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CHARACTERIZING THE NASH AND RAIFFA BARGAINING SOLUTIONS BY DISAGREEMENT POINT AXIOMS*

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We provide a new characterization of the n-person Nash bargaining solutions which does not involve Nash's Independence of Irrelevant Alternatives axiom, but mainly uses axioms which concern changes in the disagreement point and leave the feasible set fixed. The main axiom requires a convex combination of a disagreement point and the corresponding solution point to give rise to that same solution point. Further, we describe how the disagreement point approach can be applied to other bargaining solutions. The main result of the latter part is a first characterization of the so-called Continuous Raiffa solution.

1. Introduction. Starting with Nash (1950) it has become customary to formulate an (n-person) bargaining problem as a pair (S, d) where S is a convex set in n-dimensional Euclidean space, consisting of all utility n-tuples that the players can obtain by cooperating, and d is a distinguished element of S, representing the outcome that will prevail when the players do not cooperate. Nash (1953) proposed two procedures to solve the bargaining problem: the noncooperative approach and the axiomatic method. In the first, one explicitly models the negotiation process as a noncooperative game and solves for the Nash equilibria. Unfortunately, this approach yields indeterminate results since the solution in general depends on the rules of the negotiation process (Rubinstein 1982, Moulin 1984, Binmore 1987). Nash describes the second method as "one states as axioms several properties that it would seem natural for the solution to have and then one discovers that the axioms actually determine the solution uniquely" (Nash 1953, p. 129). As is well known, also this approach is indeterminate as the solution depends on which axioms one considers natural (see Roth 1979 for a survey). Thus, both methods have their drawbacks. In this paper the axiomatic method is adopted, but our results may also prove relevant for the strategic approach.

The various bargaining solutions (i.e. functions that assign an outcome $f(S, d)$ to any bargaining problem (S, d)) that have been proposed generally share a number of properties, such as individual rationality, Pareto optimality, symmetry, and scale invariance (independence of the utility representations chosen). Further, for each solution there is a characterizing axiom which, in combination with the standard properties, determines the solution uniquely. Usually, this distinguishing axiom describes how the solution should change when the feasible set S varies while the disagreement outcome d remains fixed. For example, Nash's independence of irrelevant alternatives axiom states that, when T shrinks to S while d remains fixed and $f(T, d)$ remains feasible in S, then $f(S, d) = f(T, d)$. The individual monotonicity axiom of Kalai and Smorodinsky (1975) demands that, when more attractive payoffs become available for a player, then, with fixed disagreement outcome, this player

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should not receive less utility in the new situation. Perles and Maschler (1981) consider the situation in which d is given but it is still uncertain what the feasible set will be, although the players agree on the various possibilities and the probabilities with which these arise. Their axiom of superadditivity demands that in this case all players prefer to reach an agreement immediately (formally, when $S = \frac{1}{2}S_1 + \frac{1}{2}S_2$, then $f(S,d) \geqslant \frac{1}{2}f(S_1,d) + \frac{1}{2}f(S_2,d)$).

In our view, it is at least as natural to require that the solution be well-behaved when the disagreement outcome varies while the feasible set remains fixed. In Peters (1986b), Livne (1986, 1989), Chun (1988), Chun and Thomson (1990a, b) and Thomson (1987), several axioms of this variety have been proposed and also our axioms are mainly of this type. Our new approach of characterizing bargaining solutions by axioms concerning the disagreement point has the obvious advantages of leading to new characterizations of solutions while making the role of the disagreement point more explicit. Maybe even more important is the advantage of having characterizations based on the existence of only *one* feasible set together with some sets derived from it in a natural way. To see why this is an advantage, notice that implicit in the axiomatic approach to bargaining as proposed by Nash is the assumption that the solution outcome should be independent of all characteristics of a bargaining situation that are not captured by the pair (S,d). For instance, the conclusion of the independence of irrelevant alternatives axiom formulated above is supposed to hold for all bargaining games satisfying the premises, even though the shrunken set S may well have arisen from an underlying physical situation that cannot be obtained by deleting physical alternatives in the situation leading to T. Roemer (1986) calls this implicit assumption the "Welfarist Axiom" and shows that the classical results of axiomatic bargaining theory no longer hold without it. Since in this paper it is not our main purpose to propose another theory without the "Axiom of Welfarism," we confine ourselves to pointing out that our disagreement point approach may well serve as a basis for such a theory.

Finally, let us note that the disagreement point approach can serve as a starting point of a dynamic theory of bargaining. See, in particular, Furth (1988).

The main result of the paper will be a new characterization of the family of n-person weighted Nash bargaining solutions in which Nash's independence of irrelevant alternatives axiom is avoided and the main axioms concern changes in the disagreement point. The axioms and motivations are introduced in §2, while §3 is centered around the characterization result. This characterization result is proved for the n-person case: In §4 we prove a few additional results which are or seem to be special for the 2-person case. Finally, in §5, we further exploit our approach of looking at axioms involving changes in the disagreement point. In particular, we provide a, as far as we know, first characterization of the so-called Continuous Raiffa solution (Raiffa 1953).

2. The axioms. An (n-person) *bargaining set* is a set $S \subset \mathbf{R}^n$ which is closed, convex, comprehensive, and bounded from above, i.e., there exist $p \in \mathbf{R}^n$, $c \in \mathbf{R}$, with[1] $p > 0$ such that $\Sigma p_i x_i \leqslant c$ for all $x \in S$.[2] By **BS** we denote the family of all bargaining sets.

[1]Vector inequalities: $x \geqslant (>)\ y$ if and only if $x_i \geqslant (>)\ y_i$ for all i. Similarly for $\leqslant$ and $<$.

[2]We call a set T *comprehensive* if $y \in T$ whenever $x \leqslant y \leqslant z$ and $x, z \in T$. Note that we do *not* require: $y \leqslant z\ \ \&\ \ z \in T \Rightarrow y \in T$. Comprehensiveness of a bargaining set can be interpreted as free disposability of utility. In some instances, our results would be different without this assumption. See, in particular, the second part of §3, and also §4.

An (n-person) *bargaining problem* is a pair (S, d) where $S \in \mathbf{BS}$ and $d \in S$. The point d is called the *disagreement outcome*. Note that d is allowed to be on the boundary bd(S) of S; this will be the case in particular if S has empty interior. The set of all n-person bargaining problems is denoted by $\mathbf{B}$. By (S, d) we denote a generic bargaining problem. If $x \in S$ we write

$$S_x = \{y \in S: y \geqslant x\}$$

for the points in S that weakly Pareto dominate x. The (strong) *Pareto boundary* of S is denoted by $P(S)$:

$$P(S) = \{x \in S: S_x = \{x\}\}.$$

An (n-person) *bargaining solution* is a map f: $\mathbf{B} \to \mathbf{R}^n$ with $f(S, d) \in S$ for every $(S, d) \in \mathbf{B}$. We use f to denote a generic bargaining solution.

The first axiom we want to introduce is in a sense the "dual" of the superadditivity axiom of Perles and Maschler described in the Introduction. Suppose that S is given, and that d^i will be the disagreement point with probability p^i (i from a finite index set). Suppose that all solution outcomes $f(S, d^i)$ are equal. So, if the players only meet after the disagreement point uncertainty has been resolved, they will always agree on the same outcome. Will they also agree on this outcome ex ante? Answering this question affirmatively corresponds to imposing the following axiom:

Convexity (CONV). $\{d \in S$: $f(S, d) = x\}$ is convex.

This axiom is actually well known: In 2-person bargaining with variable threat point, together with a few standard axioms it suffices to guarantee that Nash's threat game (Nash 1953) has an equilibrium (see, for example, Tijs and Jansen 1982). In the 2-person case, Nash's solution satisfies this axiom; however, CONV is a surprisingly strong axiom. If there are more than two bargainers, CONV is inconsistent with the following two axioms:

Pareto Optimality (PO). $f(S, d) \in P(S)$.

Strong Individual Rationality (SIR). For all i we have $f_i(S, d) \geqslant d_i$, with strict inequality whenever $x_i > d_i$ for some $x \in S$.

The PO axiom needs no further comment, and SIR adds to the usual individual rationality requirement an incentive to cooperate for those players who have something positive to gain.

To prove our claim that CONV, PO, and SIR are inconsistent, let f be a solution satisfying these three axioms, and let S be the convex hull of the points $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, and $(1,1,0)$. By PO and SIR the points $d = (1,0,0)$ and $d = (0,1,0)$ of S result in the outcome $f(S, d) = (1,1,0)$, but then CONV implies $f(S, d) = (1,1,0)$ for $d = (\frac{1}{2}, \frac{1}{2}, 0)$ and this contradicts SIR. Note that this example uses the fact that the disagreement point may be a boundary point; it is an open problem whether PO, CONV, and SIR are inconsistent if this is not allowed.

In our characterization of the weighted Nash solutions we will use an axiom considerably weaker than CONV. Specifically, we will require that a convex combination of a disagreement point and the corresponding solution point give rise to the same solution point, hence:

Disagreement Point Convexity (DVEX). For all $0 \leqslant \mu \leqslant 1$ we have

$$f(S, \mu d + (1 - \mu) f(S, d)) = f(S, d).$$

This requirement can be motivated as above by referring to exogenous uncertainty about the disagreement point.

(Another motivation is obtained by the following informal argument concerning endogenous (strategic) uncertainty. Consider a 2-person bargaining problem (S, d) and suppose player 1 firmly adheres to the (PO, SIR) solution f. If $f_2(S, e) > f_2(S, d)$ for $e = (d + f(S, d))/2$, then player 2 has an incentive to behave strategically: he could threaten to toss a coin and to accept $f(S, d)$ if heads come up and to walk away from the bargaining table in case of tails. By this behaviour the disagreement point is effectively converted to e, so player 1 will offer $f(S, e)$ in order to avoid disagreement, which is to the advantage of player 2. DVEX excludes manipulating behaviour of this kind.)

We will show that, given a list of other reasonable requirements, Nash solutions are the only ones to satisfy DVEX.

We now introduce our other main axioms.

Invariance (INV). $f(A(S), A(d)) = A(f(S, d))$ for all positive affine transformations A of $\mathbf{R}^n$.[3]

Disagreement Point Continuity (DCONT). For each bargaining set S and every sequence $d, d^1, d^2, \ldots$ in S, if $d^n \to d$ then $f(S, d^n) \to f(S, d)$.

The Disagreement Point Continuity axiom is a mild regularity requirement. Note that all axioms hitherto defined essentially refer to only *one* set of feasible outcomes S. As usually, axiom INV is motivated by the assumption that the players have von Neumann-Morgenstern utility functions, and hence it only refers to an equivalence class of problems without comparing two essentially different problems. The axiom would be acceptable even if one would not accept Roemer's "Welfarist Axiom" described in the Introduction. More precisely, the axiom would be acceptable as long as it is understood that $A(S)$ and S in its formulation refer to the same physical situation.

Our final axiom no longer refers to just one feasible set: it also refers to S_d but this set arises naturally from S.

Independence of Nonindividually Rational outcomes (INIR). $f(S, d) = f(S_d, d)$.

The INIR axiom was first formally discussed in Peters (1986b), and amounts to a very weak form of Nash's "independence of irrelevant alternatives." Still it is far from being harmless, although many authors assume it to hold implicitly by their choices of the domain of bargaining problems. For example, Kalai and Smorodinsky (1975, p. 514) defend the axiom, or rather their restriction to bargaining problems (S, d) with $S = S_d$, on the ground that "*if this* (i.e., $S = S_d$) *is not the case, we can disregard all the points of S that fail to satisfy this condition* (i.e., of dominating d), *because it is impossible that both players will agree to such a solution*" (i.e. a nonindividually rational solution outcome). Note that actually Kalai and Smorodinsky need a stronger argument to defend the criterion "$S = S_d$": nonindividually rational outcomes should not only never occur as solution outcomes, but they should also never influence the solution outcome. This argument exactly amounts to INIR.[4]

[3] $A(x) = (a_1 x_1 + b_1, \ldots, a_n x_n + b_n)$ with $a_i > 0$ for all i; $A(S) = \{A(x)\colon x \in S\}$.

[4] In more general games INIR may not be entirely convincing. Cf. for example the discussion on the NTU-value between Roth (1986) and Aumann (1985, 1986). For further discussions in the case of bargaining problems, see Peters (1986c) and Perles and Maschler (1981).

For completeness' sake we give here the formal definition of

Independence of Irrelevant Alternatives (IIA). For all (S, d) and (T, d) in **B** with $S \subset T$ and $f(T, d) \in S$, we have $f(S, d) = f(T, d)$.

For $t \in \mathbf{R}^n$ with $t > 0$ and $\sum_i t_i = 1$ we define the n-person *weighted Nash bargaining solution with weights* t by

$$N^t(S, d) = \underset{x \in S_d}{\operatorname{argmax}} \prod_{i \in M} (x_i - d_i)^{t_i}, \quad \text{where} \tag{2.1}$$

$$M = \{i: y_i > d_i \text{ for some } y \in S\}. \tag{2.2}$$

It is straightforward to verify that N^t is a well-defined solution which satisfies IIA, INV, SIR, PO, INIR, and DCONT (for verification of the last axiom a little bit more effort may be required). We have already seen that CONV is inconsistent with PO and SIR for $n > 2$. Furthermore, also if we restrict attention to bargaining problems where the disagreement point is not allowed to be a boundary point, then the Nash solutions still do not satisfy CONV; this can be seen by taking, in the previous example, points of the form $(1 - \alpha, \alpha, 0)$ and $(\beta, 1 - \beta, 0)$ instead of $(1, 0, 0)$ and $(0, 1, 0)$ as disagreement points, for suitably chosen α and β, sufficiently small. However, with the aid of Lemma 2.1 below it can be shown that N^t satisfies DVEX. This lemma may be well known (cf. Peters 1986a, Lemma 28.14) but for completeness' sake we provide a proof. It will be convenient to introduce the following notation for $z, d, t \in \mathbf{R}^n$ with t as above and with $z > d$.

$$L^t(d, z) = \left\{x \in \mathbf{R}^n: \sum_i t_i (x_i - d_i)(z_i - d_i)^{-1} = 1\right\}, \tag{2.3}$$

$$H^t(d, z) = \left\{x \in \mathbf{R}^n: \sum_i t_i (x_i - d_i)(z_i - d_i)^{-1} \leqslant 1\right\}. \tag{2.4}$$

$L^t(d, z)$ is a hyperplane through z and $H^t(d, z)$ is an associated halfspace. Let us call a bargaining problem (S, d) nondegenerate if $x > d$ for some $x \in S$. The following lemma characterizes N^t for nondegenerate bargaining problems. Specifically, we have that $z = N^t(S, d)$ if and only if $L^t(d, z)$ supports S at z.

LEMMA 2.1. *For a nondegenerate bargaining problem (S, d) and $z \in S$ we have that $z = N^t(S, d)$ if and only if $S \subset H^t(d, z)$.*

PROOF. Let z solve the program (2.1) with $M = \{1, 2, \dots, n\}$. Then $L^t(d, z)$ is the unique tangent hyperplane at z of the set

$$\left\{x \in \mathbf{R}^n: \prod_i (x_i - d_i)^{t_i} \geqslant \prod_i (z_i - d_i)^{t_i}\right\}.$$

Hence, $S \subset H^t(d, z)$ by a separating hyperplane theorem. To prove the converse, note that $N^t(H^t(d, z), d) = z$. Consequently, if $z \in S$ and $S \subset H^t(d, z)$, then $N^t(S, d) = z$ since N^t satisfies IIA. ■

In the next sections we use the following notation. If A is an axiom, then we also write A for the class of bargaining solutions satisfying A.

3. Characterization of Nash solutions. We first state and prove the main result of this section, and of the paper. In the second part of the section, we will discuss our choice of domain (what happens, in particular, if also noncomprehensive problems are considered?), and provide a sensitivity analysis with respect to the axioms used in the main theorem.

THEOREM 3.1. *$f \in$ INV $\cap$ SIR $\cap$ INIR $\cap$ DCONT $\cap$ DVEX if and only if there exists $t \in \mathbf{R}^n$, $t > 0$ with $\Sigma_i t_i = 1$ such that $f = N^t$ on* **B**. *In particular*, IIA $\subset$ INV $\cap$ SIR $\cap$ INIR $\cap$ DCONT $\cap$ DVEX.

For the proof of Theorem 3.1, we need some lemmas.

LEMMA 3.2. DVEX $\cap$ SIR $\subset$ PO.

PROOF. Let $(S, d) \in \mathbf{B}$. DVEX implies $f(S, f(S, d)) = f(S, d)$. Pareto optimality now immediately follows from applying SIR. ∎

The following axiom is a considerable strengthening of DVEX.

Disagreement Point Linearity (DLIN). If $e = \mu d + (1 - \mu) f(S, d)$ with $\mu \in \mathbf{R}$ and $e \in S$ then $f(S, e) = f(S, d)$.

LEMMA 3.3. DVEX $\cap$ DCONT $\cap$ SIR $\subset$ DLIN.

PROOF. Let f satisfy DVEX, DCONT, and SIR, and let (S, d) be a bargaining problem. Let e be as in the statement of DLIN with $\mu > 1$ (the case $\mu \leqslant 1$ follows from DVEX). Let M be the subset of players such that $i \in M$ if and only if $x_i > d_i$ for some $x \in S$. Then note that $f_i(S, e) = f_i(S, d) = d_i$ for all $i \notin M$, in view of SIR. We want to show that $f_i(S, e) = f_i(S, d)$ also for all $i \in M$.

For all $x, y \in S$, $x \neq y$, write $f(x)$ instead of $f(S, x)$, and let $l(x, y)$ be the straight line through x and y; by $[x, y]$ denote the line segment with endpoints x and y. Then, for $x \in S_e$, define $g(x)$ as the other (i.e. $\neq f(x)$) point of intersection of $l(d, f(x))$ with relbd(S_e), i.e. with the boundary of S_e relative to the $|M|$-dimensional subspace containing S_e. This map is well defined since d is in the interior, relint(S_e), of S_e relative to that same $|M|$-dimensional subspace, and continuous since f is continuous by DCONT. Since S_e is compact and convex, by Brouwer's fixed point theorem there exists $z \in S_e$ such that $g(z) = z$. Then $z \in$ relbd(S_e) and $d \in l(z, f(z))$. By PO, $d \geqslant f(z)$ would imply $d = f(z)$ which contradicts $d \in$ relint(S_e). Further, $d \leqslant z$ with $d \neq z$ would imply $z \in$ relint(S_e), also a contradiction. So we must have $d \in [z, f(z)]$, so $f(d) = f(z)$ by DVEX. In particular we also have $e \in l(z, f(z))$. But then we must have $z = e$ since $e \in$ relbd(S_e). So $f(e) = f(z) = f(d)$. ∎

In his 1953 paper on 2-person cooperative games, Nash justifies his IIA-axiom as follows: "This axiom is equivalent to an axiom of 'localization' of the dependence of the solution point on the shape of the set S.[5] The localization of the solution point on the upper right boundary of S is determined only by the shape of any small segment of the boundary that extends to both sides of it. It does not depend on the rest of the boundary curve" (Nash 1953, p. 138). Formally, one may state this "localization" axiom as follows (see also Lensberg 1987, p. 953):

Localization (LOC). For problems (S, d) and (T, d), if $U \cap S = U \cap T$ for an open neighbourhood U of $f(S, d)$, then $f(T, d) = f(S, d)$.

Clearly this axiom is closely related to IIA. However, it is neither weaker nor stronger than IIA. The relationship between the two axioms will be discussed in more detail at the end of this section.

[5]Actually, Nash uses "B" instead of our "S".

It is easy to prove that our basic axioms imply LOC:

LEMMA 3.4. DVEX $\cap$ DCONT $\cap$ SIR $\cap$ INIR $\subset$ LOC.

PROOF. Assume (S, d) and (T, d) satisfy the condition stated in LOC, and let e be a convex combination of d and $f(S, d)$ sufficiently close to $f(S, d)$ such that $S_e = T_e$. Such a point e exists in particular since f satisfies PO, see Lemma 3.2. Then from INIR and $f(S_e, e) = f(T_e, e)$ we conclude $f(S, e) = f(T, e)$, so $f(S, d) = f(T, d)$ follows from DLIN (Lemma 3.3). ∎

We are now sufficiently equipped to prove Theorem 3.1.

PROOF OF THEOREM 3.1. Let $V = \{x \in \mathbf{R}^n\colon \Sigma_i x_i \leqslant 1\}$ and write $t = f(V, 0)$. By SIR and Lemma 3.2, $t > 0$ and $\Sigma_i t_i = 1$. Let $(S, d) \in \mathbf{B}$. We show that $f(S, d) = N^t(S, d)$. We may suppose that S is full-dimensional, i.e. has nonempty interior; if not, by INIR of f and N^t, we can replace S by $\{x \in \mathbf{R}^n\colon x \leqslant y$ for some $y \in S\}$. We may further suppose that $d \in S \setminus \mathrm{bd}(S)$, for boundary points are taken care of by DCONT of f and N^t: boundary points can be approximated by interior points, so it is sufficient to give the proof for interior disagreement points. In view of axioms INV and SIR we may then assume that $d = 0$ and $f_i(S, d) = 1$ for all i. We write $z = N^t(S, d) > 0$, and suppress $d = 0$ from notation: so we write $f(S)$, $L^t(x), \ldots$ instead of $f(S, 0)$, $L^t(0, x), \ldots$.

By Lemma 2.1, $S \subset H^t(z)$, so $f(S) \in H^t(z)$. We will first show that actually $f(S) \in L^t(z)$. Suppose not (see Figure 1). Then $\Sigma_i t_i f_i(S) z_i^{-1} = \Sigma t_i z_i^{-1} < 1$, so we can choose an $\alpha > 0$ such that $\Sigma_i t_i z_i^{-1} < 1 - \alpha$, and let $T = S \cap H^t((1 - \alpha)z)$. Then T coincides with S in a neighbourhood of $f(S)$ so that, by LOC (Lemmas 3.3 and 3.4), $f(S) = f(T)$. Now Lemma 2.1 implies that $N^t(T) = (1 - \alpha)z$, so that $N^t(T)$ is an interior point of S. This in turn implies that, in a neighbourhood of $N^t(T)$, T coincides with the halfspace $H^t((1 - \alpha)z)$. Then INV and the fact that $f(V) = N^t(V)$ imply $f(H^t((1 - \alpha)z)) = N^t(H^t((1 - \alpha)z)) = N^t(T)$, hence, by LOC of f, $f(T) = N^t(T)$. However, $N^t(T) \notin P(S)$, in contradiction with PO of f, since $N^t(T) = f(T) = f(S)$. Consequently, $f(S) \in L^t(z)$ and $\Sigma_i t_i z_i^{-1} = 1$.

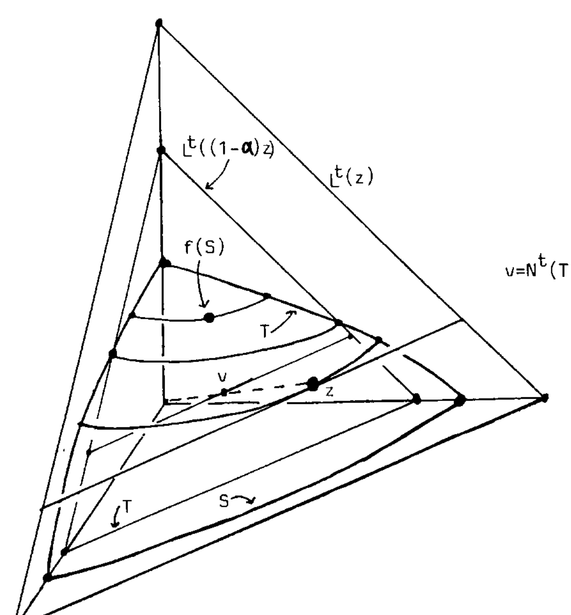

FIGURE 1.

Now assume $z \neq f(S)$ $(= (1,1,\ldots,1))$ and let $y = \frac{1}{2}z + \frac{1}{2}f(S) \in S$. Since the function h on $\mathbf{R}_+$ with $h(\beta) = \beta^{-1}$ is strictly convex we have $h(\frac{1}{2}(1) + \frac{1}{2}(z_t)) < \frac{1}{2}h(1) + \frac{1}{2}h(z_t)$ i.e.

$$y_t^{-1} = \left(\tfrac{1}{2} + \tfrac{1}{2}z_t\right)^{-1} < \tfrac{1}{2} + \tfrac{1}{2}z_t^{-1} \quad \text{if } z_t \neq 1 = f_t(S)$$

so that

$$\sum_t t_t y_t^{-1} < \tfrac{1}{2} + \tfrac{1}{2}\sum_t t_t z_t^{-1} = 1 \quad \text{if } z \neq f(S). \tag{3.1}$$

Consider the set $S' = S \cap H^t(y)$. Then Lemma 2.1 implies $N^t(S') = y$. Furthermore, S' and S coincide in a neighbourhood of $f(S) = (1,1,\ldots,1)$ in view of (3.1), so that $f(S) = f(S')$ by LOC. However, the previous part of the proof applied to S' implies $f(S') \in L^t(y)$, so $f(S) \in L^t(y)$ but this contradicts (3.1). Consequently, we must have $N^t(S) = z = f(S)$. ■

The remainder of this section will be devoted to a discussion on, firstly, our choice of domain, in particular the assumption of comprehensiveness; secondly, the relation between IIA and the localization axiom LOC; thirdly, the nonredundancy of the axioms in Theorem 3.1; and fourthly, related literature. The second point in particular gives rise to a few unsolved problems.

The domain. Our least standard domain requirement is comprehensiveness. Although this requirement seems fairly reasonable, it is not innocent. Let us denote by $\mathbf{B}^{\sim}$ the family of bargaining problems without the comprehensiveness assumption. The following example shows that, on $\mathbf{B}^{\sim}$, Lemma 3.3 no longer holds. Another notation: "conv" denotes "the convex hull of".

EXAMPLE 3.5. Let $S = \text{conv}\{(0,0),(2,2),(1,3)\} \in \mathbf{B}^{\sim}$. Let $D = \{x: x = (\alpha,\alpha) \text{ with } \alpha \leqslant 1\}$. For $x = (\alpha,\alpha)$, $x \in D \cap S$, define $f(S,x) = (1+\alpha, 3-\alpha)$. Note that $\{\text{conv}\{x, f(S,x)\}: x \in D\}$ is a partition of S so that f can be extended from D to S by the requirement of DVEX. Then f satisfies DVEX, DCONT and SIR but not DLIN. (This solution f can be extended to $\mathbf{B}^{\sim}$ in any arbitrary way, as long as it satisfies DVEX, DCONT, and SIR.)

Let us note that Theorem 3.1 still holds on $\mathbf{B}^{\sim}$ for $n = 2$, despite the violation of Lemma 3.3. For a proof of this fact we refer to the next section (Corollary 4.4). If $n > 2$, then Theorem 3.1 no longer holds without the comprehensiveness assumption, as is shown by the following example.

EXAMPLE 3.6. For every $d \in \mathbf{R}^3$ let H^d be the collection of all hyperplanes in $\mathbf{R}^3$ through d which contain an $x > d$. If (S,d) is a bargaining problem in $\mathbf{B}^{\sim}$ with $S \subset h$ for some $h \in H^d$ then we define $f(S,d) = N^p(S,d)$ with $p = (\frac{1}{2},\frac{1}{4},\frac{1}{4})$. Otherwise let $f(S,d) = N^q(S,d)$ with $q = (\frac{1}{3},\frac{1}{3},\frac{1}{3})$. This f satisfies INV, SIR, INIR, DCONT, and DVEX; the easiest way to see this is to note that both N^p and N^q satisfy these axioms, and that none of the axioms involves a comparison between bargaining problems belonging to both classes distinguished in the definition of f. For the opposite reason, f does not satisfy IIA: e.g. take an (S,d) contained in some hyperplane in H^d so that $f(S,d) = N^p(S,d)$ and such that $f(S,d) \neq N^q(S,d)$, and next consider (T,d) where $T = \{x \in \mathbf{R}^3: x \leqslant y \text{ for some } y \in S\}$. Then $S \subset T$ and $f(T,d) = N^q(T,d) = N^q(S,d)$ where the last point is in S, but unequal to $f(S,d)$. So we have a violation of IIA.

As a final remark concerning our choice of domain, we note that adding the following axiom makes Theorem 3.1 hold also on the extended class $\mathbf{B}^{\sim}$:

Independence of Non-Pareto Optimal Outcomes (INPO). For all $(S, d), (T, e) \in \mathbf{B}^{\sim}$, $d = e$ and $P(S) = P(T)$ imply $f(S, d) = f(T, e)$.

IIA and LOC. We next examine more closely the relation between IIA and LOC. Under certain lists of additional assumptions, the two axioms are equivalent, e.g. the list of conditions in Theorem 3.1. In general, however, this is not the case.

EXAMPLE 3.7. Let f be the 2-person bargaining solution on $\mathbf{B}$ defined as follows. Let $(S, d) \in \mathbf{B}$ and let $v, w \in P(S) \cap S_d$ be the points with maximal first and maximal second coordinates, respectively. Then let $f(S, d) = v$ if $v_1 - d_1 \geqslant w_2 - d_2$, and let $f(S, d) = w$ otherwise. It is easily verified that f satisfies IIA but not LOC.

Note that, on the extended class $\mathbf{B}^{\sim}$, LOC does not imply IIA: see Example 3.6. Also, it is not hard to prove that for a solution satisfying the following axiom, IIA and LOC are equivalent, see e.g. Lensberg (1987, p. 953, Lemma 15).

Feasible Set Continuity (FCONT). Let $(S, d), (S^1, d), (S^2, d), \ldots$ be a sequence of bargaining problems with $S^n \to S$ in the Hausdorff metric. Then $f(S^n, d) \to f(S, d)$.

Although often taken for granted, FCONT is a quite powerful axiom, and it does not fit in our setup of considering axioms for fixed feasible sets. Our next example shows that, also on the class $\mathbf{B}$, LOC does not imply IIA. The example, though, is rather trivial, and we conjecture that, at least for the 2-person case, under very mild additional conditions LOC does imply IIA, but as yet we have not been able to make this statement precise. For instance, we do not know whether PO suffices to obtain the implication LOC $\Rightarrow$ IIA.

EXAMPLE 3.8. To every 2-person problem (S, d) in $\mathbf{B}$ such that S is a subset of a horizontal or vertical straight line, let f assign the unique Pareto optimal point. To every other problem (S, d) let f assign d. This f satisfies LOC but not IIA.

The axioms. The following example briefly shows that none of the axioms in Theorem 3.1 can be omitted: for SIR, DCONT, DVEX, INIR, and INV, see Example 3.9(a), (b), (c), (d), and (e), respectively.

EXAMPLE 3.9. (a) $f(S, d) = d$ for all $(S, d) \in \mathbf{B}$.

(b) Let $n = 3$ and $(S, d) \in \mathbf{B}$: if d is an interior point of S then let $f(S, d) = N^p(S, d)$ with $p = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, otherwise let $f(S, d) = N^q(S, d)$ with $q = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$.

(c) Let $n = 2$ and let f be the Kalai-Smorodinsky solution, which assigns to a problem (S, d) the unique Pareto optimal point on the line segment connecting d and the utopia point $(\max\{x_1\colon x \in S_d\}, \max\{x_2\colon x \in S_d\})$.

(d) Let $n = 3$ and let $f(S, d) = N^p(S, d)$ if there is a constant c with $x_3 = c$ for all $x \in P(S)$, let $f(S, d) = N^q(S, d)$ otherwise.

(e) Let f be the "lexicographic egalitarian solution" (cf. Chun and Peters 1988). This solution does satisfy IIA.

Adding an axiom of symmetry (see §5 for the definition in the 2-person case) to the list of axioms in Theorem 3.1 would of course single out the symmetric Nash solution. The question then arises whether one or more of the axioms can be dropped. The answer to this question is negative as regards SIR, DVEX, or INV: cf. Example 3.9(a), (c), or (e). However, the question is open as regards DCONT and INIR since the corresponding examples above are nonsymmetric.

Note that the conditions in Theorem 3.1, i.e. the five axioms mentioned above, imply IIA. In the next section (Theorem 4.3) we will show that, at least for $n = 2$, the Invariance axiom INV may be weakened without destroying this implication, even on the larger class $\mathbf{B}^{\sim}$. Whether a similar result holds if $n > 2$ (cf. Example 3.9(e)) is another unsolved problem.

Related literature. Two papers most immediately related to the present one are Chun and Thomson (1990a, b). In the first paper, the authors consider an axiom called Disagreement Point Concavity (DCAV) which can be seen as the proper "dual" of the Superadditivity axiom of Perles and Maschler (1981). In the presence of PO and SIR (or also Individual Rationality: $f(S,d) \geqslant d$ always), this axiom implies our CONV and DVEX axioms. Under weak additional assumptions, DCAV is inconsistent with Invariance; in particular, it is not satisfied by the Nash solutions. In the second paper, the same authors obtain a characterization of the (symmetric) Nash solution which is closely related to ours. The main differences are, firstly, that they use a different relaxation of the Convexity axiom, and secondly, that they use a stronger continuity axiom (called Pareto continuity); the latter enables them to give an elementary proof that avoids the use of a fixed point argument.

4. The 2-person case. We start with a result for the 2-person case which does not generalize to the n-person case: this claim can be verified by considering the example in the text following the definition of SIR in §2. We give Theorem 4.1 not only because it is special for the 2-person case, but also because it may have interesting consequences in the theory of arbitration games; there, CONV together with a few other conditions guarantee the existence of a value. See, e.g., Tijs and Jansen (1982).

THEOREM 4.1. *Let $n = 2$ and let f be a solution on $\mathbf{B}^{\sim}$ satisfying* DVEX, DCONT, *and* SIR. *Then f satisfies* CONV.

PROOF. First note that f satisfies PO: this follows from Lemma 3.2 which can easily be seen to hold also on $\mathbf{B}^{\sim}$. Let (S, d^1) and (S, d^2) be 2-person bargaining problems in $\mathbf{B}^{\sim}$, with $d^1 \neq d^2$. Let us write $f(x)$ instead of $f(S,x)$, and suppose $f(d^1) = f(d^2) =: y \in P(S)$. Let $0 < \mu < 1$ and $e = \mu d^1 + (1-\mu)d^2$. We have to prove that $f(e) = y$. Suppose this is not true. Then, in view of DVEX of f, we must have

$$[e, f(e)] \cap [d^1, y] = \varnothing = [e, f(e)] \cap [d^2, y] \tag{4.1}$$

where $[\cdot, \cdot]$ denotes again a line segment. First note that in view of PO and SIR of f, (4.1) can only hold if $d^1 > d^2$ or $d^2 > d^1$. Suppose that the latter holds, and that $f_2(e) > y_2$ ($f_1(e) > y_1$ can be taken care of in the same way); this can only hold without violation of (4.1) if $f(e)$ lies above $l(d^1, d^2)$. Then, by DCONT of f, there must be a point z on $[e, d^2]$ with $f(z) \in l(d^1, d^2)$. So by DVEX, $f(d^2) = f(z) = y$; and again by DVEX, $f(e) = f(d^1) = y$, contrary to what we assumed. ■

Our next result shows that, for the 2-person case and on $\mathbf{B}^{\sim}$, the five axioms occurring in Theorem 3.1 still imply IIA, even with INV weakened. The weakening of Invariance that we will use is usually called *homogeneity* (HOM) and is obtained from INV by requiring $a_1 = a_2 = \cdots = a_n$ in its definition (see footnote 5). We first prove the following lemma, which for $n = 2$ partially extends Lemmas 3.3 and 3.4 to $\mathbf{B}^{\sim}$.

LEMMA 4.2. *Let f be a 2-person bargaining solution satisfying* SIR, DVEX, *and* DCONT *on $\mathbf{B}^{\sim}$. Then $f(S,e) = f(S,d)$ for every problem (S,d) in $\mathbf{B}^{\sim}$ and every $e \in S$ on the straight line through d and $f(S,d)$ whenever $d \neq f(S,d)$ and $f(S,d)$ is not an endpoint of $P(S)$. Under the same conditions, if f additionally satisfies* INIR *and (T,d) is another problem in $\mathbf{B}^{\sim}$ such that S and T coincide in a neighbourhood of $f(S,d)$, then $f(T,d) = f(S,d)$.*

PROOF. Let (S, d) and e be as above. Note that f satisfies PO as well (cf. Lemma 3.2 which also holds on $\mathbf{B}^{\sim}$). The proof of the first statement uses the same ideas as the proof of the corresponding Lemma 3.3. First assume that d is an interior point of S and without loss of generality assume that S is compact. For $x \in S$ let $g(x)$ be again the boundary point of S on the straight line through d and $f(S, x)$ and different from $f(S, x)$. Then it follows that $f(S, e) = f(S, d)$ by using a fixed point argument in exactly the same way as in the proof of Lemma 3.3. If d is not an interior point of S but $f(S, d)$ is not an endpoint of $P(S)$, then there is an interior point d' of S on the line segment connecting d and $f(S, d)$, so that the result follows from applying the first step to d' rather than d.

In order to prove the second statement of the lemma, we just note that it is sufficient to modify the proof of Lemma 3.4 by replacing DLIN there with the first statement in this lemma. ■

(Actually, we do not really need to use a fixed point argument in the above proof, but it makes the proof more elegant.) It will be convenient to refer to the two implications in Lemma 4.2 as Weak DLIN (WDLIN) and Weak LOC (WLOC), respectively.

THEOREM 4.3. *For $n = 2$ and on $\mathbf{B}^{\sim}$ we have*:

$$\text{HOM} \cap \text{DVEX} \cap \text{DCONT} \cap \text{SIR} \cap \text{INIR} \subset \text{IIA}.$$

PROOF. Let the 2-person bargaining solution f on $\mathbf{B}^{\sim}$ satisfy the five axioms in the condition of the theorem. Then, in view of Lemmas 3.2 (which can be seen to hold also on $\mathbf{B}^{\sim}$) and 4.2, f satisfies PO, WDLIN, and WLOC as well. Let $(S, d), (T, d) \in \mathbf{B}^{\sim}$ with $S \subset T$ and $f(T, d) \in S$. In view of HOM we may assume without loss of generality that $d = 0$ and write $f(T)$ instead of $f(T, d)$, etc. We have to prove that $f(S) = f(T)$. For this, we may also assume that $f(T) > 0$ for otherwise we are done in view of PO and SIR. We will assume $f(S) \neq f(T)$, say $f_2(S) > f_2(T)$, and derive a contradiction.

First note that we may assume that $f(S)$ is not an endpoint of $P(S)$: otherwise we continue the proof with some point d' on the line segment connecting 0 and $f(T)$ with $f(S, d')$ an interior point of $P(S)$ (by DCONT) and with $f(T, d') = f(T)$ (by DVEX).

Next, we start by considering the case where $\theta f(S)$ is an interior point of T for some $\theta > 1$ (see Figure 2). Then $\theta S \cap T$ coincides locally with θS in a neighbourhood of $\theta f(S)$ so that $f(\theta S \cap T) = f(\theta S) = \theta f(S)$ by WLOC and HOM (note that

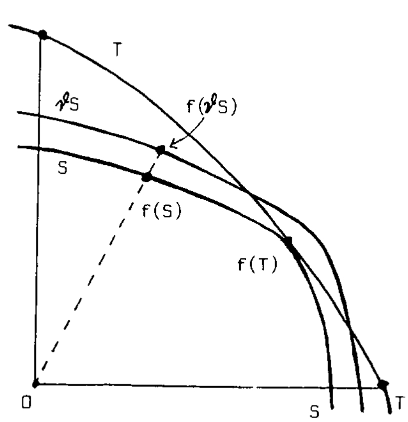

FIGURE 2.

we may apply WLOC since $f(\theta S)$ is not an endpoint of $P(\theta S)$). On the other hand, $f(T)$ is an interior point of θS so that $\theta S \cap T$ coincides locally with T around $f(T)$. If $f(T)$ is not an endpoint of $P(T)$, then we may conclude $f(\theta S \cap T) = f(T)$ by WLOC. If $f(T)$ happens to be the lower endpoint of $P(T)$, then we can find some point e on the line segment connecting 0 and $f(\theta S)$ such that still $f(\theta S \cap T, e) = f(\theta S, e) = \theta f(S)$ by the first part of this paragraph and DVEX, and such that $f(T, e)$ is above $f(T)$ but still an interior point of θS. Then, by WLOC, we conclude $f(T, e) = f(\theta S \cap T, e)$. Altogether we have $f(T) = \theta f(S)$ in the first case, or $f(T, e) = \theta f(S)$ in the second case, with $\theta f(S)$ an interior point of T, so a violation of PO.

Next, continue to assume $f(S) \neq f(T)$ and let $d(\alpha) = \alpha f(T)$ for $0 \leqslant \alpha \leqslant 1$. The previous paragraph of the proof and DVEX imply that, if $f(S, d(\alpha)) \neq f(T) = f(T, d(\alpha))$, then $\theta f(s, d(\alpha)) \notin T$ for all $\theta > 1$. Therefore, DCONT implies that the whole segment of $P(S)$ in between $f(S)$ and $f(T)$ must belong to $P(T)$. Now let $0 \leqslant \alpha \leqslant 1$ be such that $f(S, d(\alpha))$ lies strictly in between $f(S)$ and $f(T)$. Then WLOC implies that $f(S, d(\alpha)) = f(T, d(\alpha)) = f(T, d)$, a contradiction. So we must have $f(S) = f(T)$, and IIA of f. ■

COROLLARY 4.4. *On* $\mathbf{B}^{\sim}$ *we have*: $f \in$ INV $\cap$ DVEX $\cap$ DCONT $\cap$ INIR $\cap$ SIR *if and only if* $f = N^t$ *for some* $t > 0$ *with* $t_1 + t_2 = 1$.

PROOF. Every N^t satisfies the axioms in the corollary. For the converse, note that f satisfies IIA by Theorem 4.3. The desired conclusion is then a standard result of axiomatic bargaining theory (see, e.g., Roth 1979, p. 16). ■

5. The continuous Raiffa solution. In Peters (1986b), the disagreement point axiom approach was used to obtain characterizations of some of the most well-known solutions in literature, within one system of axioms. Apart from the symmetric Nash solution $N = N^{(1/2, 1/2)}$, the following solutions were characterized: the *Kalai-Smorodinsky solution* (Kalai and Smorodinsky 1975), the *Kalai-Rosenthal solution* (Kalai and Rosenthal 1978), the *egalitarian solution* (e.g., Kalai 1977), and the so-called *Continuous Raiffa solution* (Raiffa 1953). In this section, we limit attention to this last solution and its characterization. All the solutions mentioned here are 2-person, symmetric solutions.

Let CR denote the Continuous Raiffa solution. It is defined as follows. Let $(S, d) \in \mathbf{B}$ and let $h(S, d)$ denote the *utopia point* of (S, d), where $h_i(S, d) = \max\{x_i \colon x \geqslant d\}$ for $i = 1, 2$. If $d < h(S, d)$, then let R_S be the (unique) solution of the differential equation $dx_2/dx_1 = r_S(x)$ (x in the interior of S) with $R_S(d_1) = d_2$, where $r_S(x)$ is the slope of the straight line through x and $h(S, x)$. For this case $\mathrm{CR}(S, d) \in P(S)$ is defined to be the limit point of the graph of R_S. Otherwise, let $\mathrm{CR}(S, d)$ be equal to the unique Pareto optimal point weakly dominating d. For technical details, see Peters (1986b) or Livne (1989).[6]

Next, we present some additional axioms needed for the characterization of the Continuous Raiffa solution CR. Again, f denotes a generic 2-person bargaining solution and (S, d) a generic 2-person bargaining problem in $\mathbf{B}$. The *disagreement point set* of (S, d) with respect to f is the set $D(S, d, f)$ defined by:

$$D(S, d, f) = \{x \in S \colon f(S, x) = f(S, d)\}.$$

[6]These technical details are, in particular, well definedness and uniqueness. Further, we note that the CR-solution is the "continuation" of the Kalai-Smorodinsky-solution. See also Furth (1988).

Disagreement point Monotonicity (DMON). If $e \in S$ with $e_i = d_i$ and $e_j > d_j$, then $f_j(S, e) \geqslant f_j(S, d)$, for $i, j = 1, 2$ with $i \neq j$.

Strong Disagreement point Monotonicity (SDMON). As DMON with " $>$ " instead of " $\geqslant$ ", but only if such a point $f(S, e)$ exists.

Differentiability (DIFF). If $D(S, d, f)$ is the graph of a function on some interval $(\alpha, f_1(S, d)]$ (where possibly $\alpha = -\infty$), then this function is differentiable.

Disagreement Point Set Invariance (DPSI). If $(T, d) \in \mathbf{B}$ is problem with $\{x \in T: x \ngtr d\} = \{x \in S: x \ngtr d\}$, then $\{x \in D(T, d, f): x \ngtr d\} = \{x \in D(S, d, f): x \ngtr d\}$.

Symmetry (SYM). If (S, d) is a symmetric problem, i.e. (S, d) is invariant under interchanging the coordinates, then $f_1(S, d) = f_2(S, d)$.

A few comments on these axioms are in order. The interpretations of (S)DMON and SYM are obvious; DMON was first formulated by Thomson (1986) and Wakker (1987). DIFF is mainly a technical requirement, although some justification might be found by considering the disagreement point set as a kind of negotiation path: DIFF then requires smoothness, no sudden changes of direction. The DPSI axiom says that, if two problems differ "only" in their individually rational subsets, then their disagreement point sets should be the same as far as nonindividually rational points are concerned. An informal interpretation might read as follows. Imagine that the set of disagreement points leading to the same solution outcome constitutes some time path; then DPSI claims that negotiations ("intermediate" disagreement points) in an early stage are independent of the precise options attainable in the end, but only depend on the momentary utopia point. The Nash solution $N^{(1/2, 1/2)}$ typically does not have this property (recall that it satisfies LOC), all the other solutions mentioned above do have it. Note that the DPSI axiom involves an implicit comparison of different feasible sets in a certain class, and hence does not exclusively concern changes in the disagreement point.

We need the following lemma for the characterization of the Continuous Raiffa solution. We call a solution f *Individually Rational* (IR) if always $f(S, d) \geqslant d$.

LEMMA 5.1. *Let f satisfy* PO, INIR, DCONT, *and* SDMON. *Let (S, d) be a 2-person bargaining problem in* $\mathbf{B}$ *containing more than one Pareto optimal point. Then f satisfies* IR, *and if $h(S, d) > d$, then on the interval $[d_1, f_1(S, d)]$, $D(S, d, f)$ is the graph of a strictly increasing function.*

PROOF. By INIR, $f(S, d) = f(S_d, d)$, which proves the first statement in the lemma. For the second statement, suppose $h(S, d) > d$. By PO, IR, and SDMON, $f(S, d)$ must be an interior Pareto optimal point. Let α be an interior number in the mentioned interval. By SDMON, $f_1(S, (\alpha, d_2)) > f_1(S, d)$, and by PO and Individual Rationality, we have $f_2(S, m(\alpha)) > f_2(S, d)$ where $m(\alpha)$ is the upper boundary point of S with first coordinate α. So by DCONT, PO, and SDMON, there is a unique number $g(\alpha)$ with $f(S, (\alpha, g(\alpha))) = f(S, d)$. Then g is the desired function. ■

THEOREM 5.2. *The Continuous Raiffa solution* CR: $\mathbf{B} \Rightarrow \mathbf{R}$ *is the unique solution satisfying the axioms* DCONT, SDMON, DIFF, PO, INIR, DPSI, INV, *and* SYM.

PROOF. We leave it to the reader to verify that CR satisfies the eight axioms in the theorem. See also Peters (1986b) or Livne (1987). Now let f be a solution satisfying the eight axioms, and let (S, d) be a 2-person problem in $\mathbf{B}$. Let $h(S, d) > d$; otherwise, we are done in view of PO and the first statement in Lemma 5.1. By INIR, we may suppose that $S = \{x \in \mathbf{R}^2: x \leqslant y$ for some $y \in S_d\}$, and by Lemma 5.1, we have that on the interval $[d_1, f_1(S,d)]$, $D(S,d,f)$ is the graph of a strictly increasing function, say j. By DIFF, j must be differentiable.

Next consider $T = \{x \in \mathbf{R}^2\colon x \leqslant y$ for some y in the convex hull of the points $v = (d_1, h_2(S,d))$ and $w = (h_1(S,d), d_2)\}$. By SYM, INV, PO, and SDMON, we have that $D(T,d,f)$ is the straight line through d and $\frac{1}{2}(v+w)$. So by DPSI and DIFF, the slope of this straight line is exactly the right derivative of the function j in d_1. By noting that this slope is equal to $r_S(d)$ in the definition of the Continuous Raiffa solution, and so $j = R_S$, we complete the proof. ∎

In the above theorem we use eight axioms to characterize one solution. If we omit SYM, then still there are 2 axioms more than in Theorem 3.1. However, the result is tight in the sense that there is no nontrivial relaxation of the axioms without violating the result (see Peters 1986b for most of the details). Every axiom in itself is quite weak, and is satisfied by a number of nontrivially different solutions. Furthermore, all axioms are explicit, whereas many authors rely on one or more implicit axioms, notably INIR (see our discussion in §2). Livne (1989) provides another characterization which avoids the use of the technical axiom DIFF. On the other hand, he implicitly uses the INIR axiom by his choice of domain. Also Livne uses eight axioms; the intersection with our set of axioms consists of: INIR, SDMON, PO, INV, SYM. He further appeals to a stronger continuity axiom, namely FCONT.

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