

Testing for Parameter Stability in Dynamic Models across Frequencies*

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Abstract

This paper contributes to the econometric literature on structural breaks by proposing a test for parameter stability in vector autoregressive (VAR) models at a particular frequency ω , where $\omega \in [0, \pi]$. When a dynamic model is affected by a structural break, the new tests allow for detecting which frequencies of the data are responsible for parameter instability. If the model is locally stable at the frequencies of interest, the whole sample size can then be exploited despite the presence of a break. The methodology is applied to analyse the productivity slowdown in the US, and the outcome is that local stability concerns only the higher frequencies of data on consumption, investment and output.

I. Introduction

Tests for structural changes are important tools in the statistical analysis of economic time series. In this respect, the well-known Chow (1960) test still constitutes a standard reference. It consists of splitting the sample into two sub-periods, before and after the break, and testing the equality of the parameters between the two sub-samples, using an asymptotic chi-squared

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distribution. Because of its simplicity of implementation, it is still used in many empirical studies. Nonetheless, this test has been extended in several directions.¹

First, instead of considering the date of the break as known, the testing procedure should treat it as an unknown parameter to be estimated. Following the seminal paper of Quandt (1960), a recursive sequence of Chow tests could be performed, dating the break at the point where the test statistic takes the largest value. Andrews (1993) delivers the most important contribution for this extension by defining the asymptotic distribution of the sup-Chow test, which is no longer a chi-squared distribution. A further extension in this direction is developed by Bai and Perron (1998, 2003), who consider the case of multiple structural breaks with unknown dates. These authors propose several iterative methods to test for the number of breaks, and derive the asymptotic distributions of the relevant test statistics. All these procedures are valid for single-equation models with no trending regressors, such as deterministic trends or $I(1)$ processes.

The above approaches have recently been extended to multivariate regression models. Bai, Lumsdaine and Stock (1998) generalized the single-break framework in Andrews (1993) to multiple time series that are either stationary or cointegrated in the regimes of parameter stability. They showed that statistical inference is more precise when series with a common break are analysed jointly. Bai (2000) considered the issue of multiple breaks in a segmented stationary vector autoregressive (VAR) model and proved that the number of change points can be consistently estimated via information criteria, whereas Qu and Perron (2004) proposed a quasi-maximum likelihood approach to analyse multiple breaks in multivariate regression models.

Secondly, some papers have been devoted to the application of the standard Chow test to the vector error correction model (VECM). Hansen (2003), *inter alia*, provided tests for a break in the coefficients of the VECM, though his results are restricted to the case of known break dates. In particular, a partial structural change can be present in the cointegration parameters or in the adjustment coefficients. Such an extension has interesting economic implications, as it is possible to interpret a structural break as affecting the long run (partial change in the cointegration relationships) or the short run (partial change in the adjustment coefficients).

The present paper generalizes the previous idea by proposing a test for parameter stability in a segmented stationary or cointegrated VAR model at a particular frequency ω , where $\omega \in [0, \pi]$. Hence, if a VAR model is affected by a structural break, it is then possible to detect which frequencies of the data are responsible for parameter instability. Moreover, if a researcher wishes to

¹See Hansen (2001) for a detailed survey of the current state of the art.

concentrate on a subset of the frequencies of the data, the proposed test allows one to check whether the whole sample size can be exploited for the analysis, despite the presence of a break. Although the null hypothesis of local stability at frequency ω implies that the spectral density matrix at frequency ω is stable over time, the testing procedure is easily implemented in the time domain,² as it is based on a set of linear hypotheses on the autoregressive parameters. The test statistic for local stability at a given frequency has a limiting chi-squared distribution when the break date is either known or estimated by means of the sup-Chow test for a full structural change. For the latter case, a bootstrap procedure is also offered. We evaluate the finite-sample behaviour of our testing procedure through a Monte Carlo study.

The test procedure is applied to analyse the slowdown of the post-war United States output growth. Following King *et al.* (1991), we consider a trivariate system with consumption, investment and output to get a clearer view on this issue. Similar to Bai, Lumsdaine and Stock (1998), a structural break is detected in the late 1960s, and our local stability tests reveal that the system is stable only at high frequencies. This evidence is consistent with the view that a negative productivity shock is at the origin of the break.

The paper is organized as follows. In section II, the concept of local stability at frequency ω is developed for segmented stationary VAR systems and known break dates. The extensions for cointegrated systems and unknown break dates are proposed in section III. In section IV, a simulation study is performed to investigate the properties of the tests. Section V presents the empirical application, and section VI concludes.

II. Local stability in stationary VAR models

Let us consider an n -vector time series $\{X_t, t = 1, \dots, T\}$ generated by the following stationary linear stochastic process

$$X_t = \Theta D_t + C(L)\varepsilon_t, \quad \text{where } C(L) = I_n + \sum_{i=1}^{\infty} C_i L^i \quad (1)$$

such that

$$\sum_{j=1}^{\infty} j |C_j| < \infty,$$

ε_t are independent and identically distributed (i.i.d.) $N_n(0, \Sigma)$ innovations, and D_t is an m -vector of deterministic terms that may contain a constant and various trigonometric functions of time.

²Similar approaches can be found in Breitung and Candelon (2006) for Granger-causality tests, Centoni and Cubadda (2003) for measuring the cyclical effects of permanent–transitory shocks, and Christiano and Vigfusson (2003) for maximum likelihood analysis of business cycle models.

We assume that the series X_t admits the following VAR(p) representation:

$$A(L)X_t = \Phi D_t + \varepsilon_t, \quad t = 1, \dots, T, \quad (2)$$

where

$$A(L) = I_n - \sum_{i=1}^p A_i L^i$$

is such that $\det [A(c)] = 0$ implies that $|c| > 1$, and $\Phi D_t = A(L)\Theta D_t$.³

By expanding the polynomial matrix $A(L)$ on the complex conjugate points z and z^{-1} , where $z = \exp(-i\omega)$ and $\omega \in [0, \pi]$, we obtain

$$A(L) = \Delta_\omega(L) - \Pi_\omega(L)L - \Gamma_\omega(L)\Delta_\omega(L)L, \quad (3)$$

where $\Gamma_\omega(L)$ is an $n \times n$ polynomial matrix of order $(p - 3)$ if $\omega \in (0, \pi)$, $(p - 2)$ if $\omega = 0$ or $\omega = \pi$,⁴ and

$$\Delta_\omega(L) = \begin{cases} 1 - 2\cos(\omega)L + L^2 & \text{if } \omega \in (0, \pi) \\ (1 - zL) & \text{if } \omega = 0 \text{ or } \omega = \pi. \end{cases}$$

Comparing both sides of equation (3) for $L = z$ yields

$$A(z) = -\Pi_\omega(z)z, \quad (4)$$

and, by equating real and imaginary parts of equation (4), we find

$$\Pi_\omega(L) = \begin{cases} -\text{Im}[A(z)]/\sin(\omega) + (\text{Re}[A(z)] \\ \quad + \text{Im}[A(z)]\cos(\omega)/\sin(\omega))L & \text{if } \omega \in (0, \pi) \\ -zA(z) & \text{if } \omega = 0 \text{ or } \omega = \pi. \end{cases}$$

Finally, by inserting equation (3) into equation (2), we rewrite the VAR model as follows:

$$\Delta_\omega(L)X_t = \Phi D_t + \Pi_\omega(L)X_{t-1} + \Gamma_\omega(L)\Delta_\omega(L)X_{t-1} + \varepsilon_t, \quad (5)$$

As the filter $\Delta_\omega(L)$ annihilates at $L = z$, the filtered series $\Delta_\omega(L)(X_t - \Theta D_t)$ have null spectra at frequency ω . Hence the parameters $\Pi_\omega(L)$ fully characterize the stochastic behaviour of series $(X_t - \Theta D_t)$ at frequency ω . Indeed, the spectral density matrix of the stochastic process $(X_t - \Theta D_t)$ at frequency ω is given by $C(z)\Sigma C(z^{-1})'$, where $C(z) = -(\Pi_\omega(z)z)^{-1}$.⁵

The frequency-domain properties of model (2) are also determined by the nature of the deterministic vector D_t . Indeed, a linear combination of the

³The reason why we assume stationarity is twofold. First, the spectral density matrix is well defined only for stationary VAR processes. Secondly, the asymptotic theory of structural break tests does not generally allow for unit or explosive roots (see, *inter alia*, Andrews, 1993; Bai and Perron, 1998, 2003; Bai, 2000).

⁴For reasons that will be clarified later, we are assuming that $p > 2$.

⁵Note that similar reparametrizations of the VAR model are widely used in the context of seasonal cointegration analysis (see, e.g. Cubadda, 2001).

trigonometric functions $[\cos(\omega t), \sin(\omega t)]$ has its spectral mass entirely concentrated at frequency ω . Hence, let us write

$$\Phi D_t = \Phi_1 D_{1,t} + \Phi_2 D_{2,t},$$

where $D_{1,t}$ and $D_{2,t}$ are composed of m_1 and m_2 elements, respectively, such that

$$\Delta_\omega(L)D_{1,t} = 0,$$

$$\Delta_\omega(L)D_{2,t} \neq 0.$$

It is clear that the parameters Φ_1 fully characterize the deterministic behaviour of series X_t at frequency ω .

We now allow for a possible structural break at time $T_b = \lfloor bT \rfloor$, where $b \in (0, 1)$. Let us assume, for the moment, that there is only a single break and its date T_b is known. Model (2) is then generalized by the following sub-sample models:

$$A^-(L)X_t = \Phi^- D_t + \varepsilon_t, \quad t = 1, \dots, T_b, \tag{6}$$

$$A^+(L)X_t = \Phi^+ D_t + \varepsilon_t, \quad t = T_b + 1, \dots, T, \tag{7}$$

where

$$A^-(L) = I_n - \sum_{i=1}^p A_i^- L^i \quad \text{and} \quad A^+(L) = I_n - \sum_{i=1}^p A_i^+ L^i.$$

Note that we can expand both the polynomial matrices $A^-(L)$ and $A^+(L)$ on 0 and the complex conjugate points z and z^{-1} , thus obtaining the sub-sample analogues of model (5). Hence, let us consider the following particular cases of the sub-sample models (6) and (7):

$$\begin{aligned} \Delta_\omega(L)X_t &= \Phi_1 D_{1,t} + \Phi_2^- D_{2,t} + \Pi_\omega(L)X_{t-1} + \Gamma_\omega^-(L)\Delta_\omega(L)X_{t-1} + \varepsilon_t, \\ t &= 1, \dots, T_b, \end{aligned} \tag{8}$$

$$\begin{aligned} \Delta_\omega(L)X_t &= \Phi_1 D_{1,t} + \Phi_2^+ D_{2,t} + \Pi_\omega(L)X_{t-1} + \Gamma_\omega^+(L)\Delta_\omega(L)X_{t-1} + \varepsilon_t, \\ t &= T_b + 1, \dots, T, \end{aligned} \tag{9}$$

where $\Phi_1^- = \Phi_1^+ \equiv \Phi_1$ and $\Pi_\omega^-(L) = \Pi_\omega^+(L) \equiv \Pi_\omega(L)$.

As the structural break does not affect the components of series X_t that are associated with fluctuations at frequency ω , the sub-sample models (8) and (9) are said to be locally stable at that frequency. Notice that local stability is possible only if the polynomial parameter matrices $\Gamma_\omega^-(L)$ and $\Gamma_\omega^+(L)$ can freely vary from $\Pi_\omega(L)$. Therefore, a necessary condition for

local stability at frequency ω is that $p > 2$ if $\omega \in (0, \pi)$ and $p > 1$ if $\omega = 0$ or $\omega = \pi$.

The statistical problem consists of testing for each of the following null hypotheses:

$$H_0 \text{ (global stability): } [A^-(L) = A^+(L)] \cap [\Phi^- = \Phi^+],$$

$$H_* \text{ (local stability): } [\Pi_\omega^-(L) = \Pi_\omega^+(L)] \cap [\Phi_1^- = \Phi_1^+],$$

vs. the alternative hypothesis:

$$H_1 \text{ (global instability): } [A^-(L) \neq A^+(L)] \cup [\Phi^- \neq \Phi^+].$$

In particular, the sample-split Chow test statistics (see, e.g. Doornik and Hendry, 1997) for the systems of hypotheses H_0 vs. H_1 and H_* vs. H_1 are, respectively, the following:

$$\begin{aligned} \Xi_{0|1}(b) &= (T - 2p - q_0) \\ &\times \frac{\det\left(\sum_{t=p+1}^T \hat{\varepsilon}_t' \hat{\varepsilon}_t\right) - \det\left(\sum_{t=p+1}^{T_b} \hat{\varepsilon}_t' \hat{\varepsilon}_t^- + \sum_{t=T_b+1}^T \hat{\varepsilon}_t' \hat{\varepsilon}_t^+\right)}{\det\left(\sum_{t=p+1}^{T_b} \hat{\varepsilon}_t' \hat{\varepsilon}_t^- + \sum_{t=T_b+1}^T \hat{\varepsilon}_t' \hat{\varepsilon}_t^+\right)} \xrightarrow{d} \kappa^2(q_0), \end{aligned} \quad (10)$$

$$\begin{aligned} \Xi_{*|1}(b, \omega) &= (T - 2p - q_*) \\ &\times \frac{\det\left(\sum_{t=p+1}^{T_b} \tilde{\varepsilon}_t' \tilde{\varepsilon}_t^- + \sum_{t=T_b+1}^T \tilde{\varepsilon}_t' \tilde{\varepsilon}_t^+\right) - \det\left(\sum_{t=p+1}^{T_b} \hat{\varepsilon}_t' \hat{\varepsilon}_t^- + \sum_{t=T_b+1}^T \hat{\varepsilon}_t' \hat{\varepsilon}_t^+\right)}{\det\left(\sum_{t=p+1}^{T_b} \hat{\varepsilon}_t' \hat{\varepsilon}_t^- + \sum_{t=T_b+1}^T \hat{\varepsilon}_t' \hat{\varepsilon}_t^+\right)} \\ &\xrightarrow{d} \kappa^2(q_*), \end{aligned} \quad (11)$$

where $\{\hat{\varepsilon}_t, t = 1, \dots, T\}$ are the residuals resulting from ordinary least squares (OLS) estimation of the fixed-parameter model (2), $\{\hat{\varepsilon}_t^-, t = 1, \dots, T_b\}$, $\{\hat{\varepsilon}_t^+, t = T_b + 1, \dots, T\}$, $\{\tilde{\varepsilon}_t^-, t = 1, \dots, T_b\}$, $\{\tilde{\varepsilon}_t^+, t = T_b + 1, \dots, T\}$ are the residuals resulting from OLS estimation of the sub-sample models (6), (7), (8), (9), respectively, and

$$q_0 = n^2 p + mn,$$

$$q_* = \begin{cases} 2n^2 + m_1 n & \text{if } \omega \in (0, \pi) \\ n^2 + m_1 n & \text{if } \omega = 0 \text{ or } \omega = \pi. \end{cases}$$

The statistic (10) is the usual Chow test statistic for global stability, whereas (11) is the suggested test statistic for local stability at frequency ω . These statistics may be used in a sequential fashion; starting with running the

test based on the statistic $\Xi_{0|1}(b)$. If the null hypothesis of global stability is not rejected, the sequence stops. Otherwise, one can test for local stability at frequencies

$$\omega_j = \omega_0 \left(\frac{k-j}{k} \right) + \omega_k \left(\frac{j}{k} \right), \quad \text{for } 0 \leq \omega_0 < \omega_k \leq \pi \text{ and } j = 0, 1, \dots, k,$$

by means of the test statistics $\Xi_{*|1}(b, \omega_j)$.⁶

Remark 1. As correctly pointed out by a referee, the definition of the polynomial matrix $\Pi_\omega(L)$ depends on the parameterization that is considered. We can, for instance, use an alternative representation to model (5) such as the following

$$\Delta_\omega(L)X_t = \Phi D_t + \tilde{\Pi}_\omega(L)X_{t-p+1} + \tilde{\Gamma}_\omega(L)\Delta_\omega(L)X_{t-1} + \varepsilon_t, \quad (12)$$

in which the parameters of interest for local stability are the coefficients of $[X_{t-p-1}, X_{t-p}]$ and not those of $[X_{t-1}, X_{t-2}]$. However, in the Appendix we show that constancy of $\Pi_\omega(L)$ is equivalent to that of $\tilde{\Pi}_\omega(L)$. Hence, tests for local stability are invariant to isomorphic representations of the VAR.

Remark 2. We must notice that local stability can only occur at a finite set of frequencies. Indeed, local stability at frequency ω requires that both the following conditions hold

$$A^-(L) \neq A^+(L), \quad (13)$$

$$A^-(z) = A^+(z). \quad (14)$$

Given that $A^-(L)$ and $A^+(L)$ are polynomial matrices of order p , it is clear that there can exist, at most, p different points on the complex unit circle that satisfy equation (14) without violating equation (13). Since we are considering real-valued processes, this implies that local stability may occur, at most, at $(\lfloor p/2 \rfloor + 1)$ frequencies in $[0, \pi]$.

Remark 3. It may be of interest to test for the stability of a subset of parameters only. In this case, let us write the polynomial matrix $A(L)$ in equation (2) as $A(L) = A_1(L) + A_2(L)$. If we assume that the break may solely affect the parameters in $A_2(L)$, the model (2) can be generalized as:

$$\begin{aligned} [A_1(L) + A_2^-(L)]X_t &= \Phi^- D_t + \varepsilon_t, & t = 1, \dots, T_b, \\ [A_1(L) + A_2^+(L)]X_t &= \Phi^+ D_t + \varepsilon_t, & t = T_b + 1, \dots, T, \end{aligned}$$

where

⁶Note that the choice of the interval $[\omega_0, \omega_k]$ reflects the researcher's *a priori* knowledge of the frequencies at which local stability can occur. An agnostic option is to fix $\omega_0 = 0$ and $\omega_k = \pi$.

$$A_2^-(L) = I_n - \sum_{i=1}^p A_2^- L^i \quad \text{and} \quad A_2^+(L) = I_n - \sum_{i=1}^p A_2^+ L^i.$$

We can then expand both the polynomial matrices $A_2^-(L)$ and $A_2^+(L)$ on 0 and the complex conjugate points z and z^{-1} and perform tests for both global and local stability of the parameters of interest.

III. Various extensions

This section extends the above framework in various directions. In particular, we consider the cases of the cointegrated VAR and unknown break dates.

3.1. Cointegrated time series

Let us now consider an n -vector of cointegrated time series $\{Y_t, t = 1, \dots, T\}$ of order (1, 1) that is generated by the following VAR(p) model:

$$B(L)Y_t = \Phi D_t + \varepsilon_t, \quad (15)$$

where

$$B(L) = I_n - \sum_{i=1}^p A_i L^i$$

is such that $\det [B(c)] = 0$ implies that $|c| > 1$ or $c = 1$, $B(1) = -\alpha\beta'$, α and β are $n \times r$ -matrices with rank equal to r , and the matrix $\alpha'_\perp \Gamma \beta_\perp$ has full rank, where β_\perp are $n \times (n - r)$ -matrices with rank equal to $(n - r)$ such that $\alpha'_\perp \alpha = \beta'_\perp \beta = 0$,

$$\Gamma = I_n - \sum_{i=1}^{p-1} \Gamma_i \quad \text{and} \quad \Gamma_i = - \sum_{j=i+1}^p A_j \quad \text{for } i = 1, 2, \dots, p - 1.$$

Series Y_t also admits the following Wold representation:

$$\Delta Y_t = \Theta D_t + F(L)\varepsilon_t, \quad \text{where } F(L) = I_n + \sum_{i=1}^{\infty} F_i L^i \quad (16)$$

is such that

$$\sum_{j=1}^{\infty} j |F_j| < \infty,$$

and $\Theta D_t = F(L)\Phi D_t$.

In this case, a difficulty emerges in testing for local stability at the zero frequency. As $F(1) = \beta_\perp (\alpha'_\perp \Gamma \beta_\perp)^{-1} \alpha'_\perp$ (see e.g. Johansen, 1996), the spectral density matrix of series ΔY_t is singular at $\omega = 0$. Thus, the coefficient

matrix $B(1)$ does not fully characterize the long-run behaviour of series Y_t . However, we can reparameterize model (15) in order to avoid such singularity.

Suppose that the cointegration matrix β is fixed over time and is known. Then we can transform series Y_t such that $X_t = T(L)Y_t$, where $T(L) = (\beta', \Delta\beta_{\perp}')$. Model (15) can thus be written as in equation (2), where $A(L) = B(L)T(L)^{-1}$. Notice that if a super-consistent estimate of the cointegration matrix is available, one can simply use the estimate of β instead of the unknown population values without affecting the asymptotic distributions of the test statistics (10) and (11).

However, the cointegration matrix may be affected by the structural break at time T_b as well. In this case, series Y_t can be transformed as $X_t = T(L, t)'Y_t$, where

$$T(L, t) = \begin{cases} (\beta^-, \Delta\beta_{\perp}^-)', & t = 1, \dots, T_b \\ (\beta^+, \Delta\beta_{\perp}^+)', & t = T_b + 1, \dots, T. \end{cases}$$

Again, one can substitute the matrices β^- and β^+ with their super-consistent estimates. Inference on time-varying cointegration relationships is discussed, *inter alia*, by Hansen (2003), and Andrade, Bruneau and Gregoir (2005).

3.2. Unknown break date

In the previous sections of the paper, the date of the break was considered as known beforehand. However, it is especially relevant to extend our procedure to the case where the break date is determined by means of the data itself. In such a case, Quandt (1960) proposed performing the Chow (1960) test recursively, using the supremum of the statistics. It is possible to apply this approach to the test based on equation (10) by considering the following statistic:

$$\Xi_{0|1}(\hat{b}) = \sup[\Xi_{0|1}(b)],$$

where $t = [bT]$ and $b \in (0, 1)$. Based on Andrews (1993), Bai *et al.* (1998) provided the asymptotic distribution of the above test statistic in the multivariate case.

We recommend testing for local stability at the various Fourier frequencies fixing $b = \hat{b}$. A rationale for this procedure lies in the fact that the limit distribution of the break date estimator is unaffected by the imposition of valid restrictions on the other parameters of the model, see Qu and Perron (2004). This implies that imposing local stability at a given frequency provides no efficiency gains for the break date estimation in large samples. Formally, we then propose using the test statistics:

$$\Xi_{*|1}(\hat{b}, \omega_j) \quad (17)$$

where

$$\omega_j = \omega_0 \left(\frac{k-j}{k} \right) + \omega_k \left(\frac{j}{k} \right) \quad \text{for } 0 \leq \omega_0 < \omega_k \leq \pi \text{ and } j = 0, 1, \dots, k.$$

Since Bai *et al.* (1998) proved that the estimators of the segmented VAR parameters have the same asymptotic distribution when the break date is either known or estimated, the test statistic (17) converges in distribution to the same as that of equation (11) under the null hypothesis. Nevertheless, the chi-squared distribution is sometimes a poor approximation of the exact distribution even when the break date is known (see, e.g. Candelon and Lütkepohl, 2001). Hence, we propose the following bootstrap procedure:

- 1 Compute the usual Chow test statistic $\Xi_{0|1}(b)$ and find $\hat{b} = \arg\{\sup[\Xi_{0|1}(b)]\}$ for $b \in [0.15, 0.85]$.
- 2 Save the unrestricted residuals of the sub-sample models under H_1 conditional on $b = \hat{b}$. Then obtain one matrix of residuals $\hat{\varepsilon}$.
- 3 Save the estimated parameters of the full-sample model under H_0 .
- 4 Save the estimated parameters of the sub-sample models under H_* conditional on $b = \hat{b}$.
- 5 Sample from $\hat{\varepsilon}$ h times. Then, take the estimated parameters in (3) to rebuild the data that are used to bootstrap $\Xi_{0|1}(\hat{b})$, and use the estimated parameters in (4) to rebuild the data that are used to bootstrap $\Xi_{*|1}(\hat{b}, \omega_j)$.
- 6 Obtain the bootstrap distributions of $\Xi_{0|1}(\hat{b})$, and $\Xi_{*|1}(\hat{b}, \omega_j)$ for $j = 1, \dots, k$.

The testing procedure for local stability can be extended to the case of multiple breaks with unknown dates. As shown by Bai and Perron (2003), a dynamic programming algorithm can be used to search for an optimal partition that globally maximizes the likelihood function for any given number of breaks. The number of breaks can then be determined by means of either information criteria (see Bai, 2000) or testing procedures (see Qu and Perron, 2004). After fixing the number and dates of the breaks to their estimated values, the tests for local stability can be applied to any pair of adjacent regimes. In principle, a bootstrap procedure could also be used for the case of multiple breaks. However, the combined use of dynamic programming algorithms and resampling techniques is, admittedly, rather time consuming.

IV. Simulation study

In this section, a Monte Carlo experiment is conducted to evaluate the finite-sample performances of the proposed testing procedure. In particular, we examine in a simple univariate framework, the size and power of both the asymptotic and bootstrap tests for local stability, at frequency ω (H_*) vs. global instability (H_1).⁷

To this aim, we start by considering the following simple stationary AR(3) model:

$$X_t = a_1X_{t-1} + a_2X_{t-2} + a_3X_{t-3} + \varepsilon_t,$$

where $\varepsilon_t \sim N(0, \sigma^2)$. We assume that the date-generating process (DGP) under the hypothesis H_1 has the following form:

$$X_t = a_1X_{t-1} + a_2X_{t-2} + a_3X_{t-3} + \varepsilon_t, \quad t = 1, \dots, T_b. \tag{18}$$

$$X_t = \tau(\mu + a_1X_{t-1} + a_2X_{t-2} + a_3X_{t-3}) + \varepsilon_t, \quad t = T_b + 1, \dots, T. \tag{19}$$

While the DGP under the null hypothesis H_* of local stability at frequency ω is of the form:

$$\Delta_\omega(L)X_t = \Pi_\omega(L)X_{t-1} + a_3\Delta_\omega(L)X_{t-1} + \varepsilon_t, \quad t = 1, \dots, T_b, \tag{20}$$

$$\Delta_\omega(L)X_t = \Pi_\omega(L)X_{t-1} + \tau_*[\mu + a_3\Delta_\omega(L)X_{t-1}] + \varepsilon_t, \quad t = T_b + 1, \dots, T \tag{21}$$

where $\Pi_\omega(L) = a_1 - a_3 - 2\cos(\omega) + [1 + a_2 + 2a_3\cos(\omega)]L$.

The design parameters are set at the following values: $a_1 = 0.15$, $a_2 = -0.05$, $a_3 = 0.1$, $\sigma = 1$, $\mu = 0.15$, $T = 200, 500$, $b \equiv T_b/T = 0.25, 0.50, 0.75$, $\tau = 1, 2, 3, 4$, $\tau_* = 2, 3$, and $\omega = \pi/4, \pi/2, 3\pi/4$.

Some comments on the choices of the parameter values are in order. We let the breaks occur at three different fractions of the sample and take three different sizes. Indeed, previous results in the literature suggest that the parameters b and τ are the most important in determining the performances of structural change tests (see, *inter alia*, Candelon and Lütkepohl, 2001; Bai and Perron, 2006). Notice that the break fraction b is treated as an unknown parameter to be estimated, and we use a trimming parameter equal to 15%. The AR parameters are chosen such that the process is stationary in both the regimes and for all the considered sizes of the break. We let the constant term vary across the regimes because both the models, (18, 19) and (21, 22), are locally unstable at the zero frequency.

⁷For a detailed analysis of the bootstrapped version of the global stability test (H_0 vs. H_1), the reader can refer to Diebold and Chen (1996) or Candelon and Lütkepohl (2001).

TABLE 1

Rejection rates of 5% level tests under the null hypothesis of local stability at frequency ω

T_b/T :	Asymptotic test			Bootstrap test		
	0.25	0.50	0.75	0.25	0.50	0.75
$T = 200; \tau_* = 2$						
$\omega = \pi/4$	0.192	0.164	0.186	0.052	0.050	0.054
$\omega = \pi/2$	0.172	0.160	0.184	0.056	0.078	0.076
$\omega = 3\pi/4$	0.134	0.134	0.160	0.068	0.064	0.068
$T = 200; \tau_* = 3$						
$\omega = \pi/4$	0.136	0.100	0.108	0.056	0.060	0.052
$\omega = \pi/2$	0.130	0.082	0.102	0.060	0.058	0.062
$\omega = 3\pi/4$	0.070	0.056	0.070	0.068	0.054	0.062
$T = 500; \tau_* = 2$						
$\omega = \pi/4$	0.166	0.120	0.140	0.064	0.054	0.058
$\omega = \pi/2$	0.132	0.136	0.124	0.054	0.074	0.062
$\omega = 3\pi/4$	0.092	0.086	0.090	0.046	0.054	0.054
$T = 500; \tau_* = 3$						
$\omega = \pi/4$	0.088	0.076	0.074	0.058	0.056	0.056
$\omega = \pi/2$	0.072	0.076	0.072	0.062	0.056	0.062
$\omega = 3\pi/4$	0.073	0.060	0.062	0.062	0.064	0.060

The rejection rates of the tests for local stability are based on both the asymptotic and bootstrap critical values at the 5% level. In each experiment, 500 series of length $T + 50$ are generated with initial values set to zero. The first 50 observations are discarded to eliminate dependence resulting from the starting conditions. For the bootstrap tests, 500 bootstrap draws are performed in each of the 500 replications.

Table 1 shows the rejection rates of the tests at the 5% level when the DGP is given by the processes (20, 21). We see that the rejection rates of the bootstrap test are always quite close to the nominal size, while we also see that the asymptotic test tends to be oversized, especially when $T = 200$, $\tau_* = 2$, and T_b/T differs from 0.50. With $T = 200$, the bootstrap test is better sized than the asymptotic one for all the 18 experiments and 15 differences between the rejection rates are indeed significant.⁸ Even with $T = 500$ the bootstrap test is less size-distorted in 17 experiments and 12 differences between the rejection rates are significant. Interestingly, the empirical sizes of the two tests are more similar when the model is locally stable at frequency $3\pi/4$.

In order to evaluate the effects of the break at frequencies $\pi/4$, $\pi/2$ and $3\pi/4$ under the alternative hypothesis H_1 , Table 2 reports the spectra of the processes (18, 19) at those frequencies. It emerges that the effect of the break,

⁸We consider a difference between the rejection frequencies as insignificant when it is smaller than twice the Monte Carlo standard error at the nominal 5% level, i.e. 0.02.

TABLE 2
Spectrum of model (19)

Break size	Frequencies		
	$\pi/4$	$\pi/2$	$3\pi/4$
$\tau = 1$	0.992	1.105	0.696
$\tau = 2$	0.972	1.220	0.494
$\tau = 3$	0.942	1.342	0.362
$\tau = 4$	0.905	1.471	0.274

as measured by the relative change in the spectrum at the frequency of interest, is the strongest at frequency $3\pi/4$.

We report in Table 3 the rejection rates of both the asymptotic and bootstrap tests at the 5% level when the DGP is given by the processes (18, 19). Given the size distortions of the asymptotic test, caution is needed in comparing the empirical power of the two tests. However, the asymptotic test rejects more often in almost all the experiments. The two tests tend to have similar power as the parameters τ and T increase, as well as when the null hypothesis is local stability at frequency $\pi/4$. For both the tests, it appears that for a break size of two or three times the standard deviation of the errors, the power is relatively low even if a large sample is considered. This result indicates that the size of the break should be large enough to make the distinction between local and global stability. As in Candelon and Lütkepohl (2001), it is observed that the rejection frequency is generally lower when the break is located at the borders of the sample (i.e. $T_b/T = 0.25, 0.75$). The results reveal to us that the frequency at which the break occurs is also important for the empirical power. As expected in view of Table 2, the power is the highest when the break occurs at frequency $3\pi/4$ as the relative change in the spectrum under H_1 is the strongest at that frequency.

A practical problem arises when one ignores the frequencies at which local stability may occur. In such a situation, it is necessary to apply the test at several different frequencies to determine those at which the model is potentially locally stable.⁹ Hence, it is important to evaluate the power of the test for local stability at frequency ω_* when the model is locally stable at a frequency $\omega \neq \omega_*$. In order to save space, only the results relative to $\tau_* = 3$ are reported in Table 4. As expected, the larger the difference between ω and ω_* , the larger the rejection rates. Indeed, the power is the lowest when $\omega = \pi/2$. Similar to the results for the alternative hypothesis of global instability, the asymptotic and bootstrap tests tend to perform more similarly when $T = 500$.

⁹A referee suggested an iterative estimation procedure of the frequencies at which local stability occurs that is similar in spirit to the one that Bai and Perron (1998) proposed to detect multiple breaks. We leave to future research a formal analysis of the properties of such a procedure.

TABLE 3

Rejection rates of 5% level tests under the alternative hypothesis of global instability

$T_b/T:$	<i>Asymptotic test</i>			<i>Bootstrap test</i>		
	0.25	0.50	0.75	0.25	0.50	0.75
$T = 200; \tau = 2$						
$\omega = \pi/4$	0.222	0.206	0.200	0.068	0.064	0.068
$\omega = \pi/2$	0.170	0.176	0.198	0.066	0.070	0.078
$\omega = 3\pi/4$	0.282	0.256	0.234	0.144	0.136	0.112
$T = 200; \tau = 3$						
$\omega = \pi/4$	0.278	0.286	0.206	0.114	0.184	0.146
$\omega = \pi/2$	0.154	0.158	0.174	0.084	0.104	0.116
$\omega = 3\pi/4$	0.360	0.400	0.296	0.274	0.336	0.230
$T = 200; \tau = 4$						
$\omega = \pi/4$	0.432	0.538	0.404	0.272	0.512	0.372
$\omega = \pi/2$	0.134	0.198	0.180	0.120	0.198	0.170
$\omega = 3\pi/4$	0.636	0.672	0.456	0.608	0.690	0.456
$T = 500; \tau = 2$						
$\omega = \pi/4$	0.264	0.236	0.192	0.126	0.136	0.088
$\omega = \pi/2$	0.168	0.160	0.188	0.082	0.090	0.074
$\omega = 3\pi/4$	0.312	0.348	0.286	0.172	0.226	0.158
$T = 500; \tau = 3$						
$\omega = \pi/4$	0.502	0.614	0.438	0.416	0.596	0.412
$\omega = \pi/2$	0.158	0.216	0.206	0.134	0.196	0.174
$\omega = 3\pi/4$	0.692	0.800	0.600	0.638	0.778	0.572
$T = 500; \tau = 4$						
$\omega = \pi/4$	0.888	0.968	0.863	0.878	0.968	0.851
$\omega = \pi/2$	0.306	0.432	0.335	0.296	0.442	0.297
$\omega = 3\pi/4$	0.954	0.966	0.942	0.944	0.972	0.934

In empirical applications, the order of the AR process is unknown. It is thus of interest to investigate the robustness of the local stability test at frequency ω when the model dynamics are misspecified.¹⁰ The test would clearly be inconsistent if the true order is underestimated. Hence, it is of interest to examine the implications of choosing the AR order in a liberal fashion. However, simulations would become too time-consuming if we allow for estimating the AR order within the bootstrap procedure. Hence, we analyse the effects of using an AR order which has one lag more than the true one, on the size and power of our stability tests. The experiments are performed for the case $T = 200$ and $\tau = \tau^* = 3$, which is quite representative of the other DGPs, and the results on the size and power of the test are reported in Tables 5 and 6, respectively. It turns out that the effect of over-parametrization on size is rather limited. Indeed, only few rejection rates in Table 5 are significantly larger than

¹⁰We thank an anonymous referee for pointing out this issue.

TABLE 4

Rejection rates of 5% level tests for local stability at frequency ω_* under local stability at frequency $\omega \neq \omega_*$ with $\tau_* = 3$

T_b/T :	Asymptotic test			Bootstrap test			
	0.25	0.50	0.75	0.25	0.50	0.75	
<i>T = 200</i>							
$\omega = \pi/4$	$\omega_* = \pi/2$	0.324	0.456	0.412	0.200	0.334	0.268
$\omega = \pi/4$	$\omega_* = 3\pi/4$	0.526	0.566	0.418	0.386	0.460	0.294
$\omega = \pi/2$	$\omega_* = \pi/4$	0.266	0.268	0.188	0.120	0.156	0.124
$\omega = \pi/2$	$\omega_* = 3\pi/4$	0.194	0.200	0.164	0.126	0.160	0.118
$\omega = 3\pi/4$	$\omega_* = \pi/2$	0.740	0.854	0.700	0.424	0.814	0.658
$\omega = 3\pi/4$	$\omega_* = \pi/4$	0.344	0.346	0.238	0.288	0.320	0.238
<i>T = 500</i>							
$\omega = \pi/4$	$\omega_* = \pi/2$	0.630	0.810	0.718	0.550	0.758	0.652
$\omega = \pi/4$	$\omega_* = 3\pi/4$	0.858	0.930	0.816	0.794	0.900	0.754
$\omega = \pi/2$	$\omega_* = \pi/4$	0.554	0.656	0.474	0.450	0.612	0.408
$\omega = \pi/2$	$\omega_* = 3\pi/4$	0.316	0.384	0.296	0.276	0.360	0.258
$\omega = 3\pi/4$	$\omega_* = \pi/2$	0.996	0.998	1.000	0.992	0.998	0.998
$\omega = 3\pi/4$	$\omega_* = \pi/4$	0.730	0.802	0.620	0.706	0.796	0.618

TABLE 5

Rejection rates of 5% level tests under the null hypothesis of local stability at frequency ω based on an AR(4) model

T_b/T :	Asymptotic test			Bootstrap test		
	0.25	0.50	0.75	0.25	0.50	0.75
<i>T = 200; $\tau_* = 3$</i>						
$\omega = \pi/4$	0.156	0.116	0.136	0.072	0.072	0.064
$\omega = \pi/2$	0.102	0.070	0.082	0.066	0.056	0.050
$\omega = 3\pi/4$	0.088	0.072	0.060	0.074	0.080	0.052

TABLE 6

Rejection rates of 5% level tests under the alternative hypothesis of global instability based on an AR(4) model

T_b/T :	Asymptotic test			Bootstrap test		
	0.25	0.50	0.75	0.25	0.50	0.75
<i>T = 200; $\tau = 3$</i>						
$\omega = \pi/4$	0.248	0.258	0.206	0.126	0.172	0.142
$\omega = \pi/2$	0.144	0.132	0.240	0.086	0.108	0.098
$\omega = 3\pi/4$	0.414	0.412	0.274	0.314	0.344	0.206

the corresponding ones in Table 1. It also appears that when the model is over-parametrized, the bootstrap version has lower size distortion than the asymptotic one.

We also notice that power is slightly affected by the over-parametrization of the AR model, in particular for the case of the bootstrap test. Overall, these results suggest that local stability tests are quite robust to a liberal choice of the AR order.

V. Empirical application: output, consumption and investment

Several studies have been devoted to the analysis of the productivity slowdown in postwar US output. As the univariate analysis of the output series by Bai *et al.* (1998) lead to inconclusive results, these authors considered a trivariate system composed of consumption (C), investment (I) and output (Y). Following King *et al.* (1991), the rationale behind this idea is that a break in the productivity process should also be present in variables possessing strong long-run links with output, in particular, consumption and investment. Indeed, Bai *et al.* (1998) proved that if the stochastic growth model by King *et al.* (1988) is augmented with a break in the average growth rate of productivity, such a break will affect the three variables $c = \ln(C)$, $i = \ln(I)$, and $y = \ln(Y)$, but not the 'great ratios' ($c - y$) and ($i - y$).

We thus investigate local and global stability in the following dynamic model:

$$A_3(L) \begin{pmatrix} c_t - y_t \\ i_t - y_t \\ \Delta(c_t + y_t + i_t) \end{pmatrix} = \Phi_3 + \varepsilon_{3,t},$$

where $A_3(L)$ is a polynomial matrix of order 4,¹¹ Φ_3 is a vector of constant terms and $\varepsilon_{3,t}$ are $N_3(0, \Sigma_3)$ innovations. Quarterly data are obtained from the Saint-Louis Federal Reserve Bank and cover the period 1954Q1–2004Q4. Y_t is the private GDP per capita, C_t the real personal consumption expenditures per capita and I_t the private fixed investment per capita. The variables are seasonally adjusted and divided by the civilian non-institutional population aged 16 and over.

The parameter stability of the above model will be investigated using the bootstrap procedure for unknown breaks that was discussed in section 3.2. Following Andrews (1993), the trimming region is [0.15, 0.85], and the results, reported in Table 7, show that a break is detected in 1968Q3. Similar to Bai *et al.* (1998), the break is dated earlier than the first oil shock.

¹¹The lag length is chosen according to the Akaike information criterion. Other choices of the lag length do not qualitatively modify the results.

TABLE 7
Stability test H_0 vs. H_1

Break date	Statistic	Bootstrap p -value	Asymptotic p -value
1968q3	51.576	3.92%	<1.00%

Note: The bootstrap p -value is obtained after 5,000 replications.

In order to gain a deeper insight into the origin of the break, the local stability tests are performed for the frequencies $\omega_j = (j/100)\pi$, for $j = 1, 2, \dots, 99$. The tests statistics, along with their bootstrap 95% and asymptotic critical values, are plotted in Figure 1.

It turns out that this system appears to be unstable at the lower frequencies, in particular when ω approaches zero. It must be noticed that the empirical results seem to indicate that local stability holds for all the frequencies higher than 1, which is contradicted by Remark 2. In the light of the simulation results documented in Table 4, this phenomenon is likely due to a sort of leakage problem of local stability tests, namely power is low for frequencies that are close to the ones for which the model is stable. Breitung and Candelon

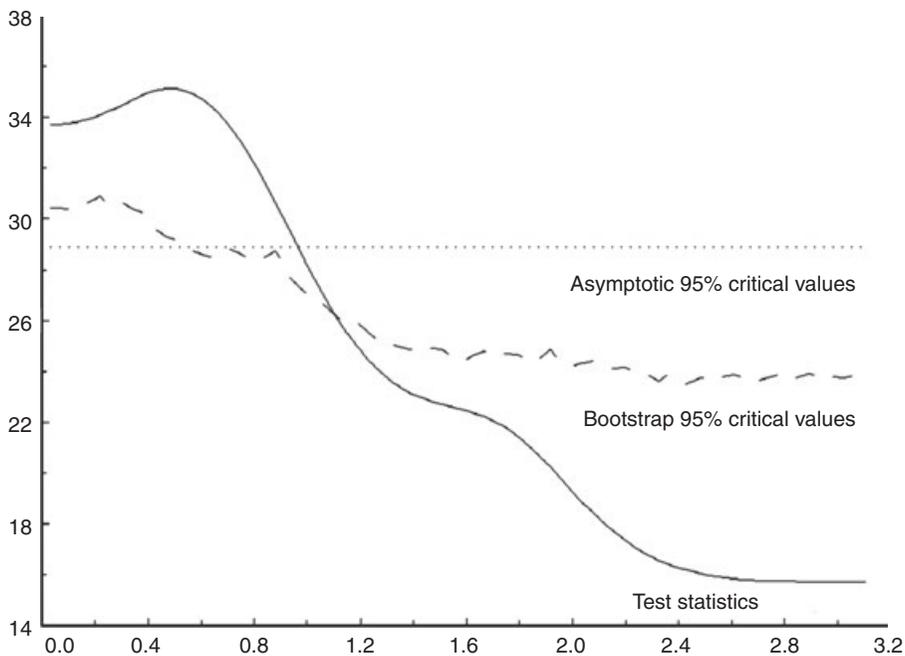


Figure 1. Tests for local stability

(2006) documented a similar problem for their causality test via a local power analysis. Nevertheless, the testing results clearly indicate that global instability concerns the low frequencies.

The empirical evidence suggests that the break in 1968Q3 can be labelled as 'real' as it affects the long-run properties of the variables. However, unlike the prediction of the theoretical model by Bai *et al.* (1998), the stochastic components of the data are also unstable at low frequencies. Therefore, a simple break in the productivity average growth rate is not, apparently, the only origin of this break.

VI. Conclusions

In this paper, we develop a new testing procedure for parameter stability at a particular frequency in a segmented stationary or cointegrated VAR model. By doing so, it is possible to determine the frequencies which are responsible for the parameter instability in a dynamic model. The local stability tests can provide a deeper insight into the origin of a structural break. The example presented in this study highlights the practical value of this procedure in empirical studies. A structural break is detected at 1968Q3 for an output–consumption–investment system, and the application of the new tests reveals that local stability exclusively concerns the high frequency components of the data. This evidence suggests that a real productivity shock is likely to be at the origin of the structural break.

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Appendix

In this appendix we show that tests for local stability are invariant to reparametrizations of the VAR such as that in equation (12). Indeed, the considered alternative representation is based on the following expansion

$$A(L) = \Delta_\omega(L) - \tilde{\Pi}_\omega(L)L^{p-1} - \tilde{\Gamma}_\omega(L)\Delta_\omega(L)L,$$

which yields at $L = z$

$$A(z) = -\tilde{\Pi}_\omega(z)z^{p-1}.$$

By comparing the above equation with equation (4), we obtain the following relation

$$\tilde{\Pi}_\omega(z) = z^{2-p}\Pi_\omega(z), \quad (22)$$

which immediately reveals that $\tilde{\Pi}_\omega(L) = \Pi_\omega(L)$ when $\omega = 0$ or $\omega = \pi$ and p is even, and $\tilde{\Pi}_\omega(L) = -\Pi_\omega(L)$ when $\omega = \pi$ and p is odd. Hence, in the following we concentrate on the case $\omega \in (0, \pi)$.

By equating real and imaginary parts of both sides of equation (22) we find

$$\operatorname{Re}\{\tilde{\Pi}_\omega(z)\} = \operatorname{Re}\{\Pi_\omega(z)\} \cos[(2-p)\omega] + \operatorname{Im}\{\Pi_\omega(z)\} \sin[(2-p)\omega], \quad (23)$$

$$\operatorname{Im}\{\tilde{\Pi}_\omega(z)\} = \operatorname{Im}\{\Pi_\omega(z)\} \cos[(2-p)\omega] - \operatorname{Re}\{\Pi_\omega(z)\} \sin[(2-p)\omega]. \quad (24)$$

Moreover, by writing $\Pi_\omega(L) = \Pi_{\omega,0} + \Pi_{\omega,1}L$, we obtain for $L = z$

$$\operatorname{Re}\{\Pi_\omega(z)\} = \Pi_{\omega,0} + \Pi_{\omega,1} \cos(\omega),$$

$$\operatorname{Im}\{\Pi_\omega(z)\} = -\Pi_{\omega,1} \sin(\omega).$$

Substituting the above equations into (23) and (24) yields

$$\begin{aligned} \operatorname{Re}\{\tilde{\Pi}_\omega(z)\} &= [\Pi_{\omega,0} + \Pi_{\omega,1} \cos(\omega)] \cos[(2-p)\omega] \\ &\quad - \Pi_{\omega,1} \sin(\omega) \sin[(2-p)\omega], \end{aligned} \quad (25)$$

$$\begin{aligned} \operatorname{Im}\{\tilde{\Pi}_\omega(z)\} &= -\Pi_{\omega,1} \sin(\omega) \cos[(2-p)\omega] \\ &\quad - [\Pi_{\omega,0} + \Pi_{\omega,1} \cos(\omega)] \sin[(2-p)\omega]. \end{aligned} \quad (26)$$

Similarly, by writing $\tilde{\Pi}_\omega(L) = \tilde{\Pi}_{\omega,0} + \tilde{\Pi}_{\omega,1}L$, we obtain for $L = z$

$$\operatorname{Re}\{\tilde{\Pi}_\omega(z)\} = \tilde{\Pi}_{\omega,0} + \tilde{\Pi}_{\omega,1} \cos(\omega),$$

$$\operatorname{Im}\{\tilde{\Pi}_\omega(z)\} = -\tilde{\Pi}_{\omega,1} \sin(\omega).$$

Substituting the above equations into (26) and (25) yields the following linear system

$$\begin{bmatrix} \tilde{\Pi}_{\omega,0} \\ \tilde{\Pi}_{\omega,1} \end{bmatrix} = \Upsilon \begin{bmatrix} \Pi_{\omega,0} \\ \Pi_{\omega,1} \end{bmatrix},$$

where

$$\Upsilon = \begin{bmatrix} \cos[(2-p)\omega] + \frac{\sin(\omega)\sin[(2-p)\omega]}{\sin(\omega)} & \frac{[\cos(\omega)]^2 \sin[(2-p)\omega]}{\sin(\omega)} \\ \cos(\omega)\sin[(2-p)\omega]/\sin(\omega) & \cos[(2-p)\omega] - \cos(\omega)\sin[(2-p)\omega]/\sin(\omega) \end{bmatrix}.$$

Since $|\Upsilon|=1$, we conclude that coefficients of $\tilde{\Pi}_\omega(L)$ are a non-singular linear transformation of those of $\Pi_\omega(L)$. Hence, constancy of $\Pi_\omega(L)$ is equivalent to that of $\tilde{\Pi}_\omega(L)$.