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A NOTE ON ADDITIVE UTILITY AND BARGAINING*

Hans PETERS

University of Limburg, 6200 MD Maastricht, the Netherlands

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Necessary and sufficient conditions are given under which a decision maker's von Neumann–Morgenstern utility function on the Cartesian product of two prospect spaces can be expressed as a sum of coordinate utility functions, assuming that all preferences are given. A main motivation for this result is an application in axiomatic bargaining theory.

1. Introduction

Keeney and Raiffa (1976, p. 231), following Fishburn (1965), give a necessary and sufficient condition under which a von Neumann–Morgenstern utility function on the product of two given prospect spaces can be written as a scaled sum of coordinate utility functions. We shall extend this result to the case where these coordinate utility functions represent given preferences.

We adopt the following notational conventions. Capital Latin letters will always denote prospect spaces. Small Latin letters (possibly with superscipts) denote elements of prospect spaces or their lottery sets (see below), e.g., a, a', a^0, $a^i, \ldots \in A$ or $\in L(A)$. Small Greek letters denote numbers in [0,1]; indexed, they are supposed to sum up to 1. The expression 'for all ...' is omitted when confusion is improbable.

For a prospect space P, we denote by $L(P)$ the *set of finite lotteries on* P. A typical element of $L(P)$ is denoted $\Sigma_{i=1}^{m}\mu_i p^i$ which is to be interpreted as the prospect p^i resulting with probability μ_i. The lottery operation is supposed to satisfy the familiar laws of commutativity and associativity. The sure prospect $p \in P$ will be identified with any lottery resulting in p with probability 1.

A *preference relation* $\geqslant_P$ on $L(P)$ is a complete and transitive binary relation on $L(P)$. By $>_P$ and $=_P$ we denote the corresponding strict (antisymmetric) preference and indifference relations, respectively. The meaning of $\leqslant_P$ and $<_P$ should be obvious. For any P and $\geqslant_P$ we assume, in the sequel, that $p >_P p'$ for some p, $p' \in P$. Herstein and Milnor (1953) provide a set of necessary and sufficient axioms for $\geqslant_P$ to be representable by a *von Neumann–Morgenstern* (vNM) *utility function* u: $L(P) \to R$, i.e., u satisfies

$$u(p) > u(p') \quad \text{iff} \quad p >_P p', \tag{1}$$

$$u\left(\sum_{i=1}^{m} \mu_i p^i\right) = \sum_{i=1}^{m} \mu_i u(p^i). \tag{2}$$

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Further, the following statements then hold: If u and v both represent $\geqslant_P$, then

$$v = ku + l \quad \text{where} \quad k, l \in \mathbb{R}, \quad k > 0. \tag{3}$$

If $p, p', p'' \in L(P)$ with $p \geqslant_P p' \geqslant_P p''$ and $p >_P p''$, then there exists a unique μ with

$$p' =_P \mu p + (1 - \mu) p''. \tag{4}$$

Any preference relation occurring in the sequel is assumed to be representable by a vNM utility function. Let A, B, and $C = A \times B$ be prospect spaces for a decision maker. Keeney and Raiffa (1976, p. 231) show that under the assumption of additive independence (see section 2) on $\geqslant_C$, a vNM utility function w for $\geqslant_C$ can be written as $k_A w_A + k_B w_B$ where k_A and k_B are positive constants and w_A and w_B are *induced* utility functions on $L(A)$ and $L(B)$. In the present note we shall extend this result to the case where w_A and w_B represent *given* preference relations on $L(A)$ and $L(B)$. Section 2 introduces the axioms and section 3 contains the main result. A motivation for this result is an application, in section 4, to axiomatic bargaining theory.

2. The axioms

Let A, B, and $C = A \times B$ be prospect spaces with $\geqslant_A$, $\geqslant_B$ and $\geqslant_C$ a decision maker's preference relations on the corresponding lottery sets. We start with a weaker version of the *additive independence* axiom [cf. Keeney and Raiffa (1976, p. 230)]:

A.1. For any a, a', b, b' we have $\frac{1}{2}(a, b) + \frac{1}{2}(a', b') =_C \frac{1}{2}(a, b') + \frac{1}{2}(a', b)$.

Notice that, by the natural identification

$$\left(\sum_{i=1}^{m} \mu_i a^i, \sum_{j=1}^{n} \nu_j b^j \right) = \sum_{i=1,}^{m} \sum_{j=1}^{n} \mu_i \nu_j (a^i, b^j), \tag{5}$$

we may put $L(A) \times L(B) \subset L(C)$.

If $\geqslant_C$ satisfies A.1, then for $\Sigma_{i=1}^m \mu_i(a^i, b^i) \in L(C)$ we have

$$\begin{aligned}
\sum_{i=1}^{m} \mu_i (a^i, b^i) &= \sum_{j>i=1}^{m} \mu_i \mu_j \left((a^i, b^i) + (a^j, b^j) \right) + \sum_{i=1}^{m} \mu_i^2 (a^i, b^i), \\
&=_C \sum_{j>i=1}^{m} 2\mu_i \mu_j \left(\tfrac{1}{2}(a^i, b^j) + \tfrac{1}{2}(a^j, b^i) \right) + \sum_{i=1}^{m} \mu_i^2 (a^i, b^i), \\
&= \sum_{i,j=1}^{m} \mu_i \mu_j (a^i, b^j) = \left(\sum_{i=1}^{m} \mu_i a^i, \sum_{i=1}^{m} \mu_i b^i \right),
\end{aligned} \tag{6}$$

where the second step follows, by using (2), from A.1, the last step from (5), and all the other steps from properties of lotteries. So (5) and (6) together enable us to identify $L(A) \times L(B)$ with $L(C)$ if $\geqslant_C$ satisfies A.1. The obvious interpretation of A.1. is that the decision maker only cares for what he gets in $L(A)$ and $L(B)$, and not for the specific combination.

The second axiom relates $\geqslant_C$ with $\geqslant_A$ and $\geqslant_B$, and is an axiom of *weak monotonicity*:

A.2. There exist a^0 and b^0 with $(a^0, b) \geqslant_C (a^0, b') \Rightarrow b \geqslant_B b'$ and $(a, b^0) \geqslant_C (a', b^0) \Rightarrow a \geqslant_A a'$ for all a, a' and b, b'.

3. Main result

Our main result is the following extension of Keeney and Raiffa (1976, Theorem 5.1).

Theorem 1. Let A, B, C, $\geqslant_A$, $\geqslant_B$, and $\geqslant_C$ be as in section 2. The following two statements are equivalent:

(i) $\geqslant_C$ satisfies A.1, and $\geqslant_A$, $\geqslant_B$, and $\geqslant_C$ satisfy A.2.
(ii) There exist vNM representations u, v and w for $\geqslant_A$, $\geqslant_B$ and $\geqslant_C$, respectively, and positive constants k_u and $k_{v'}$, with $w(a, b) = k_u u(a) + k_v v(b)$ for all a, b.

Proof. The implication (ii) $\Rightarrow$ (i) is straightforward. For (i) $\Rightarrow$ (ii), let a^0 and b^0 be as in A.2. Take $\hat{a} \in A$, $\hat{b} \in B$ with $\hat{a} \neq_A a^0$ and $\hat{b} \neq_B b^0$. In view of (3), we can choose vNM representations u and v for $\geqslant_A$ and $\geqslant_B$ such that $u(a^0) = v(b^0) = 0$, and $u(\hat{a})$ and $v(\hat{b})$ arbitrary [but consistent with (1)]. Also, fix w for $\geqslant_C$ by $w(a^0, b^0) = 0$ and $w(\hat{a}, b^0) = u(\hat{a})$, noting that $w(\hat{a}, b^0)$ and $u(\hat{a})$ must have the same sign by A.2. Similarly, $k_v := w(a^0, \hat{b})/v(\hat{b}) > 0$.

By applying A.1 and (2), we have for all a and b: $\frac{1}{2}w(a, b) + \frac{1}{2}(a^0, b^0) = \frac{1}{2}w(a, b^0) + \frac{1}{2}w(a^0, b)$, hence $w(a, b) = w(a, b^0) + w(a^0, b)$. The proof is finished (with $k_u = 1$) if we show $w(a, b^0) = u(a)$ and $w(a^0, b) = k_v v(b)$ for all a and b. We only prove the first equality, and distinguish three cases: (1) $(a, b^0) \leqslant_C (a^0, b^0)$ and $(a, b^0) \leqslant_C (\hat{a}, b^0)$, (2) $(a, b^0) \geqslant_C (a^0, b^0)$ and $(a, b^0) \geqslant_C (\hat{a}, b^0)$, and (3) the remaining case in which, by (4) $(a, b^0) =_C \mu(\hat{a}, b^0) + (1 - \mu)(a^0, b^0)$ for a unique $\mu \neq 0$.

We only consider the last case, the other ones are similar. In that case, $w(a, b^0) = \mu w(\hat{a}, b^0) = \mu u(\hat{a}) = \mu u(a)/\mu = \mathrm{u}(a)$. Here, the third equality follows from $a =_A (1 - \mu)a^0 + \mu\hat{a}$, which again follows by A.2 from $(a, b^0) =_C (\mu\hat{a} + (1 - \mu)a^0, b^0)$, which again, by (6), follows from our starting point $(a, b^0) =_C \mu(\hat{a}, b^0) + (1 - \mu)(a^0, b^0)$. Q.E.D.

Remark 1. Suppose, in Theorem 1, that (i) holds, and that there are already given vNM utility functions u and v with $u(a^0) = v(b^0) = 0$. Suppose further that $\hat{a}$ and $\hat{b}$ as in the above proof exist such that $u(\hat{a}) = v(\hat{b})$ and $(\hat{a}, b^0) =_C (a^0, \hat{b})$. Then $k_u = k_v$, in particular we may set $k_u = k_v = 1$.

4. An application in axiomatic bargaining theory

Let P be a prospect space with d, $\bar{p} \in P$, and u and v vNM utility functions on $L(P)$, such that $u(d) \neq u(\bar{p}) = 0 = v(\bar{p}) \neq v(d)$. A *bargaining situation* is a set A with $\{\bar{p}, d\} \subset A \subset P$, of which the interpretation is that there are two bargainers with utility functions u and v restricted to $L(A)$, who may reach an agreement $a \in L(A)$, or get the *conflict point* $\bar{p} \in A$. The point d is interpreted as an always available alternative. Let G denote the family of all such bargaining situations. Further, we assume that for any bargainer and any A, $B \in G$ and $C = A \times B$, axioms A.1 and A.2 are satisfied for $\geqslant_A$, $\geqslant_B$ and $\geqslant_C$, with $a^0 = b^0 = \bar{p}$, and that moreover $(\bar{p}, d) =_C (d, \bar{p})$. Here C is called the *simultaneous bargaining situation corresponding to A and B*. A *solution* φ is a map assigning, for all A, $B \in G$, elements $\varphi(A) \in L(A)$, $\varphi(B) \in L(B)$, $\varphi(A \times B) \in L(C)$.

Profitability of simultaneous bargaining can be expressed by the following axiom for φ:

S.1. For all A, $B \in G$, both bargainers (weakly) prefer $\varphi(A \times B)$ to $(\varphi(A), \varphi(B))$.

This axiom can be translated into utility space by means of Theorem 1 and especially Remark 1 (with $\hat{a} = \hat{b} = d$). It is not difficult to verify, then, that S.1 translates into:

S.2. $\tilde{\varphi}(S_A + S_B) \geqslant \tilde{\varphi}(S_A) + \tilde{\varphi}(S_B)$ for all $A, B \in G$.

Here $S_A := \{(u(a), v(a)) : a \in L(A)\}$ and $\tilde{\varphi}(S_A) := (u(\varphi(A)), v(\varphi(A)))$, and $\tilde{\varphi}(S_A + S_B) := (u_C(\varphi(C)), v_C(\varphi(C)))$ where $u_C(a, b) = u(a) + u(b)$ and $v_C(a, b) = v(a) + v(b)$ for all a and b.

Axiom S.2 is known in axiomatic bargaining theory as *super-additivity* [Peters (1983), Perles and Maschler (1981)]. In the present note we hope to have succeeded in giving a foundation, in underlying bargaining situations A rather than in bargaining games S_A for the use of the super-additivitity axiom.

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