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Computing equilibria in finance economies with incomplete markets and transaction costs*

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Summary. Transaction costs on financial markets may have important consequences for volumes of trade, asset pricing, and welfare. This paper introduces an algorithm for the computation of equilibria in the general equilibrium model with incomplete asset markets and transaction costs. We show that economies with transaction costs can be analyzed with differentiable homotopy techniques and thus in the same framework as frictionless economies despite the existence of non-differentiabilities of agents' asset demand functions and the existence of locally non-unique equilibria. We introduce an equilibrium selection concept into the computation of economic equilibria that picks out a specific equilibrium in the presence of a continuum of equilibria.

Keywords and Phrases: Transaction costs, Incomplete markets, Computational methods, Asset pricing.

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1 Introduction

The purpose of this paper is to extend the work of Brown et al. (1996a,b), DeMarzo and Eaves (1996), and Schmedders (1998, 1999) on the computation of GEI equilibria to a setting with transaction costs. We lay the theoretical foundation for a computational analysis of the impact of linear transaction costs in GEI economies. We present a homotopy algorithm for the computation of equilibria in finance economies with such transaction costs. Finding equilibria in models with transaction costs is frequently thought of as a combinatorial problem (see Duffie and Jackson, 1989), which leads to the conclusion that the search for an equilibrium is quite cumbersome. We show that instead we can phrase the problem as finding a solution to a single small system of equations.

The theoretical development of our algorithm reveals difficulties that are of economic importance. In the presence of linear transaction costs, excess demand functions are typically not differentiable and equilibria do not need to be locally unique. In particular, some markets get closed endogenously and equilibria with closed markets exhibit a continuum of asset prices in the closed market. This fundamental difference between models with and without transaction costs implies that computing equilibria in transaction costs models is not a simple extension of the known methods for models without such a friction. The way we take care of these problems are of general interest and not restricted to the computation of equilibria in GEI economies with linear transaction costs. In particular we show how to embed the model with transaction costs into the same differentiable framework that is popular for standard GEI finance economies. Crucial to the development of our arguments is the introduction of an equilibrium selection concept that allows us to pick out particular equilibria if there is indeed a continuum of equilibrium asset prices. Moreover, this concept allows us to characterize the entire continuum of equilibria.

Transaction costs are still important features of financial markets and are likely to have a big impact on volumes of trade, asset pricing and agents' welfare. In addition to the well-known reasons for market incompleteness such as informational asymmetries and moral hazard problems, transaction costs are sometimes given as another explanation of market incompleteness, see, for example, Geanakoplos (1990). This potential consequence of transaction costs is just one striking example of their impact on the volume of trade for financial assets. Although commission fees have decreased substantially over the past decade, other forms of transaction costs like the ones caused by bid-asks spreads remain substantial. For the importance of bid-ask spreads, as well as for some numerical assessments of transaction costs caused by them, see Aiyagari and Gertler (1991) and Jouini and Kallal (1995). The transaction costs in our model could also be interpreted as a form of securities transaction tax. Summers and Summers (1989) propose such a tax (see also Stiglitz, 1989) to reduce excessive speculation by "throwing sand in the gears" of financial markets, similar to Tobin's proposal to impose a tiny tax on currency trades. Again the intuition is that increasing transaction costs should lead to decreasing volumes of trade.

The remainder of the paper is organized as follows. In Section 2 we describe the model of a finance economy with transaction costs and characterize the set of no-arbitrage prices. Section 3 introduces the homotopy and outlines the main ideas of our homotopy approach. We introduce the equilibrium selection concept and show how to define an appropriate homotopy. Section 4 reports numerical results.

2 A finance economy with transaction costs

We consider the standard model of a finance economy with the additional feature of transaction costs on the financial markets. There are two dates, $t = 0, 1$, with uncertainty at $t = 0$ about the state of nature that realizes at $t = 1$. We identify date 0 with state of nature 0. At date 1, exactly one out of S possible states of nature realizes.

There are H agents in the economy. An agent h is characterized by his initial income stream $e^h \in \mathbb{R}^{1+S}$ and his preferences. The future initial income stream, which is uncertain, is denoted by $e^h_{\mathbb{1}} \in \mathbb{R}^S$. Agent h has a preference over income spent for consumption in the various states, $c^h \in \mathbb{R}^{1+S}_+$. Preferences of agent h are represented by a utility function $u^h : \mathbb{R}^{1+S}_+ \to \mathbb{R}$.

Agents have the possibility to use J assets in order to modify their income stream across time and across states. Asset j pays a dividend d^j_s in state of nature s. The stream of dividends is $d^j = (d^j_1, \ldots, d^j_S)^\top$ and the matrix of asset payoffs is $A = (d^1, \ldots, d^J)$. Without loss of generality, assets are in zero net supply. With a slight abuse of notation we also use H, J and S to denote the set of agents, the set of assets and the set of future states of nature, respectively.

Prices $q = (q_1, \ldots, q_J)^\top$ of assets are denoted in units of income. Sales of assets by agent h are denoted by $\theta^{h,-} \in \mathbb{R}^J_+$ and purchases by $\theta^{h,+} \in \mathbb{R}^J_+$. The net trade in assets then leads to an asset portfolio $\theta^h = \theta^{h,+} - \theta^{h,-}$. Both buyers and sellers of assets incur real transaction costs.

Agent h's trade $(\theta^{h,-}, \theta^{h,+})$ leads to transaction costs $\sum_{j\in J} k^{h,-}_j \theta^{h,-}_j + \sum_{j\in J} k^{h,+}_j \theta^{h,+}_j$. Here, $k^{h,-}_j$ and $k^{h,+}_j$ are nonnegative transaction costs, denoted in units of income per unit of trade in asset j. For notational simplicity, we assume $k^{h,-}_j = k^{h,+}_j = k^{h',+}_j$ for all agents h and h', and denote these costs by k_j. This approach to modeling transaction costs is identical to the one in Arrow and Hahn (1999). A finance economy with incomplete markets and transaction costs is given by $\mathcal{E} = \{(e^h, u^h)_{h\in H}, A, k\}$.

Throughout the paper we make the following assumptions.

A1 For all $h \in H$, the initial income stream e^h belongs to $\mathbb{R}^{1+S}_{++}$. We define the open set $E = \mathbb{R}^{(1+S)H}_{++}$.

A2 For all $h \in H$, the utility function u^h is three times continuously differentiable on $\mathbb{R}^{1+S}_{++}$, $\partial u^h \gg 0$, $\partial^2 u^h$ is negative definite, and interior income streams are preferred to income streams with zero components, $u^h(c^h) > u^h(\bar{c}^h)$, when $c^h \in \mathbb{R}^{1+S}_{++}$ and $\bar{c}^h \in \mathbb{R}^{1+S}_+ \setminus \mathbb{R}^{1+S}_{++}$.

A3 The rank of the matrix of asset payoffs A is J.

A4 For all $j \in J$, transaction costs are positive, $k_j > 0$.

Assumptions A1 and A2 are standard in the literature, when differentiability of demand functions is needed for the analysis. The assumption that interior income streams are preferred is standard as well, but can easily be dispensed with when using the techniques developed in this paper. The assumption that the matrix of asset payoffs has full rank, A3, is not without loss of generality in the case with transaction costs. We need this assumption to avoid degeneracies in the agents' optimization problems.

At an asset price system q the decision problem of agent h consists of choosing an asset trade $(\theta^{h,-}, \theta^{h,+}) \in \mathbb{R}^J_+ \times \mathbb{R}^J_+$ and a compatible consumption pattern. The agent chooses an element of his budget set,

$$B^h(q) = \left\{ (\theta^{h,-}, \theta^{h,+}, c^h) \in \mathbb{R}^{2J+1+S}_+ \middle| c^h_0 + \sum_{j \in J} (q_j + k_j)\theta^{h,+}_j \right.$$
$$\left. \leq e^h_0 + \sum_{j \in J} (q_j - k_j)\theta^{h,-}_j c^h_{\mathbb{1}} \leq e^h_{\mathbb{1}} + A(\theta^{h,+} - \theta^{h,-}) \right\},$$

that maximizes utility.

We restrict ourselves in this paper to transaction costs on the units of assets traded. Of course, we could instead consider transaction costs on the value of assets traded. Our algorithm can easily be extended to models with such transaction costs. Note also that we consider transaction costs in a broad sense, so they incorporate effort, fees, taxes, or bid-ask spreads.

2.1 No-arbitrage asset prices

Some asset prices $q \in \mathbb{R}^J$ may lead to arbitrage in which case an agent's decision problem does not have a solution. This fact creates the need to characterize the set of no-arbitrage prices. Because of the presence of transaction costs, the set of no-arbitrage prices gets larger than in the model of a finance economy without transaction costs.

Definition 2.1. A vector $q \in \mathbb{R}^J$ is a *no-arbitrage asset price system* when there does not exist an asset portfolio $(\theta^-, \theta^+) \in \mathbb{R}^J_+ \times \mathbb{R}^J_+$ such that $(q - k) \cdot \theta^- - (q + k) \cdot \theta^+ > 0$ and $A(\theta^+ - \theta^-) \geq 0$, or $(q - k) \cdot \theta^- - (q + k) \cdot \theta^+ \geq 0$ and $A(\theta^+ - \theta^-) > 0$.[1]

A vector q admits arbitrage possibilities whenever it is possible to make strictly positive profits in at least one state and nonnegative profits in all states. We have the following characterization of no-arbitrage prices.

Proposition 2.2. *A vector $q \in \mathbb{R}^J$ is a no-arbitrage asset price system if and only if there is $\pi \in \mathbb{R}^S_{++}$ such that $\pi^\top A - k \leq q \leq \pi^\top A + k$.*

[1] For a vector $x \in \mathbb{R}^n$ we define $x > 0$ as $x_i \geq 0$ for all i and $x_j > 0$ for at least some j.

Proof. We define the matrix M by

$$M = \begin{bmatrix} (q-k)^\top & -(q+k)^\top \\ -A & A \\ I & 0 \\ 0 & I \end{bmatrix},$$

where I is a $(J \times J)$-identity matrix and 0 a $(J \times J)$-zero matrix. By definition, a vector $q \in \mathbb{R}^J$ is a no-arbitrage price system if and only if for each $s = 0, \ldots, S$, there is no solution $\theta^s = (\theta^{s,-}, \theta^{s,+}) \in \mathbb{R}^J \times \mathbb{R}^J$ to $M\theta^s \geq 0$ and $(M\theta^s)_s > 0$. By the variant of Farkas' lemma given in Rockafellar (1970), Theorem 22.2, page 198, the latter condition is equivalent to: for every $s = 0, \ldots, S$, there exists $\lambda^s \in \mathbb{R}_+^{1+S+2J}$ with $\lambda_s^s > 0$ and ${\lambda^s}^\top M = 0$, which is the case if and only if there exists $\lambda \in \mathbb{R}_{++}^{1+S} \times \mathbb{R}_+^{2J}$ such that $\lambda^\top M = 0$. Now the theorem follows immediately after some elementary algebra. □

Proposition 2.2 reduces to the fundamental theorem on the pricing of financial securities for $k = 0$. The set of no-arbitrage prices is denoted by Q, and it is straightforward to show that Q is an open set if A has full column rank. Proposition 2.2 implies that Q is no longer a cone with vertex zero. The next proposition easily follows from our assumptions.

Proposition 2.3. *When $q \in Q$, the budget set $B^h(q)$ is compact and convex, and the agent's decision problem has a solution $(\theta^{h,-}, \theta^{h,+}, c^h)$ that is unique and that satisfies $\theta^{h,-} \cdot \theta^{h,+} = 0$.*

2.2 Agents' demand and competitive equilibrium

Proposition 2.3 implies that a single agent is never active simultaneously on the demand side and the supply side of an asset market. There is no ambiguity when we do not consider supply and demand of assets separately, but instead use the net asset portfolio purchased, $\theta^h = \theta^{h,+} - \theta^{h,-}$. Proposition 2.3 implies that the (net) demand of agent h for assets at prices $q \in Q$ is a function $g^h : Q \to \mathbb{R}^J$. The demand of all agents for all assets is given by the HJ-vector $g(q) = (g^1(q), \ldots, g^H(q))$. Total asset demand is a function $G : Q \to \mathbb{R}^J$, where $G(q) = \sum_{h=1}^H g^h(q)$.

Definition 2.4. A *competitive equilibrium* for an economy $\mathcal{E}$ is a collection of portfolio holdings $\theta^* = (\theta^{*1}, \ldots, \theta^{*H}) \in \mathbb{R}^{HJ}$ and asset prices $q^* \in \mathbb{R}^J$ such that

1. θ^{*h} is a utility maximizing asset portfolio for agent h at prices q^*,
2. $\sum_{h \in H} \theta^{*h} = 0$.

The price vector q^* is a competitive equilibrium price system if and only if $G(q^*) = 0$. Equilibrium asset portfolios and equilibrium incomes spent on consumption in each state are completely determined by equilibrium prices, since, due to the monotonicity of the utility function, the solution to an agent's optimization problem satisfies all inequalities in the definition of the budget set with equality.

Using standard methods one can now proceed to prove the existence of a competitive equilibrium. Extending the work of Hens (1991) to our model with transaction costs one can show that the individual asset demand functions g^h and the total asset demand function G are continuous on the set Q and satisfy a properness condition. We refer to Herings and Schmedders (2001) for a detailed proof.

3 Computation of equilibria

A large part of the literature on GEI finance economies in the past (for a recent survey, see Hens and Pilgrim, 2002) focusses on differentiable economies. Differentiability assumptions have been widely made in general equilibrium ever since Debreu (1972) (see also Mas-Colell, 1985). One reason why GEI models with transaction costs have been rarely examined in much detail is that transaction costs are commonly believed to be incompatible with the usual differentiability assumption.

The main purpose of this paper is to show that the computation of equilibria for models with transaction costs can be embedded in the standard differentiable framework, even though there may be locally non-unique equilibria. We can compute equilibria for our model in the tradition of Brown et al. (1996), Schmedders (1998, 1999), and Herings and Kubler (2002). Crucial in our approach is the introduction of an equilibrium selection concept that allows us to pick out particular equilibria if there is indeed a continuum. Moreover, this concept allows us to characterize the entire continuum of equilibria. The homotopy approach allows us to nicely relate a constructive existence proof to the computation of equilibria. Moreover, Kubler and Schmedders (2000) show that other algorithms such as Newton-based methods do very poorly computing GEI equilibria, and because of differentiability problems one would expect such methods to do even worse in models with transaction costs.

3.1 The common homotopy approach

A natural homotopy for computing equilibria in our model would be the function $F : [0,1] \times Q \to \mathbb{R}^J$ which is defined by

$$F(t,q) = tG(q) + (1-t)(q^0 - q), \quad (t,q) \in [0,1] \times Q,$$

where q^0 may be any price system in Q. Note that for $t = 0$, there is a unique solution, $q = q^0$. For $t = 1$, the problem $F(1,q) = 0$ is equal to the problem $G(q) = 0$, and therefore solving $F(1,q) = 0$ amounts to finding an equilibrium asset price vector q. The idea now would be to start at the unique solution for $F(0,q) = 0$ and to follow a path of solutions to $F(t,q) = 0$ until eventually a solution to $F(1,q) = 0$ is reached. In order to formalize this idea we would like to show that F is a twice continuously differentiable function, that $F^{-1}(\{0\})$ is a compact 1-dimensional differentiable manifold with boundary, and that the boundary of $F^{-1}(\{0\})$ equals the transversal intersection of $F^{-1}(\{0\})$ and the boundary of $[0,1] \times Q$. Usually it is impossible to prove the transversality conditions for all economies. The standard approach is then to invoke methods from transversality theory such as Sard's theorem to show that transversality conditions hold for a generic set of economies.

When we try to apply this standard homotopy approach to finding equilibria in our model we encounter two significant problems. The first difficulty is the existence of non-differentiabilities of the homotopy F, which are due to non-differentiabilities in the market excess demand function G. We approach this problem by subdividing the domain of F into subsets where the excess demand G and thus the homotopy F are differentiable. The zero sets of F on the different subsets are then nicely tied together to guarantee the convergence of the algorithm. The second difficulty is that even the application of transversality theory cannot rule out the occurrence of robust degeneracies which occur at $t = 1$. With transaction costs it cannot be ruled out that certain assets are robustly non-traded by every trader, even at equilibrium prices. In that case there is a continuum of equilibrium prices, as small perturbations in prices of assets that nobody trades in do not affect market clearing. The solutions to the homotopy equations cannot even be expected to constitute a 1-dimensional topological manifold. For example, if two or more assets are robustly non-traded, then the prices of all these assets can vary within some neighborhood without affecting the equilibria.

3.2 Equilibrium selection

We solve the robust degeneracy problem by making an equilibrium selection. We analyze two alternatives for equilibrium selection. The first alternative considers only those competitive equilibria where for each asset market there is either non-zero trade, or there is zero trade and at least one agent is indifferent between selling an asset and not selling an asset. The second alternative considers those competitive equilibria where for each asset market there is non-zero trade, or at least one agent is indifferent between buying and not buying an asset.

We say that an agent h is indifferent at q between selling asset j and being inactive in asset market j if $g_j^h(q) = 0$ and the relaxation of the non-negativity constraint on $\theta_j^{h,-}$ would not affect the optimal decision of household h. Under our differentiability assumptions on utilities the latter condition is equivalent to the requirement that the Lagrange multiplier corresponding to the inequality $\theta_j^{h,-} \geq 0$ equals zero at q. This Lagrange multiplier is denoted by $\lambda_j^{h,-}(q)$ and equals $-\partial_{c_0^h} u^h(c^h(q))(q_j - k_j) + \sum_{s=1}^S \partial_{c_s^h} u^h(c^h(q))d_s^j$. A similar definition applies for an agent to be indifferent between buying asset j and being inactive in asset market j. The Lagrange multiplier corresponding to the inequality $\theta_j^{h,+} \geq 0$ is denoted by $\lambda_j^{h,+}(q)$ and equals $\partial_{c_0^h} u^h(c^h(q))(q_j + k_j) - \sum_{s=1}^S \partial_{c_s^h} u^h(c^h(q))d_s^j$. The set of agents which is indifferent between selling asset j and not selling asset j is denoted by $I_j^-(q)$ and the set of agents which is indifferent between buying asset j and not buying asset j by $I_j^+(q)$. The demand function of agents in $I_j^-(q) \cup I_j^+(q)$ displays a non-differentiability at the asset price system q. It is due to this non-differentiability that robust non-degeneracies may occur.

Definition 3.1. A competitive equilibrium (θ^*, q^*) of $\mathcal{E}$ is *demand-perfect*, if in each asset market j there is non-zero trade, $\theta_j^{*h} \neq 0$ for some h, or at least one agent is indifferent between selling asset j and not selling asset j, $I_j^-(q^*) \neq \emptyset$.

The terminology demand-perfect comes from the fact that generically demand-perfect equilibria are the ones that are robust to the introduction of a trader that demands all assets. Indeed, generically, demand-perfect equilibria are obtained by perturbing the total excess demand function of the economy by an excess demand function of a trader that demands all assets, considering the set of competitive equilibria that results, and taking the limit of the set of competitive equilibria for a perturbation going to zero. For each asset market, it holds in the limit either that there is non-zero trade or that one agent will be on the limit of supplying the asset or not supplying the asset. A similar motivation can be given for supply-perfect equilibria. From an economic point of view, no information is lost by restricting attention to supply-perfect or demand-perfect equilibria, in the sense that no competitive equilibrium allocations are lost.

Proposition 3.2. For each competitive equilibrium (θ^*, q^*) of $\mathcal{E}$ there is exactly one allocationally equivalent supply-perfect equilibrium $(\theta^*, q^{\mathrm{s}})$ and exactly one allocationally equivalent demand-perfect equilibrium $(\theta^*, q^{\mathrm{d}})$.

Proof. Let (θ^*, q^*) be a competitive equilibrium of $\mathcal{E}$, inducing income streams used for consumption c^*. If for all $j \in J$ there exists $h \in H$ such that $\theta_j^{*h} \neq 0$, then (θ^*, q^*) is both a supply-perfect and a demand-perfect equilibrium and the proposition holds.

Suppose asset market $j \in J$ is such that $\theta_j^{*h} = 0$ for all $h \in H$. We give the argument for the existence of a demand-perfect equilibrium that is allocationally equivalent to the competitive equilibrium; the argument for the existence of an allocationally equivalent supply-perfect equilibrium is similar. If $\min_{h \in H} \lambda_j^{h,-}(q^*) = 0$, then we define $q_j^{\mathrm{d}} = q_j^*$. Otherwise, $\min_{h \in H} \lambda_j^{h,-}(q^*) = \min_{h \in H} -\partial_{c_0^{*h}} u^h(c^{*h})(q_j^* - k_j) + \sum_{s=1}^S \partial_{c_s^h} u^h(c^{*h}) d_s^j > 0$. Since $\lambda_j^{h,-}(q)$ is a function that is linearly decreasing in q_j, we may define q_j^{d} unambiguously by

$$\min_{h \in H} -\partial_{c_0^{*h}} u^h(c^{*h})(q_j^{\mathrm{d}} - k_j) + \sum_{s=1}^{S} \partial_{c_s^h} u^h(c^{*h}) d_s^j = 0.$$

If asset market $j \in J$ is such that $\theta_j^{*h} \neq 0$ for some $h \in H$, then we define $q_j^{\mathrm{d}} = q_j^*$.

Using the first-order conditions for the decision problem of household h, it is easily verified that θ^{*h} is an optimal asset portfolio at prices q^{d}. For all asset markets j for which $\theta_j^{*h} = 0$ for all $h \in H$, it holds that $I_j^-(q^{\mathrm{d}}) \neq \emptyset$. It follows that $(\theta^*, q^{\mathrm{d}})$ is a demand-perfect equilibrium. □

The proof of Proposition 3.2 yields another important aspect of supply-perfect and demand-perfect equilibria. These equilibria give a lower bound, respectively an upper bound, on prices that sustain a certain allocation. We exploit this property in our numerical example, to give the equilibrium interval of asset prices in case there is no trade in a certain asset.

We modify the excess demand function for the computation of demand-perfect equilibria. To this end we define, for $q \in Q$,

$$\Lambda_j^-(q) = \prod_{h=1}^{H} \lambda_j^{h,-}(q), \quad j = 1, \ldots, J.$$

We now add the term $(\varphi(\Lambda_j^-(q)))/(1 + e^{q_j})$ to the total excess demand function G_j, where $\varphi : \mathbb{R} \to \mathbb{R}$ is any bounded, differentiable function with $\varphi(0) = 0$ and $\varphi' > 0$. Any function of q_j with an everywhere nonnegative derivative that diverges to plus infinity as q_j tends to plus infinity would suffice. This addition results in the function $\widetilde{G}$, defined by

$$\widetilde{G}_j(q) = G_j(q) + \frac{\varphi(\Lambda_j^-(q))}{1 + e^{q_j}}, \quad j \in J,\ q \in Q.$$

Proposition 3.3. It holds that $(g^1(q^{\mathrm{d}}), \ldots, g^H(q^{\mathrm{d}}), q^{\mathrm{d}})$ is a demand-perfect competitive equilibrium of $\mathcal{E}$ if and only if $\widetilde{G}(q^{\mathrm{d}}) = 0$.

Proof. Consider a demand-perfect equilibrium induced by prices q^{d}. It holds that $G(q^{\mathrm{d}}) = 0$ and, for every asset j, either there is a household h' such that $g_j^{h'}(q^{\mathrm{d}}) \neq 0$, or for all h, $g_j^h(q^{\mathrm{d}}) = 0$ and there is a household h' such that $\lambda_j^{h',-}(q^{\mathrm{d}}) = 0$. In the first case it follows by the definition of a competitive equilibrium that without loss of generality $g_j^{h'}(q^{\mathrm{d}}) < 0$ and therefore $\lambda_j^{h',-}(q^{\mathrm{d}}) = 0$. In both cases it is then immediate that $\widetilde{G}(q^{\mathrm{d}}) = 0$.

Consider a price system q^{d} such that $\widetilde{G}(q^{\mathrm{d}}) = 0$. For every asset j, either $g_j^{h'}(q^{\mathrm{d}}) \neq 0$ for some agent h', or $g_j^h(q^{\mathrm{d}}) = 0$ for all agents $h = 1, \ldots, H$. Since $\Lambda_j^-(q^{\mathrm{d}}) \geq 0$, $\varphi(0) = 0$ and $\varphi' > 0$, it holds that $\varphi(\Lambda_j^-(q^{\mathrm{d}})) \geq 0$. It follows that $g_j^{h'}(q^{\mathrm{d}}) < 0$ for some agent h' in the former case, so $\lambda_j^{h',-}(q^{\mathrm{d}}) = 0$ and $\Lambda_j^-(q^{\mathrm{d}}) = 0$. Then it is immediate that $G_j(q^{\mathrm{d}}) = 0$. In the latter case it holds that $G_j(q^{\mathrm{d}}) = 0$, so $\Lambda_j^-(q^{\mathrm{d}}) = 0$, which implies that $\lambda_j^{h',-}(q^{\mathrm{d}}) = 0$ for some agent h'. Combining the two cases implies that q^{d} induces a demand-perfect equilibrium. □

3.3 Bid-ask structures

Now we address the issue of non-differentiabilities of $\widetilde{G}$. The basic idea is to subdivide the domain of $\widetilde{G}$ into subsets on which $\widetilde{G}$ is differentiable, and to jump from one such subset to the next when tracking the solution curve of the homotopy. This approach is supported by the fact that generically these subsets are a covering of Q, that is, the relative interiors of any two such subsets have an empty intersection, and the union of all those subsets equals the set Q. The standard homotopy approach allows us to follow a path of solutions in any of these subsets. Once we hit a boundary of our domain subset we switch over to the next subset and continue the path. In order to formalize this idea we need some further notation.

We define a set R of sign vectors,

$$R = \{r \in \mathbb{R}^{HJ} \mid r_j^h \in \{-1, 0, +1\}\}.$$

A sign vector $r \in R$ determines a subset of Q where the sign of the trades being made, the bid-ask structure, is determined by r. If $r_j^h = -1$, then agent h supplies asset j, if $r_j^h = 0$ then agent h does not trade in asset market j, and if $r_j^h = +1$, then agent h is buying asset j.

Formally, for $r \in R$,

$$\begin{aligned} Q(r) = \{q \in Q \mid\ & g_j^h(q) < 0, \text{ or } g_j^h(q) = 0 \text{ and } h \in I_j^-(q), \text{ when } r_j^h = -1, \\ & g_j^h(q) = 0, \text{ when } r_j^h = 0, \\ & g_j^h(q) > 0, \text{ or } g_j^h(q) = 0 \text{ and } h \in I_j^+(q), \text{ when } r_j^h = +1\}, \end{aligned}$$

and $g^{h,r} : Q(r) \to \mathbb{R}^J$ and $G^r : Q(r) \to \mathbb{R}^J$ denote the restrictions of the individual demand functions for assets and the total demand function for assets to $Q(r)$. For all $q \in Q(r)$ an agent h is always taking a long position, a short position, and not trading at all in the same set of assets. The fact that the long or short position could go to zero complicates the definition somewhat. We define $\Lambda^{-r} : Q(r) \to \mathbb{R}^J$ as the restriction of Λ^- to $Q(r)$. Notice that Λ_j^{-r} is identically equal to zero if $r_j^h = -1$ for at least one household h.

Proposition 3.4. *For $r \in R$, the asset demand functions $g^{h,r}$ and G^r, and the function Λ^{-r} are twice continuously differentiable.*[2]

The proof of Proposition 3.4 is given in Herings and Schmedders (2001). The main step of the proof consists of applying the implicit function theorem to the system of necessary and sufficient first-order conditions that characterize $g^{h,r}$ on $Q(r)$.

3.4 Homotopy

We can now define our homotopy function $\widetilde{F} : [0, 1] \times Q \to \mathbb{R}^J$ by

$$\widetilde{F}(t, q) = t\widetilde{G}(q) + (1 - t)(q^0 - q), \quad (t, q) \in [0, 1] \times Q,$$

where q^0 may be any price system in Q. Note that for $t = 0$, there is a unique solution, $q = q^0$. For $t = 1$, the problem $\widetilde{F}(1, q) = 0$ is equal to the problem $\widetilde{G}(q) = 0$, and therefore, by Proposition 3.3, solving $\widetilde{F}(1, q) = 0$ amounts to finding an asset price vector q that induces a demand-perfect equilibrium. The function $\widetilde{F}$ is differentiable on all subsets $[0, 1] \times Q(r)$.

Next we define the set $P(r)$ consisting of the pairs of the homotopy parameter and the asset price vectors that satisfy the homotopy equation, together with the requirement that r be compatible with the bid-ask structure in all markets, that is,

$$P(r) = \{(t, q) \in [0, 1] \times Q(r) | \widetilde{F}(t, q) = 0\}.$$

[2] A function with domain a subset of Euclidean space which is not necessarily open is differentiable if it has a differentiable extension to an open neighborhood of its domain of definition.

It is a well-known problem that it is usually impossible to find an analytical solution for the function G, and as a consequence for $\widetilde{F}$. We are not aware of any utility function satisfying the standard monotonicity and concavity assumptions for which analytical solutions are available in the presence of both transaction costs and incomplete markets. The homotopy approach makes it possible to tackle this problem in an elegant way. Instead of characterizing the set $P(r)$ by means of the total demand function for assets G we will make use of the first-order equations that characterize demand. In the literature this is called an extended system, see also Citanna (2000).

Given a sign vector $r \in R$ we are interested in the collection of assets for which agent h is a supplier, the assets in which he does not trade, and the assets for which agent h acts as a buyer. These sets are denoted by $J_h^-(r)$, $J_h^0(r)$, and $J_h^+(r)$, so

$$J_h^-(r) = \{j \in J \mid r_j^h = -1\}, \quad J_h^0(r) = \{j \in J \mid r_j^h = 0\},$$
$$J_h^+(r) = \{j \in J \mid r_j^h = +1\}.$$

The following notation indicates for each sign vector $r \in R$ all combinations of agents and assets where supply, inactivity or demand occurs,

$$R^-(r) = \{(h,j) \in H \times J \mid r_j^h = -1\}, \qquad R^0(r) = \{(h,j) \in H \times J \mid r_j^h = 0\},$$
$$R^+(r) = \{(h,j) \in H \times J \mid r_j^h = +1\}.$$

Consider any sign vector $r \in R$ and any $(t,q) \in \mathbb{R} \times Q$, then $(t,q) \in P(r)$ if and only if there is $(\lambda^-, \lambda^+, \theta, c) \in \mathbb{R}^K$, where $K = 3HJ + H(1+S)$, such that

$$\lambda_j^{h,-} = 0, \quad (h,j) \in R^-(r), \quad (1)$$

$$\lambda_j^{h,+} = 0, \quad (h,j) \in R^+(r), \quad (2)$$

$$\theta_j^h = 0, \quad (h,j) \in R^0(r), \quad (3)$$

$$c_0^h - e_0^h + \sum_{j \in J_h^-(r)} \theta_j^h (q_j - k_j) + \sum_{j \in J_h^+(r)} \theta_j^h (q_j + k_j) = 0, \quad h \in H, \quad (4)$$

$$c_s^h - e_s^h - \sum_{j \in J_h^-(r) \cup J_h^+(r)} \theta_j^h d_s^j = 0, \quad h \in H,\ s \in S, \quad (5)$$

$$\lambda_j^{h,-} + \partial_{c_0^h} u^h(c^h)(q_j - k_j) - \sum_{s=1}^{S} \partial_{c_s^h} u^h(c^h) d_s^j = 0, \quad h \in H,\ j \in J, \quad (6)$$

$$\lambda_j^{h,+} - \partial_{c_0^h} u^h(c^h)(q_j + k_j) + \sum_{s=1}^{S} \partial_{c_s^h} u^h(c^h) d_s^j = 0, \quad h \in H,\ j \in J, \quad (7)$$

$$t \sum_{h \in H} \theta_j^h + (1-t)(q_j^0 - q_j) + t \frac{\varphi(\prod_{h=1}^{H} \lambda_j^{h,-})}{1 + e^{q_j}} = 0, \quad j \in J, \quad (8)$$

$$\lambda_j^{h,-} \geq 0, \quad (h,j) \in R^0(r), \quad (9)$$

$$\lambda_j^{h,+} \geq 0, \quad (h,j) \in R^0(r), \quad (10)$$

$$-\theta_j^h \geq 0, \quad (h,j) \in R^-(r), \quad (11)$$

$$\theta_j^h \geq 0, \quad (h,j) \in R^+(r), \quad (12)$$
$$t \geq 0, \quad (13)$$
$$1 - t \geq 0. \quad (14)$$

Equations (8) are the perturbed market-clearing conditions that correspond to the homotopy equations $\widetilde{F}(t,q) = 0$, only that the asset-demand functions are replaced by the portfolio choices θ_j^h. Note that equations (8) are the only equations containing the homotopy parameter t that is constrained to lie between 0 and 1 by inequalities (13)–(14). Equations (1)–(7) and inequalities (9)–(12) are the first-order conditions of the agents' utility maximization problems. These conditions are necessary and sufficient since the agents' utility maximization problems are convex programming problems with linear constraints, so a constraint qualification is satisfied. In fact, the first-order conditions of the agents' utility maximization problems also lead to the inequalities $\lambda_j^{h,-} \geq 0$ for $(h,j) \in R^+(r)$ and $\lambda_j^{h,+} \geq 0$ for $(h,j) \in R^-(r)$. These inequalities are redundant, as they follow with strict inequality from equations (1), (6) and (7), and (2), (6) and (7), respectively, making use of the assumption that $\partial_{c_0^h} u^h(c^h)$ and k_j are strictly positive. These inequalities are therefore omitted. Equations (4) and (5) are the budget constraints, equations (6) and (7) are the derivatives with respect to the decision variables θ_j^h, and equations (1)–(3) are the complementary slackness conditions for the multipliers corresponding to the sign constraints on the decision variables. Note that for $(h,j) \in R^0(r)$ the complementarity condition reduces simply to $\theta_j^h = 0$, that is, to equation (3). If $\theta_j^h < 0$, then $\lambda_j^{h,-}$ must be 0, and the complementarity condition is just equation (1). Inequalities (9)–(12) are the sign restriction on the decision variables and multipliers.

For $r \in R$, the solutions to the system of equations (1)–(8) and the inequalities (9)–(14) are denoted by $\widetilde{P}(r)$.

Proposition 3.5. *For $r \in R$, $P(r)$ and $\widetilde{P}(r)$ are C^2 diffeomorphic.*

Proof. We define the function $f : \mathbb{R} \times Q(r) \to \mathbb{R} \times Q(r) \times \mathbb{R}^{HJ} \times \mathbb{R}^{HJ} \times \mathbb{R}^{HJ} \times \mathbb{R}^{H(1+S)}$ by

$$f(t,q) = (t, q, \lambda^-(q), \lambda^+(q), g(q), c(q)),$$

where $c_0^h(q) = e_0^h - \sum_{j \in J} q_j g_j^h(q) + \sum_{j \in J_h^-(r)} k_j g_j^h(q) - \sum_{j \in J_h^+(r)} k_j g_j^h(q)$, and $c_{\mathbb{1}}^h = e_{\mathbb{1}}^h + A g^h(q)$. Then $(t,q) \in P(r)$ if and only if $f(t,q) \in \widetilde{P}(r)$. That the function f is C^2 follows easily, see Herings and Schmedders (2001). Obviously, f^{-1} is C^∞. □

Define the open set $E = \mathbb{R}_{++}^{(1+S)H}$.

Theorem 3.6. *There is a subset E^* of E such that $E \setminus E^*$ has a closure with Lebesgue measure zero and for all $e \in E^*$, for all $r \in R$, $\widetilde{P}(r)$ is a compact, 1-dimensional C^2 manifold with boundary. A point $(t, q, \lambda^-, \lambda^+, \theta, c)$ in the boundary of $\widetilde{P}(r)$ is either not a boundary point of $\widetilde{P}(\overline{r})$ for all $\overline{r} \neq r$ and belongs to $\{0,1\} \times Q \times \mathbb{R}^K$, or is a boundary point of exactly one $\widetilde{P}(\overline{r})$ with $\overline{r} \neq r$ and belongs to $(0,1) \times Q \times \mathbb{R}^K$. Moreover, r and $\overline{r}$ differ in exactly one element which changes from -1 to 0 or from $+1$ to 0, or the reverse.*

For almost all economies, for all $r \in R$, the set $\widetilde{P}(r)$ is a compact, 1-dimensional differentiable manifold with boundary, so it is a finite collection of disjoint paths and loops. It follows that each component of $\widetilde{P}(r)$, i.e. a maximally connected subset of $\widetilde{P}(r)$, is either a path or a loop. We write $\widetilde{P}(r) = \widetilde{P}(r, 1) \cup \cdots \cup \widetilde{P}(r, c(r))$, where $\widetilde{P}(r, c)$, $c = 1, \ldots , c(r)$, is a component of $\widetilde{P}(r)$ and $c(r)$ is the number of components in $\widetilde{P}(r)$. The set $\widetilde{P} = \cup_{r \in R} \widetilde{P}(r)$ is C^2 diffeomorphic to the set of all solutions to the homotopy equations, so $\widetilde{P}$ is C^2 diffeomorphic to $\widetilde{F}^{-1}(\{0\})$.

The proof of Theorem 3.6 is given in Herings and Schmedders (2001) and consists of three parts. First, consider the system of equations (1)–(8). The number of variables $(t, q, \lambda^-, \lambda^+, \theta, c)$ in the system of equations (1)–(8) equals $1 + J + K$, one more than the number of equations which is given by $J + K$. Using transversality theory, it is therefore indeed possible to show that there is generically a one-dimensional set of solutions. Second, if in addition to (1)–(8), we require exactly one of the inequalities in (9)–(14) to hold with equality, using transversality theory, one can show that generically one obtains a finite set of locally unique solutions. Third, it can be shown, generically, that a system of equations consisting of (1)–(8) and two or more equations of (9)–(14) has no solutions. Although in principle standard, the proof is very tedious because of the large number of cases that have to be verified in the transversality arguments.

The following result confirms that the non-differentiabilities of $\widetilde{F}$ are well-behaved and do lead to well-behaved non-differentiabilities of $\widetilde{F}^{-1}(\{0\})$ that allow us to prove convergence of our algorithm.

Theorem 3.7. *There is a subset E^* of E such that $E \setminus E^*$ has a closure with Lebesgue measure zero and such that for all $e \in E^*$ the following statements hold. The set $\widetilde{F}^{-1}(\{0\})$ is a compact 1-dimensional piecewise C^2 manifold with boundary.[3] The boundary of $\widetilde{F}^{-1}(\{0\})$ equals the intersection of $F^{-1}(\{0\})$ and $\{0, 1\} \times Q$ and is a compact 0-dimensional manifold. There is a unique boundary point in $\{0\} \times Q$ that is connected by $\widetilde{F}^{-1}(\{0\})$ to a uniquely determined boundary point in $\{1\} \times Q$, i.e. the homotopy path is well-defined.*

Proof. Consider the set E^* of Theorem 3.6. For all $e \in E^*$, for all $r \in R$, the set $\widetilde{P}(r)$ consists of a finite number of paths and loops. Each path in $\widetilde{P}(r)$ has two boundary points. If it has a boundary point in $\{0, 1\} \times Q \times \mathbb{R}^K$, then the boundary point does not belong to any $\widetilde{P}(\overline{r})$ for $\overline{r} \neq r$. It is then a boundary point of $\widetilde{P}$. If a path has a boundary point in $(0, 1) \times Q$, then it is a boundary point of exactly one $\widetilde{P}(\overline{r})$ with $\overline{r} \neq r$. So it is a boundary point of a path in $\widetilde{P}(\overline{r})$. This path has another boundary point, either in $\{0, 1\} \times Q \times \mathbb{R}^K$ or in $(0, 1) \times Q \times \mathbb{R}^K$. In the former case, we have found a boundary point of $\widetilde{P}$. In the latter case, there is exactly one $\widetilde{r}$ such that the boundary point is also a boundary point of an arc in $\widetilde{P}(\widetilde{r})$, etc.

Since the cardinality of the set R is finite, and each $\widetilde{P}(r)$ consists of finitely many paths and loops, it will either be the case that eventually a path is generated with a boundary point in $\{0, 1\} \times Q \times \mathbb{R}^K$, or a path is generated that has been generated before. In the latter case, we have found a piecewise C^2 loop of $\widetilde{P}$. In the former case,

[3] A manifold is a 1-dimensional piecewise C^2 manifold if it is a 1-dimensional topological manifold that is a finite union of C^2 manifolds.

the finite chain of paths constitutes a piecewise C^2 path of $\widetilde{P}$ with boundary points belonging to $\{0,1\} \times Q$. It follows that $\widetilde{P}$ is a compact 1-dimensional piecewise C^2 manifold with boundary, where the boundary is given by the intersection of $\widetilde{P}$ and $\{0,1\} \times Q \times \mathbb{R}^K$. As a consequence it follows that $\widetilde{F}^{-1}(\{0\})$ is a compact 1-dimensional piecewise C^2 manifold with boundary, where the boundary is given by the intersection of $\widetilde{F}^{-1}(\{0\})$ and $\{0,1\} \times Q$. Notice that the argument above is nothing but a nonlinear version of the door-in door-out principle of Lemke and Howson (1964).

It is easy to see that there is a unique boundary point in $\{0\} \times Q$, since $\widetilde{F}(0,q) = q^0 - q$, which has $q = q^0$ as the unique solution in $\widetilde{F}^{-1}(\{0\})$. Since $\widetilde{F}^{-1}(\{0\})$ is a 1-dimensional manifold with boundary, the unique solution in $\{0\} \times Q$ is connected by $\widetilde{F}^{-1}(\{0\})$ to a uniquely determined boundary point in $\{1\} \times Q$. □

The homotopy approach now consists of numerically following the homotopy path in $\widetilde{F}^{-1}(\{0\})$ that connects the unique boundary point in $\{0\} \times Q$ to a uniquely determined boundary point in $\{1\} \times Q$. The latter point induces a demand-perfect equilibrium.

3.5 Bounds on asset prices

Adding equations (6) and (7) yields the equation $\lambda_j^{h,-} + \lambda_j^{h,+} = 2\partial_{c_0^h} u^h(c^h) k_j$. This equation shows that $\lambda_j^{h,-} + \lambda_j^{h,+} > 0$ implying the last statement of Proposition 2.3, namely that $\theta_j^{h,-} \cdot \theta_j^{h,+} = 0$, it can never be optimal for an agent to be both long and short in a financial security. Moreover, this equation sheds light on what happens along the homotopy path when an agent changes sides on a security market, for example, from being long, to being inactive, to being short. When the agent's long position in asset j is reduced down to zero, and the homotopy path is at the boundary of two sets $\widetilde{P}(r)$ and $\widetilde{P}(r')$ the shadow prices are $\lambda_j^{h,-} = 2\partial_{c_0^h} u^h(c^h) k_j$ and $\lambda_j^{h,+} = 0$. As the homotopy path moves through the interior of $\widetilde{P}(r')$ both shadow prices are positive indicating by complementary slackness that the asset variable is zero, that is $\theta_j^h = 0$, or equivalently $\theta_j^{h,-} = \theta_j^{h,+} = 0$. As the path hits a set $\widetilde{P}(r'')$ where the agent is short in asset j (in its interior) the shadow prices reach the point where $\lambda_j^{h,-} = 0$ and $\lambda_j^{h,+} = 2\partial_{c_0^h} u^h(c^h) k_j$.

The shadow prices always satisfy $\lambda_j^{h,-}, \lambda_j^{h,+} \geq 0$ resulting in the following inequalities for all $h \in H$,

$$q_j \leq \; k_j + \sum_{s=1}^{S} \left(\frac{\partial_{c_s^h} u^h(c^h)}{\partial_{c_0^h} u^h(c^h)} d_s^j \right), \quad \text{and} \quad q_j \geq -k_j + \sum_{s=1}^{S} \left(\frac{\partial_{c_s^h} u^h(c^h)}{\partial_{c_0^h} u^h(c^h)} d_s^j \right).$$

Hence, the price range for asset j in equilibrium equals

$$-k_j + \max_{h \in H} \left\{ \sum_{s=1}^{S} \frac{\partial_{c_s^h} u^h(c^h)}{\partial_{c_0^h} u^h(c^h)} d_s^j \right\} \leq q_j \leq k_j + \min_{h \in H} \left\{ \sum_{s=1}^{S} \frac{\partial_{c_s^h} u^h(c^h)}{\partial_{c_0^h} u^h(c^h)} d_s^j \right\}.$$

An immediate consequence of the last inequalities is that the price difference between the asset prices in a demand-perfect equilibrium and a supply-perfect equilibrium never exceeds $2k_j$. In a demand-perfect equilibrium the upper bound $k_j + \min_{h\in H}\{\sum_{s=1}^{S} \frac{\partial_{c_s^h} u^h(c^h)}{\partial_{c_0^h} u^h(c^h)} d_s^j\}$ and in a supply-perfect equilibrium the lower bound $-k_j + \max_{h\in H}\{\sum_{s=1}^{S} \frac{\partial_{c_s^h} u^h(c^h)}{\partial_{c_0^h} u^h(c^h)} d_s^j\}$ is computed.

3.6 A differentiable homotopy

A drawback of our algorithm is that a large number of bid-ask structures might be generated before reaching a demand-perfect equilibrium. To avoid this problem we exploit the complementarity between the portfolio variables $\theta_j^{h,-}$ $(\theta_j^{h,+})$ and the shadow prices $\lambda_j^{h,-}$ $(\lambda_j^{h,+})$ of the nonnegativity constraints for the portfolio variables. See Garcia and Zangwill (1981) for a discussion of this approach. Equations (1)–(3) and inequalities (9)–(12) imply the standard complementarity conditions

$$\theta_j^{h,-} \cdot \lambda_j^{h,-} = 0 \text{ and } \theta_j^{h,+} \cdot \lambda_j^{h,+} = 0.$$

Therefore, we actually can represent, for example, $\theta_j^{h,+}$ and $\lambda_j^{h,+}$ by a single variable. We introduce two vectors $\alpha^-, \alpha^+ \in \mathbb{R}^{HJ}$ and substitute the following functions for the portfolio variables and shadow prices

$$\lambda_j^{h,-} = (\max\{0, \alpha_j^{h,-}\})^l, \qquad \lambda_j^{h,+} = (\max\{0, \alpha_j^{h,+}\})^l,$$
$$\theta_j^{h,-} = (\max\{0, -\alpha_j^{h,-}\})^l, \qquad \theta_j^{h,+} = (\max\{0, -\alpha_j^{h,+}\})^l,$$

where l can be any integer greater than or equal to two. Note that the functions are $l-1$ times continuously differentiable in the variables $\alpha_j^{h,-}$ and $\alpha_j^{h,+}$, respectively. By definition of these functions inequalities (9)–(12) are always automatically satisfied and we can drop them from consideration.

A bid-ask structure r corresponds to $(\alpha^-, \alpha^+) \in A^-(r) \times A^+(r)$, where

$$\begin{aligned} A^-(r) = \{\alpha^- \in \mathbb{R}^{HJ} \mid\ & \alpha_j^{h,-} \leq 0 \text{ if } (h,j) \in R^-(r) \\ & \alpha_j^{h,-} \geq 0 \text{ if } (h,j) \in R^0(r) \cup R^+(r)\} \\ A^+(r) = \{\alpha^+ \in \mathbb{R}^{HJ} \mid\ & \alpha_j^{h,+} \leq 0 \text{ if } (h,j) \in R^+(r) \\ & \alpha_j^{h,+} \geq 0 \text{ if } (h,j) \in R^-(r) \cup R^0(r)\}. \end{aligned}$$

Given a particular bid-ask structure, equations (1)–(3) are also automatically satisfied. There is a solution $(t, q, \lambda^-, \lambda^+, \theta, c) \in \widetilde{P}(r)$ if and only if there is $(t, q, \alpha^-, \alpha^+, c)$, where $(\alpha^-, \alpha^+) \in A^-(r) \times A^+(r)$, such that

$$\partial_{c_0^h} u^h(c)(q_j - k_j) - \sum_{s=1}^{S} \partial_{c_s^h} u^h(c) d_s^j + (\max\{0, \alpha_j^{h,-}\})^l = 0, \qquad (15)$$
$$(h,j) \in H \times J,$$

$$\partial_{c_0^h} u^h(c)(q_j + k_j) - \sum_{s=1}^{S} \partial_{c_s^h} u^h(c) d_s^j - (\max\{0, \alpha_j^{h,+}\})^l = 0, \tag{16}$$
$$(h, j) \in H \times J,$$

$$c_0^h - e_0^h + \sum_{j \in J} q_j((\max\{0, -\alpha_j^{h,+}\})^l - (\max\{0, -\alpha_j^{h,-}\})^l)$$
$$+ \sum_{j \in J} k_j((\max\{0, -\alpha_j^{h,-}\})^l + (\max\{0, -\alpha_j^{h,+}\})^l) = 0, \quad h \in H, \tag{17}$$

$$c_s^h - e_s^h - \sum_{j \in J}((\max\{0, -\alpha_j^{h,+}\})^l - (\max\{0, -\alpha_j^{h,-})^l\}) d_s^j = 0, \tag{18}$$
$$h \in H,\ s \in S,$$

$$t \sum_{h \in H}((\max\{0, -\alpha_j^{h,+}\})^l - (\max\{0, -\alpha_j^{h,-}\})^l)$$
$$+(1-t)(q_j^0 - q_j) + t\frac{\varphi(\prod_{h \in H}(\max\{0, \alpha_j^{h,-}\})^l)}{1 + e^{q_j}} = 0, \quad j \in J. \tag{19}$$

The advantage of this reduced system is the absence of inequality constraints and the independence of the system of the sign vector r. We can use standard path-following methods to follow the path generated by this particular homotopy. Obviously the values of all variables along the path generated by this homotopy are identical to those along the path generated by the homotopy $\widetilde{F}$.

4 Numerical example

We implemented our homotopy algorithm on a 450 MHz PCPentiumII using the software package HOMPACK. This software package is a collection of FORTRAN 77 subroutines for solving systems of nonlinear equations using homotopy methods (Watson et al., 1987). From the three methods available in HOMPACK we selected the most robust path-following algorithm, which tracks the homotopy path by solving an ordinary differential equation. The starting point of the homotopy can be found using a standard nonlinear equation solver. We use a variation of the penalty approach of Schmedders (1998). We approximate the Jacobian of the homotopy function with a one-sided difference formula. When the path-following routine finds a solution we use a Newton routine to refine the solution to further reduce the error. In all our examples the maximum relative errors are of the order of magnitude of 10^{-10}. The running time of the computer implementation of our algorithm is less than two seconds for the examples below.

4.1 Simple economy

Consider an economy with $H = 2$ agents, $S = 4$ possible states in period $t = 1$, and $J = 2$ assets, called a bond and a stock. Both agents have identical von-Neumann-Morgenstern CRRA utility functions with identical uniform beliefs. That is, agent

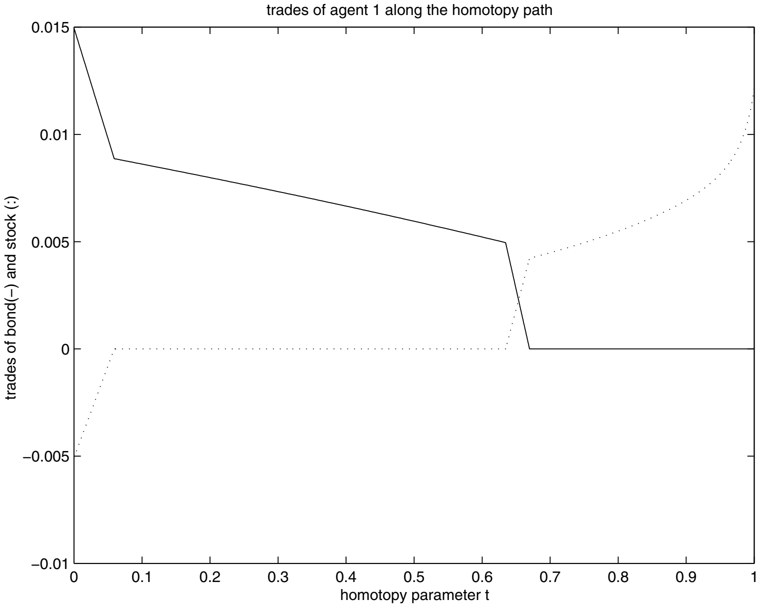

Figure 1. Asset trades of agent 1 along the homotopy path

i's utility function equals:

$$u^i(c) = \frac{c_0^{1-\gamma_i}}{1-\gamma_i} + \sum_{s=1}^{4} \frac{1}{4} \frac{c_s^{1-\gamma_i}}{1-\gamma_i}.$$

The two agents have coefficients of risk-aversion of $\gamma_1 = 5$ and $\gamma_2 = 1$, respectively. Both agents have an endowment of $e_0^1 = e_0^2 = 1$ in period 0. Agent 1 has an endowment (labor income) $e_1^1 = (0.9, 1.1, 0.9, 1.1)$ at date $t = 1$; agent 2 has zero endowment at $t = 1$, but he owns the entire stock paying dividends $d^{\mathrm{st}} = (0.5, 1.0, 1.5, 2.0)$. The stock is in unit net supply. (Agent 2's endowment at $t = 0$ can be thought of as the stock's dividend $d_0^{\mathrm{st}} = 1$ at $t = 0$.) The bond pays one unit in the second period regardless of the state of nature and is in zero net supply. Agents trading the bond and the stock have to pay identical transaction cost $k = k_{\mathrm{b}} = k_{\mathrm{st}} = 0.05$ for both securities.

Using the differentiable homotopy of Section 3.6 we need to solve a system of $2HJ + H + HS + J = 20$ equations with 21 unknowns. By solving equations (17) and (18) for the consumption variables and substituting the obtained values into the first-order conditions (15) and (16) we can reduce the system to $2HJ + J = 10$ equations and 11 unknowns. We use the expected payoffs of the two assets as values for the “starting" prices q^0, that is, $q_{\mathrm{b}}^0 = 1$ and $q_{\mathrm{st}}^0 = 1.25$.

We depict the nature of the homotopy path in a few figures. Figures 1 and 2 show the change of agent 1's and agent 2's portfolio, respectively, as a function of the homotopy parameter. The behavior of the two portfolios is extremely different along the path. For small values of t the first agent is long in the bond and short in

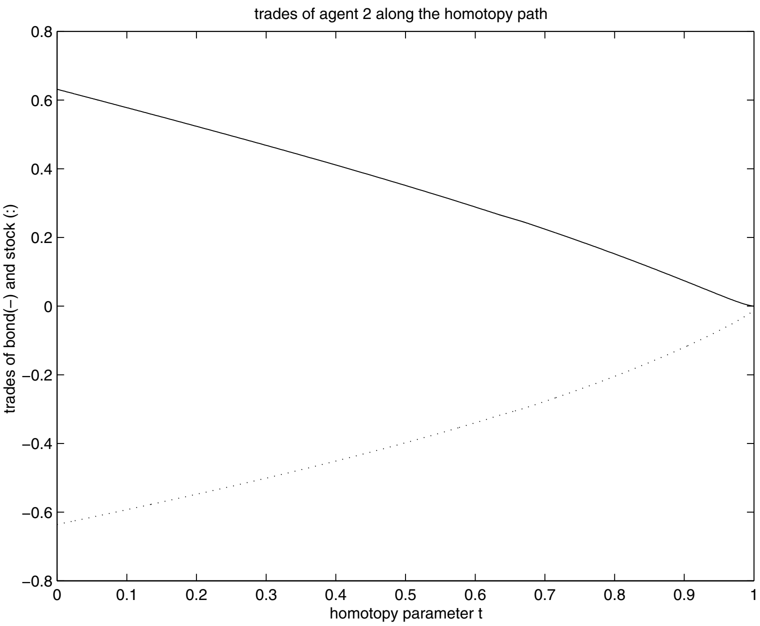

Figure 2. Asset trades of agent 2 along the homotopy path

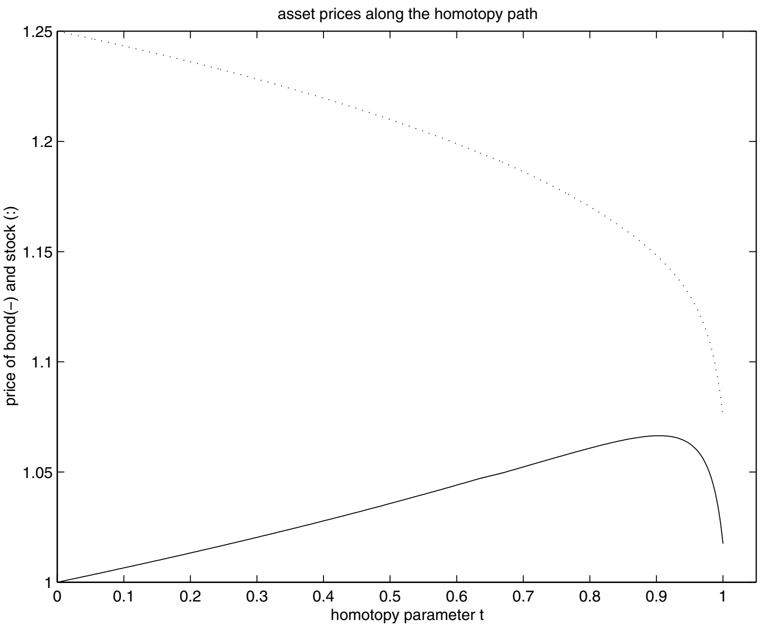

Figure 3. Asset prices along the homotopy path

the stock. The short position in the stock decreases quickly to zero as t increases and the path of θ^1_{b} exhibits a kink which in turn leads also to a kink in the path of θ^1_{st}. This type of behavior of the portfolio functions is typical; whenever one function exhibits a kink with a function value of zero then the other portfolio function also has a non-differentiability. However, these kinks in the first agent's portfolio functions do not affect the second agent's portfolio functions. Along the homotopy path no holding of the second agent hits zero for $t < 1$ resulting in smooth portfolio functions. Note that equation (19) does not enforce market clearing for $t < 1$. Only as t hits 1 the variable θ^2_{b} decreases to zero. In equilibrium the bond market is closed. The equilibrium trade on the stock market equals a sale of 0.012 shares by agent 2 to agent 1. Figure 3 displays the behavior of the asset prices along the homotopy path. Both price functions are smooth and are unaffected by the kinks in the portfolio functions of agent 1.

4.2 The need for an equilibrium selection

With this simple economy we can also show the importance of an equilibrium selection. Figure 4 shows the prices along the homotopy path for the homotopy without the term used for the equilibrium selection, that is, for the homotopy where equation (19) is replaced by the following equation:

$$t\sum_{h\in H}((\max\{0,-\alpha_j^{h,+}\})^l-(\max\{0,-\alpha_j^{h,-}\})^l)+(1-t)(q_j^0-q_j)=0, \quad j\in J.$$

Until t hits 1 the price paths are identical. But in equilibrium the bond is not traded due to the large transaction costs resulting in a continuum of equilibrium bond

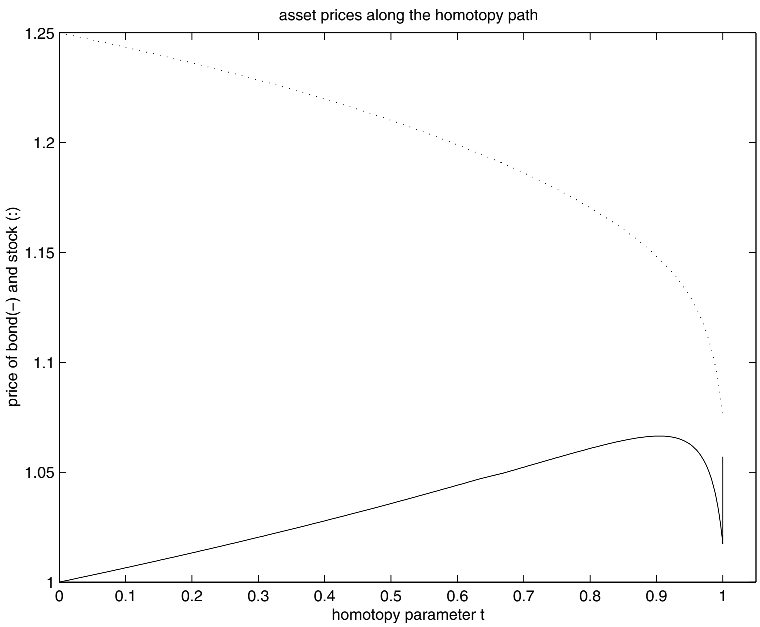

Figure 4. Continuum of equilibrium prices

prices. If we don't force the homotopy to make an equilibrium selection then the path runs into this continuum and the homotopy exhibits a drop in rank at $t = 1$ which causes numerical problems. (In this example the homotopy solver cannot find a unique stable solution at $t = 1$ and instead finds many solutions with varying bond prices. Eventually the solver reports one of the found solutions and indicates a numerical problem.)

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