Econ. Theory 7, 291–306 (1996)

Economic Theory

Two characterizations of the uniform rule for division problems with single-peaked preferences*

Gert-Jan Otten,[1] Hans Peters,[2] and Oscar Volij[3]

[1] Department of Econometrics, CentER for Economic Research, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, THE NETHERLANDS

[2] Department of Quantitative Economics, University of Limburg, P.O. Box 616, 6200 MD Maastricht, THE NETHERLANDS

[3] CentER for Economic Research, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, THE NETHERLANDS

Received: June 20, 1994

Summary. The uniform rule is considered to be the most important rule for the problem of allocating an amount of a perfectly divisible good between agents who have single-peaked preferences. The uniform rule was studied extensively in the literature and several characterizations were provided. The aim of this paper is to provide two different formulations and corresponding axiomatizations of the uniform rule. These formulations resemble the Nash and the lexicographic egalitarian bargaining solutions; the corresponding axiomatizations are based on axioms of independence of irrelevant alternatives and restricted monotonicity.

1. Introduction

We consider the problem of distributing a non-negative amount of a perfectly divisible good among a finite set of agents who have single-peaked preferences, i.e., up to a certain amount an agent likes to consume more of the good, beyond this amount the opposite holds. This problem has been studied extensively in the literature. Sprumont (1991) initiated the axiomatic analysis by characterizing the uniform rule. He showed that the uniform rule is the unique rule which satisfies Pareto optimality, strategy-proofness and either envy-freeness or anonymity. Ching (1994) shows that the anonymity property can be replaced by the weaker property of equal treatment and provides an alternative proof. Other axiomatizations of the uniform rule are given in Thomson (1994a) using the well-known principles of consistency and converse consistency. As a result of this extensive analysis, the uniform rule is now considered to be the most interesting rule for this type of problems.

* The authors are grateful to Jeffrey Banks, Steve Ching, Vincent Feltkamp, Bezalel Peleg, Sang-Young Sonn, William Thomson and an anonymous referee for helpful comments.

Correspondence to. H. Peters

In this paper we give two new characterizations of the uniform rule, both of which are inspired by the axiomatizations of two different bargaining solutions. In section 2 we associate with each economy an auxiliary bargaining problem in such a way that the set of Pareto optimal allocations of the bargaining problem coincides with the set of efficient divisions in the original economy. Next we show that the division recommended by the uniform rule to each economy, coincides with the allocation recommended both by the Nash and the lexicographic egalitarian bargaining solutions to the associated bargaining problem. The proofs are interesting because they use the principles of consistency and converse consistency in different contexts, namely in the context of bargaining problems on the one hand, and of the allocation of a commodity among agents with single-peaked preferences on the other hand. Moreover, they illustrate that consistency and converse consistency, which have been employed in axiomatic characterizations of game theoretic solution concepts (for example, Sobolev (1975), Peleg (1985, 1986), Lensberg (1988), Peleg and Tijs (1992) to mention just a few), can be helpful for other purposes as well. Both our results suggest that the uniform rule might be characterized by means of some suitably adapted set of axioms that characterize the bargaining solutions mentioned above. Section 3 provides two characterizations of the uniform rule. One uses axioms reminiscent of those used by Nash (1950) to axiomatize the Nash bargaining solution and the other uses axioms inspired by the axiomatization of the lexicographic egalitarian bargaining solution by Chun and Peters (1988). More specifically, the first characterization is based on an independence of irrelevant alternatives axiom, and the second one is based on a restricted monotonicity axiom.

2. The uniform rule

2.1 The model

Let $I \subset \mathbb{N}$ be a non-empty set of agents and let $\bar{M}$ be some fixed positive number. A *coalition* is a finite, non-empty subset of I. Given any *preference relation* R over $[0, \bar{M}]$, i.e., a complete and transitive binary relation, we denote $x \; R \; y$ if $(x, y) \in R$, $x \; P \; y$ if $x \; R \; y$ and not $y \; R \; x$, and $x \; I \; y$ if $x \; R \; y$ and $y \; R \; x$. R is called *single-peaked* if there exists a number $p(R) \in [0, \bar{M}]$ such that for all $x, y \in [0, \bar{M}]$, with $x < y \leq p(R)$ or $p(R) \leq y < x$, we have $y \; P \; x$. $p(R)$ is called the *peak* of the relation R. By $\mathscr{R}$ we denote the set of all single-peaked preferences over $[0, \bar{M}]$. The introduction of $\bar{M}$ is just for notational convenience: It allows us to define peaks as a function only of the preferences, i.e., independently of the amount to be divided. An alternative way would be to define preferences over $[0, \infty)$, but then monotonic increasing preferences would be excluded from the definition of single-peaked preferences unless we say that in this case the peak is infinity.

An *economy* is a tuple $E = \langle M, (R_i)_{i \in N} \rangle$, where $0 \leq M \leq \bar{M}$, N is a coalition, and for each $i \in N$, $R_i \in \mathscr{R}$. Denote $p(E) := (p(R_i))_{i \in N}$. The class of all economies is denoted by $\mathscr{E}$. An economy represents the problem of allocating a positive amount of a perfectly divisible good, which cannot be disposed of, among a group of agents who have single-peaked preferences over $[0, \bar{M}]$.

Let N be a coalition and $x \in \mathbb{R}^N$. If $S \subset N$, $S \neq \emptyset$, then we denote $x(S) := \sum_{i \in S} x_i$, and $x_S := (x_i)_{i \in S} \in \mathbb{R}^S$. For $x, y \in \mathbb{R}^S$ we denote $x \leq y \, (x < y)$ if $x_i \leq y_i \, (x_i < y_i)$ for all $i \in S$.

Let $E = \langle M, (R_i)_{i\in N}\rangle$ be an economy. An *allocation* for E is a vector $x\in\mathbb{R}^N_+$ such that $x(N) = M$. By $X^*(E)$ we denote the set of all allocations for E. An allocation $x\in X^*(E)$ is called *efficient* if there is no $y\in X^*(E)$ such that $y_i R_i x_i$ for all $i\in N$ and $y_i P_i x_i$ for some $i\in N$. $X(E)$ denotes the set of all efficient allocations for E.

Sprumont (1991) showed that an allocation for an economy is efficient if and only if there are no two agents such that one gets more than his peak and the other gets less than his peak. This means that an allocation is efficient if and only if all agents are on the "same side" of their peaks. Formally, for an economy $E = \langle M, (R_i)_{i\in N}\rangle\in\mathscr{E}$ and $x\in X^*(E)$,

$$x\in X(E) \Leftrightarrow \begin{cases} x \le p(E) & \text{if } M \le \sum_{i\in N} p(R_i) \\ x \ge p(E) & \text{if } M \ge \sum_{i\in N} p(R_i). \end{cases}$$

A *rule* is a function ϕ which assigns to each economy $E\in\mathscr{E}$ an allocation $\phi(E)\in X^*(E)$, which can be interpreted as a recommendation for economy E.

A rule which plays a central role in the literature of economies with single-peaked preferences is the uniform rule, see Sprumont (1991), Thomson (1991, 1994a, b, 1992a, b), Ching (1992, 1994).

The *uniform rule*, U, is defined as follows. Let $E = \langle M, (R_i)_{i\in N}\rangle\in\mathscr{E}$ and $i\in N$. Then

$$U_i(E) := \begin{cases} \min\{p(R_i), \lambda\} & \text{if } M \le \sum_{i\in N} p(R_i) \\ \max\{p(R_i), \lambda\} & \text{if } M \ge \sum_{i\in N} p(R_i), \end{cases}$$

where λ is such that $U(E)\in X^*(E)$.

For the case in which there is too little to divide, i.e., $M < \sum_{i\in N} p(R_i)$, the uniform rule chooses appropriately an amount λ and allocates it to every agent with peak above this amount while all other agents obtain their peak. Here, appropriately means that the resulting division is indeed an allocation. Note that the uniform rule takes into account only the amount M and the peaks of the preferences of the individual agents.

One of the reasons why the uniform rule is interesting, is that it is the only rule which satisfies many desirable properties. For example, it always recommends envy-free allocations. Moreover, the uniform rule is strategy-proof, i.e., if it is applied on the basis of declared preferences, it is a (weakly) dominant strategy for each player to declare his true preferences. We now discuss four other properties, which are also satisfied by the uniform rule.

Let ϕ be a rule.

Pareto optimality: ϕ is Pareto optimal if $\phi(E)\in X(E)$ for all $E\in\mathscr{E}$.

M-Monotonicity: ϕ is M-monotonic if for all economies $E = \langle M, (R_i)_{i\in N}\rangle\in\mathscr{E}$, and $E' = \langle M', (R_i)_{i\in N}\rangle\in\mathscr{E}$, with $M \le M'$, we have $\phi(E) \le \phi(E')$.[1]

Let $E = \langle M, (R_i)_{i\in N}\rangle\in\mathscr{E}$ be an economy, $x\in X^*(E)$, and $S \subset N$, $S \ne \varnothing$. The *reduced economy* w.r.t. S and x is

$$E^{S,x} := \langle x(S), (R_i)_{i\in S}\rangle.$$

[1] M-monotonicity is different from the 1-sided resource monotonicity introduced in Thomson (1994b). But if Pareto optimality is imposed both properties are equivalent.

Remark 2.1 Note that $E^{S,x} \in \mathscr{E}$. Further, if $\varnothing \neq T \subset S$, then $E^{T,x} = [E^{S,x}]^{T,x_S}$.

A rule ϕ is *consistent* if for all economies $E = \langle M, (R_i)_{i \in N} \rangle \in \mathscr{E}$, and all $S \subset N$, $S \neq \varnothing$ we have, if $x = \phi(E)$, then $x_S = \phi(E^{S,x})$.

Roughly speaking, consistency of a rule means that, if a subgroup of agents would decide to pool their parts of the allocation prescribed by the rule and apply the same rule to redistribute this total, then the agents in that group would end up each with the same amount as before. Thomson (1994a) proved that the uniform rule is consistent. For more details on the consistency principle the reader is referred to Thomson (1990, 1994a).

A rule ϕ is *converse consistent* if for all economies $E \in \mathscr{E}$ and all $x \in X^*(E)$ we have, if $x_S = \phi(E^{S,x})$ for all $S \subset N$ with $|S| = 2$, then $x = \phi(E)$.

Converse consistency means that, given a certain allocation x for an economy, if the restriction of x is recommended for every reduced economy with respect to a subgroup of two agents and x, then the allocation x is recommended in the large economy. As a consequence of the following lemma we obtain that the uniform rule is converse consistent.

Lemma 2.2 *Let ϕ be an M-monotonic rule. Then ϕ is consistent if and only if (i) for every economy $E \in \mathscr{E}$ there exists an $x \in X^*(E)$ such that $x_S = \phi(E^{S,x})$ for all $S \subset N$ with $|S| = 2$, and (ii) ϕ is converse consistent.*

Proof. ($\Rightarrow$) Let $E \in \mathscr{E}$. Take $x := \phi(E)$. Consistency of ϕ yields that $x_S = \phi(E^{S,x})$ for all $S \subset N$ with $|S| = 2$. In order to prove that ϕ is converse consistent, it suffices to show that there is no allocation $y \in X^*(E)$, $y \neq x$, such that $y_S = \phi(E^{S,y})$ for all $S \subset N$ with $|S| = 2$. Suppose that there exists such a y. Since $x(N) = y(N) = M$, it follows that there are $i, j \in N$ such that $x_i < y_i$ and $x_j > y_j$. Take $S := \{i,j\}$. W.l.o.g. we assume that $x(S) \geq y(S)$. M-Monotonicity of ϕ yields that $\phi(E^{S,x}) \geq \phi(E^{S,y})$. Hence, $x_S \geq y_S$, which yields a contradiction.

($\Leftarrow$) Let $E \in \mathscr{E}$. Let $\varnothing \neq T \subset N$, and $x = \phi(E)$. We have to prove that $x_T = \phi(E^{T,x})$. By assumption there exists a $y \in X^*(E)$ such that $y_S = \phi(E^{S,y})$ for all $S \subset N$ with $|S| = 2$. Converse consistency of ϕ yields that $y = \phi(E) = x$. Hence, $x_S = \phi(E^{S,x})$ for all $S \subset N$ with $|S| = 2$. By remark 2.1, $x_S = \phi([E^{T,x}]^{S,x_T})$ for all $S \subset T$, with $|S| = 2$. Clearly, $x_T \in X^*(E^{T,x})$. Hence, converse consistency of ϕ yields $x_T = \phi(E^{T,x})$. □

2.2 *Bargaining solutions*

Before we state the main results of section 2, we first recall some notions from cooperative bargaining theory. Those who are acquainted with this theory may skip this subsection.

Let $N \subset I$ be a coalition. A *bargaining problem* for N is a subset B of $\mathbb{R}^N_+$ which satisfies the following properties:

(i) B is compact and convex.
(ii) There exists a $y \in B$ with $y > 0$.
(iii) B is comprehensive, i.e., if $x \in B$, and $y \in \mathbb{R}^N_+$, with $y \leq x$, then $y \in B$.

Let $\mathscr{B}$ denote the set of all bargaining problems.[2]

A (*bargaining*) *solution* is a function $\mathscr{F}$ which assigns to each $B \in \mathscr{B}$ an element $\mathscr{F}(B)$ of B.

A prominent solution is the Nash bargaining solution introduced by Nash (1950). Let $B \in \mathscr{B}$ be a bargaining problem for N. The *Nash bargaining solution* is defined by

$$\mathscr{N}(B) := \operatorname{argmax}\left\{\prod_{i \in N} x_i \mid x \in B\right\}.$$

Another bargaining solution is the lexicographic egalitarian solution. To define it we need some notation.

Let $\alpha : \mathbb{R}^N \to \mathbb{R}^N$ be a function such that for each $x \in \mathbb{R}^N$ the vector $\alpha(x)$ is a reordering of the coordinates of x in a non-decreasing order. So if $i, j \in N$ with $i < j$, then $\alpha_i(x) \leq \alpha_j(x)$. The lexicographic maximin ordering $>^{lm}$ on $\mathbb{R}^N$ is defined by $x >^{lm} y$ if $\alpha(x) >^{l} \alpha(y)$, where $>^{l}$ denotes the lexicographic order on $\mathbb{R}^N$, i.e., for $a, b \in \mathbb{R}^N$, $a >^{l} b$ if there exists an $i \in N$ such that $a_i > b_i$ and $a_j = b_j$ for all $j < i$.

The *lexicographic egalitarian solution*, $\mathscr{L} : \mathscr{B} \to \mathbb{R}^N$ assigns to each bargaining problem $B \in \mathscr{B}$ the unique point which is maximal with respect to the lexicographic maximin ordering $>^{lm}$.

It is well-known that both $\mathscr{N}$ and $\mathscr{L}$ satisfy the three properties listed below.

A solution $\mathscr{F}$ is *Pareto optimal* if for all $B \in \mathscr{B}$, and all $y \in B$ we have, if $y \geq \mathscr{F}(B)$, then $y = \mathscr{F}(B)$.

A solution $\mathscr{F}$ satisfies *strict individual rationality* if $\mathscr{F}(B) > 0$ for all $B \in \mathscr{B}$.

Thomson and Lensberg (1989) and Lensberg (1988) characterized the lexicographic egalitarian solution and the Nash bargaining solution respectively, using a consistency property. In order to introduce it we need the following definition.

Let $B \in \mathscr{B}$ be a bargaining problem for N, let $x \in B$, and let $S \subset N$, $S \neq \varnothing$. The *reduced bargaining problem* w.r.t. S and x is

$$B^{S,x} := \{y_S \in \mathbb{R}^S_+ \mid (y_S, x_{N \setminus S}) \in B\}.$$

Note that not necessarily, $B^{S,x} \in \mathscr{B}$. However, if $x = \mathscr{N}(B)$ or if $x = \mathscr{L}(B)$, then $B^{S,x} \in \mathscr{B}$. This is a consequence of the fact that both $\mathscr{N}$ and $\mathscr{L}$ satisfy strict individual rationality.

The consistency property is now defined as follows.

A solution $\mathscr{F}$ is *consistent* if for all bargaining problems $B \in \mathscr{B}$ for N, and all $S \subset N$, $S \neq \varnothing$ we have, if $B^{S,x} \in \mathscr{B}$ where $x = \mathscr{F}(B)$, then $x_S = \mathscr{F}(B^{S,x})$.

For the results in this section we are going to make use of the fact that both $\mathscr{N}$ and $\mathscr{L}$ satisfy the consistency property. The results in the next section are based on the characterizations of $\mathscr{N}$ and $\mathscr{L}$ by Nash (1950) and Chun and Peters (1988).

---

[2] Usually, a bargaining problem is defined by a set B and a disagreement outcome $d \in B$. For our analysis the disagreement outcome does not play an explicit role: The reader may think of the disagreement outcome as being $d = 0$

2.3 *Two formulations of the uniform rule*

Let $E = \langle M, (R_i)_{i\in N}\rangle$ be an economy. Let $\rho(E)$ denote the set of agents $i\in N$ for which there is an $x\in X(E)$ such that $x_i > 0$. Note that $\rho(E) \neq \emptyset$ if and only if $M > 0$. If one is interested in Pareto optimal rules, it is clear that the problem is, how to divide the total amount M among the agents in $\rho(E)$, for all efficient allocations give zero to the agents not in $\rho(E)$. In other words, the set of agents which are *relevant* for economy E is $\rho(E)$.

We now state the main results of this section.

Theorem 2.3 *Let* $E = \langle M, (R_i)_{i\in N}\rangle$ *be an economy. Then* $U(E)$ *is the unique element of* $argmax\{\prod_{i\in\rho(E)} y_i \mid y\in X(E)\}$, *if* $\rho(E) \neq \emptyset$, *and* $U(E) = 0$, *otherwise.*

Theorem 2.4 *Let* $E = \langle M, (R_i)_{i\in N}\rangle$ *be an economy. Then* $U(E)$ *is the unique efficient allocation for E which is maximal with respect to* $>^{lm}$.

Instead of giving a direct proof of both theorems, we will give an indirect one based on some properties of the uniform rule and the consistency property of both the Nash solution and the lexicographic egalitarian solution.

Proof of theorems 2.3 and 2.4.
Clearly, both theorems hold if the economy consists of only one agent or if $M = 0$. So from now on attention is restricted to economies with at least two agents and $M > 0$. For any such an economy E define $B(E)$:= comp $X(E)$.[3] (See figure 1).

Case 1: All agents are relevant.
Let $E = \langle M, (R_i)_{i\in N}\rangle$ be an economy with $\rho(E) = N$ and $|N| \geq 2$. Since, $\rho(E) = N$, and $X(E)$ is a convex set, there exists a point $y\in X(E)$ with $y > 0$. Hence, $B(E)$ is a bargaining problem. $B(E)$ is called the *bargaining problem associated with E*.[4]

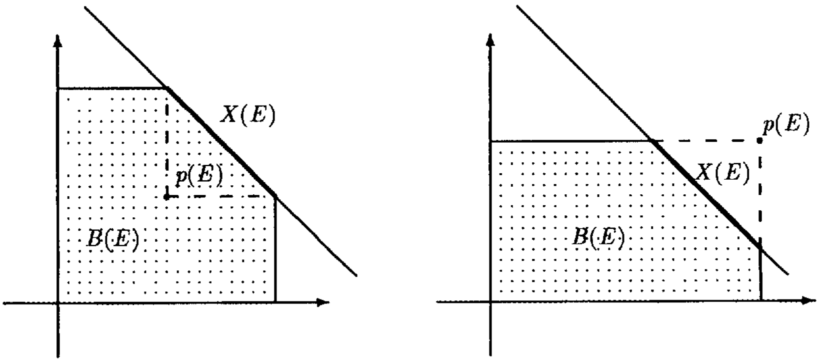

Figure 1. The set B(E) *in case E is an economy with two agents*

[3] comp$X(E)$ denotes the *comprehensive hull* of $X(E)$, i.e., the set of all $y\in\mathbb{R}^N_+$ such that $y \leq x$ for some $x\in X(E)$.
[4] It should be noted that $B(E)$ represents a set of physical allocations, whereas a bargaining problem in the usual sense represents a set of utility n-tuples.

The following lemma shows that the operation of reducing an economy commutes with the operation of reducing an associated bargaining problem. It also implies that, within this context, the consistency requirements for bargaining problems and economies coincide.

Lemma 2.5 *Let* $E = \langle M, (R_i)_{i\in N}\rangle$ *be an economy with* $|N| \geq 2$ *and* $\rho(E) = N$. *Let* $S \subset N$, $S \neq \emptyset$, *and* $x \in X(E)$. *Then*

$$B(E^{S,x}) = B^{S,x}(E).$$

Proof. We only prove the case $\sum_{i\in N} p(R_i) \leq M$. The other case is easier.

Since $x \in X(E)$, it follows that $\sum_{i\in S} p(R_i) \leq x(S)$. Hence,

$$X(E^{S,x}) = \{y \in \mathbb{R}^S_+ \mid y(S) = x(S),\ y_i \geq p(R_i) \forall i \in S\}.$$

Let $y \in B(E^{S,x}) = \text{comp}X(E^{S,x})$. Then there exists a $z \in X(E^{S,x})$ with $z \geq y$. This means that $z(S) = x(S)$, and $z_i \geq p(R_i)$ for all $i \in S$, which implies $(z, x_{N\setminus S}) \in X(E) \subset B(E)$. Hence, by definition of the reduced bargaining problem, it follows that $z \in B^{S,x}(E)$. Since $B^{S,x}(E)$ is comprehensive, we have $y \in B^{S,x}(E)$.

Now take $y \in B^{S,x}(E) = \{y \in \mathbb{R}^S_+ \mid (y, x_{N\setminus S}) \in \text{comp}X(E)\}$. Then there exists a $t \in X(E)$ with $t \geq (y_S, x_{N\setminus S})$ and $t_i \geq p(R_i)$ for all $i \in S$. Since $t_{N\setminus S} \geq x_{N\setminus S}$ and $t(N) = x(N)$, it follows that $t(S) \leq x(S)$. Hence, $t_S \in \text{comp}\{z \in \mathbb{R}^S_+ \mid z(S) = x(S),\ z_i \geq p(R_i)$ for all $i \in S\} = B(E^{S,x})$. Since $t_S \geq y_S$, comprehensiveness of $B(E^{S,x})$ implies that $y_S \in B(E^{S,x})$. Hence, $B^{S,x}(E) \subset B(E^{S,x})$. □

In order to prove theorems 2.3 and 2.4 for this case it is sufficient to show that

$$U(E) = \mathcal{N}(B(E)) = \mathcal{L}(B(E)). \tag{1}$$

First note that in case $|N| = 2$, it is immediately clear that $U(E) = \mathcal{L}(B(E))$. Furthermore, it is also straightforward to show that $\mathcal{N}(B(E)) = \mathcal{L}(B(E))$.

Hence, it remains to show that (1) holds if $|N| > 2$. This will follow from lemma 2.6 below.

Let $\mathcal{E}' \subset \mathcal{E}$ be the family of economies E with $\rho(E) = N$.

Lemma 2.6 *Let* $\mathcal{F}$ *be a bargaining solution which satisfies Pareto optimality, strict individual rationality, and consistency. If* $\mathcal{F}(B(E)) = U(E)$ *for all* $E = < M, (R_i)_{i\in N} > \in \mathcal{E}'$ *with* $|N| = 2$, *then* $\mathcal{F}(B(E)) = U(E)$ *for all* $E = \mathcal{E}'$.

Proof. Let $x := \mathcal{F}(B(E))$. From strict individual rationality we know that $x > 0$ and therefore, $B^{S,x}(E) \in \mathcal{B}$. Moreover, by consistency of $\mathcal{F}$

$$x_S = \mathcal{F}(B^{S,x}(E)) \quad \text{for all } S \subset N,\ |S| = 2.$$

Furthermore, from Pareto optimality of $\mathcal{F}$ and the definition of $B(E)$, it follows that $x \in X(E)$. So by lemma 2.5, $B(E^{S,x}) = B^{S,x}(E)$. Hence,

$$x_S = \mathcal{F}(B(E^{S,x})) \quad \text{for all } S \subset N,\ |S| = 2.$$

Since $B(E^{S,x}) = \text{comp}X(E^{S,x}) \in \mathcal{B}$, it follows that there exists an $y \in X(E^{S,x})$ with $y > 0$. So $\rho(E^{S,x}) = S$ for all $S \subset N$, $|S| = 2$. Hence, by assumption

$$x_S = U(E^{S,x}) \quad \text{for all } S \subset N,\ |S| = 2.$$

Converse consistency of the uniform rule now yields

$$x = U(E). \quad \square$$

Since both $\mathcal{N}$ and $\mathcal{L}$ are consistent, strict individually rational and Pareto optimal bargaining solutions, which satisfy (1) in case E is an economy with two agents, it immediately follows from lemma 2.6 that (1) holds for all $E \in \mathcal{E}'$. This ends the proof of case 1.

Case 2: Not all agents are relevant.
To complete the proof of theorems 2.3 and 2.4 we consider an economy $E = \langle M, (R_i)_{i\in N}\rangle$ with $\rho(E) \neq N$.

Let $x := U(E)$ and $S := \rho(E)$. $S \neq \varnothing$ since $M > 0$. Pareto optimality of U implies that $x_{N\setminus S} = 0_{N\setminus S}$. Consistency of U implies that $x_S = U(E^{S,x})$. Clearly, $\rho(E^{S,x}) = S$. So by case 1, we have $x_S = \text{argmax}\{\prod_{i\in S} y_i \mid y \in X(E^{S,x})\}$, and moreover, we have that x_S is maximal with respect to $>^{lm}$ in $X(E^{S,x})$. Since $X(E) = X(E^{S,x}) \times \{0_{N\setminus S}\}$, it immediately follows that $U(E) = (x_S, 0_{N\setminus S}) = \text{argmax}\{\prod_{i\in S} y_i \mid y \in X(E)\}$, and that x is maximal with respect to $>^{lm}$ in $X(E)$. $\square$

Clearly, the fact that the Nash solution and the lexicographic egalitarian solution coincide is due to the narrowness of the class of bargaining problems that arise as associated with economies.

A similar kind of proof can be found in Aumann and Maschler (1985), who showed that one bankruptcy rule, the contested garment consistent rule, can be defined as the nucleolus of an appropriately chosen TU-game. Theorem 2.3 can be seen as a generalization of Dagan and Volij (1993) who showed that the constrained equal award rule for bankruptcy problems corresponds to the Nash bargaining solution of an appropriately chosen bargaining problem.

3. Two characterizations of the uniform rule

It is clear from the previous section that, at least formally, there is a relation between the uniform rule on the one hand, and the Nash and the lexicographic egalitarian bargaining solutions on the other hand. This suggests that the uniform rule might be characterized by means of a suitable adaptation of some properties that characterize these bargaining solutions. Before we go into axiomatic characterizations of the uniform rule, we present some properties, most of which are satisfied by the uniform rule.

Let ϕ be a rule.

Equal treatment: ϕ satisfies equal treatment if for all $E = \langle M, (R_i)_{i\in N}\rangle \in \mathcal{E}$ and all $i, j \in N$, if $R_i = R_j$, then $\phi_i(E) I_i \phi_j(E)$.

It is easy to see that together with Pareto optimality, equal treatment implies that any two agents with identical preferences get the same physical amount of the good.

Peak only: ϕ satisfies peak only if for all economies $E = \langle M, (R_i)_{i\in N}\rangle$, $E' = \langle M, (R'_i)_{i\in N}\rangle \in \mathcal{E}$, we have, if $p(E) = p(E')$, then $\phi(E) = \phi(E')$.

This property requires from a rule to take into consideration only the peaks of the preference profile when dividing a certain amount M. Peak only is a natural axiom for rules which satisfy Pareto optimality. To see this recall that a Pareto optimal rule selects allocations which are characterized by the fact that either all agents get more than their peaks or all agents get less than their peaks. Once restricted to the relevant side of the peak all preferences with this peak are identical. Hence, the peak contains all the 'relevant' information.

The following property, though different, is reminiscent of the one used by Nash (1950) in his characterization of the Nash bargaining solution.

Independence of irrelevant alternatives (*IIA*): ϕ satisfies IIA if for all economies $E=\langle M, (R_i)_{i\in N}\rangle$, $E'=\langle M, (R'_i)_{i\in N}\rangle\in\mathscr{E}$, with $X(E)\subset X(E')$, we have, if $\phi(E')\in X(E)$, then $\phi(E)=\phi(E')$.

The IIA axiom makes sense only if ϕ is Pareto optimal. The idea behind this axiom is the following. If some efficient allocations which were not selected by the rule become inefficient, then this should not result in a change of the recommended outcome if this outcome is still efficient. For our results we need only a weaker version of IIA which requires independence only in cases where in both economies either there is too much to divide or there is too little to divide.

One-sided independence of irrelevant alternatives: ϕ satisfies one-sided IIA if for all $E=\langle M, (R_i)_{i\in N}\rangle$, $E'=\langle M, (R'_i)_{i\in N}\rangle\in\mathscr{E}$, with $X(E)\subset X(E')$ such that $\max\{\sum_{i\in N}p(R_i), \sum_{i\in N}p(R'_i)\}<M$ or $\min\{\sum_{i\in N}p(R_i), \sum_{i\in N}p(R'_i)\}\geq M$ the following condition holds: If $\phi(E')\in X(E)$, then $\phi(E)=\phi(E')$.

Consider the following property:

Monotonicity: ϕ satisfies monotonicity if for all economies $E=\langle M,(R_i)_{i\in N}\rangle$ and $E'=\langle M',(R'_i)_{i\in N}\rangle$, such that for each $x\in X(E)$ there exists an $x'\in X(E')$ with $x'_iR'_ix_i$ for all $i\in N$ we have $\phi_i(E')R'_i\phi_i(E)$ for all $i\in N$.

This axiom states that if for every efficient allocation x in E we can find an efficient allocation x' in E' such that x' is weakly preferred to x by all agents in E', then the same must be true for the recommendations $\phi(E')$ and $\phi(E)$, namely $\phi(E')$ must be weakly preferred to $\phi(E)$ by all agents in E'. This axiom is similar in spirit to the monotonicity axiom of bargaining theory, and, like in bargaining theory, monotonicity is incompatible with Pareto optimality, as the following lemma shows.

Lemma 3.1 *There is no Pareto optimal rule ϕ that satisfies monotonicity.*

Proof. Assume by contradiction that ϕ satisfies both properties. Let E, E' and E'' be three two-agent economies in which there are 3 units to be divided. The peaks of the preference relations are respectively, $p=(1,2)$, $p'=(2,1)$ and $p''=(3,3)$. By Pareto optimality of ϕ we have that $\phi(E)=(1,2)$. It is clear that $X(E)\subset X(E'')$ so E and E'' trivially satisfy the condition in the monotonicity property. Hence by monotonicity, we must have $\phi(E'')=(1,2)$. A similar argument shows that $\phi(E'')=(2,1)$, which is a contradiction. □

Lemma 3.1 shows that if we want to keep Pareto optimality, we must, as in bargaining theory, weaken the monotonicity requirement. We are going to weaken the monotonicity axiom in two different ways. First, we are going to allow for non-monotonicity only if one of the agents that got his peak in the smaller problem, strictly prefers the recommendation for the bigger problem. Second, we are going to require this restricted form of monotonicity only when comparing some very specific economies.

One-sided restricted monotonicity: ϕ satisfies one-sided restricted monotonicity if for all economies $E=\langle M, (R_i)_{i\in N}\rangle, E'=\langle M, (R'_i)_{i\in N}\rangle \in \mathscr{E}$, satisfying $X(E)\subset X(E')$ and either $\max\{\sum_{i\in N} p(R_i), \sum_{i\in N} p(R'_i)\} < M$ or $\min\{\sum_{i\in N} p(R_i), \sum_{i\in N} p(R'_i)\} \geq M$ the following condition holds: If $\phi_i(E)\ R'_i\ \phi_i(E')$ for all $i\in N$ such that $\phi_i(E)=p(R_i)$, then $\phi_i(E')\ R'_i\ \phi_i(E)$ for all $i\in N$.

In order to understand this axiom, note that $\phi_i(E)=p(R_i)$ means that it is physically impossible to make agent i better off in economy E. In this case we say that i's peak is binding at $\phi(E)$. One-sided restricted monotonicity says that given two economies E and E' satisfying the conditions in the definition of this property, if ϕ does not behave monotonically, i.e., there is some agent in E' who strictly prefers $\phi(E)$ to $\phi(E')$, then there must be some other agent in E', whose peak was binding at $\phi(E)$, who strictly prefers $\phi(E')$ to $\phi(E)$. In other words, if the peaks of the agents' preferences change in the same direction, then no agent's award should follow the opposite direction unless there is an agent whose peak was binding in the original situation and whose award followed the direction of his peak in the transition to the new situation. The motivation for this axiom is the same as the one for the restricted monotonicity satisfied by the lexicographic egalitarian bargaining solution (Chun and Peters (1988)): In some situations an agent may benefit from the fact that it is physically impossible to make other agents better off. If this impossibility disappears due to the fact that the peaks change, it may be bad news for those who benefitted from the previous situation, i.e., their awards may go farther away from their peaks. It is only this kind of non-monotonic behavior that is allowed by the restricted monotonicity axiom.

The following lemma shows that there is a relation between the one-sided monotonicity axiom and the one-sided IIA.

Lemma 3.2 *If a rule ϕ satisfies Pareto optimality and one-sided restricted monotonicity, then it satisfies one-sided IIA.*

Proof. Let ϕ be a rule which satisfies Pareto optimality and one-sided restricted monotonicity and let $E=\langle M, (R_i)_{i\in N}\rangle$, $E'=\langle M, (R'_i)_{i\in N}\rangle \in \mathscr{E}$, be two economies satisfying $X(E)\subset X(E')$. We distinguish two cases.

Case 1: $\min\{\sum_{i\in N} p(R_i), \sum_{i\in N} p(R'_i)\} \geq M$.
Assume $\phi(E')\in X(E)$. Then by Pareto optimality of ϕ, we have

$$\max\{\phi_i(E), \phi_i(E')\} \leq \min\{p(R_i), p(R'_i)\} \quad \text{for all } i\in N. \tag{2}$$

Since $X(E)\subset X(E')$, it follows that

$$\min\{M, p(R_i)\} \leq p(R'_i) \quad \text{for all } i\in N. \tag{3}$$

Let $i\in N$ be such that $\phi_i(E)=p(R_i)$. Then it follows from (2) and (3) that $\phi_i(E')\leq p(R_i)=\phi_i(E)\leq p(R'_i)$. This implies that $\phi_i(E)\ R'_i\ \phi_i(E')$ for all $i\in N$ with $\phi_i(E)=p(R_i)$.

Since ϕ satisfies one-sided restricted monotonicity, if follows that $\phi_i(E')\ R_i'\ \phi_i(E)$ for all $i \in N$. Since $\phi(E) \in X(E')$, we must have $\phi_i(E')\ I_i'\ \phi_i(E)$ for all $i \in N$. Since both $\phi(E)$ and $\phi(E')$ are efficient in E' it follows that $\phi(E) = \phi(E')$.

Case 2: $\max\{\sum_{i \in N} p(R_i), \sum_{i \in N} p(R_i')\} \leq M$.
In this case $X(E) \subset X(E')$ implies that $p(E') \leq p(E)$. Let $i \in N$ be such that $\phi_i(E) = p(R_i)$. Then, since $\phi(E') \in X(E)$, it follows from Pareto optimality of ϕ that $\phi_i(E') \geq p(R_i) = \phi_i(E) \geq p(R_i')$. This implies that $\phi_i(E)\ R_i'\ \phi_i(E')$ for all $i \in N$ with $\phi_i(E) = p(R_i)$.

Since ϕ satisfies one-sided restricted monotonicity, it follows that $\phi_i(E')\ R_i'\ \phi_i(E)$ for all $i \in N$. Since both $\phi(E)$ and $\phi(E')$ are efficient in E', it follows that $\phi(E) = \phi(E')$. □

The following lemma will allow us to considerably simplify notation.

Lemma 3.3 *If a rule ϕ satisfies Pareto optimality and one-sided IIA, then it satisfies peak only.*

Proof. Let ϕ satisfy Pareto optimality and one-sided IIA and let $E = \langle M, (R_i)_{i \in N} \rangle$, $E' = \langle M, (R_i')_{i \in N} \rangle \in \mathscr{E}$ be two economies with $p(E) = p(E')$. Then $X(E) = X(E')$ and since ϕ is Pareto optimal, we have $\phi(E') \in X(E) = X(E')$. Hence, by one-sided IIA, $\phi(E) = \phi(E')$.

Lemmas 3.2 and 3.3 imply

Corollary 3.4 *Let ϕ be a Pareto optimal rule which satisfies one-sided restricted monotonicity. Then it also satisfies peak only.*

It will follow from theorem 3.5 and from example (iii) below that the converses of lemmas 3.2, 3.3, and corollary 3.4 are not true.

The following property imposes a restriction only when the solution satisfies peak only.

> *Conditional p-continuity*: A solution ϕ is conditional p-continuous if the following holds: If ϕ is peak only, then it is continuous with respect to the peaks.

Note that conditional p-continuity is weaker than the continuity with respect to preferences introduced by Sprumont (1991).

We are now ready to state the two main results of this section, which are characterizations of the uniform rule, based on axioms inspired by the results of the previous section.

Theorem 3.5 *The uniform rule is the unique rule which satisifes*

(i) Pareto optimality
(ii) Equal treatment
(iii) One-sided IIA
(iv) Conditional p-continuity.

Proof. It is clear that the uniform rule satisfies properties (i), (ii) and (iv). That the uniform rule satisfies (one-sided) IIA follows immediately from theorem 2.3 above. For each $M \in [0, \bar{M}]$, let $\mathscr{E}(M)$ be the class of economies in which M is the amount to be divided. Furthermore, let $\Delta(M) := \{x \in \mathbb{R}_+^N \mid x(N) = M\}$. For $p \in \mathbb{R}_+^N$ let $S(p) := \{x \in \Delta(M) \mid x \le p \text{ or } x \ge p\}$.

Now let ϕ be a rule satisfying the foregoing axioms. By lemma 3.3 ϕ is peak only. Let $M \in [0, \bar{M}]$, and let N be a coalition. Define the following function $f: \mathbb{R}_+^N \to \Delta(M)$ by

$$f(p) = \phi(E) \text{ for some } E \in \mathscr{E}(M) \text{ with } p(E) = p.$$

Since ϕ is peak only, f is well-defined.

Since ϕ satisfies (i)–(iv), the reader can easily verify that f satisfies the following properties.

(A.1) $f(p) \in S(p)$ for all $p \in \mathbb{R}_+^N$.

(A.2) $f_i(p) = f_j(p)$ for all $p \in \mathbb{R}_+^N$ with $p_i = p_j$.

(A.3) For all $p, q \in \mathbb{R}_+^N$ such that either $\max\{p(N), q(N)\} < M$ or $\min\{p(N), q(N)\} \ge M$, we have, if $f(q) \in S(p) \subset S(q)$, then $f(p) = f(q)$.

(A.4) f is continuous in p.

To conclude the proof of theorem 3.5 it suffices to show that for all $p \in \mathbb{R}_+^N$

$$f_i(p) = \begin{cases} \min\{p_i, \lambda\} & \text{if } p(N) \ge M \\ \max\{p_i, \lambda\} & \text{if } p(N) \le M, \end{cases} \tag{4}$$

where λ is such that $f(p) \in \Delta(M)$.

Let $p \in \mathbb{R}_+^N$. Assume $p(N) < M$ (the case $p(N) > M$ is similar, and the case $p(N) = M$ is trivial). (A.1) implies $f(p) \ge p$.

Define the following set of agents:

$$K := \{i \in N \mid f_i(p) > p_i\}.$$

Since $p(N) < M$, $K \neq \emptyset$.

The proof of (4) follows from the following four steps.

Step A: Let $i \in K$ and let $0 \le q_i \le p_i$. Define $q \in \mathbb{R}_+^N$ by

$$q_j = \begin{cases} p_j & \text{if } j \in N \setminus \{i\} \\ q_i & \text{if } j = i. \end{cases}$$

Then $f(q) = f(p)$.

Proof. Let

$$\alpha := \inf\{z_i \in [q_i, p_i] \mid f(z_i, p_{-i}) = f(p)\}.$$

Here, p_{-i} denotes the vector $p_{N \setminus \{i\}}$.

By (A.4), if follows that $f(\alpha, p_{-i}) = f(p)$. We prove that $\alpha = q_i$. Suppose, on the contrary, that $\alpha > q_i$. Since $f_i(\alpha, p_{-i}) = f_i(p) > p_i$, it follows from (A.4) that there exists a $q_i < a < \alpha$ close enough to α, such that $f_i(a, p_{-i}) > p_i$. By (A.1), we have $f_j(a, p_{-i}) \ge p_j$ for all $j \in N \setminus \{i\}$. Hence, $f(a, p_{-i}) \in S(p)$. Clearly, $S(p) \subset S(a, p_{-i})$. Therefore, by (A.3) we have $f(a, p_{-i}) = f(p)$, contradicting the definition of α. We conclude that $\alpha = q_i$, and so it follows that $f(q) = f(q, p_{-i}) = f(p)$. □

Step B: for all $i, j \in K$ we have $f_i(p) = f_j(p)$.

Proof. Let $i, j \in K$, and let $0 \leq v = \min\{p_i, p_j\}$. Define $q \in \mathbb{R}_+^N$ by

$$q_k = \begin{cases} p_k & \text{if } k \in N \backslash \{i, j\} \\ v & \text{if } k = i, j. \end{cases}$$

(A.2) yields that $f_i(q) = f_j(q)$. From Step A it now follows that $f_i(p) = f_j(p)$. □

Step C: For all $i, j \in N$ we have, if $p_i \leq p_j$, then $f_i(p) \leq f_j(p)$.

Proof. Suppose that there exist $i, j \in N$, with $p_i \leq p_j$ and $f_i(p) > f_j(p)$. Define $q \in \mathbb{R}_+^N$ by

$$q_k = \begin{cases} p_k & \text{if } k \in N \backslash \{i\} \\ p_j & \text{if } k = i. \end{cases}$$

From (A.1) and the definition of q it follows that $q_k = p_k \leq f_k(p)$ for $k \in N \backslash \{i\}$. Moreover, from the assumption it follows that $q_i = p_j \leq f_j(p) < f_i(p)$. Hence, $f(p) \geq q$, and $\sum_{k \in N} q_k < \sum_{k \in N} f_k(p) = M$. Therefore, $f(p) \in S(q) \subset S(p)$. (A.3) now yields that $f(p) = f(q)$. Hence using (A.2), we obtain $f_i(p) = f_i(q) = f_j(q) = f_j(p)$, which contradicts the assumption $f_i(p) > f_j(p)$. □

According to Step B all agents in K obtain the same amount. Denote this amount by λ, i.e., $f_i(p) = \lambda$ for all $i \in K$.
We now have

Step D: $p_i \geq \lambda$ for all $i \in N \backslash K$.

Proof. Suppose that there exists an $i \in N \backslash K$, with $p_i < \lambda$. Take $j \in K$. By definition of K and λ we have $f_i(p) = p_i < \lambda = f_j(p)$. Hence by Step C, we have $p_i < p_j$. Define $q \in \mathbb{R}_+^N$ by

$$q_k = \begin{cases} p_k & \text{if } k \in N \backslash \{j\} \\ p_i & \text{if } k = j. \end{cases}$$

By Step A it follows that $f(q) = f(p)$. (A.2) yields $f_j(p) = f_j(q) = f_i(q) = f_i(p) = p_i < p_j$, which contradicts (A.1). □

Now we show that (4) holds.
From Step B and the definition of K and λ it follows that

$$f_i(p) = \max\{p_i, \lambda\} \quad \text{for all } i \in K.$$

From Step D and the definition of K we obtain

$$f_i(p) = \max\{p_i, \lambda\} \quad \text{for all } i \in N \backslash K.$$

Since $f(p) \in \Delta(M)$, (4) holds. This completes the proof of theorem 3.5. □

The following examples show that the properties (i)–(iv) in theorem 3.5 are independent.

(i) The egalitarian rule ϕ^1 defined by $\phi^1(E) = \left(\frac{M}{|N|}, \ldots, \frac{M}{|N|}\right)$ for all economies $E = \langle M, (R_i)_{i \in N} \rangle \in \mathscr{E}$ satisfies equal treatment, (one-sided) IIA, and conditional p-continuity, but not Pareto optimality.

(ii) Let ϕ^2 be defined as follows: For each $E = \langle M, (R_i)_{i\in N}\rangle \in \mathscr{E}$

$$\phi^2(E) := \begin{cases} U(E) & \text{if } |N| \neq 2 \\ \operatorname{argmax}\{x_i^{1/4}\, x_j^{3/4} \,|\, x \in X(E)\} & \text{if } N = \{i,j\},\ i<j. \end{cases}$$

ϕ^2 satisfies Pareto optimality, (one sided) IIA and conditional p-continuity, but not equal treatment.

(iii) Let ϕ^3 be defined as follows: For each $E = \langle M, (R_i)_{i\in N}\rangle \in \mathscr{E}$ and $i \in N$

$$\phi_i^3(E) := \begin{cases} U_i(E) & \text{if } \sum_{j\in N} p(R_j) \geq M \\ p(R_i) + \frac{1}{|N|}(M - \sum_{j\in N} p(R_j)) & \text{otherwise.} \end{cases}$$

ϕ^3 satisfies Pareto optimality, equal treatment, and conditional p-continuity, but not (one-sided) IIA.

(iv) Let ϕ^4 be defined as follows: For each $E = \langle M, (R_i)_{i\in N}\rangle \in \mathscr{E}$

$$\phi^4(E) := \begin{cases} U(E) & \text{if } |N| \neq 2 \\ \operatorname{argmin}\{x_i x_j \,|\, x \in X(E)\} & \text{if } N = \{i,j\} \text{ and } (\frac{M}{2}, \frac{M}{2}) \notin X(E) \\ (\frac{M}{2}, \frac{M}{2}) & \text{otherwise.} \end{cases}$$

ϕ^4 satisfies Pareto optimality, equal treatment and (one-sided) IIA, but not conditional p-continuity.

The following theorem shows that if one-sided IIA is replaced by one-sided restricted monotonicity in theorem 3.5, then we can drop conditional p-continuity.

Theorem 3.6 *The uniform rule is the unique rule on $\mathscr{E}$* which satisfies

(i) Pareto optimality
(ii) Equal treatment
(iii) One-sided restricted monotonicity.

Proof. It is clear that the uniform rule satisfies properties (i) and (ii). The following lemma shows that it satisfies (iii).

Lemma 3.7 *The uniform rule satisfies one-sided restricted monotonicity.*

Proof. Let $E = \langle M, (R_i)_{i\in N}\rangle$, $E' = \langle M, (R_i')_{i\in N}\rangle \in \mathscr{E}$, be two economies satisfying $\max\{\sum_{i\in N} p(R_i), \sum_{i\in N} p(R_i')\} < M$ (the other case is similar) and assume $X(E) \subset X(E')$. Then $p(E) \geq p(E')$. For all $i \in N$, let $U_i(E) = \max\{p(R_i), \lambda\}$ and $U_i(E') = \max\{p(R_i'), \lambda'\}$. Define $K := \{i \in N \,|\, U_i(E) > p(R_i)\}$ and assume $U_i(E)\ R_i'\ U_i(E')$ for all $i \in N\backslash K$, i.e.,

$$U_i(E) \leq U_i(E'), \quad \text{for all } i \in N\backslash K. \tag{5}$$

We need to show that $U_i(E')\ R_i'\ U_i(E)$ for all $i \in N$. Since

$$M = \sum_{i\in N} \max\{p(R_i'), \lambda'\} = \sum_{i\in N} \max\{p(R_i), \lambda\} \geq \sum_{i\in N} \max\{p(R_i'), \lambda\},$$

it follows that $\lambda' \geq \lambda$.

Take $i \in K$. It follows from the definition of K that $p(R_i') \leq p(R_i) \leq \lambda \leq \lambda'$. Hence,

$$U_i(E) \leq U_i(E'). \tag{6}$$

This together with assumption (5) implies that (6) holds for all $i\in N$. But since $\sum_{i\in N} U_i(E)=\sum_{i\in N} U_i(E')$, we have $U_i(E)=U_i(E')$ for all $i\in N$, which in turn implies that $U_i(E')\, R_i'\, U_i(E)$ for all $i\in N$. □

Now let ϕ be a rule satisfying the axioms (i)–(iii). By corollary 3.4 ϕ is peak only. Let $M\in[0,\bar{M}]$, and let N be a coalition. Analogously to the proof of theorem 3.5, define the function $f:\mathbb{R}_+^N\to\Delta(M)$ by

$$f(p)=\phi(E) \text{ for some } E\in\mathscr{E}(M) \text{ with } p(E)=p.$$

Since ϕ is peak only, f is well-defined.

The reader can easily verify that (i)–(iii) together with lemma 3.2 imply that f satisfies, besides (A.1), (A.2), and (A.3) (see the proof of theorem 3.5), the following property.

(A.5) For all $p,q\in\mathbb{R}_+^N$, with $S(p)\subset S(q)$, and such that either $\max\{p(N),q(N)\}<M$ or $\min\{p(N),q(N)\}\geq M$, we have, if $|f_i(p)-q_i|\leq|f_i(q)-q_i|$ for all $i\in N$ such that $f_i(p)=p_i$, then $|f_i(q)-q_i|\leq|f_i(p)-q_i|$ for all $i\in N$.

To conclude the proof of theorem 3.6 it suffices to show that for all $p\in\mathbb{R}_+^N$

$$f_i(p)=\begin{cases}\min\{p_i,\lambda\} & \text{if } p(N)\geq M\\ \max\{p_i,\lambda\} & \text{if } p(N)\leq M,\end{cases}$$

where λ is such that $f(p)\in\Delta(M)$.
Let $p\in\mathbb{R}_+^N$. Assume $p(N)<M$ (the case $p(N)>M$ is similar, and the case $p(N)=M$ is trivial). (A.1) implies $f(p)\geq p$.
Define the following set of agents:

$$K:=\{i\in N\,|\,f_i(p)>p_i\}.$$

Since $\sum_{i\in N} p_i<M$, $K\neq\emptyset$.
Analogously to the proof of theorem 3.5 we now have

Step A′: Let $i\in K$ and let $0\leq q_i\leq p_i$. Define $q\in\mathbb{R}_+^N$ by

$$q_k=\begin{cases}p_k & \text{if } k\in N\backslash\{i\}\\ q_i & \text{if } k=i.\end{cases}$$

Then $f(q)=f(p)$.

Proof. From the definition of q it follows that $S(p)\subset S(q)$. Suppose $f(p)\neq f(q)$. Since $p,q\in\Delta(M)$, it is not true that $f(q)\leq f(p)$. Hence by (A.5), it follows that there exists a $j\in N$ such that $f_j(p)=p_j$ and $f_j(p)>f_j(q)$. Since $j\notin K$, it follows that $j\neq i$. Hence, $q_j=p_j=f_j(p)>f_j(q)\geq q_j$ which is a contradiction. □

The proof of theorem 3.6 now follows from the remark that in the proofs of Steps B, C, D above only (A.1), (A.2), and (A.3) are used. So the proof of theorem 3.6 can proceed in the same way as that of theorem 3.5. □

The examples (i), (ii), and (iii) above show that the axioms in theorem 3.6 are independent.

References

Aumann, R. J., Maschler, M.: Game theoretic analysis of a bankruptcy problem from the Talmud. J. Econ. Theory **36**, 195–213 (1985)

Ching, S.: A simple characterization of the uniform rule. Econ. Lett. **40**, 57–60 (1992)

Ching, S : An alternative characterization of the uniform rule Soc. Choice Welfare **11**, 131–136 (1994)

Chun, Y., Peters, H.: The lexicographic egalitarian solution. Cah. Centre Etud. Rech Opérat. **30**, 149–156 (1988)

Dagan, N., Volij, O.: The bankruptcy problem: a cooperative bargaining approach Math Soc. Sci **26**, 287–297 (1993)

Lensberg, T.: Stability and the Nash solution. J. Econ. Theory **45**, 330–341 (1988)

Nash, J.: The bargaining problem. Econometrica **18**, 155–162 (1950)

Peleg, B.. An axiomatization of the core of cooperative games without side payments. J. Math. Econ. **14**, 203–214 (1985)

Peleg, B.: On the reduced game property and its converse. Int. J. Game Theory **15**, 187–200 (1986) A correction: Int. J. Game Theory **16**, 290 (1987)

Peleg, B., Tijs, S. H.: The consistency principle for games in strategic form. Discussion paper 19, Center for Rationality and Interactive Decision Theory, The Hebrew University of Jerusalem. Jerusalem, Israel (1992)

Sobolev, A. I.: The characterization of optimality principles in cooperative games by functional equations. In: Vorobev, N. N. (ed.) Mathematical methods in the social sciences, vol. 6, pp 94–151 (Vilnius. in Russian, 1975)

Sprumont, Y.: The division problem with single-peaked preferences: A characterization of the uniform rule. Econometrica **59**, 509–519 (1991)

Thomson, W.: The consistency principle, In Ichiishi, T., Neyman, A , Tauman, Y. (eds.) Game theory and applications, pp. 187–215. New York. Academic Press 1991

Thomson, W.: Population-monotonic solutions to the problem of fair division when preferences are single-peaked. Econ. Theory **5**, 229–246 (1995)

Thomson, W.. Consistent solutions to the problem of fair division when preferences are single-peaked. J. Econ. Theory **63**, 219–245 (1994a)

Thomson, W.: Resource-monotonic solutions to the problem of fair division when preferences are single-peaked. Soc Choice Welfare **11**. 205–223 (1994b)

Thomson, W.: Manipulation and implementation of solutions to the problem of fair division when preferences are single-peaked. Discussion paper, University of Rochester. 1992a

Thomson, W.: The replacement principle in private good economies when preferences are single-peaked. Discussion paper, University of Rochester, 1992b

Thomson, W., Lensberg, T : Axiomatic theory of bargaining with a variable number of agents. Cambridge: Cambridge University Press 1989