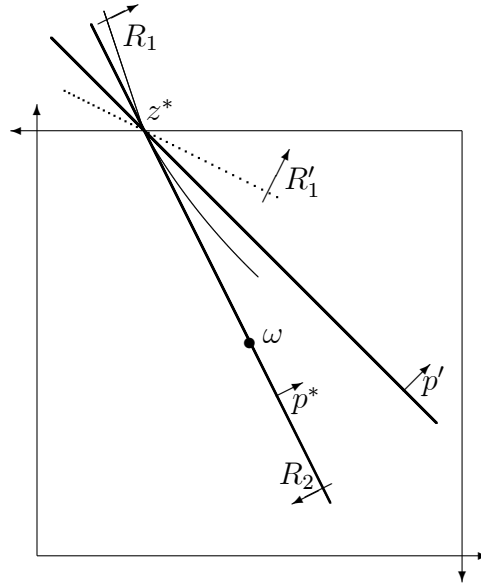


Figure 3

and  $p^* \cdot y_1 > p^* \cdot \omega_1$ . Hence,  $1 \in J(z^*)$  and  $p' \in T_1(p^*, z^*)$ . When agent 1 has preferences  $R_1$ , there exists  $y_1 \in (B_1(p', z_1^*) \cap A_1) \setminus B_1(p, \omega_1)$  such that  $y_1 P_1 z_1^*$ . Notice that we use a new constrained budget set  $B_1(p', z_1^*) \cap A_1$  where  $z_1^*$  plays the role of agent 1's endowment—hence the idea of retrading. But when agent 1 has preferences  $R'_1$ , there does not exist  $y_1 \in B_1(p', z_1^*) \cap A_1$  such that  $y_1 P_1 z_1^*$ . Thus  $(z^*, p^*) \notin WE(R')$ . For if  $(z^*, p^*) \in WE(R')$ , each  $p' \in T_1(p, z^*)$  would create profitable retrading opportunities for agent 1 in  $B_1(p', z_1^*) \cap A_1$ .

Whenever agents agree on an allocation  $x \in \partial A$ , the game continues to Stage 2 and agent  $f_1(\pi)$  is given the opportunity to reveal whether  $x$  is Walrasian for the true preference profile. The revelation is made possible because, for each  $\{R, R'\} \subset \mathcal{R}^D$  and each  $(x, p) \in \partial A \times \Delta_+$  such that  $(x, p) \in WE(R) \setminus WE(R')$ , there exist  $i \in J(x)$  and  $p' \neq p$  such that  $SUC(R_i, x_i) \cap \{y_i \in A_i : p' \cdot y_i = p' \cdot x_i\} \neq \emptyset$  while  $UC(R'_i, x_i) \cap \{y_i \in A_i : p' \cdot y_i = p' \cdot x_i\} = \emptyset$ . Interestingly, in contrast to implementation in Nash equilibrium, for the boundary problem, a change in the property of an allocation  $x \in \partial A$  when going from one preference profile  $R$  to another  $R'$ —from being Walrasian to not being Walrasian—is revealed through differences in the upper contour sets at that allocation.

We now present formally our mechanism. It is defined for  $n \geq 3$  agents.

For each  $i \in N$ , select an  $\epsilon_i \in \mathbb{R}_+^\ell \setminus \{0\}$  such that  $\omega_i - \epsilon_i \in \mathbb{R}_+^\ell \setminus \{0\}$ .<sup>20</sup>

**Mechanism  $\Gamma$ :**

**Stage 1:** Agents simultaneously choose  $(m_i^1)_{i=1}^n$  with  $m_i^1 = (x^i, p^i, \pi^i) \in F \times \Delta_+ \times \mathcal{P}$  such that, for each  $i \in N$  and each  $j \neq i$ ,  $p^i \cdot x_j^i = p^j \cdot \omega_j$ .

1) If for each  $i \in N$ ,  $(x^i, p^i) = (\bar{x}, \bar{p})$  and  $\bar{x} \in A^\circ$ , the game stops and the outcome is  $\bar{x}$ .

2) If for each  $i \in N$ ,  $(x^i, p^i) = (\bar{x}, \bar{p})$  and  $\bar{x} \in \partial A$ , then go to Stage 2.

3) If for each  $j \in N \setminus \{i\}$ ,  $(x^j, p^j) = (\bar{x}, \bar{p})$ ,  $i \neq f_n(\pi)$ , and  $m_i = (x', p') \neq (\bar{x}, \bar{p})$ , then (i) if  $\bar{p} \cdot x'_i = \bar{p} \cdot \omega_i$ , agent  $i$  gets  $x'_i$ , each  $j \in N \setminus \{i, f_n(\pi)\}$  gets  $\frac{\bar{\omega} - x'_i}{n-2}$ , and  $f_n(\pi)$  gets the 0 bundle, or (ii) if  $\bar{p} \cdot x'_i \neq \bar{p} \cdot \omega_i$ , then each  $j \in N$  gets his endowment  $\omega_j$ .

4) In all other cases, the game stops,  $f_1(\pi)$  gets  $\omega_{f_1(\pi)} + \epsilon_{f_n(\pi)}$ , each  $j \in N \setminus \{f_1(\pi), f_n(\pi)\}$  gets  $\omega_j$ , and  $f_n(\pi)$  gets  $\omega_{f_n(\pi)} - \epsilon_{f_n(\pi)}$ .

**Stage 2:** Agent  $f_1(\pi)$  chooses  $m_{f_1(\pi)}^2 = (k, p') \in N \setminus \{f_1(\pi)\} \times \Delta_+$ .

1) if  $\bar{x}_k \in \partial A_k$  and  $p' \in T_k(\bar{p}, \bar{x})$ , go to Stage 3.<sup>21</sup>

2) In all other cases, the game stops and the outcome is  $\bar{x}$ .

**Stage 3:** Agent  $k$  chooses  $m_k^3 = y_k \in \{y'_k \in A_k : p' \cdot y'_k = p' \cdot \bar{x}_k, \bar{p} \cdot y_k > \bar{p} \cdot \omega_k\} \cup \{\bar{x}_k\}$ .

1) If he chooses  $y_k = \bar{x}_k$ , he gets it and agent  $f_1(\pi)$  gets  $\bar{x}_{f_1(\pi)} + \frac{1}{n-2}(\bar{\omega} - \bar{x}_k - \bar{x}_{f_1(\pi)})$ . If  $k \in N \setminus \{f_n(\pi)\}$ , then each  $j \in N \setminus \{f_1(\pi), k, f_n(\pi)\}$  gets  $\frac{1}{n-2}(\bar{\omega} - \bar{x}_k - \bar{x}_{f_1(\pi)})$ , and  $f_n(\pi)$  gets the 0 bundle. Otherwise, if  $k = f_n(\pi)$ , then each  $j \in N \setminus \{f_1(\pi), k, f_{n-1}(\pi)\}$  gets  $\frac{1}{n-2}(\bar{\omega} - \bar{x}_k - \bar{x}_{f_1(\pi)})$  and agent  $f_{n-1}(\pi)$  gets the 0 bundle.

2) If he chooses  $y_k \neq \bar{x}_k$ , he gets it. Agent  $f_1(\pi)$  gets the 0 bundle and each  $j \in N \setminus \{f_1(\pi), k\}$  gets  $\frac{\bar{\omega} - y_k}{n-2}$ .

We can now proceed to the main theorem of the paper.

**Theorem 1:** *Suppose the domain is  $\mathcal{R}^D$  and that  $n \geq 3$ . Then, the Walrasian correspondence is doubly implemented by  $\Gamma$  in subgame perfect and strong subgame perfect equilibrium.*

<sup>20</sup>Individual epsilons are parameters of this mechanism.

<sup>21</sup>That is,  $p' \in \Delta_+$  is such that there exist feasible bundles  $y_k \neq \bar{x}_k$ , with  $\bar{p} \cdot y_k = \bar{p} \cdot \omega_k$  and  $p' \cdot y_k < p' \cdot \bar{x}_k$

*Proof:* Let  $R \in \mathcal{R}^D$ . The proof is divided into two parts.

*First part:*  $SPE\mathcal{O}(\Gamma, R) \subseteq W(R)$ .<sup>22</sup>

Let  $m \in SPE(\Gamma, R)$  with  $m_i^1 = (x^i, p^i, \pi^i)$ . We show that  $g(m) \in W(R)$ . The proof is divided in several lemma.

**Lemma 1:** *There exists  $(\bar{x}, \bar{p})$  such that, for each  $i \in N$ ,  $(x^i, p^i) = (\bar{x}, \bar{p})$*

*Proof:* The proof is by contradiction. There are two cases.

*Case 1:* There exist  $(\bar{x}, \bar{p}) \in A \times \Delta_+$  and  $i \in N \setminus \{f_n(\pi)\}$  such that for each  $j \neq i$ ,  $(x^j, p^j) = (\bar{x}, \bar{p})$ , and  $(x^i, p^i) \neq (\bar{x}, \bar{p})$ .

First, if  $\bar{p} \cdot x'_i = \bar{p} \cdot \omega_i$ , the game stops and  $g_{f_n(\pi)}(m) = 0$ . Let  $k \equiv f_n(\pi)$ . Let  $m_k^1 = (x'^k, p'^k, \pi'^k)$  be such that  $k = f_1(\pi'^k, \pi^{-k})$ , and  $(x'^k, p'^k) \neq (\bar{x}, \bar{p})$ . Then  $g_k(m'_k, m_{-k}) = \omega_k + \epsilon_{f_n(\pi'^k, \pi^{-k})}$ . Since preferences are strongly monotonic,  $\omega_k > 0$ , and for each  $j \in N$ ,  $\epsilon_j > 0$ , this is a profitable deviation for agent  $k$ . Thus,  $m \notin SPE(\Gamma, R)$ .

Next, if  $\bar{p} \cdot x'_i \neq \bar{p} \cdot \omega_i$ , the game stops and each  $j \in N$  gets  $g_j(m) = \omega_j$ . Let  $k \in N \setminus \{i\}$ . Let  $m_k^1 = (x'^k, p'^k, \pi'^k)$  be such that  $k = f_1(\pi'^k, \pi^{-k})$ , and  $(x'^k, p'^k) \neq (\bar{x}, \bar{p})$ . Then  $g_k(m'_k, m_{-k}) = \omega_k + \epsilon_{f_n(\pi'^k, \pi^{-k})}$ . Since preferences are strongly monotonic,  $\omega_k > 0$ , and for each  $j \in N$ ,  $\epsilon_j > 0$ , this is a profitable deviation for agent  $k$ . Thus  $m \notin SPE(\Gamma, R)$ .

*Case 2:* Either (i) There exist  $(\bar{x}, \bar{p}) \in A \times \Delta_+$  and  $k \equiv f_n(\pi)$  such that for each  $j \in N \setminus \{f_n(\pi)\}$ ,  $(x^j, p^j) = (\bar{x}, \bar{p})$ , and  $(x^k, p^k) \neq (\bar{x}, \bar{p})$ ; or (ii) There exists  $\{i, j, k\} \subset N$  such that  $(x^i, p^i) \neq (x^j, p^j) \neq (x^k, p^k)$ .

In either case, the game stops, each  $j \in N \setminus \{f_1(\pi), f_n(\pi)\}$  gets  $g_j(m) = \omega_j$ , and  $g_{f_n(\pi)}(m) = \omega_{f_n(\pi)} - \epsilon_{f_n(\pi)}$ . Let  $k \in N \setminus \{f_1(\pi), f_n(\pi)\}$ . Let  $m_k^1 = (x^k, p^k, \pi^k)$  be such that  $k = f_1(\pi^k, \pi^{-k})$  and  $f_n(\pi^k, \pi^{-k}) = f_n(\pi)$ . Then  $g_k(m'_k, m_{-k}) = \omega_k + \epsilon_{f_n(\pi^k, \pi^{-k})}$ . Since preferences are strongly monotonic,  $\omega_k > 0$ , and for each  $j \in N$ ,  $\epsilon_j > 0$ , this is a profitable deviation for agent  $k$ . Thus,  $m \notin SPE(\Gamma, R)$ . ■

**Lemma 2:** *Let  $\bar{x} \in A^\circ$ . Then  $(\bar{x}, \bar{p}) \in WE(R)$*

*Proof:* Suppose not. The game stops at Stage 1 with  $g(m) = \bar{x} \in A^\circ$  but  $(\bar{x}, \bar{p}) \notin WE(R)$ .

By definition of a Walrasian equilibrium, convexity of preferences and the fact that  $\bar{x} \in A^\circ$ , there exist  $k \in N$  and  $x'_k \in A_k$  such that  $\bar{p} \cdot x'_k = \bar{p} \cdot \omega_k$  and  $x'_k P_k \bar{x}_k$ . Let  $m_k^1 = (x', \bar{p}, \pi^k)$  be such that  $k = f_1(\pi^k, \pi^{-k})$ , and for each  $j \in$

<sup>22</sup>Since  $SSPE(\Gamma, R) \subseteq SPE(\Gamma, R)$ , it is enough, for the first part of the proof, to show that  $SPE\mathcal{O}(\Gamma, R) \subseteq WE(R)$ .

$N \setminus \{k\}$ ,  $x'_j = \frac{\bar{\omega} - x'_k}{n-1}$ . Then  $g_k(m'_k, m_{-k}) = x'_k$ . Since  $x'_k P_k \bar{x}_k$  by construction, this is a profitable deviation for agent  $k$ . Thus,  $m \notin SPE(\Gamma, R)$ . ■

**Lemma 3:** *Let  $\bar{x} \in \partial A$ . Then the game stops at Stage 2 with  $\bar{x}$  as outcome*

*Proof:* Suppose not. There exist  $i \in N$  and  $l, m \in L$  such that  $\bar{x}_{il} = \bar{\omega}_l$ ,  $\bar{x}_{im} > 0$ , and the game goes beyond Stage 2.

Given the rules of the game, there is  $k \in \{f_1(\pi), f_n(\pi), f_{n-1}(\pi)\}$  who gets  $g_k(m) = 0$ . Let  $m'_k = (\bar{x}, \bar{p}, \pi'^k)$  be such that  $k = f_1(\pi'^k, \pi^{-k})$  and  $m'_k = (f_2(\pi'^k, \pi^{-k}), \bar{p})$ . The game stops with  $g(m'_k, m_{-k}) = \bar{x}$  as outcome.

Remember that for each  $j \in N$ ,  $\bar{x}_j > 0$  since  $\omega_j > 0$ ,  $\bar{p} \cdot \bar{x}_j = \bar{p} \cdot \omega_j$  and  $\bar{p} \in \Delta_+$ . Hence, by deviating, agent  $k$  gets  $\bar{x}_k > 0$ . By strong monotonicity of preferences, this is a profitable deviation for agent  $k$ . Thus,  $m \notin SPE(\Gamma, R)$ . ■

**Lemma 4:** *Let  $\bar{x} \in \partial A$ . Then  $(\bar{x}, \bar{p}) \in WE(R)$*

*Proof:* Suppose not. The game stops at stage 2 with  $g(m) = \bar{x} \in \partial A$  but  $(\bar{x}, \bar{p}) \notin WE(R)$ . There are two cases to consider.

*Case 1:*  $g(m) = \bar{x}$  but there exists  $i \in N$  for whom,

$$(B_i(\bar{p}, \omega_i) \cap A_i) \cap SUC(R_i, \bar{x}_i) \neq \emptyset.$$

Let  $k \equiv i$ . Let  $m'_k = (x', \bar{p}, \pi'^k)$  be such that  $k = f_1(\pi'^k, \pi^k)$ ,  $x'_k \in (B_k(\bar{p}, \omega_k) \cap A_k) \cap SUC(R_k, \bar{x}_k)$ ,  $\bar{p} \cdot x'_k = \bar{p} \cdot \omega_k$ , and for each  $j \in N \setminus \{k\}$ ,  $x'_j = \frac{\bar{\omega} - x'_k}{n-1}$ . The game stops at Stage 1 and  $g_k(m'_k, m_{-k}) = x'_k$ . Since  $x'_k P_k \bar{x}_k$  by construction, this is a profitable deviation for agent  $k$ . Thus,  $m \notin SPE(\Gamma, R)$ .

*Case 2:*  $g(m) = \bar{x}$  but there exists  $i \in N$  for whom,

$$B_i(\bar{p}, \omega_i) \cap SUC(R_i, \bar{x}_i) \neq \emptyset.$$

Since *Case 1* is ruled out, we have that if  $x'_i \in B_i(\bar{p}, \omega_i) \cap SUC(R_i, \bar{x}_i)$ , then for some  $l \in L$ ,  $x'_{il} > \bar{\omega}_l$ . Note that  $\bar{x}_i \in \partial A_i$ . Let  $k \in N \setminus \{i\}$ . Let  $m'_k = (\bar{x}, \bar{p}, \pi'^k)$  be such that  $k = f_1(\pi'^k, \pi^k)$ , and  $m'_k = (f_j(\pi), p')$  be such that  $i = f_j(\pi)$ ,  $p' \in T_i(\bar{p}, \bar{x})$ , and  $(B_i(p', \bar{x}_i) \cap A_i) \cap UC(R_i, \bar{x}_i) = \{\bar{x}_i\}$ . By construction, the unique best response of  $i$  at Stage 3 is  $\bar{x}_i$ . Hence,  $g_k(m'_k, m_{-k}) = \bar{x}_k + \frac{1}{n-2}(\bar{\omega} - \bar{x}_i - \bar{x}_k)$ . Since for each  $j \in N$ ,  $\bar{x}_j > 0$ , and preferences are strongly monotonic, this is a profitable deviation for agent  $k$ . Thus,  $m \notin SPE(\Gamma, R)$ . ■



*Second part:*  $W(R) \subseteq SSPEO(\Gamma, R)$ .

Let  $(x, p) \in WE(R)$ . Then the following profile of strategy supports  $x$  as SSPE outcome of  $(\Gamma, R)$ .

(i) At Stage 1, each  $i \in N$  announces  $m_i^1 = (x, p, \pi^I)$ , where  $\pi^I$  is the identity permutation.

Let  $(\bar{x}, \bar{p})$  be the unanimously agreed price-allocation pair. Let  $C \equiv \{i \in J(\bar{x}) \setminus \{f_1(\pi)\} : \text{(a) } (B_i(\bar{p}, \omega_i) \cap A_i) \cap SUC(R_i, \bar{x}_i) = \emptyset, \text{ and (b) there is } x'_i \in B_i(\bar{p}, \omega_i) \cap SUC(R_i, \bar{x}_i) \text{ with } x'_{il} > \bar{\omega}_l \text{ for some } l \in L\}$ .

(ii) At Stage 2, agent  $f_1(\pi)$  makes the following announcement:

*Case 1:*  $C \neq \emptyset$

If  $|C| = 1$ ,  $m_{f_1(\pi)}^2 = (k, p') \in C \times T_k(\bar{p}, \bar{x})$  such that  $(B_k(p', \bar{x}_k) \cap A_k) \cap UC(R_k, \bar{x}_k) = \{\bar{x}_k\}$  and  $(B_k(p', \bar{x}_k) \cap A_k) \cap SUC(R_k, \bar{x}_k) = \emptyset$ .

If  $|C| > 1$ ,  $m_{f_1(\pi)}^2 = (k, p') \in C \times T_k(\bar{p}, \bar{x})$  such that for each  $j \in C \setminus \{k\}$ ,  $\bar{x}_k \leq \bar{x}_j$ ,  $(B_k(p', \bar{x}_k) \cap A_k) \cap UC_k(R_k, \bar{x}_k) = \{\bar{x}_k\}$ , and  $(B_k(p', \bar{x}_k) \cap A_k) \cap SUC_k(R_k, \bar{x}_k) = \emptyset$ .<sup>23</sup>

*Case 2:*  $C = \emptyset$

Then,  $m_{f_1(\pi)}^2 = (f_2(\pi), \bar{p})$ .

(iii) Following the choice of agent  $k$  by  $f_1(\pi)$  and the announcement of  $p$ , at Stage 3, then  $m_k^3 = y_k$  such that,

$$\begin{cases} y_k \neq \bar{x}_k \text{ if there exists } y_k \in \{y'_k \in A_k : y_k P_k \bar{x}_k, p' \cdot y_k = p' \cdot \bar{x}_k, \bar{p} \cdot y_k > \bar{p} \cdot \omega_k\}, \\ y_k = \bar{x}_k \text{ otherwise.} \end{cases}$$

It is clear that agent  $k$  is playing a best response at Stage 3. Agent  $k$  chooses the bundle he prefers between  $\bar{x}_k$  and every possible  $y_k$  on the budget hyperplane of  $B_k(p', \bar{x}_k)$ . In case of indifference, he favors agent  $f_1(\pi)$  and chooses  $\bar{x}_k$ .<sup>24</sup> Given  $(\bar{x}, \bar{p})$  agreed upon at Stage 1 and the protocol  $f(\pi)$ , notice that  $f_1(\pi)$  is also playing a best response at Stage 2. He announces  $p' \neq \bar{p}$  only if there exists  $k \in N \setminus \{f_1(\pi)\}$  for whom  $\bar{x}_{f_1(\pi)} \in \partial A_k$ ,  $(B_k(\bar{p}, \omega_k) \cap A_k) \cap UC_k(R_k, \bar{x}_k) = \{\bar{x}_k\}$ , and such that there is  $x'_k \in B_k(\bar{p}, \omega_k) \cap SUC_k(R_k, \bar{x}_k)$

<sup>23</sup>Since the outcome that  $f_1(\pi)$  gets at stage 3 can depend on the bundle that  $k$  gets, the optimal choice for  $f_1(\pi)$  is to choose  $k \in C$  who is awarded the smallest bundle at  $\bar{x}$ .

<sup>24</sup>This situation could happen off the equilibrium path.

and  $l \in L$  with  $x'_{kl} > \bar{\omega}_l$ . By doing so, agent  $f_1(\pi)$  obtains  $\bar{x}_{f_1(\pi)} + \frac{1}{n-2}(\bar{\omega} - \bar{x}_k - \bar{x}_{f_1(\pi)}) > \bar{x}_{f_1(\pi)}$ . Whenever this condition is not satisfied, one of the best response of  $f_1(\pi)$  is to announce  $p' = \bar{p}$  and to choose  $f_2(\pi)$ . Moreover, observe that a joint deviation by  $f_1(\pi)$  and  $k$  cannot make both agents better off at Stage 3.

Finally, recall that for each Walrasian equilibrium  $(x, p)$ , allocation  $x$  is individually rational, efficient, and such that for each  $i \in N$ ,  $B_i(p, \omega_i) \cap SUC_i(R_i, x_i) = \emptyset$ . Thus, each deviation by a coalition  $H \subseteq N$  results in the same outcome –for instance if agents modify the permutation they each announce– or in an outcome at which not all members of  $H$  are strictly better-off than at the Walrasian allocation. Therefore, the profile of strategies described is a strong subgame perfect equilibrium of  $(\Gamma, R)$ .

Hence, on the equilibrium path, each  $i \in N$  announces  $(x^i, p^i) = (x, p) \in WE(R)$ . If  $x \in A^\circ$ , the game stops at Stage 1 and  $g(m) = x$ . Otherwise, it goes to Stage 2 where  $m_{f_1(\pi)}^2 = (f_2(\pi), p)$ . The game stops at Stage 2 and  $g(m) = x$ .

*Q.E.D.*

**Remark 1** *In the previous section, we underlined that the issue raised in Proposition 1 applies to the Lindhal correspondence. The mechanism we use here can be adapted to the public goods case to implement the Lindhal correspondence.*

## 5 Conclusion

We have shown that, without differentiability of preferences, the Walrasian correspondence is not implementable in any of the game theoretical solution concepts commonly considered in the implementation literature. The boundary problem generates an impossibility result. Imposing differentiability of preferences, we constructed a mechanism that takes care of the boundary problem and doubly implements the Walrasian correspondence in subgame perfect and strong subgame perfect equilibrium. Our mechanism is based on price-allocation announcements and corresponds closely to the description of Walrasian equilibrium: moves along price hyperplanes are at the heart of the Walrasian equilibrium concept. The mechanism allows for a better understanding of the strategic issues attached to the boundary problem.

Finally, our mechanism can be extended to incomplete information settings by incorporating announcements of types at Stage 1 as well as state-contingent prices and allocations. An interesting issue would be to analyze the information transmission that occurs across stages and to characterize an extension of the Walrasian correspondence to incomplete information settings. Bochet (2006) follows a similar approach using a static mechanism. He finds ambiguous connections between Walrasian and Rational Expectations equilibria.

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