On the computation of stable sets for bimatrix games

Dries Vermeulen[*], Mathijs Jansen

*Department of Quantitative Economics, University of Maastricht, PO Box 616,
MD-6200 Maastricht, The Netherlands*

Abstract

In this paper, an alternative definition of stable sets, defined by Mertens [Mertens, 1989. Stable equilibria – a reformulation. Part I. Definitions and basic properties. Mathematics of Operations Research 14, 575–625], is given where perturbations are interpreted as restrictions on the strategy space instead of perturbations of the payoffs. This alternative interpretation is then used to compute a special type of stable sets –called standard stable sets – in the context of bimatrix games, exclusively using linear optimization techniques and finite enumerations.
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1. Introduction

The first systematic investigation concerning the definition of stability of a normal form equilibrium was executed by Kohlberg and Mertens (1986). Their approach differed from what had been done before. Up till then usually ad hoc remedies were introduced for specific shortcomings of Nash equilibrium. Kohlberg and Mertens simply started with the formulation of a list of desiderata that should be satisfied by any reasonable interpretation of what a stable equilibrium is. Unfortunately, despite several efforts, they did not find

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* Corresponding author.
  E-mail addresses: d.vermeulen@ke.unimaas.nl (D. Vermeulen), m.jansen@ke.unimaas.nl (M. Jansen).

a definition of stability of equilibrium that featured all desiderata. Several attempts were subsequently made to find an interpretation of stability that did satisfy all their requirements. Eventually Mertens (1989, 1991) presented a definition that satisfied all these conditions, along with a couple of new additions to the list of desiderata.

Original definition: We will first briefly explain how Mertens (1989) defined stable sets. Since, for reasons we will explain in a moment, we will restrict ourselves to a two-person context, we will present the terminology only for bimatrix games. The basic notion in the definition of stable sets is that of a perturbation. For a bimatrix game, a perturbation is in fact a pair of non-negative vectors, one for each player. For each player the number of coordinates of the corresponding perturbation equals the number of pure strategies of that player. Given such a perturbation, we can define a new game, that is played as follows. First let the players of the original bimatrix game each choose a strategy. Given these choices we add to each player's choice the corresponding perturbation and normalize the result. Now the payoff to a player in the perturbed game is simply the payoff he would get in the original game if the perturbed strategies thus constructed were played.

Thus each perturbation induces a perturbed game. Such a perturbed game will have a non-empty set of Nash equilibria. The graph of the correspondence that assigns to each perturbation its set of perturbed Nash equilibria is denoted by $\mathcal{E}$.

Now stable sets are determined with the aid of the notion of an essential germ. Loosely speaking, a germ is a connected chunk of the graph $\mathcal{E}$, and such a germ is called essential when it satisfies some essentiality condition when considered sufficiently close to the zero perturbation. In this paper, the essentiality condition itself is phrased in terms of singular homology groups. It states that the projection from the graph $\mathcal{E}$ onto the perturbation space should induce a homomorphism between homology groups (to be made precise in the definition) that is not the trivial map. (This is a slight deviation from the definition in Mertens (1989), but it has the advantage that we need not add a statement concerning Hausdorff limits of semi-algebraic sets. This way we immediately get a purely topological notion of an essential germ for arbitrary compact parts of the graph $\mathcal{E}$.) Now a set T is called stable if there is an essential germ in $\mathcal{E}$ for which T is the part of the germ directly above the zero perturbation.

1.1. Aim of the paper

In Mertens (1989) the author is already concerned with the question of computability of this type of stability in Remark 1, pp 590–593. In this remark, the author sketches an algorithm for the computation of semi-algebraic stable sets. This algorithm though will in general, even for bimatrix games, involve finding solutions to systems of higher-order polynomial equations. This effect is basically due to the rescaling factor in the definition of a perturbed game. The algorithm is also based on fairly involved procedures such as the elimination algorithm of Tarski and the triangulation algorithm for semi-algebraic sets.

In this paper, we will present an algorithm that is capable of computing a (or all) stable set(s) exclusively using addition and scalar multiplication. Both the algorithm and the proof of its validity only use elementary techniques.

1.2. Two provisos

The above assertion is subject to the following two provisos. First of all, we will only consider bimatrix games. The reason for this is that, for normal form games with more than two players, even the inequalities that determine the Nash equilibrium set are in terms of higher-order polynomials. Thus it cannot be expected that linear techniques will be adequate tools to solve these games.

Secondly, we will restrict ourselves to a special type of stable sets. If the task is: compute one stable set, then this proviso is not relevant (one could after all simply compute a stable set of this special type, and leave it at that). However, for tasks like: compute all stable sets or, given a set, check whether or not it is stable, we need some restrictions. This is basically due to the fact that the only a priori restriction for a stable set is that it be compact and connected. However, the class of all compact and connected sets is way too general to be handled effectively only by linear computation techniques. For this reason we will restrict our domain of sets to a specific class that we will specify below in the introduction and in Section 6.

1.3. Contents of the paper

The results: Basically we will do two things. First, we will show that there is an alternative definition of stable sets that does not involve rescaling. This makes the alternative definition more appropriate for being handled by linear computation techniques.

Given this alternative definition we will construct an algorithm that, given the primitive data of the game (i.e. the bimatrix) and for the special type of sets we consider, decides in a finite number of linear operations whether or not the set is stable.

Alternative definition: The alternative definition is based on a reinterpretation of perturbations. Algebraically speaking, a perturbation is still a vector like we described above, but the game induced by a perturbation is going to be different. In the alternative definition a perturbation is simply a restriction of the strategy space. Given a perturbation, the players in this new game are only allowed to play strategies that put a minimum amount of weight on each pure strategy, these minimum amounts being specified by the perturbation in question. Thus we get a new, perturbed, game with its own set of equilibria. The graph of the correspondence that assigns its set of equilibria to each perturbation is denoted by $\mathcal{F}$. Now we can redefine stable sets by requiring that the essential germs are supposed to be chunks taken from $\mathcal{F}$ instead of $\mathcal{E}$. As it turns out, this new notion of stability yields the same collection of stable sets as the original notion of Mertens.

STANDARD STABLE SETS: The advantage of the alternative definition is that, in the case of bimatrix games, it preserves the linear structure of the inequalities that define the collection of Nash equilibria. Thus, given a bimatrix game, the graph $\mathcal{F}$ can be written as the union of a finite number of chunks of this graph, each of which is determined by a finite number of a specific type of linear (in)equalities. Such a chunk will be called a *polyhedral chunk* of $\mathcal{F}$.

Nevertheless, it cannot be expected that *all* stable sets of the bimatrix game can be computed since basically the only ex ante restriction on a candidate-stable set is that it be

compact and connected (in a strong sense). This still leaves a lot of freedom of choice in degenerate games like the zero-payoff game in which basically *every* compact and connected set is stable. Thus, the freedom of choice in the selection of (different but essentially identical) strategies for a stable set leads to a multitude of (different but essentially identical) stable sets.

Therefore we restrict our attention to a special type of set. We will only consider sets that are the part above the zero-perturbation of the union of a number of polyhedral chunks of $\mathcal{F}$. Roughly speaking, we eliminate the choice problem this way and simply add *all* possible choices of strategies to our stable set. Stable sets of this form are called *standard stable sets*.

It turns out that a candidate-stable set in question is a standard stable set if and only if the union of the polyhedral chunks involved is an essential germ. We will show that it only takes a finite number of linear operations to either compute all essential germs of this form (and thus also all standard stable sets) or, given a number of polyhedral chunks of $\mathcal{F}$, to decide whether or not it is an essential germ.

Computation: The heart of the algorithm consists of two procedures. The first procedure checks connectedness of the candidate germ under consideration. This is done by explicitly constructing a combinatorial graph that is connected if and only if the candidate germ is connected. Checking connectedness of a graph is of course a finite task.

The second procedure concerns the essentiality condition. We show that, sufficiently close to the zero perturbation, the homomorphism induced by the projection map from the graph $\mathcal{F}$ to the perturbation space can be determined in a finite number of steps.

Together these two procedures can be used to check whether or not a set in standard form is an essential germ. Thus, e.g., by a simple enumeration procedure, it is possible to determine all standard stable sets of the bimatrix game under consideration.

1.4. Organization of the paper

Section 2 summarizes the notation used in this paper as well as several elementary facts about Abelian groups and bimatrix games. In Section 3 the, slightly adapted, original definition of stable sets from Mertens (1989) is presented. In Section 4 we present our alternative definition and prove its equivalence with the original one. In Section 5 the notion of a standard stable set is introduced and the relation with arbitrary stable sets and maximal stable sets is explained. Finally, in Section 6 the algorithm to compute all standard stable sets is presented together with a proof of its validity.

Appendix A contains a review of the definition of simplicial homology groups. Appendix B discusses a specific homeomorphism needed in the proof of the validity of our algorithm.

2. Preliminaries

In this section we introduce the notation we will use throughout this paper. The cardinality of a finite set M is denoted by $|M|$. For a set X in $\mathbb{R}^n$, ext(X) denotes the set of extreme points of X. A set is called a polytope if it is the convex hull of a finite number of points. If the dimension of a polytope is one less than the number of its extreme points it is called

a simplex. A non-empty subset F^1 of P is called a face if for any two points x and y in P and any positive number $\lambda < 1$ the event that $\lambda x + (1 - \lambda)y$ is an element of F implies that both x and y are elements of F. If F consists of one single point, this point is called an extreme point or vertex of P. If F is not equal to P it is called a proper face of P. A set is called polyhedral if it is the set of solutions to a finite number of linear inequalities. Given a topology on a set X and a point x in X, any set containing an open set that contains x is called a neighborhood of x. X is called connected if it cannot be written as a disjoint union of two non-empty and closed sets. For a subspace Y of X, the (topological) boundary ∂Y of Y is the collection of points x in X with the property that each neighborhood of x has a non-empty intersection with both Y and $X \setminus Y$. The closure cl(Y) of Y is the union of Y and ∂Y. The set $\mathring{Y} := Y \setminus \partial Y$ is called the interior of Y.

2.1. Abelian groups

For an element g in an Abelian group G and a positive integer n the element ng in G is defined to be the n-fold sum of g. Furthermore, $0g := 0$ where the 0 on the right-hand side of the equality sign denotes the neutral element of G, and for $n \leq -1$, the expression ng denotes the inverse of $(-n)g$. A family $B = \{g_\alpha\}_{\alpha \in I}$ of elements of G is called a basis when each element g of G can be written uniquely as a finite sum

$$g = \sum n_\alpha g_\alpha$$

where each n_α is an integer. Given an arbitrary set S, the Abelian group generated by S is the set of all functions $\varphi : S \to \mathbb{Z}$ that take values different from zero only on a finite number of elements of S. It is clear that each element φ in this group can be written uniquely as

$$\varphi = \sum n_\alpha \mathbb{1}_{s_\alpha}$$

where each n_α is an integer and $\mathbb{1}_{s_\alpha}$ is the characteristic function of $\{s_\alpha\}$. By abuse of notation we will identify s_α with its characteristic function and write

$$s = \sum n_\alpha s_\alpha$$

Note in particular that, in case S is finite, the Abelian group generated by S equals $\mathbb{Z}^S$. Now suppose we have two Abelian groups G and H. A homomorphism from G to H is a map $f : G \to H$ such that

$$f(a + b) = f(a) + f(b)$$

for all $a, b \in G$. If f has an inverse map f^{-1} it is called an isomorphism. A homomorphism f is called trivial if $f(a) = 0$ for all $a \in G$.

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[1] Non-emptiness is not a strict requirement. It is however customary in the definition of homology groups. Admittance of the empty face would yield reduced homology.

2.2. Bimatrix games

Throughout this paper we will only concern ourselves with bimatrix games. So, we assume that there are two players, player I and player II. Player I has a finite set M and player II has a finite set N of pure strategies. The payoff matrices $(a_{ij})_{i \in M, j \in N}$ of player I and $(b_{ij})_{i \in M, j \in N}$ of player II are denoted by A and B, respectively. Furthermore,

$$\Delta(M) := \left\{ p \in \mathbb{R}^M \,|\, p_i \geq 0 \text{ for all } i \in M \text{ and } \sum_{i \in M} p_i = 1 \right\}$$

is the set of mixed strategies of player I and

$$\Delta(N) := \left\{ q \in \mathbb{R}^N \,|\, q_j \geq 0 \text{ for all } j \in N \quad \text{and } \sum_{j \in N} q_j = 1 \right\}$$

is the set of mixed strategies of player II. The payoff for player I is pAq and the payoff for player II is pBq when the strategy pair (p, q) is played. For $i \in M$ the ith unit vector is denoted by e_i and is interpreted as the situation in which player I is playing pure strategy i with probability one. Similarly a pure strategy $j \in N$ of player II is identified with e_j. We will also write $\Delta := \Delta(M) \times \Delta(N)$.

Definition 1. A *Nash equilibrium* of the game (A, B) is a strategy pair (p, q) such that

$$pAq \geq p'Aq \quad \text{for all } p' \in \Delta(M)$$

and

$$pBq \geq pBq' \quad \text{for all } q' \in \Delta(N).$$

The collection of equilibria of the game (A, B) is denoted by $E(A, B)$.

3. The definition of stable sets

In this section we will present a slightly modified version of the definition of stable sets given in Mertens (1989). First we will introduce some notation.

A *perturbation* for player I is a vector $\delta = (\delta_i)_{i \in M}$ with $\delta_i \geq 0$ and $\sum_{i \in M} \delta_i \leq 1$. The collection of all perturbations is denoted by P_1. Similarly we can define the collection P_2 of perturbations $\varepsilon = (\varepsilon_j)_{j \in N}$ for player II. A pair (δ, ε) is also called a perturbation. The collection of all such perturbations is $P := P_1 \times P_2$. A perturbation (δ, ε) in P is called *completely mixed* if $\delta_i > 0$ for all i and $\varepsilon_j > 0$ for all j. For some real number $\eta > 0$, write

$$P_1(\eta) := \left\{ \delta \in P_1 \,|\, \sum_{i \in M} \delta_i \leq \eta \right\}$$

and $P_2(\eta)$ is similarly defined. Furthermore, $P(\eta) := P_1(\eta) \times P_2(\eta)$.

3.1. Payoff perturbations

A perturbation (δ, ε) defines a perturbed game in the following way. The payoff-perturbed game associated with the perturbation (δ, ε) is the game $(A(\delta, \varepsilon), B(\delta, \varepsilon))$ with

$$A(\delta, \varepsilon)_{i,j} := \sigma(e_i, \delta) \cdot A \cdot \tau(e_j, \varepsilon)$$

where

$$\sigma(p, \delta) := \frac{p + \delta}{1 + \sum_i \delta_i} \quad \text{and} \quad \tau(q, \varepsilon) := \frac{q + \varepsilon}{1 + \sum_j \varepsilon_j}.$$

The payoff matrix $B(\delta, \varepsilon)$ is defined analogously. The set of equilibria of the perturbed game is simply $E(A(\delta, \varepsilon), B(\delta, \varepsilon))$. We write $\mathcal{E}$ for the graph of the correspondence that assigns the collection $E(A(\delta, \varepsilon), B(\delta, \varepsilon))$ of perturbed equilibria to the perturbation (δ, ε). Notice that the choice $\delta = 0$ and $\varepsilon = 0$ returns the original bimatrix game (A, B). Hence, $E(A(0), B(0)) = E(A, B)$.

3.2. Stable sets

Let S be a closed subset of the product space $P \times \Delta$. For $\eta > 0$,

$$S(\eta) = \{(\delta, \varepsilon, p, q) \in S | (\delta, \varepsilon) \in P(\eta)\}$$

is the part of S above $P(\eta)$ and

$$\partial_v S(\eta) = \{(\delta, \varepsilon, p, q) \in S(\eta) | (\delta, \varepsilon) \in \partial P(\eta)\}$$

is the part of S above $\partial P(\eta)$. Usually $\partial_v S(\eta)$ is called the *vertical boundary* of $S(\eta)$. Furthermore, let $S^i(\eta)$ be the set $S(\eta) \setminus \partial_v S(\eta)$. This is the set of points $(\delta, \varepsilon, p, q)$ in $S(\eta)$ for which (δ, ε) is completely mixed, $\sum_i \delta_i < \eta$ and $\sum_j \varepsilon_j < \eta$.

Now notice that the canonical projection π that assigns the perturbation (δ, ε) to $(\delta, \varepsilon, p, q)$ is a map from $S(\eta)$ to $P(\eta)$ that maps $\partial_v S(\eta)$ into $\partial P(\eta)$. So, the projection π is a map from the topological pair $(S(\eta), \partial_v S(\eta))$ to the topological pair $(P(\eta), \partial P(\eta))$. Hence, as is, e.g., explained in Munkres (1984), it induces a homomorphism π_* from the relative singular homology group $H_d(S(\eta), \partial_v S(\eta))$ to the relative singular homology group $H_d(P(\eta), \partial P(\eta))$. The question we now ask ourselves is: how might this homomorphism π_* look like. And in fact there is not much choice as the next remark shows.

Remark. Let C be any convex and compact set of dimension d and let ∂C be its relative topological boundary (relative w.r.t. its affine hull, that is). Then it is a well-known fact that the relative singular homology groups $H_k(C, \partial C)$ are all trivial, except when $k = d$, in which case it is isomorphic to the group $\mathbb{Z}$ of integers.

In particular we see that $H_d(P(\eta), \partial P(\eta))$ is the trivial group, except when $d = |M| + |N|$, in which case the group is isomorphic with $\mathbb{Z}$. So, for each dimension the induced homomorphism π_* is necessarily trivial, except perhaps in case $d = |M| + |N|$.

These observations are the main motivation for the following definitions.

Definition 2. non-empty, closed set S in $P \times \Delta$ is called a *germ* if for sufficiently small $\eta > 0$:

(1) the set $S^i(\eta)$ is connected, and
(2) $S(\eta) = \mathrm{cl}(S^i(\eta))$.
     When for sufficiently small $\eta > 0$ it even holds that
(3) for dimension $d = |M| + |N|$ the homomorphism π_* induced by the projection π from the topological pair $(S(\eta), \partial_v S(\eta))$ to $(P(\eta), \partial P(\eta))$ is not the trivial map,

we say that the germ S is *essential*.

Even though essentiality of a germ, based on a homomorphism between homology groups, is a rather abstract notion, it has very intuitive geometrical implications. For example, when a germ S is essential, then, for sufficiently small η, there does not exist a continuous homotopy from $S(\eta)$ to $P(\eta)$ that constantly maps the vertical boundary of $S(\eta)$ into the boundary of $P(\eta)$. Roughly speaking this means that, when the germ is viewed as a plastic foil above the perturbation space that is glued to the vertical boundary, it cannot be moved to the vertical boundary without either tearing the material apart or ungluing the germ above the boundary.

Definition 3. A closed set T in Δ is called *stable* if there exists an essential germ $S \subset \mathcal{E}$ such that

$$T = \{(p, q)|(0, 0, p, q) \in S\}.$$

The above definition of stable sets differs slightly from the definition in Mertens in several aspects. First of all, Mertens based his definition on simplicial instead of singular homology groups. However, simplicial homology is only defined for triangulable sets. As a consequence of this, Mertens initially uses the above definition, but with the additional requirement that the germ involved is semi-algebraic (and therefore triangulable). Subsequently he also considers the Hausdorff limits of the stable sets thus obtained to be stable sets. The advantage of using singular homology is that the above definition can be used directly for arbitrary closed sets. This does not make much difference, because for semi-algebraic (and more generally triangulable) sets both types of homology groups coincide by Theorem 34.3 in Munkres (1984) and the topological invariance of homology groups. Finally, another difference is that Mertens considers different coefficient modules, but that can also be done in singular homology.

Nevertheless, the above definition preserves all major results of the original definition, such as existence, perfection, backward induction and ordinality. Existence easily follows from the observation that any semi-algebraic set that is stable in the sense of Mertens (1989) is also stable according to our definition. Mertens (1989) has shown the existence of such a set. Perfection is fairly straightforward. Backward induction follows from the observation that the proof of Hillas et al. (2001) can be applied directly to our definition to show that stable sets in the sense used here are also stable in the sense of Hillas (1990). Ordinality can be shown by proving that our definition satisfies both invariance and admissible-best-reply invariance. These conditions are sufficient for ordinality as is shown in Mertens (2004) and Vermeulen and Jansen (2000).

4. An alternative definition of stable sets

Even though one can obtain results on computability using the original definition (see, e.g. Mertens, 1989, Remark 1, pp. 590–593) this definition is not suited for our purposes. The problem is that, even for bimatrix games, the linear structure of the inequalities that characterize the equilibrium set is lost when payoffs are perturbed. This is basically due to the rescaling factor in the denominator of the perturbation map. However, there is an alternative way to interpret perturbations in terms of restrictions of the strategy spaces. We will first show that the resulting notion of stable sets under this interpretation is equivalent with the original one. In the next section we will also show that the linear structure of the equilibrium correspondence is preserved under this interpretation, and how this fact can be exploited for computational purposes.

4.1. Strategy perturbations

We will first give a reinterpretation of a perturbation. More precisely, given a perturbation, we will construct an alternative way to associate a perturbed game with this perturbation. So, let (δ, ε) be a perturbation. The perturbed game $(A, B, \delta, \varepsilon)$ is played as follows. The players are only allowed to play strategy pairs (p, q) in the restricted strategy space $\Delta(\delta) \times \Delta(\varepsilon)$ where

$$\Delta(\delta) := \{p \in \Delta(M) | p_i \geq \delta_i \quad \text{for all } i \in M\}$$

and $\Delta(\varepsilon)$ is similarly defined. The payoffs in this game remain pAq and pBq. An *equilibrium* of the perturbed game $(A, B, \delta, \varepsilon)$ is a strategy pair (p, q) in the restricted strategy space such that

$$pAq \geq p'Aq \quad \text{for all } p' \in \Delta(\delta)$$

and

$$pBq \geq pBq' \quad \text{for all } q' \in \Delta(\varepsilon).$$

The collection of equilibria of the perturbed game $(A, B, \delta, \varepsilon)$ is denoted by $E(A, B, \delta, \varepsilon)$. We write $\mathcal{F}$ for the graph of the correspondence that assigns the collection $E(A, B, \delta, \varepsilon)$ of perturbed equilibria to the perturbation (δ, ε).[2]

Definition 4. A closed set T in Δ is called *strategy-stable* if there exists an essential germ $S \subset \mathcal{F}$ such that

$$T = \{(p, q) | (0, 0, p, q) \in S\}$$

Remark. So, the only difference with the previous definition is that in this case we require the germ to be a subset of $\mathcal{F}$ instead of $\mathcal{E}$.

The remainder of this section is devoted to the proof that the above definition of stability is equivalent to Mertens' definition presented in the previous section. The proof is based on the existence of a particular homeomorphism from $\mathcal{E}$ to $\mathcal{F}$. We will start with a description of this homeomorphism. Consider the sets $\mathcal{C} := \mathcal{C}_1 \times \mathcal{C}_2$ and $\mathcal{D} := \mathcal{D}_1 \times \mathcal{D}_2$ defined by

$$\mathcal{C}_1 := \left\{ (p, \delta) \in \mathbb{R}^M \times \mathbb{R}^M | \delta_i \geq 0 \text{ and } \sum_{i \in M} \delta_i \leq 1 \right\}$$

and

$$\mathcal{C}_2 := \left\{ (q, \varepsilon) \in \mathbb{R}^N \times \mathbb{R}^N | \varepsilon_j \geq 0 \text{ and } \sum_{j \in N} \varepsilon_j \leq 1 \right\}$$

$$\mathcal{D}_1 := \left\{ (p, \delta) \in \mathbb{R}^M \times \mathbb{R}^M | \delta_i \geq 0 \text{ and } \sum_{i \in M} \delta_i \leq \frac{1}{2} \right\}$$

and

$$\mathcal{D}_2 := \left\{ (q, \varepsilon) \in \mathbb{R}^N \times \mathbb{R}^N | \varepsilon_j \geq 0 \text{ and } \sum_{j \in N} \varepsilon_j \leq \frac{1}{2} \right\}.$$

Define the functions $I_1 : \mathcal{C}_1 \to \mathcal{D}_1$ and $J_1 : \mathcal{D}_1 \to \mathcal{C}_1$ by

$$I_1(p, \delta) := \frac{1}{1 + \sum_i \delta_i} \cdot (p + \delta, \delta) \quad \text{and} \quad J_1(p, \delta) := \frac{1}{1 - \sum_i \delta_i} \cdot (p - \delta, \delta).$$

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[2] For reasons that will become clear in a moment we restrict this correspondence to those perturbations (δ, ε) for which $\sum_i \delta_i \leq \frac{1}{2}$ and $\sum_j \varepsilon_j \leq \frac{1}{2}$.

It is straightforward to show that I_1 is the inverse map of J_1.[3] Similarly we can define the map I_2 from $\mathcal{C}_2$ to $\mathcal{D}_2$ with inverse map J_2. So, $I := (I_1, I_2)$ is a continuous map from $\mathcal{C}$ to $\mathcal{D}$ with inverse map $J := (J_1, J_2)$.

Lemma 1. *The restriction of I to $\mathcal{E}$ is a homeomorphism from $\mathcal{E}$ to $\mathcal{F}$ and the restriction of J to $\mathcal{F}$ is its inverse.*

Proof. Since I is clearly continuous with inverse J, it is sufficient to show that I maps $\mathcal{E}$ into $\mathcal{F}$ and vice versa. So, let $(\delta, \varepsilon, p, q)$ be an element of $\mathcal{E}$. In other words, (p, q) is an equilibrium of the perturbed game $(A(\delta, \varepsilon), B(\delta, \varepsilon))$. Write

$$p^* := \frac{p + \delta}{1 + \sum_i \delta_i} \quad \text{and} \quad q^* := \frac{q + \varepsilon}{1 + \sum_j \varepsilon_j}$$

as well as

$$\delta^* := \frac{\delta}{1 + \sum_i \delta_i} \quad \text{and} \quad \varepsilon^* := \frac{\varepsilon}{1 + \sum_j \varepsilon_j}$$

We want to show that (p^*, q^*) is an equilibrium of the game $(A, B, \delta^*, \varepsilon^*)$. First notice that p^* is indeed an element of $\Delta(\delta^*)$ and q^* is an element of $\Delta(\varepsilon^*)$. Now take any other strategy p' in $\Delta(\delta^*)$. Define the strategy (!) p'' by

$$p'' := \frac{p' - \delta^*}{1 - \sum_i \delta_i^*}.$$

Then $p' = \sigma(p'', \delta)$, $p^* = \sigma(p, \delta)$ and $q^* = \tau(q, \varepsilon)$. So,

$$p'Aq^* = \sigma(p'', \delta) \cdot A \cdot \tau(q, \varepsilon) = \sum_i p_i'' \sum_j q_j A(\delta, \varepsilon)_{i,j}$$

$$\leq \sum_i p_i \sum_j q_j A(\delta, \varepsilon)_{i,j} = \sigma(p, \delta) \cdot A \cdot \tau(q, \varepsilon) = p^*Aq^*$$

where the inequality follows from the fact that (p, q) is an equilibrium of $(A(\delta, \varepsilon), B(\delta, \varepsilon))$. This shows that p^* is a best reply against q^* within $\Delta(\delta^*)$. In the same way we find that q^* is a best reply against p^* within $\Delta(\varepsilon^*)$. Hence, (p^*, q^*) is an equilibrium of $(A, B, \delta^*, \varepsilon^*)$.

Conversely, let $(\delta, \varepsilon, p, q)$ be an element of $\mathcal{F}$. In other words, (p, q) is an equilibrium of the perturbed game $(A, B, \delta, \varepsilon)$. We have to show that $J(\delta, \varepsilon, p, q)$ is an element of $\mathcal{E}$. This though follows from an analogous line of reasoning.

Now that we have this homeomorphism from $\mathcal{E}$ to $\mathcal{F}$ the proof of the equivalence of the two definitions of stability presented previously is elementary and discussed below. $\quad\square$

Theorem 1. *A set T in Δ is stable if and only if it is strategy-stable.*

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[3] For this particular reason we do not require p to be a strategy. When we do include this requirement, this statement is no longer true.

Proof. Suppose that T is stable. We will show that T is also strategy-stable. To this end, let $S \subset \mathcal{E}$ be an essential germ for T. Since $I(S)$ is a subset of $\mathcal{F}$ by the previous lemma, it is sufficient to show that it is an essential germ for T.

To this end, first notice that, for $1 > \eta > 0$, $I(S(\eta))$ equals $I(S)(\eta/(1+\eta))$ and $I(\partial_v S(\eta))$ equals $\partial_v I(S)(\eta/(1+\eta))$. So, I is a map between the pairs $(S(\eta), \partial_v S(\eta))$ and $(I(S)(\eta/(1+\eta)), \partial_v I(S)(\eta/(1+\eta)))$. Furthermore, the map b from $P(\eta/(1+\eta))$ to $P(\eta)$ defined by

$$b(\delta, \varepsilon) := \left(\frac{\delta}{1-\delta}, \frac{\varepsilon}{1-\varepsilon} \right)$$

is a map between pairs $(P(\eta/(1+\eta)), \partial P(\eta/(1+\eta)))$ and $(P(\eta), \partial P(\eta))$. Finally, the composition of the maps I, π and b $\qquad\qquad\square$

$$
\begin{array}{ccc}
\left(S(\eta), \partial_v S(\eta) \right) & \xrightarrow{\ \ I\ \ } & \left(I(S)(\tfrac{\eta}{1+\eta}), \partial_v I(S)(\tfrac{\eta}{1+\eta}) \right) \\
\rho \downarrow & & \pi \downarrow \\
\left(P(\eta), \partial P(\eta) \right) & \xleftarrow{\ \ b\ \ } & \left(P(\tfrac{\eta}{1+\eta}), \partial_v P(\tfrac{\eta}{1+\eta}) \right)
\end{array}
$$

equals the projection ρ from $(S(\eta), \partial_v S(\eta))$ to $(P(\eta), \partial P(\eta))$. So, $b_* \circ \pi_* \circ I_*$ equals ρ_* by Theorem 30.1 of Munkres (1984). Hence, since ρ_* is not the trivial map by assumption, π_* cannot be the trivial map either. The proof of the converse implication in the statement of the theorem is virtually identical to the above proof.

Although we only presented the equivalence of both notions of stability in the context of bimatrix games, the same can be shown for arbitrary n-person normal form games.

5. Standard stable sets

From a topological perspective stable sets can still take on many forms. Essentially the only restrictions are compactness and connectedness. Therefore it cannot be expected that, given an arbitrary (bimatrix) game, all stable sets can be computed. If we consider, e.g., the 2×2 bimatrix game in which both players receive zero payoffs regardless of the strategies chosen, it is easy to check that *any* compact and connected set is a stable set. This huge degree of freedom of choice is mainly due to the high degree of degeneracy of this game: it does not matter *which* (essentially identical) strategies you choose to be part of your stable set, and consequently *any* choice is indeed allowed!

In this section we will introduce a specific type of stable set, called standard stable set, that turns out to be sufficiently well-behaved for purposes of computability involving solely linear optimization techniques. Roughly speaking, standard stable sets avoid the problem we just discussed by simply selecting *all* strategies in case we have freedom of choice.

In this section we will show that in the case of bimatrix games the collection of standard stable sets is fairly large and still captures the spirit of the notion of stability pretty well.

5.1. The linear structure of $\mathcal{F}$

The structure of standard stable sets derives from the linear structure of the graph $\mathcal{F}$ of the equilibrium correspondence. Let (δ, ε) be a perturbation of a bimatrix game (A, B). For a strategy p of player I in the restricted strategy space $\Delta(\delta)$ the δ-*carrier* $C_\delta(p)$ of p is defined as

$$\mathcal{C}_\delta(p) := \{i \in M \,|\, p_i > \delta_i\}.$$

Analogously we can define the ε-carrier $\mathcal{C}_\varepsilon(q)$ of a strategy of player II in the strategy space restricted by the perturbation ε. For a strategy p of player I the set $PB_2(p)$ of *pure best replies* of player II to p is defined by

$$PB_2(p) := \{j \in N \,|\, pBe_j \geq pBe_l \text{ for all } l \in N\}.$$

Again we can do something similar for player I and define $PB_1(q)$. Using this terminology we have the following key lemma. Its proof can, e.g., be found in Vermeulen (1996).

Lemma 2. *The strategy pair (p, q) is an equilibrium of the perturbed game $(A, B, \delta, \varepsilon)$ if and only if the δ-carrier of p is a subset of $PB_1(q)$ and the ε-carrier of q is a subset of $PB_2(p)$.*

Even though we will not prove this lemma here, we will try to give some intuition for what it says. Consider the following situation. In the left-hand side picture below, player I's pure best responses against q are e_2 and e_3. So, in the unperturbed case his set of best responses – represented by the fat line – is simply the convex hull of these two pure best responses. In the $(\delta_1, \delta_2, \delta_3)$-perturbed case on the right, his set of best responses is simply the convex hull of the "perturbed versions" $(\delta_1, 1 - \delta_1 - \delta_3, \delta_3)$ and $(\delta_1, \delta_2, 1 - \delta_1 - \delta_2)$ of e_2 and e_3.
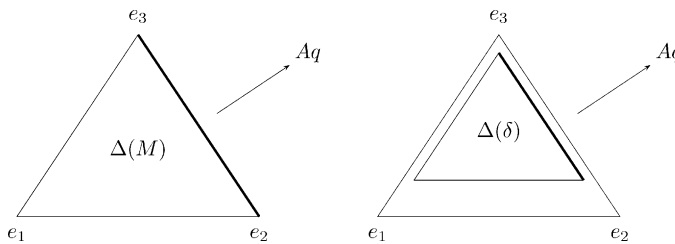
This behavior of best responses of course carries over immediately to perturbed Nash equilibria, hence Lemma 2. From this geometrical intuition it is also clear that, once the objective Aq for optimization is kept fixed, the graph of the best response sets over perturbations is linear.

This insight can be used as follows to decompose the graph $\mathcal{F}$ into a finite number of polytopes. Let $I \subset M$ be a set of pure strategies of player I and let $J \subset N$ be a set of pure strategies of player II. With these two sets of pure strategies we can associate a subset $S(I, J)$ of the collection $\mathcal{D}_1$ defined above. This set $S(I, J)$ is formally defined as the collection of

solutions (p, δ) in $\mathbb{R}^M \times \mathbb{R}^M$ of the system of linear (in)equalities

$$
\begin{aligned}
pBe_j - pBe_k &\geq 0 &\quad &\text{for all } j \in J \text{ and all } k \in N \\
p_i &\geq \delta_i &\quad &\text{for all } i \in I \\
p_i &= \delta_i &\quad &\text{for all } i \notin I.
\end{aligned}
$$

$$
\begin{aligned}
0 &\leq \delta_i &\quad &\text{for all } i \in M \\
\textstyle\sum_{i \in M} p_i &= 1 \\
\textstyle\sum_{i \in M} \delta_i &\leq \tfrac{1}{2}
\end{aligned}
$$

The group of (in)equalities after the blank line are merely added to guarantee that p is a strategy in $\Delta(\delta)$ and that (p, δ) is indeed an element of $\mathcal{D}_1$ as soon as (p, δ) is a solution of the above system of inequalities. The first group of inequalities states that every pure strategy in J is a best reply against p. The second and third groups of (in)equalities guarantee that p the δ-carrier of p is a subset of I. In $\mathcal{D}_2$ we can analogously define the set $T(I, J)$ by a system of linear (in)equalities. We will frequently encounter sets of the form

$$
S(I, J) \times T(I, J)
$$

in the remainder of this paper, and we will therefore give these sets a name.

Definition 5. A set of the form described above is called a *polyhedral chunk* of $\mathcal{F}$. This name is justified by the following straightforward consequence of the previous lemma.

Lemma 3. *Each polyhedral chunk of $\mathcal{F}$ is a subset of $\mathcal{F}$.*

Notice that, since each equilibrium is indeed an element of some polyhedral chunk of $\mathcal{F}$, this lemma states that $\mathcal{F}$ is the union of the collection of polyhedral chunks.

5.2. Standard stable sets

Now we have done enough preliminary work to be able to define the notion of a standard stable set. The idea is that, in order to construct a stable set, one first needs to decide which polyhedral chunks are needed, and secondly one needs to select within each of these polyhedral chunks a collection of equilibria that is sufficiently robust. For a standard stable set we leave out the second step and only decide which polyhedral chunks go into the stable set, and which do not. Thus we get the following definition.

Definition 6. A germ $S \subset \mathcal{F}$ is said to be in *standard form* if it can be written as the union of a number of polyhedral chunks. A stable set T is called *standard stable* if there is an essential germ $S \subset \mathcal{F}$ for T that is in standard form.

The next theorem shows that the class of standard stable sets is a sufficiently rich class of stable sets to capture the flavor of stability pretty well. It also immediately implies that the maximal elements of the (finite) collection of standard stable sets coincide with the maximal stable sets (w.r.t. set inclusion) defined in Govindan and Wilson (2002).

Theorem 2. *Each stable set is contained in a standard stable set.*

Proof. Suppose that T is a stable set and let $S \subset \mathcal{F}$ be an essential germ for it. Now let $\mathcal{A}$ be the collection of those sets $S(I, J) \times T(I, J)$ that have a sequence $(\delta^k, \varepsilon^k, p^k, q^k)_{k=1}^{\infty}$ in common with S for which $(\delta^k, \varepsilon^k)_{k=1}^{\infty}$ is completely mixed and convergent to $(0, 0)$. Let V be the union of these sets. We will show that V is an essential germ in $\mathcal{F}$ that contains $S(\eta)$ for sufficiently small η. For if we can prove that, it immediately follows that

$$W := \{(p, q) | (0, 0, p, q) \in V\}$$

is a standard stable set that contains T.

First note that V is a subset of $\mathcal{F}$ by Lemma 3. Next we will show by contradiction that, for sufficiently small η, V contains $S(\eta)$.[4] Suppose this is not the case. Then there is a sequence $(\delta^k, \varepsilon^k, p^k, q^k)_{k=1}^{\infty}$ in S for which $(\delta^k, \varepsilon^k)_{k=1}^{\infty}$ converges to $(0, 0)$ and none of the $(\delta^k, \varepsilon^k, p^k, q^k)$ are elements of V. Moreover, since $S(\eta) = \mathrm{cl}(S^i(\eta))$ for sufficiently small η, we may even assume that all $(\delta^k, \varepsilon^k)$ are completely mixed. Next, by taking a subsequence if necessary, we can make sure that there is a pair (I, J) such that for all k

$$\mathcal{C}_{\delta^k}(p^k) = I \quad \text{and} \quad \mathcal{C}_{\varepsilon^k}(q^k) = J$$

Then however $S(I, J) \times T(I, J)$ must be a subset of V by the definition of V. Contradiction.

Now we will show that V is an essential germ. Take an $\eta > 0$ such that the requirements for an essential germ are fulfilled for $S(\eta)$ and moreover $S(\eta)$ is a subset of V. We will check the three requirements for an essential germ one by one for $V(\eta)$.

(1) The set $V^i(\eta)$ is connected. To see this, suppose that there are two closed sets F and G such that $F \cap V^i(\eta)$ and $G \cap V^i(\eta)$ are not empty, mutually disjoint and their union equals $V^i(\eta)$. We will derive a contradiction.

   Since $S(\eta)$ is a subset of V, also $F \cap S^i(\eta)$ and $G \cap S^i(\eta)$ are mutually disjoint and their union is $S^i(\eta)$. So, it suffices to show that $F \cap S^i(\eta)$ is not empty. Suppose it is empty. Then $S^i(\eta)$ must be contained in G. Now take a polytope $Q = S(I, J) \times T(I, J)$ in $\mathcal{A}$. So, by definition of $\mathcal{A}$, there is a sequence $(\delta^k, \varepsilon^k, p^k, q^k)_{k=1}^{\infty}$ in $Q \cap S$ for which $(\delta^k, \varepsilon^k)_{k=1}^{\infty}$ is completely mixed and convergent to $(0, 0)$. In particular this implies that the intersection of Q and $S^i(\eta)$ is not empty. So, since $S^i(\eta)$ is contained in G, this implies that $Q^i(\eta)$ must have a non-empty intersection with G. Therefore, since $Q^i(\eta)$ is a connected set, $Q^i(\eta) \cap F$ must be empty. Then however $Q^i(\eta)$ must be contained in G. This though, since Q was chosen arbitrarily in $\mathcal{A}$, implies that $V^i(\eta)$ has an empty intersection with F. Contradiction.

(2) $V(\eta) = \mathrm{cl}(V^i(\eta))$. This immediately follows from the fact that V is the union of a finite number of polytopes Q in $\mathcal{A}$ for each of which $Q^i(\eta)$ is not empty.

(3) For dimension $d = |M| + |N|$ the homomorphism π_* induced by the projection π from the topological pair $(V(\eta), \partial_v V(\eta))$ to $(P(\eta), \partial P(\eta))$ is not the trivial map. To see this,

---

[4] This automatically implies that $\mathcal{A}$ cannot be empty.

first notice that $S(\eta)$ is a subset of V by the choice of η. Then the inclusion map

$$\iota : (S(\eta), \partial_v S(\eta)) \rightarrow (V(\eta), \partial_v V(\eta))$$

is a map between topological pairs. Furthermore, $\pi_{|S(\eta)} := \pi_{|V(\eta)} \circ \iota$ where $\pi_{|S(\eta)}$ and $\pi_{|V(\eta)}$ denote the respective restrictions of the projection π to $S(\eta)$ and $V(\eta)$. Thus we get that $(\pi_{|S(\eta)})_* := (\pi_{|V(\eta)})_* \circ \iota_*$ and $(\pi_{|V(\eta)})_*$ cannot be trivial since $(\pi_{|S(\eta)})_*$ is not trivial by assumption. $\qquad\square$

6. Computability of standard stable sets

All standard stable sets can be computed in finite time. There are several ways to see this. We will explain one of them. We selected our method of choice not on grounds of computational speed, but merely for ease of exposition.

First we will show that we can restrict ourselves to germs of a special form. Consider a fixed pair (I, J) of sets of pure strategies for the moment. Let

$$\text{ext}(I, J) := \text{ext}(S(I, J) \times T(I, J))$$

denote the set of extreme points of the associated polyhedral chunk $S(I, J) \times T(I, J)$.

Definition 7. We say that the pair (I, J) is *admissible* if

(1) there exists a point $(0, 0, p, q)$ in $\text{ext}(I, J)$,
(2) there is no pure strategy i in M such that $\delta_i = 0$ for all $(\delta, \varepsilon, p, q)$ in $\text{ext}(I, J)$, and
(3) there is no pure strategy j in N such that $\varepsilon_j = 0$ for all $(\delta, \varepsilon, p, q)$ in $\text{ext}(I, J)$.

Requirement (1) excludes chunks of the graph of the equilibrium correspondence that are not present directly above the zero perturbation. Such parts of the graph are clearly not needed in an essential germ. Thus, this requirement is not really crucial, it is only convenient. Requirements (2) and (3) are crucial. They guarantee that the associated polyhedral chunk contains at least one point $(\delta, \varepsilon, p, q)$ for which (δ, ε) is completely mixed. Together these requirements guarantee, e.g., that

$$[S(I, J) \times T(I, J)]^i(\eta)$$

is not empty for all $\eta > 0$. It is easy to see that every standard stable set has an essential germ in standard form that consists entirely of polyhedral chunks $S(I, J) \times T(I, J)$ for which (I, J) is admissible. Thus, since admissibility is evidently a finitely computable property, we can from now on assume that only admissible pairs (I, J) are used to construct germs.

Now we have made enough precautions to explain our algorithm. Let $\mathcal{J}$ be a set of admissible pairs and let V be the union over all chunks $S(I, J) \times T(I, J)$ for (I, J) in $\mathcal{J}$.

Since V is automatically a subset of $\mathcal{F}$, the set

$$W := \{(p, q)|(0, 0, p, q) \in V\}$$

is stable if and only if V is an essential germ. First notice that, by the admissibility of $\mathcal{J}$, the requirement

$$V(\eta) = \mathrm{cl}(V^i(\eta))$$

automatically holds for all η. We will explain how to test in finite time whether or not V features the remaining two requirements for an essential germ. We will basically show that there exists an $\eta^* > 0$ such that for all $\eta \leq \eta^*$,

(1) $V^i(\eta)$ is connected if and only if a certain finite graph $(\mathcal{J}, E)$ is connected, and
(2) $\pi(\eta)_*$ is not trivial $\Leftrightarrow \pi(\eta^*)_*$ is not trivial (where $\pi(\eta)$ indicates the projection from the topological pair $(V(\eta^*), \partial_v V(\eta^*))$ to the topological pair $(P(\eta^*), \partial P(\eta^*))$).

Given these two results it evidently suffices to check whether the graph $(\mathcal{J}, E)$ is connected and whether $\pi(\eta^*)_*$ is not trivial. Thus, the test itself consists of three different procedures, namely

(1) a procedure that computes $\eta^* > 0$,
(2) a procedure that checks in finite time whether the graph $(\mathcal{J}, E)$ is connected, and
(3) a procedure that checks in finite time whether the homomorphism π_* induced by the projection π from the topological pair $(V(\eta^*), \partial_v V(\eta^*))$ to $(P(\eta^*), \partial P(\eta^*))$ is not the trivial map.

We will consider these three procedures one by one. The computation of η^* is fairly simple. First, for a polytope $S(I, J) \times T(I, J)$ with (I, J) in $\mathcal{J}$, compute the collection $\mathrm{ext}(I, J)$ of extreme points of this polytope. Next, compute

$$\eta(I, J)$$

$$:= \min \left\{ \sum_i \delta_i + \sum_j \varepsilon_j \, \middle| \, (\delta, \varepsilon, p, q) \in \mathrm{ext}(I, J) \text{ for some } (p, q) \text{ and } (\delta, \varepsilon) \neq (0, 0) \right\}$$

Notice that $\eta(I, J) > 0$ because (I, J) is assumed to be admissible. Now take

$$\eta^* := \tfrac{1}{4} \min\{\eta(I, J)|(I, J) \in \mathcal{J}\}.$$

This number η^* will be fixed for the remainder of this paper. The only thing left to do is to tackle the issues of connectedness and non-triviality.

6.1. How to check connectedness

Define the undirected graph $(\mathcal{J}, E)$ as follows. Its vertex set is $\mathcal{J}$. For two distinct elements (I, J) and (I', J') in $\mathcal{J}$ the edge $\{(I, J), (I', J')\}$ between these two vertices is an element of E if and only if the two polyhedral chunks

$$S(I, J) \times T(I, J) \quad \text{and} \quad S(I', J') \times T(I', J')$$

have in common both a point $(0, 0, p, q)$ and a point $(\delta, \varepsilon, p, q)$ for which (δ, ε) is completely mixed.

Theorem 3. *For $\eta \leq \eta^*$, the set $V^i(\eta)$ is connected if and only if the graph $(\mathcal{J}, E)$ is connected.*

Proof. Suppose that $(\mathcal{J}, E)$ is connected. Since each intersection of the two elements in an edge have a point $(\delta, \varepsilon, p, q)$ (with (δ, ε) completely mixed) in common, it is easy to show that $V^i(\eta)$ is (path-)connected.

Conversely, suppose that $(\mathcal{J}, E)$ is not connected. So, we can take write $(\mathcal{J}, E)$ as the disjoint union of two graphs $(\mathcal{J}_1, E_1)$ and $(\mathcal{J}_2, E_2)$. Let F be the union over all sets $S(I, J) \times T(I, J)$ with (I, J) in $\mathcal{J}_1$ and G be the union over all sets $S(I, J) \times T(I, J)$ with (I, J) in $\mathcal{J}_2$. Clearly F and G are closed, non-empty sets and $V^i(\eta)$ is the union of $V^i(\eta) \cap F$ and $V^i(\eta) \cap G$. So, it is sufficient to show that the intersection of $V^i(\eta) \cap F$ and $V^i(\eta) \cap G$ is empty. Suppose on the contrary that the intersection $V^i(\eta) \cap F$ and $V^i(\eta) \cap G$ is not empty. We will derive a contradiction.

Since the intersection of $V^i(\eta) \cap F$ and $V^i(\eta) \cap G$ is not empty there must be sets $(I, J) \in \mathcal{J}_1$ and $(I', J') \in \mathcal{J}_2$ such that the intersection $Q \cap R$ of

$$Q := S(I, J) \times T(I, J) \quad \text{and} \quad R := S(I', J') \times T(I', J')$$

has a point $(\delta, \varepsilon, p, q)$ in $V^i(\eta)$. Now notice that, since this point is contained in the face $Q \cap R$ of Q and R, it must be a convex combination of the points in

$$\text{ext}(I, J) \cap \text{ext}(I', J')$$

However, since $\eta < \eta^*$, we know that at least one of these points must be of the form $(0, 0, p, q)$. Thus, $Q \cap R$ contains the point $(\delta, \varepsilon, p, q)$ with (δ, ε) completely mixed as well as a point of the form $(0, 0, p, q)$. Hence, there is an edge between (I, J) and (I', J') and that contradicts the assumption that $(\mathcal{J}_1, E_1)$ and $(\mathcal{J}_2, E_2)$ are disjoint. $\square$

Finally notice that, given $\mathcal{J}$, the graph $(\mathcal{J}, E)$ can be constructed in a finite number of operations and that the connectedness of this graph can also be checked in finite time.

6.2. How to check non-triviality

Let $\pi(\eta)_*$ denote the homomorphism that is induced by the projection $\pi(\eta)$ from the topological pair $(V(\eta), \partial_v V(\eta))$ to the topological pair $(P(\eta), \partial P(\eta))$ between the corresponding singular homology groups. The task is to check whether $\pi(\eta)_*$ is not trivial for

sufficiently small η. This though is not a finite task because of the clause "for sufficiently small η" in the above condition. As said before, in fact we bypass this problem by showing it is sufficient to merely check that $\pi(\eta^*)_*$ is not trivial.

Theorem 4. *For all $\eta < \eta^*$, $\pi(\eta)_*$ is not trivial if and only if $\pi(\eta^*)_*$ is not trivial.*

Proof. We will apply the results from Appendix B to this situation. Take $\mathbb{R}^m = \mathbb{R}^n = \mathbb{R}^M \times \mathbb{R}^N$. Perturbations (δ, ε) will be interpreted as the x-variable and strategy pairs (p, q) as the y-variable. Notice that this does indeed place our setting within the non-negative orthant. Take

$$\mathcal{P} := \{S(I, J) \times T(I, J) | (I, J) \in \mathcal{J}\}.$$

Notice that indeed each element of $\mathcal{P}$ has an element of the form $(0, y) = (0, 0, p, q)$ and an element $(x, y) = (\delta, \varepsilon, p, q)$ with $x = (\delta, \varepsilon) \neq (0, 0)$. Also, the collection of polytopes in $\mathcal{P}$ together with all their proper faces is a polyhedral complex. Thus from Appendix B we get that $\eta^* = (1/2)\eta_*$. So, for every $\eta \leq \eta^*$, $P(\eta)$ is a subset of $C(\eta_*)$. So we can apply Proposition B.1 of Appendix B taking $D = P(\eta)$ and we get homeomorphisms $f(\eta)$ from $V(\eta)$ to $U(\eta_*)$ and $g_{P(\eta)}$ from $\mathbb{R}_+^M \times \mathbb{R}_+^N$ to itself such that $f(\eta)(0, 0, p, q) = (0, 0, p, q)$ and the diagram

$$
\begin{array}{ccccc}
V(\eta) & \xrightarrow{f(\eta)} & U(\eta_*) & \xrightarrow{f(\eta^*)^{-1}} & V(\eta^*) \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\
\mathbb{R}^M \times \mathbb{R}^N & \xrightarrow{g_{P(\eta)}} & \mathbb{R}^M \times \mathbb{R}^N & \xrightarrow{(g_{P(\eta^*)})^{-1}} & \mathbb{R}^M \times \mathbb{R}^N
\end{array}
$$

commutes. Thus we get that the maps $f := f(\eta^*)^{-1} \circ f(\eta)$ and $g := (g_{P(\eta^*)})^{-1} \circ g_{P(\eta)}$ are homeomorphisms, $f(0, 0, p, q) = (0, 0, p, q)$, and the diagram

$$
\begin{array}{ccc}
V(\eta) & \xrightarrow{f} & V(\eta^*) \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\
\mathbb{R}^M \times \mathbb{R}^N & \xrightarrow{g} & \mathbb{R}^M \times \mathbb{R}^N
\end{array}
$$

commutes. Now notice that g is a homeomorphism from $P(\eta)$ to $P(\eta^*)$. So, it must be a homeomorphism from the topological pair $(P(\eta), \partial P(\eta))$ to the topological pair $(P(\eta^*), \partial P(\eta^*))$. Now the commutativity of the above diagram implies that the map f is a homeomorphism from the topological pair $(V(\eta), \partial_v(V(\eta)))$ to the topological pair $(V(\eta^*), \partial_v(V(\eta^*)))$. Hence, the diagram

$$
\begin{array}{ccc}
H_d(V(\eta), \partial_v V(\eta)) & \xrightarrow{f_*} & H_d(V(\eta^*), \partial_v V(\eta^*)) \\
\downarrow{\scriptstyle \pi(\eta)_*} & & \downarrow{\scriptstyle \pi(\eta^*)_*} \\
H_d(P(\eta), \partial P(\eta)) & \xrightarrow{g_*} & H_d(P(\eta^*), \partial P(\eta^*))
\end{array}
$$

commutes and f_* and g_* are isomorphisms. Now the theorem immediately follows. $\quad\square$

6.3. How to check non-triviality of $\pi(\eta^*)_*$

The only thing left to verify is whether we can check that the homomorphism π_* induced by the projection map $\pi \equiv \pi(\eta^*)$ from $V(\eta^*)$ to $P(\eta^*)$ is trivial or not. In order to do this, we want to show first that this can in fact be decided using simplicial homology groups, defined in Appendix A, instead of singular homology groups. To this end we need to introduce some terminology.

6.4. Polyhedral complexes

Let I be a finite[5] set of indices. A (finite) collection

$$\mathcal{P} := \{P_i | i \in I\}$$

of (non-empty) polytopes is called a polyhedral complex if the faces of each P_i are also elements of $\mathcal{P}$ and, moreover, each intersection $P_i \cap P_j$ is a face of both P_i and P_j as soon as this intersection is not empty. A polyhedral complex $\mathcal{C}$ whose elements are all simplices is called a simplicial complex. A refinement of $\mathcal{P}$ is a polyhedral complex $\mathcal{R}$ such that each polytope in $\mathcal{R}$ is a subset of some polytope in $\mathcal{P}$ and, secondly, each polytope P in $\mathcal{P}$ is the union if polytopes in $\mathcal{R}$. The refinement $\mathcal{R}$ is called simplicial if $\mathcal{R}$ happens to be a simplicial complex. The underlying space of the polyhedral complex $\mathcal{P}$ is the set

$$\cup_{P \in \mathcal{P}} P.$$

Let $\mathcal{P}$ be a polyhedral complex with underlying space X and let $\mathcal{Q}$ be a polyhedral complex with underlying space Y. A map f from X to Y is said to be polyhedral from $\mathcal{P}$ to $\mathcal{Q}$ if f maps each polytope P in $\mathcal{P}$ linearly [6] onto an element of $\mathcal{Q}$. Now let $\mathcal{P}$ be a polyhedral complex in $\mathbb{R}^m \times \mathbb{R}^n$ with underlying space X. Furthermore, let $\pi : X \to \mathbb{R}^m$ be defined by $\pi(x, y) := x$.

Lemma 4. *Using only linear optimization techniques and finite enumerations we can compute a simplicial refinement $\mathcal{C}$ of $\mathcal{P}$ together with a simplicial complex $\mathcal{D}$ whose underlying space is $\pi(X)$ such that π is a polyhedral map from $\mathcal{C}$ to $\mathcal{D}$.*

It is essential for our main assertion in this section, the computability of standard stable sets in finite time using exclusively linear optimization techniques and finite enumerations, that *all* manipulations and computations used in the proof of this lemma can indeed be executed only using linear optimization techniques and finite enumerations. Although we do not prove that here, the details can be found in Vermeulen and Jansen (2004, Appendix B).

---

[5] In most textbooks finiteness is not required. We however will encounter only finite complexes in this article, so we will make life a bit easier and develop the required machinery only for finite complexes.

[6] The map f is called linear on a polytope P if for every x and y in P and λ in $[0, 1]$ we have that $f(\lambda x + (1 - \lambda)y) = \lambda f(x) + (1 - \lambda)f(y)$.

Now we turn back to the main goal in this section. We will explain how the result from Lemma 4 can be used to check in finite time whether or not $\pi(\eta^*)_*$ is trivial. First we will explain how one can construct a polyhedral complex $\mathcal{P}$ whose underlying space is $V(\eta^*)$. Since V is the union of the sets $S(I, J) \times T(I, J)$ where the pairs (I, J) range through the set $\mathcal{J}$, it is clear that $V(\eta^*)$ is the union over all (I, J) in $\mathcal{J}$ of the sets

$$[S(I, J) \times T(I, J)](\eta^*)$$

It can easily be checked that such a set is a polytope and that, if not empty, the intersection of two such sets is a face of both. Given these facts, it is straightforward to check that the collection $\mathcal{P}$ of all sets

$$[S(I, J) \times T(I, J)](\eta^*)$$

with (I, J) in $\mathcal{J}$ together with their faces is a polyhedral complex. Also notice that, given $\mathcal{J}$, this complex can be computed in a finite number of steps.

Given the above terminology, Lemma 4 applied to the projection map π on the underlying space $V(\eta^*)$ of the polyhedral complex $\mathcal{P}$ constructed above states that there is a simplicial complex $\mathcal{C}$ with underlying space $V(\eta^*)$ and a simplicial complex $\mathcal{D}$ with underlying space $\pi(V(\eta^*)) = P(\eta^*)$ such that the projection π from $V(\eta^*)$ to $P(\eta^*)$ is a polyhedral map from $\mathcal{C}$ to $\mathcal{D}$.

In order to establish the connection with relative simplicial homology, we also need to consider the following two subcomplexes.[7] Let $\mathcal{B}$ be the simplicial subcomplex of $\mathcal{D}$ whose underlying space is $\partial P(\eta^*)$ and let $\mathcal{A}$ be the simplicial subcomplex of $\mathcal{C}$ whose underlying space is $\partial_v V(\eta^*)$.

Now notice that π is automatically a polyhedral map from $(\mathcal{C}, \mathcal{A})$ to $(\mathcal{D}, \mathcal{B})$, meaning that it is a polyhedral map from $\mathcal{C}$ to $\mathcal{D}$ such that the image under π of each element of $\mathcal{A}$ is an element of $\mathcal{B}$. Therefore π induces a homomorphism from the simplicial homology group $H_d(\mathbf{X}, \mathbf{A})$ to the simplicial homology group $H_d(\mathbf{Y}, \mathbf{B})$ as follows. Let $\pi_\# : C_d(\mathbf{X}, \mathbf{A}) \to C_d(\mathbf{Y}, \mathbf{B})$ be the homomorphism induced by the map that assigns

$$\pi_\#([v_0, \dots, v_d] + C_d(\mathbf{A}))$$

$$:= \begin{cases} [\pi(v_0), \dots, \pi(v_d)] + C_d(\mathbf{B}) & \text{if all } \pi(v_0), \dots, \pi(v_d) \text{ are distinct} \\ C_d(\mathbf{B}) & \text{else} \end{cases}$$

to each element $[v_0, \dots, v_d] + C_d(\mathbf{A})$ of $C_d(\mathbf{X}, \mathbf{A})$. It can be shown in the diagram below:

$$
\begin{array}{ccccc}
C_{d+1}(\mathbf{X}, \mathbf{A}) & \xrightarrow{\partial_{d+1}} & C_d(\mathbf{X}, \mathbf{A}) & \xrightarrow{\partial_d} & C_{d-1}(\mathbf{X}, \mathbf{A}) \\
\pi_\# \downarrow & & \pi_\# \downarrow & & \pi_\# \downarrow \\
C_{d+1}(\mathbf{Y}, \mathbf{B}) & \xrightarrow{\partial_{d+1}} & C_d(\mathbf{Y}, \mathbf{B}) & \xrightarrow{\partial_d} & C_{d-1}(\mathbf{Y}, \mathbf{B})
\end{array}
$$

---

[7] A subcomplex of a simplicial complex $\mathcal{C}$ is a simplicial complex that is a subset of $\mathcal{C}$.

that the homomorphism $\pi_\#$ commutes with the boundary operator. We can therefore define a map $\pi_{*\mathrm{sim}} : H_d(\mathbf{X}, \mathbf{A}) \to H_d(\mathbf{Y}, \mathbf{B})$ by, for all $k \in \mathrm{Ker}(\partial_d)$,

$$\pi_{*\mathrm{sim}}(k + \mathrm{Im}(\partial_{d+1})) := \pi_\#(k) + \mathrm{Im}(\partial_{d+1})$$

This map is again a homomorphism. Now Theorem 34.4 of Munkres (1984) states that there exist isomorphisms m_* and n_* such that the diagram

$$
\begin{array}{ccc}
H_d(\mathbf{C}, \mathbf{A}) & \xrightarrow{\ m_*\ } & H_d(V(\eta^*), \partial_v V(\eta^*)) \\[2pt]
{\scriptstyle \pi_{*\mathrm{sim}}}\Big\downarrow & & \Big\downarrow{\scriptstyle \pi_*} \\[2pt]
H_d(\mathbf{D}, \mathbf{B}) & \xrightarrow{\ n_*\ } & H_d(P(\eta^*), \partial P(\eta^*))
\end{array}
$$

commutes. Thus the central question in this section, whether we can in some sense check in finite time whether π_* is the trivial map or not, boils down to the question: can we check in finite time whether or not the map

$$H_d(\mathbf{C}, \mathbf{A}) \xrightarrow{\ \pi_{*\mathrm{sim}}\ } H_d(\mathbf{D}, \mathbf{B})$$

we just defined is trivial. As it turns out, this is indeed possible. First notice that, since there actually are procedures to compute the complexes $\mathcal{C}$ and $\mathcal{D}$, we can also compute bases for the groups $C_d(\mathbf{C}, \mathbf{A})$ and $C_d(\mathbf{D}, \mathbf{B})$ in finite time. Given these bases we will show how we can compute a basis for $\mathrm{Ker}(\partial_d)$ and $\mathrm{Im}(\partial_d)$.

To this end, take an enumeration $b_1, \ldots, b_k$ of the finite collection of basis elements $\upsilon + C_d(\mathbf{A})$ of $C_d(\mathbf{C}, \mathbf{A})$, where υ ranges through the collection of oriented d-simplices not contained in $\mathbf{A}$. Similarly, let $c_1, \ldots, c_m$ be an enumeration of the finite collection of basis elements $w + C_{d-1}(\mathbf{A})$ of $C_{d-1}(\mathbf{C}, \mathbf{A})$. Now note that ∂_d is a homomorphism from $C_d(\mathbf{C}, \mathbf{A})$ to $C_{d-1}(\mathbf{C}, \mathbf{A})$. So, the entire map ∂_d is determined by the images

$$\partial_d(b_1), \ldots, \partial_d(b_k)$$

in $C_{d-1}(\mathbf{C}, \mathbf{A})$ of the basis $b_1, \ldots, b_k$. Furthermore, we can compute in finite time the representation

$$\partial_d(b_i) = \sum_{j=1}^{m} n_{ij} c_j$$

of each $\partial_d(b_i)$ in terms of the basis $c_1, \ldots, c_m$ of $C_{d-1}(\mathbf{C}, \mathbf{A})$. Thus we can represent the map ∂_d by the integer-valued matrix

$$
N := \begin{bmatrix}
n_{11} & \cdots & n_{1m} \\
\vdots & & \vdots \\
n_{k1} & \cdots & n_{\mathrm{km}}
\end{bmatrix}
$$

Consider the following elementary operations that we will allow on this integer matrix.

(1) interchange row i and row k,
(2) multiply row i by -1, and
(3) replace row i by row i + row k for $k \neq i$.

Each of these three operations correspond to a transformation of the current basis $b_1, \ldots, b_k$. The first operation corresponds to an interchange of b_i and b_k. The second operation corresponds to a replacement of b_i by $-b_i$ and the third to a replacement of b_i by $b_i + b_k$.

Obviously we can define similar operations on the columns that correspond to similar operations on the current basis $c_1, \ldots, c_m$. In particular, the replacement of column j by column j + column l corresponds to the replacement of c_l by $c_l - c_j$ (!). Now from Munkres (1984, Section 11, Theorem 11.3), we get the following result.

Proposition 1. *Using the above six elementary operations we can construct in a finite number of steps bases $d_1, \ldots, d_k$ of $C_d(\mathbf{C}, \mathbf{A})$ and $e_1, \ldots, e_k$ of $C_{d-1}(\mathbf{C}, \mathbf{A})$ such that the corresponding matrix of ∂_d has the diagonal form*

$$D := \begin{bmatrix} p_1 & & & \\ & \ddots & & \ominus \\ & & p_r & \\ & \ominus & & \ominus \end{bmatrix}$$

where $p_1, \ldots, p_r$ are positive integers and each $\ominus$ is a null-matrix of appropriate dimensions.

It can be checked that the change of bases mentioned in the above proposition can be performed in a finite number of steps, each of which involves only a finite number of algebraic computations. Thus we get a basis $e_1, \ldots, e_r$ for $\mathrm{Im}(\partial_d)$ and a basis $d_{r+1}, \ldots, d_k$ for $\mathrm{Ker}(\partial_d)$. Now it can easily be seen that $\pi_{*\mathrm{sim}}$ is trivial if and only if

$$\pi_\#(d_{r+1}), \ldots, \pi_\#(d_k)$$

are all elements of the subgroup $\mathrm{Im}(\partial_{d+1})$ of $C_d(\mathbf{D}, \mathbf{B})$. This however can be tested in finite time as follows. As shown above we can use Proposition 1 to construct a basis $g_1, \ldots, g_s$ of $\mathrm{Im}(\partial_{d+1})$. Once we have computed this basis, note that

$$\pi_\#(d_{r+1}), \ldots, \pi_\#(d_k)$$

are all elements of $\mathrm{Im}(\partial_{d+1})$ if and only if we can find integers n_{ij} such that for all $r + 1 \leq i \leq k$:

$$\pi_\#(d_i) = \sum_{j=1}^{s} n_{ij} \cdot g_j.$$

However, since all $\pi_\#(d_i)$ and all g_j can be represented as vectors in $\mathbb{Z}^{\mathbf{D} \backslash \mathbf{B}}$, this is equivalent to asking whether a certain integer-valued linear system has an (and in that case automat-

ically unique) integer-valued solution. This though can easily be tested using Gaussian elimination.

Appendix A. Simplicial homology groups

Let σ be a simplex, and let $\{v_0, \ldots, v_d\}$ be its set of vertices. Then the dimension of this simplex is d and it is simply called a *d*-simplex. An *orientation* of a simplex is an ordering of its vertices modulo even permutations. A simplex together with an orientation of this simplex is called an *oriented* simplex. It is generically denoted by $v = [v_0, \ldots, v_d]$.

Let $\mathcal{C}$ be a simplicial complex. Choose for each simplex $\sigma \in \mathcal{C}$, an (arbitrary) orientation and denote the collection of all oriented simplices thus constructed by $\mathbf{C}$. Consider the Abelian group $\mathbb{Z}^{\mathbf{C}}$ generated by $\mathbf{C}$. We identified an element v of $\mathbf{C}$ with the characteristic function $\mathbf{1}_v$ of $\{v\}$ in $\mathbb{Z}^{\mathbf{C}}$. It turns out to be convenient to identify the opposite orientation of v with $-\mathbf{1}_v$ and consequently denote it by $-v$. Now let d be an integer in $\mathbb{Z}$. A *d-chain* is an element

$$c = \sum n_\alpha v_\alpha$$

in $\mathbb{Z}^{\mathbf{C}}$ in which n_α is non-zero only if v_α is an oriented *d*-simplex. The subgroup of $\mathbb{Z}^{\mathbf{C}}$ of all *d*-chains on $\mathbf{C}$ is denoted by $C_d(\mathbf{C})$ and is called the group of oriented *d*-chains of $\mathcal{C}$. It is evidently generated by the set of *elementary d-chains* $[v_0, \ldots, v_d]$.

A.1. The boundary operator

Now take an elementary *d*-chain $[v_0, \ldots, v_d]$ in $\mathbf{C}$. Define

$$\partial_d([v_0, \ldots, v_d]) := \sum_{i=0}^{d} (-1)^i [v_0, \ldots, \hat{v}_i, \ldots, v_d]$$

where $[v_0, \ldots, \hat{v}_i, \ldots, v_d] := [v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_d]$. It can be shown that this is indeed a correct definition. Since the collection of elementary *d*-chains is a basis for $C_d(\mathbf{C})$, this definition extends uniquely to a homomorphism

$$\partial_d : C_d(\mathbf{C}) \to C_{d-1}(\mathbf{C}).$$

The resulting map is called the *boundary operator* in dimension d. The kernel of ∂_d in $C_d(\mathbf{C})$ is known as the group of *d*-cycles and its image in $C_{d-1}(\mathbf{C})$ is called the group of $d-1$-boundaries. It can be shown that $\partial_d \circ \partial_{d+1} = 0$.

A.2. Relative homology groups

Now let $\mathcal{C}$ be a simplicial complex with associated collection $\mathbf{C}$ of oriented simplices. Furthermore let $\mathcal{C}_0$ be a subcomplex of $\mathcal{C}$ and denote its associated collection of oriented simplices by $\mathbf{C}_0$. The group $C_{d0} := C_d(\mathbf{C}_0)$ of those d-chains on $\mathbf{C}$ that only take non-zero values on elements of $\mathbf{C}_0$ is a subgroup of $C_d(\mathbf{C})$.[8] So, we can define the quotient group $C_d(\mathbf{C}, \mathbf{C}_0)$ whose elements are the sets of the form

$$v + C_{d0} := \{v + w | w \in C_{d0}\}$$

where v ranges through $C_d(\mathcal{C})$. This Abelian group is called the group of *relative* chains of dimension d. Define the map ∂_d from the group $C_d(\mathbf{C}, \mathbf{C}_0)$ of relative d-chains to the group $C_{d-1}(\mathbf{C}, \mathbf{C}_0)$ of relative $d-1$-chains by

$$\partial_d(v + C_{d0}) := \partial_d(v) + C_{d-1,0}$$

for all $v + C_{d0}$ in $C_d(\mathbf{C}, \mathbf{C}_0)$. One easily checks that ∂_d is a homomorphism and that $\partial_d \circ \partial_{d+1} = 0$. So, we can define the *relative* simplicial homology group $H_d(\mathbf{C}, \mathbf{C}_0)$ of dimension d by

$$H_d(\mathbf{C}, \mathbf{C}_0) := \frac{\mathrm{Ker}(\partial_d)}{\mathrm{Im}(\partial_{d+1})}$$

A.3. Some intuition

A convenient setting to see how homology groups look like is the one where $\mathcal{C}_0$ is empty – so homology groups are defined directly on $C_d(\mathbf{C})$ instead of $C_d(\mathbf{C}, \mathbf{C}_0)$ – and where $\mathcal{C}$ only contains zero-dimensional and one-dimensional oriented simplices. In the field of combinatorial optimization such a setting is called a directed flow network.
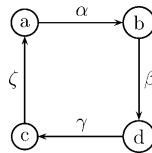
This network has four (oriented) vertices a, b, c and d, and four oriented 1-simplices, namely the arcs $[a, b]$, $[b, d]$, $[d, c]$ and $[c, a]$. The arcs can be thought of as pipelines through which water or oil is transported from one node to the other in the direction of the arrow. Negative amounts correspond to flows in the opposite direction. Now suppose we transport the amounts α from a to b, β from b to d, γ from d to c, and ζ from c to a. This corresponds to the 1-chain

$$c = \alpha[a, b] + \beta[b, d] + \gamma[d, c] + \zeta[c, a]$$

---

[8] This is a slight abuse of notation. Formally the elements of $C_d(\mathbf{C}_0)$ are elements of $\mathbb{Z}^{\mathbf{C}_0}$, not of $\mathbb{Z}^{\mathbf{C}}$.

All that the boundary operator ∂_1 does is computing the resulting net stock of fluid in the nodes (vertices) of the network. So, as can be seen in the picture, the net stock in, e.g., a is $\zeta - \alpha$ and the net stock in c is $\gamma - \zeta$. In total the resulting 0-chain when we apply the boundary operator ∂_1 to the above 1-chain is

$$\partial_1(c) = (\zeta - \alpha)a + (\alpha - \beta)b + (\beta - \gamma)c + (\gamma - \zeta)d$$

Now, the 1-chain is called a flow in the network when the resulting net stock in each node equals zero, that is, when in each node the inflow equals the outflow. This is precisely when $\alpha = \beta = \gamma = \zeta$. In the language of homology groups this means that $\partial_1(c) = 0$, in which case c is an element of $\mathrm{Ker}(\partial_1)$ and called a cycle. In the above picture it is very clear why one would like to call c a cycle (or a flow) whenever $\alpha = \beta = \gamma = \zeta$. And it is also clear that $\mathrm{Ker}(\partial_1)$ is generated by

$$[a, b] + [b, d] + [d, c] + [c, a]$$

This is actually a very general principle. Consider, e.g., the disconnected network
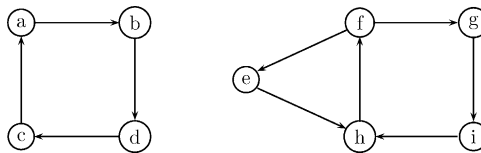
Notice that, since there are no 2-simplices, the group $\mathrm{Im}(\partial_2)$ is the trivial group. So, the one-dimensional homology group is simply $\mathrm{Ker}(\partial_1)$. And along the same line of reasoning as above we can deduce that the latter group is generated by the three cycles $[a, b] + [b, d] + [d, c] + [c, a]$, $[f, g] + [g, i] + [i, h] + [h, f]$ and $[f, e] + [e, h] + [h, f]$, and must hence be isomorphic to $\mathbb{Z}^3$.

Thus, the one-dimensional homology groups basically counts the number of "elementary" cycles in a network. Just like in the above network, there may be more cycles, e.g., $e \to h \to i \to g \to f \to e$, but these cycles can be written as integer combinations of cycles in the basis, in this case $e \to h \to f \to e$ minus $f \to g \to i \to h \to f$. Notice that this way the arc $[h, f]$ is indeed once counted in the direction of the arrow, and once backwards. The zero-dimensional homology group, even for general simplicial complexes, simply counts the number of connected components of the complex, and is in this case isomorphic to $\mathbb{Z}^2$.

Appendix B. A homeomorphism

Consider the non-negative orthant $\mathbb{R}^m_+ \times \mathbb{R}^n_+$ of the product space $\mathbb{R}^m \times \mathbb{R}^n$. A generic element of $\mathbb{R}^m \times \mathbb{R}^n$ is denoted by (x, y) with x in $\mathbb{R}^m$ and y in $\mathbb{R}^n$. Further, let

$$\mathcal{P} := \{P_\alpha | \alpha \in A\}$$

be a collection of polytopes in this non-negative orthant with the following two additional properties. First, each polytope in this collection contains at least one element of the form $(0, y)$ and at least one element of the form (x, y) with $x \neq 0$. Second, the collection of polytopes in $\mathcal{P}$ together with all their proper faces is a polyhedral complex. For each polytope P_α in $\mathcal{P}$ let $\mathrm{ext}(P_\alpha)$ be its set of extreme points. Define

$$\eta(P_\alpha) := \min \left\{ \sum_{j=1}^{m} x_j \,\middle|\, (x, y) \in \mathrm{ext}(P_\alpha) \text{ and } x \neq 0 \right\}.$$

Further define

$$\eta_* := \tfrac{1}{2} \min\{\eta(P_\alpha) \mid P_\alpha \in \mathcal{P}\}$$

For $\eta \leq \eta_*$, write $C(\eta)$ for the collection of points x in $\mathbb{R}_+^m$ for which $x_1 + \cdots + x_m \leq \eta$. Let $P_\alpha(\eta)$ be the collection of points (x, y) in P_α for which x is an element of $C(\eta)$. Let $U(\eta)$ be the union over all sets $P_\alpha(\eta)$ for P_α in $\mathcal{P}$. Let $D \subset \mathbb{R}^m$ be a compact and convex neighborhood of 0 with the additional property that d is a subset of $C(\eta_*)$. Write

$$V := \{(x, y) \in U(\eta_*) \mid x \in D\}$$

and

$$\partial_v V := \{(x, y) \in U(\eta_*) \mid x \in \partial D\}.$$

For x in $\mathbb{R}_+^m$, write $\|x\| := \sum_{i=1}^{m} x_i$, and define

$$\eta(x) := \begin{cases} \max \left\{ \eta > 0 \,\middle|\, \eta \frac{x}{\|x\|} \in D \right\} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

Obviously, η takes on positive values exclusively. Furthermore, η is continuous everywhere, except perhaps in $x = 0$. Also note that x is an element of d if and only if $\|x\| \leq \eta(x)$. Define the map g from $\mathbb{R}_+^m$ to $\mathbb{R}_+^m$ [9] by

$$g(x) := \frac{\eta_*}{\eta(x)} \cdot x$$

Even though η need not be continuous in $x = 0$, it can easily be verified that g is a homeomorphism from D to $C(\eta_*)$. Let π be the projection from $\mathbb{R}^m \times \mathbb{R}^n$ onto $\mathbb{R}^m$ defined by $\pi(x, y) := x$. In this setting the following can be shown.

---

[9] Notice that the map g actually depends on D.

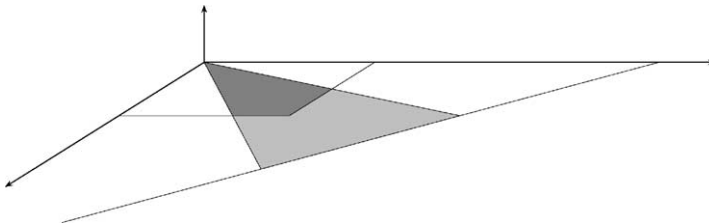**Proposition B.1.** *There exists a homeomorphism $f : V \to U(\eta_*)$ such that $f(0, y) = (0, y)$ and the diagram*

$$
\begin{array}{ccc}
V & \xrightarrow{\ f\ } & U(\eta_*) \\
\Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi} \\
\mathbb{R}^m & \xrightarrow{\ g\ } & \mathbb{R}^m
\end{array}
$$

commutes.

The proof of this proposition takes several pages and can be found in Vermeulen and Jansen (2004). The intuition though behind this construction is fairly clear. Consider a polytope P_α in the collection $\mathcal{P}$. For the moment we can act as if P_α is the only element of $\mathcal{P}$.
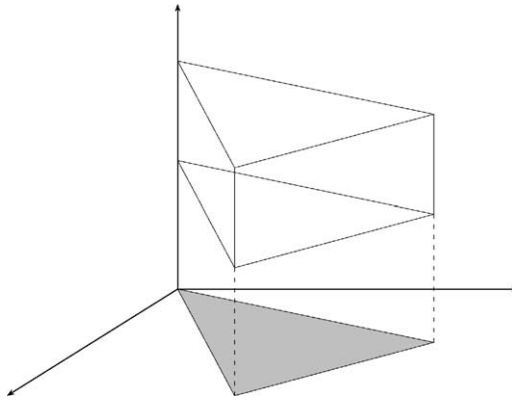
Let the large triangle that is partially displayed in the picture below be the set $C(\eta_*)$. So, the smaller, shaded, triangle depicts the set $U(\eta_*)$ as can be seen above. It is the projection of the set $P_\alpha(\eta_*)$ represented by the triangular block. Now let the square displayed below represent the set d. As one can see, it is indeed a subset of $C(\eta_*)$. Now it is clear that one can map d homeomorphically onto $C(\eta_*)$ by simply rescaling each ray emanating from the origin by an appropriate scaling factor. This is precisely what g does. It is also clear that this way g induces a homeomorphism from the darker shaded area to $U(\eta_*)$.

Now the set V is the smaller block displayed below. The proposition states that there exists a homeomorphism from this smaller block V to the larger block $P_\alpha(\eta_*)$ such that the set above the origin in the ground floor space is mapped identically onto itself and such that first using f and then projecting down is the same as first projecting down, and then applying g.
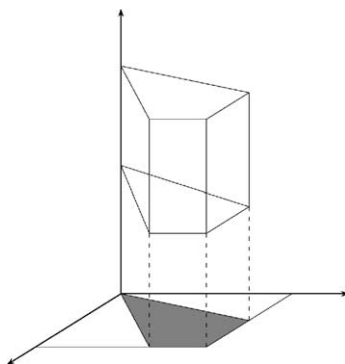
In the situation above it is actually quite clear how f should be chosen. It is simply a matter of mimicking g at each vertical level. However, when P_α is higher-dimensional, one may have many ways to choose these "vertical levels" due to the fact that the number of extreme points of $P_\alpha(\eta_*)$ counted in a "vertical" direction (this number being 2 in this case) may be much higher than the dimension of the strategy space (which is taken to be 1 in the above pictures). All that the proposition is saying is that one can choose a specific way to do this anyway, and that one can even do it in such a way that the map constructed acts identically on overlapping parts of different polyhedra of $\mathcal{P}$.

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