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Coalitional strategy-proof and resource-monotonic solutions for multiple assignment problems

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Abstract. We consider the problem of allocating indivisible objects when agents may desire to consume more than one object and monetary transfers are not possible. Each agent receives a set of objects and free disposal is allowed. We are interested in allocation rules that satisfy "appealing" properties from an economic and social point of view. Our main result shows that sequential dictatorships are the only *efficient* and *coalitional strategy-proof* solutions to the multiple assignment problem. Adding *resource-monotonicity* narrows this class down to serial dictatorships.

1 Introduction

We investigate the problem of assigning indivisible objects to a set of agents when monetary transfers are not possible. Most of the literature considers situations where each agent receives exactly one object. Such problems arise when we have to assign jobs to workers, or apartments to students. For the assignment problem (with or without property rights) where each agent may receive at most one object and monetary transfers are not allowed, rules satisfying desirable properties were recently studied; see for instance Ehlers (2002), Ehlers et al. (2002), Ehlers and Klaus (2003), Ergin (2000), Ma (1994), Miyagawa (2002), Pápai (2000a), and Svensson (1994).

We depart from the above papers and consider so-called multiple assignment problems. Each agent receives a set of objects and monetary

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transfers are not possible. As an example, one may think of a heritage consisting of indivisible objects (e.g., furniture and household items) that has to be distributed among the heirs (e.g., the children of the deceased), respecting the wish that the objects should not be sold but allocated. Since agents may receive sets of objects, there are several interesting preference domains one could consider. Pápai (2000b,2001) studies the multiple assignment problem on the domain of strict preferences and on monotonic preference domains (on the monotonic preference domain any set of objects is strictly preferred to all of its proper subsets and on the quantity-monotonic preference domain receiving more objects is always preferred to receiving fewer objects).

We consider the same domains of preference relations as in Klaus and Miyagawa (2001): the domain of strict preferences, the domain of additive and strict preferences, and the domain of responsive, separable, and strict preferences. Our main focus however is on responsive, separable, and strict preferences. For example, separability implies that if object x is a "good" (receiving x is preferred to receiving nothing), then for each set of objects not containing x, receiving this set and x is preferred to receiving only this set. So, in line with our assumption of free disposal, separability excludes the possibility that x becomes a "bad object" if an agent consumes it with another set of objects – a natural assumption in the heritage example.

We search for solutions (or assignment rules) that satisfy "desirable" properties from an economic and social point of view. Most of the literature concerned with this "axiomatic" approach to assignment problems is for models which also include the possibility of monetary transfers; for the class of multiple assignment problems see Beviá (1998) and Tadenuma (1996). Surprisingly, the results for the multiple assignment problems we study here and assignment problems where each agent may receive at most one object are very different. Whereas in the latter case natural trading mechanisms (e.g., the core) satisfy many desirable properties, this is not the case for multiple assignment problems. The results we obtain might be interpreted as negative results since, depending on the preference domain, either incompatibility of the properties result or the set of allocation rules is narrowed down to sequential or serial dictatorship rules only. However, a practical advantage of serial (sequential) dictatorships is that they are simple and can be implemented easily. Furthermore, they are *efficient*, *strategy-proof*, and satisfy other appealing properties discussed below. They can be considered to be "fair" if the ordering of the agents is fairly determined; for instance by queuing, seniority, or randomization (Abdulkadiroğlu and Sönmez 1998,1999; Bogomolnaia and Moulin 2001).

We briefly discuss the organization of the paper and our results. In Sect. 2 we introduce the model and two basic properties. First of all, we impose *efficiency*, meaning that a rule only chooses efficient assignments. Second, in order to eliminate profitable misrepresentation of only privately known preferences, we impose *strategy-proofness* (no agent ever gains by misrepresenting his preferences).

In Sect. 3 we consider the stronger non-manipulation property *coalitional strategy-proofness* (no group of agents ever gains by jointly misrepresenting their preferences). Our main result is that *efficiency* and *coalitional strategy-proofness* only allow for sequential dictatorships; i.e., there exists a first dictator who always chooses his best set of objects. Depending on the first dictator's choice, a second dictator is determined who again chooses his best subset of the remaining objects. Depending on the choices of the previous dictators, a third dictator is determined etc., Pápai (2001) shows the same result on the domain of strict preferences. However, Pápai's proof uses preference relations that are not separable and therefore having a larger preference domain makes *coalitional strategy-proofness* a stronger property. Consequently, Pápai's and our proof are completely different. Furthermore, it is surprising that her result even holds on much smaller domains, namely the domain of additive and strict preferences and the domain of responsive, separable, and strict preferences. For various domains, we are able to give a characterization of *efficient* and *coalitional strategy-proof* rules for the multiple assignment problem. When each agent receives exactly one object the characterization of the rules satisfying *efficiency* and *coalitional strategy-proofness* is still missing.

In Sect. 4 we consider the multiple assignment problem for variable sets of indivisible objects. *Resource-monotonicity* describes the effect of a change in the available resource on the welfare of the agents. A rule satisfies *resource-monotonicity*, if after such a change either all agents (weakly) lose together or all (weakly) gain together. On the domain of responsive, separable, and strict (or additive and strict) preferences, the solidarity property *resource-monotonicity* is compatible with both *efficiency* and *coalitional strategy-proofness*. All three properties together characterize the class of serial dictatorships. On the domain of strict preferences, *efficiency*, *coalitional strategy-proofness*, and *resource-monotonicity* are not compatible.

2 The model and basic properties

We consider the same multiple assignment model as Pápai (2001) and Klaus and Miyagawa (2001). Let $K \equiv \{x_1, \ldots, x_k\}$ denote the finite set of objects and $N \equiv \{1, \ldots, n\}$ the finite set of agents. We always assume that $|K| = k \geq 2$ and $|N| = n \geq 2$. Let 2^K denote the set of all subsets of K including the empty set. For subsets of K consisting of exactly one object, with some abuse of notation, we omit the brackets and write x instead of $\{x\}$. Each agent $i \in N$ has a complete and transitive preference relation R_i over 2^K. The associated strict preference relation is denoted by P_i. We assume that R_i is strict; that is, for all distinct subsets $S, S' \subseteq K$, we have either SP_iS' or $S'P_iS$. Thus, SR_iS' means that either SP_iS' or $S = S'$. We further assume that R_i is *responsive* and *separable*.

A preference relation is *responsive* if, for any two sets that differ only in one object, the set containing the more preferred object is preferred to the other: for all $S \subseteq K$ and all $x, y \in K \backslash S$,

$$xP_iy \iff (S \cup x)P_i(S \cup y).$$

A preference relation is *separable* if $x \in K$ is preferred to nothing if and only if for all sets $S \in 2^K$ not containing x, $S \cup x$ is preferred to S: for all $S \subseteq K$ and all $x \in K \backslash S$,

$$xP_i\emptyset \iff (S \cup x)P_iS.$$

Together with strictness and completeness of preferences this implies that for all $S \subseteq K$ and all $x \in K \backslash S$,

$$\emptyset P_ix \iff SP_i(S \cup x).$$

Roth (1985) introduced responsiveness of preference relations for college admission problems (Gale and Shapley, 1962). For the notion of separability we use here, we refer to Barberà et al. (1991).

Let $\mathscr{R}$ be the set of responsive, separable, and strict preference relations over 2^K.[1] Let $\mathscr{S}$ denote the class of strict preference relations over 2^K and $\mathscr{A}$ the class of additive and strict preference relations over 2^K.[2] Note that $\mathscr{A} \subsetneq \mathscr{R} \subsetneq \mathscr{S}$. If not otherwise stated, we assume that preferences are responsive, separable, and strict; that is, $\mathscr{R}$ is our default preference domain. In what follows, all definitions using the preference domain $\mathscr{R}$ also apply to the preference domains $\mathscr{A}$ and $\mathscr{S}$.

A preference profile is a list $R = (R_1, R_2, \ldots, R_n)$ such that for all $i \in N$, $R_i \in \mathscr{R}$. Let $\mathscr{R}^{\mathscr{N}}$ denote the set of preference profiles. Since, for the time being, the set of agents and the set of objects are fixed, $\mathscr{R}^{\mathscr{N}}$ completely describes the set of *multiple assignment problems*.

Given $m \leq k$ and an ordered collection of objects $\{y_1, y_2, \ldots, y_m\} \subseteq K$, let $L(y_1, y_2, \ldots, y_m)$ be the class of preference relations $R_i \in \mathscr{R}$ such that $y_1P_iy_2P_i \cdots P_iy_mP_i\emptyset$ and for all $y \in K \setminus \{y_1, \ldots, y_m\}$, $\emptyset P_iy$. Note that then responsiveness and separability imply $\{y_1, y_3\}P_i\{y_2, y_3\}P_i\emptyset P_iy_{m+1}$ and so on. Furthermore, let $L(\emptyset)$ be the set of preference relations $R_i \in \mathscr{R}$ such that for all $y \in K$, $\emptyset P_iy$. The set $L(\emptyset)$ contains the preference relations where each object is conceived to be a "bad".

Given $R_i \in \mathscr{R}$, let $B(R_i) \equiv \{x \in K \mid xP_i\emptyset\}$ be the set of objects that are each preferred to $\emptyset$. Objects belonging to $B(R_i)$ are conceived to be "goods". Separability of R_i implies that $B(R_i)$ is the most preferred set at R_i for agent i; i.e., for all $S \in 2^K$, $B(R_i)R_iS$.

[1] All results that we establish for the domain of responsive, separable, and strict preferences also remain true on the domain of separable and strict preferences.

[2] A preference relation R_1 is *additive* if there exists a function $u: K \cup \emptyset \to \mathbb{R}$ such that for all $S, S' \in 2^K$,

$$SR_iS' \iff \sum_{k \in S} u(k) \geq \sum_{k \in S'} u(k)$$

with the convention $\sum_{k \in \emptyset} u(k) = 0$.

An *assignment* is a list $(S_1, \ldots, S_n)$ such that for all $i \in N$, $S_i \subseteq K$ and for all $i, j \in N$ such that $i \neq j$, $S_i \cap S_j = \emptyset$. The set S_i is the (possibly empty) set of objects assigned to agent i. The second condition simply says that no two agents receive the same object. Note that we allow free disposal and therefore the union of all S_i's may be a strict subset of K.

An *assignment rule*, or *rule* for short, is a function φ that associates with each preference profile $R \in \mathscr{R}^{\mathscr{N}}$ an assignment $\varphi(R) = (\varphi_i(R))_{i \in N}$. We are interested in rules satisfying the following properties.

A rule φ is *efficient* if it chooses for each profile an efficient assignment.

Efficiency. For all $R \in \mathscr{R}^{\mathscr{N}}$, there is no assignment $(S_i)_{i \in N}$ such that for all $i \in N$, $S_i R_i \varphi_i(R)$, with strict preference holding for some $j \in N$.

It is straightforward to check that separability of preference relations and free disposal imply the following (Klaus and Miyagawa 2001).

Lemma 1. *If a rule φ is efficient, then for all $R \in \mathscr{R}^{\mathscr{N}}$,*

(i) *for all $i \in N$, $\varphi_i(R) \subseteq B(R_i)$ and*
(ii) $\cup_{i \in N} \varphi_i(R) = \cup_{i \in N} B(R_i)$.

It is easy to see that the converse of Lemma 1 is wrong.[3]

The following notation will be useful later on. Given $R \in \mathscr{R}^{\mathscr{N}}$ and $M \subseteq N$, let R_M denote the preference profile $(R_i)_{i \in M}$. It is the restriction of R to the set M. Given $i, j \in N$, we also use the notation $R_{-i} \equiv R_{N \setminus \{i\}}$ and $R_{-i,j} \equiv R_{N \setminus \{i,j\}}$.

A rule φ is *strategy-proof* if no agent ever benefits from misrepresenting his preferences. In game theoretical terms, a rule is *strategy-proof* if in its associated direct revelation game, it is a weakly dominant strategy for each agent to announce his true preference relation.

Strategy-Proofness. For all $R \in \mathscr{R}^{\mathscr{N}}$, all $i \in N$, and all $R'_i \in \mathscr{R}$, $\varphi_i(R) R_i \varphi_i(R'_i, R_{-i})$.

3 Coalitional strategy-proofness

In this section we investigate a stronger nonmanipulation condition than *strategy-proofness*. A rule φ satisfies *coalitional strategy-proofness* if no coalition of agents ever benefits from misrepresenting their preferences.

Coalitional strategy-proofness. For all $R \in \mathscr{R}^{\mathscr{N}}$ and all $M \subseteq N$, there exists no $R' \in \mathscr{R}^{\mathscr{N}}$ such that $R'_{N \setminus M} = R_{N \setminus M}$ and for all $i \in M$, $\varphi_i(R') R_i \varphi_i(R)$ with strict preference holding for some $j \in M$.

[3] For example, let $n = 2$, $k = 2$, $R_1 \in L(x_1, x_2)$, and $R_2 \in L(x_2, x_1)$. The assignment (x_2, x_1) satisfies (i) and (ii) of Lemma 1 but is Pareto dominated by (x_1, x_2).

Coalitional strategy-proofness excludes "bossy" behavior; that is, none of the agents can influence the allocation of some other agent by unilaterally changing his preferences without changing his own allocation. This concept of *non-bossiness* was introduced by Satterthwaite and Sonnenschein (1981).[4]

Non-bossiness. For all $R \in \mathscr{R}^{\mathscr{N}}$, all $i \in N$, and all $R'_i \in \mathscr{R}$, if $\varphi_i(R) = \varphi_i(R'_i, R_{-i})$, then $\varphi(R) = \varphi(R'_i, R_{-i})$.

The following lemma is straightforward.

Lemma 2. *If a rule φ satisfies coalitional strategy-proofness, then it satisfies strategy-proofness and non-bossiness.*

Our main result is a characterization of the class of *efficient* and *coalitional strategy-proof* rules. A rule satisfying these properties is a "sequential dictatorship": a first dictator chooses his most preferred set of objects; depending on the first dictator's choice, a second dictator is determined who again chooses his most preferred subset of the remaining objects; depending on the choices of the previous dictators, a third dictator is determined and so on. In order to formalize the class of sequential dictatorships we need some additional notation.

A *permutation π on N* is a bijective function $\pi: N \to N$. Let Π^N denote the *set of all permutations on N*. Given $\pi \in \Pi^N$ and $i \in N$, we interpret agent $\pi(i)$ to be the ith "dictator".

Sequential Dictatorship. For all $R \in \mathscr{R}^{\mathscr{N}}$, there is a fixed $\pi_R \in \Pi^N$ such that

$$\begin{aligned}
\varphi_{\pi_R(1)}(R) &= B(R_{\pi_R(1)}),\\
\varphi_{\pi_R(2)}(R) &= B(R_{\pi_R(2)}) \backslash B(R_{\pi_R(1)}),\\
\varphi_{\pi_R(3)}(R) &= B(R_{\pi_R(3)}) \backslash \left[B\left(R_{\pi_R(1)}\right) \cup B\left(R_{\pi_R(2)}\right)\right],\\
&\vdots\\
\varphi_{\pi_R(n)}(R) &= B(R_{\pi_R(n)}) \backslash \left[\cup_{i=\pi_R(1)}^{\pi_R(n-1)} B(R_i)\right].
\end{aligned}$$

We call agent $\pi_R(1)$ the *first dictator at R*, agent $\pi_R(2)$ the *second dictator at R*, etc.

For all $R, \bar{R} \in \mathscr{R}^{\mathscr{N}}$ and $\pi_R, \pi_{\bar{R}} \in \Pi^N$ as specified above, the following two conditions must be satisfied:

(i) $\pi_R(1) = \pi_{\bar{R}}(1)$.

(ii) Let $m \in \{1, \ldots, n-1\}$. If for all $i \in \{1, \ldots, m\}$, $\pi_R(i) = \pi_{\bar{R}}(i)$ and $\varphi_{\pi_R(i)}(R) = \varphi_{\pi_{\bar{R}}(i)}(\bar{R})$, then $\pi_R(m+1) = \pi_{\bar{R}}(m+1)$. ○

[4] Since preferences are strict, we could equivalently define *non-bossiness* in terms of welfare.

Note that, by (i), for any sequential dictatorship there exists a unique first dictator. In (ii) we formalize that the choice of the next dictator who is allowed to choose his most preferred set of objects from the remaining objects only depends on the previous dictators and their individual choices and not on the exact preferences of the previous dictators or the remaining agents.

A subclass of sequential dictatorships are serial dictatorships. Here, the choice of the next dictator does not depend on the sets of objects chosen by the first dictators. Formally, a rule φ is a *serial dictatorship* if φ is a sequential dictatorship and there exists $\bar{\pi} \in \Pi^N$ such that for all $R \in \mathscr{R}^{\mathcal{N}}$, $\varphi_{\pi_R}(R) = \varphi_{\bar{\pi}}(R)$.

Theorem 1. *The following statements are equivalent.*
(a) φ *is a sequential dictatorship.*
(b) φ *satisfies efficiency and coalitional strategy-proofness.*
(c) φ *satisfies efficiency, strategy-proofness, and non-bossiness.*

Proof.

(a) implies (b). It is straightforward to check that sequential dictatorships satisfy *efficiency* and *coalitional strategy-proofness.*

(b) implies (c). By Lemma 3, *coalitional strategy-proofness* implies *strategy-proofness* and *non-bossiness*.

(c) implies (a). Let φ be a rule satisfying *efficiency*, *strategy-proofness*, and *non-bossiness*. We show in four steps that φ is a sequential dictatorship.

Step 1. We show that there exists $j \in N$ such that for all $R \in \mathscr{R}^{\mathcal{N}}$, if for all $i \in N$, $B(R_i) = K$, then

$$\varphi_j(R) = K. \tag{1}$$

Let $R \in \mathscr{R}^{\mathcal{N}}$ be such that for all $i \in N$, $B(R_i) = K$. Suppose that for all $i \in N$, $\varphi_i(R) \neq K$. By *efficiency* and Lemma 1, $\cup_{i \in N} \varphi_i(R) = K$. Without loss of generality, we suppose that $x_1 \in \varphi_1(R) \neq \emptyset$ and $x_2 \in \varphi_2(R) \neq \emptyset$.

Let $(R_1, R_2, R'_{-1,2}) \in \mathscr{R}^{\mathcal{N}}$ be such that for all $i \in N \backslash \{1,2\}, B(R'_i) = \varphi_i(R)$. If we change R stepwise to $(R_1, R_2, R'_{-1,2})$, then *strategy-proofness* and *non-bossiness* imply that $\varphi(R_1, R_2, R'_{-1,2}) = \varphi(R)$. In particular, $x_1 \in \varphi_1(R_1, R_2, R'_{-1,2})$ and $x_2 \in \varphi_2(R_1, R_2, R'_{-1,2})$. Let $R'_2 \in \mathscr{R}$ be such that $B(R'_2) = \varphi_2(R_1, R_2, R'_{-1,2}) \cup x_1$ and for all $y \in \varphi_2(R_1, R_2, R'_{-1,2})$, yP'_2x_1. By *strategy-proofness*, responsiveness, and separability of $R'_2, \varphi_2(R_1, R'_{-1}) = \varphi_2(R_1, R_2, R'_{-1,2})$. By *non-bossiness*, $\varphi(R_1, R'_{-1}) = \varphi(R_1, R_2, R'_{-1,2})$. Let $R'_1 \in \mathscr{R}$ be such that $B(R'_1) = \varphi_1(R_1, R'_{-1}) \cup x_2$ and for all $y \in \varphi_1(R_1, R'_{-1}), yP'_1x_2$. Similarly as above it follows that $\varphi(R') = \varphi(R_1, R'_{-1})$.

If $\varphi_1(R') = \{x_1\}$, then let $\bar{R}_1 = R'_1$. Suppose that $|\varphi_1(R')| \geq 2$. Let $\bar{R}_1 \in L(x_1, x_2)$ be such that $\varphi_1(R')\bar{P}_1\emptyset$. By *efficiency*, $\varphi_1(\bar{R}_1, R'_{-1}) \subseteq \{x_1, x_2\}$. By *strategy-proofness* and construction of $\bar{R}_1$, $|\varphi_1(\bar{R}_1, R'_{-1})| \geq 1$. Thus, by the previous facts, *efficiency*, $x_1\bar{P}_1x_2$, and $x_2P'_2x_1$, we have $\varphi_1(\bar{R}_1, R'_{-1}) = \{x_1\}$ or $\varphi_1(\bar{R}_1, R'_{-1}) = \{x_1, x_2\}$.

Suppose that $\varphi_1(\bar{R}_1, R'_{-1}) = \{x_1, x_2\}$. Let $R''_1 \in \mathscr{R}$ be such that $B(R''_1) = \varphi_1(R') \cup x_2$, $\{x_1, x_2\} P''_1 \varphi_1(R')$, and $x_1 P''_1 x_2$. By *efficiency*, $\varphi_1(R') \backslash x_1 \subseteq \varphi_1(R''_1, R'_{-1})$. Thus, $\varphi_1(R''_1, R'_{-1}) \in \{\varphi_1(R') \backslash x_1, \varphi_1(R'), (\varphi_1(R') \backslash x_1) \cup x_2, \varphi_1(R') \cup x_2\}$.

Suppose that $\varphi_1(R''_1, R'_{-1}) = \varphi_1(R') \backslash x_1$ or $\varphi_1(R''_1, R'_{-1}) = \varphi_1(R')$. Then, $\{x_1, x_2\} = \varphi_1(\bar{R}_1, R'_{-1}) P''_1 \varphi_1(R''_1, R'_{-1})$; a contradiction to *strategy-proofness*.

Suppose that $\varphi_1(R''_1, R'_{-1}) = (\varphi_1(R') \backslash x_1) \cup x_2$. Then, by responsiveness, $\varphi_1(R') P''_1 \varphi_1(R''_1, R'_{-1})$; a contradiction to *strategy-proofness*.

Suppose that $\varphi_1(R''_1, R'_{-1}) = \varphi_1(R') \cup x_2$. Then $\varphi_1(R''_1, R'_{-1}) P'_1 \varphi_1(R')$; a contradiction to *strategy-proofness*.

Thus, we have $\varphi_1(\bar{R}_1, R'_{-1}) = \{x_1\}$. By *efficiency*, for all $i \in N \backslash \{1\}$, $\varphi_i(\bar{R}_1, R'_{-1}) = \varphi_i(R')$.

If $\varphi_2(\bar{R}'_1, R'_{-1}) = \{x_2\}$, then let $\bar{R}_2 = R'_2$. Suppose that $|\varphi_2(\bar{R}_1, R'_{-1})| \geq 2$. Let $\bar{R}_2 \in L(x_2, x_1)$ be such that $\varphi_2(\bar{R}_1, R'_{-1}) \bar{P}_2 \emptyset$. Similarly as above it follows that $\varphi_2(\bar{R}_1, \bar{R}_2, R'_{-1,2}) = \{x_2\}$ and for all $i \in N \backslash \{2\}$, $\varphi_i(\bar{R}_1, \bar{R}_2, R'_{-1,2}) = \varphi_i(\bar{R}_1, R'_{-1})$.

Let $\bar{R} \equiv (\bar{R}_1, \bar{R}_2, R'_{-1,2})$. Hence, $\bar{R}_1 \in L(x_1, x_2)$, $\bar{R}_2 \in L(x_2, x_1)$,

$$\varphi_1(\bar{R}) = \{x_1\} \text{ and } \varphi_2(\bar{R}) = \{x_2\} \ . \tag{2}$$

Let $\tilde{R} \in \mathscr{R}^{\mathscr{N}}$ be such that $B(\tilde{R}_1) = B(\tilde{R}_2) = \{x_1\}$, $\{x_1, x_2\} \tilde{P}_1 \emptyset$, $\{x_1, x_2\} \tilde{P}_2 \emptyset$, and $\tilde{R}_{-1,2} = R'_{-1,2}$. By *efficiency*, $\varphi_1(\tilde{R}) = \{x_1\}$ or $\varphi_2(\tilde{R}) = \{x_1\}$. Without loss of generality, suppose that $\varphi_1(\tilde{R}) = \{x_1\}$ and $\varphi_2(\tilde{R}) = \emptyset$.

Let $\tilde{R}'_1 \in L(x_1, x_2)$; *i.e.*, $x_1 \tilde{P}'_1 x_2$. Since $\varphi_1(\tilde{R}) = \{x_1\}$, by *strategy-proofness*, $x_1 \in \varphi_1(\tilde{R}'_1, \tilde{R}_{-1})$. Thus, by *efficiency*, $\varphi_1(\tilde{R}'_1, \tilde{R}_{-1}) = \{x_1, x_2\}$ and $\varphi_2(\tilde{R}'_1, \tilde{R}_{-1}) = \emptyset$. Let $\tilde{R}''_1 \in L(x_2, x_1)$; *i.e.*, $x_2 \tilde{P}''_1 x_1$. By *efficiency* and *strategy-proofness*, $\varphi_1(\tilde{R}''_1, \tilde{R}_{-1}) = \{x_1, x_2\}$ and $\varphi_2(\tilde{R}''_1, \tilde{R}_{-1}) = \emptyset$.

Let $\tilde{R}'_2 \in L(x_1, x_2)$; *i.e.*, $x_1 \tilde{P}'_2 x_2$. Since $x_1 \tilde{P}_2 \{x_1, x_2\} \tilde{P}_2 \emptyset$, by *strategy-proofness*, $x_1 \notin \varphi_2(\tilde{R}''_1, \tilde{R}'_2, \tilde{R}_{-1,2})$. Thus, by *efficiency*, $x_1 \in \varphi_1(\tilde{R}''_1, \tilde{R}'_2, \tilde{R}_{-1,2})$. Since $x_2 \tilde{P}''_1 x_1$ and $x_1 \tilde{P}'_2 x_2$, *efficiency* implies that $\{x_2\} \neq \varphi_2(\tilde{R}''_1, \tilde{R}'_2, \tilde{R}_{-1,2})$. Thus, $\varphi_1(\tilde{R}''_1, \tilde{R}'_2, \tilde{R}_{-1,2}) = \{x_1, x_2\}$ and $\varphi_2(\tilde{R}''_1, \tilde{R}'_2, \tilde{R}_{-1,2}) = \emptyset$. Applying *strategy-proofness* twice and since $\tilde{R}_{-1,2} = \bar{R}_{-1,2}$, it follows that $\varphi_1(\bar{R}) = \{x_1, x_2\}$ and $\varphi_2(\bar{R}) = \emptyset$, a contradiction to (2). Hence, without loss of generality, if $R \in \mathscr{R}^{\mathscr{N}}$ is such that for all $i \in N$, $B(R_i) = K$, then $\varphi_1(R) = K$.

Step 2. We prove that for all $R \in \mathscr{R}^{\mathscr{N}}$, $\varphi_1(R) = B(R_1)$.

Let $R^1 \in \mathscr{R}^{\mathscr{N}}$ be such that for all $i \in N$, $B(R^1_i) = K$. By Step 1, $\varphi_1(R^1) = B(R^1_1) = K$. Let $R^2 = (R^1_1, R_{-1}) \in \mathscr{R}^{\mathscr{N}}$. If we change profile R^1 stepwise to profile R^2, then it follows by *strategy-proofness* and *non-bossiness* that $\varphi_1(R^2) = K$ and for all $i \in N \backslash \{1\}$, $\varphi_i(R^2) = \emptyset$.

Let $R^3 = (\bar{R}_1, R_{-1}) \in \mathscr{R}^{\mathscr{N}}$ be such that $B(\bar{R}_1) = B(R_1)$ and for all $S \subsetneq B(R_1)$, $K \bar{P}_1 S$. Thus, by *efficiency* and *strategy-proofness*, $\varphi_1(R^3) = B(\bar{R}_1)$. Finally, by *strategy-proofness*, $\varphi_1(R) = B(R_1)$. Hence, without loss of generality, agent 1 is the (unique) first dictator.

Step 3. We show that for all $R \in \mathscr{R}^{\mathscr{N}}$, there exists $\pi_R \in \Pi^N$ such that

$$\varphi_{\pi_R(1)}(R) = B(R_{\pi_R(1)}),$$
$$\varphi_{\pi_R(2)}(R) = B(R_{\pi_R(2)}) \backslash B\big(R_{\pi_R(1)}\big),$$

$$\varphi_{\pi_R(3)}(R) = B(R_{\pi_R(3)}) \backslash \left[B\left(R_{\pi_R(1)}\right) \cup B\left(R_{\pi_R(2)}\right)\right],$$
$$\vdots \qquad (3)$$
$$\varphi_{\pi_R(n)}(R) = B(R_{\pi_R(n)}) \backslash \left[\cup_{i=\pi_R(1)}^{\pi_R(n-1)} B(R_i)\right].$$

Let $R \in \mathscr{R}^{\mathscr{N}}$ and set $\pi_R(1) = 1$. Since $\varphi_1(R) = B(R_1)$, it follows that for all $i \in N\backslash\{1\}$, $\varphi_i(R) \subseteq B(R_i)\backslash B(R_1)$. Keeping R_1 fixed and varying the preferences of all remaining agents, similarly as in Steps 1 and 2, we can prove that there must exist a (unique) dictator $\pi_R(2)$ over the remaining set of objects $K\backslash B(R_1)$ if $B(R_1) \neq K$. If $B(R_1) = K$, then we could choose $\pi_R(2)$ arbitrarily from $N\backslash\{\pi_R(1)\}$. Thus, $\varphi_{\pi_R(2)}(R) = B(R_{\pi_R(2)})\backslash B(R_1)$. Hence, using Steps 1 and 2 sequentially, we can derive some $\pi_R \in \Pi^N$ such that (3) holds.

Step 4. Finally, we show that for all $R \in \mathscr{R}^{\mathscr{N}}$ we can choose $\pi_R \in \Pi^N$ such that (3) holds and for all $R, \bar{R} \in \mathscr{R}^{\mathscr{N}}$ and $\pi_R, \pi_{\bar{R}} \in \Pi^N$ the following additional conditions are satisfied:

(i) $\pi_R(1) = \pi_{\bar{R}}(1)$.
(ii) Let $m \in \{1, \dots, n-1\}$. If for all $i \in \{1, \dots, m\}$, $\pi_R(i) = \pi_{\bar{R}}(i)$ and $\varphi_{\pi_R(i)}(R) = \varphi_{\pi_{\bar{R}}(i)}(\bar{R})$, then $\pi_R(m+1) = \pi_{\bar{R}}(m+1)$.

Let $R \in \mathscr{R}^{\mathscr{N}}$. By Step 2, agent 1 is the first dictator. Let $\pi_R(1) \equiv 1$ and recursively define π_R as follows:

Let $m \in \{1, \dots, n-1\}$. If $\cup_{i=\pi_R(1)}^{\pi_R(m)} B(R_i) = K$, then $\pi_R(m+1) \equiv \min(N\backslash\{\pi_R(1), \dots, \pi_R(m)\})$. Otherwise, if $\cup_{i=\pi_R(1)}^{\pi_R(m)} B(R_i) \neq K$, then by Step 2 and 3 there exists a unique dictator in $N\backslash\{\pi_R(1), \dots, \pi_R(m)\}$ over the remaining objects $K\backslash \cup_{i=\pi_R(1)}^{\pi_R(m)} B(R_i)$. Let $j \in N\backslash\{\pi_R(1), \dots, \pi_R(m)\}$ be this dictator. Define $\pi_R(m+1) \equiv j$.

By definition and Step 2, (3) and (i) hold. In order to complete the proof, we show (ii) by induction on m.

Induction basis m = 1. By (i) and Step 3, $\pi_R(1) = \pi_{\bar{R}}(1) = 1$. We show that if $\varphi_1(R) = \varphi_1(\bar{R})$, then $\pi_R(2) = \pi_{\bar{R}}(2)$.

If $\varphi_1(R) = \varphi_1(\bar{R}) = K$, then $B(R_1) = B(\bar{R}_1) = K$. By definition $\pi_R(2) = \pi_{\bar{R}}(2) = 2$, the desired conclusion.

Let $\varphi_1(R) = \varphi_1(\bar{R}) \neq K$. Let $R', \bar{R}' \in \mathscr{R}^{\mathscr{N}}$ be such that $R'_1 = R_1$, $\bar{R}'_1 = \bar{R}_1$, and for all $i \in N\backslash\{1\}$, $B(R'_i) = B(\bar{R}'_i) = K$ and $R'_i = \bar{R}'_i$. By (i), $\pi_{R'}(1) = \pi_{\bar{R}'}(1) = 1$. Hence,

$$\varphi_1(R') = B(R'_1) = B(\bar{R}'_1) = \varphi_1(\bar{R}') \ . \qquad (4)$$

By definition and Step 3 it follows that $\pi_R(2) = \pi_{R'}(2)$ and $\pi_{\bar{R}}(2) = \pi_{\bar{R}'}(2)$. Hence,

$$\varphi_{\pi_R(2)}(R') = \varphi_{\pi_{\bar{R}}(2)}(\bar{R}') = K\backslash\varphi_1(R') \ . \qquad (5)$$

Recall that $\bar{R}' = (\bar{R}'_1, R'_{-1})$. By (4), *strategy-proofness* and *non-bossiness* imply $\varphi(\bar{R}') = \varphi(\bar{R}'_1, R'_{-1}) = \varphi(R')$. Thus, by (5), $\pi_{R'}(2) = \pi_{\bar{R}'}(2)$. Hence, $\pi_R(2) = \pi_{\bar{R}}(2)$, the desired conclusion of (ii).

Induction hypothesis. Let $m \in \{1, \ldots, n-1\}$. If for all $i \in \{1, \ldots, m-1\}$, $\pi_R(i) = \pi_{\bar{R}}(i)$ and $\varphi_{\pi_R(i)}(R) = \varphi_{\pi_{\bar{R}}(i)}(\bar{R})$, then $\pi_R(m) = \pi_{\bar{R}}(m)$.

Induction Step $m \to m+1$*:* Let $m \in \{1, \ldots, n-1\}$. Suppose that for all $i \in \{1, \ldots, m\}$, $\pi_R(i) = \pi_{\bar{R}}(i)$ and $\varphi_{\pi_R(i)}(R) = \varphi_{\pi_{\bar{R}}(i)}(\bar{R})$. We have to prove that $\pi_R(m+1) = \pi_{\bar{R}}(m+1)$.

Without loss of generality, suppose that $\pi_R(1) = 1, \ldots, \pi_R(m) = m$. Hence, $\pi_{\bar{R}}(1) = 1, \ldots, \pi_{\bar{R}}(m) = m$. Let $M \equiv \{1, \ldots, m\}$. If $\cup_{i \in M} \varphi_i(R) = K$, then $\cup_{i \in M} B(R_i) = \cup_{i \in M} B(\bar{R}_i) = K$. Since $N \backslash \{\pi_R(1), \ldots, \pi_R(m)\} = N \backslash \{\pi_{\bar{R}}(1), \ldots, \pi_{\bar{R}}(m)\} = N \backslash M$, by definition $\pi_R(m+1) = \pi_{\bar{R}}(m+1) = m+1$, the desired conclusion.

Let $\cup_{i \in M} \varphi_i(R) = \cup_{i \in M} \varphi_i(\bar{R}) \neq K$. Let $R', \bar{R}' \in \mathscr{R}^{\mathcal{N}}$ be such that for all $i \in M$, $R'_i = R_i$, $\bar{R}'_i = \bar{R}_i$, and for all $i \in N \backslash M$, $B(R'_i) = B(\bar{R}'_i) = K$ and $R'_i = \bar{R}'_i$. By our induction hypothesis, $R_M = R'_M$, and $\bar{R}_M = \bar{R}'_M$, we have for all $i \in M$, $\pi_R(i) = \pi_{R'}(i)$ and $\pi_{\bar{R}}(i) = \pi_{\bar{R}'}(i)$.

By definition and Step 3 it follows that $\pi_R(m+1) = \pi_{R'}(m+1)$ and $\pi_{\bar{R}}(m+1) = \pi_{\bar{R}'}(m+1)$. Hence, for all $i \in M$,

$$\varphi_i(R') = \varphi_i(\bar{R}') \tag{6}$$

and

$$\varphi_{\pi_R(m+1)}(R') = \varphi_{\pi_{\bar{R}}(m+1)}(\bar{R}') = K \backslash \cup_{i \in M} \varphi_i(R') \ . \tag{7}$$

By Step 3 and (i), $\varphi_1(R') = B(R'_1)$ and $\varphi_1(\bar{R}') = B(\bar{R}'_1)$. By (6), *strategy-proofness* and *non-bossiness* imply $\varphi(\bar{R}'_1, R'_{-1}) = \varphi(R')$. By Step 3 and the induction hypothesis, $\varphi_2(\bar{R}'_1, R'_{-1}) = B(R'_2) \backslash B(R'_1)$ and $\varphi_2(\bar{R}') = B(\bar{R}'_2) \backslash B(\bar{R}'_1)$. Similarly as above it follows that $\varphi(\bar{R}'_1, \bar{R}'_2, R'_{-1,2}) = \varphi(\bar{R}'_1, R'_{-1})$. Since $R'_{N \backslash M} = \bar{R}'_{N \backslash M}$ and (6) holds, by changing the preferences of agents $1, \ldots, m$ at profile R' stepwise to those at $\bar{R}'$ it follows that $\varphi(R') = \varphi(\bar{R}')$. Thus, by (7), $\pi_{R'}(m+1) = \pi_{\bar{R}'}(m+1)$. Hence, $\pi_R(m+1) = \pi_{\bar{R}}(m+1)$, the desired conclusion of (ii).

Step 4 completes the proof showing that φ is a sequential dictatorship. ■

Remark 1. *The proof of Theorem 1 remains valid if we restrict the preference domain* $\mathscr{R}$ *to the domain of additive and strict preference relations* $\mathscr{A}$*. By* strategy-proofness, *it is easy to show that Theorem 1 remains valid on the larger domain of strict preference relations* $\mathscr{S}$*.*[5] *Pápai (2001) gives a direct proof of the equivalence of (a) and (c) as stated in Theorem 1 on the domain of strict preference relations* $\mathscr{S}$*. As already mentioned in the Introduction, our proof is different from Pápai's (2001) proof; her proof uses the non-separable preference relation where receiving all objects is strictly preferred to receiving nothing, but receiving nothing is strictly preferred to receiving some but not all objects.* ◁

[5] On the general domain of strict preferences, the formal definition of sequential dictatorships has to be slightly adjusted.

Remark 2. *When each agent receives exactly one object and all objects are preferred to receiving nothing, Svensson (1999) shows that serial dictatorships are the only rules satisfying efficiency, strategy-proofness, non-bossiness,and* "neutrality" *(the rule does not depend on the names of the objects).*[6] *His result does not apply to multiple assignment problems. Using Theorem 1 these properties characterize the sequential dictatorships where the choice of the next dictator depends only on the cardinalities of the sets of objects chosen by the previous dictators.* ◁

To conclude this section, we discuss the logical independence of the properties in Theorem 1.

Any constant rule satisfies *coalitional strategy-proofness* (*strategy-proofness* and *non-bossiness*), but not *efficiency*; e.g., let φ be such that for all $R \in \mathscr{R}^{\mathscr{N}}$, $\varphi_i(R) = \emptyset$.[7]

Example 1. *Let* $N = \{1, 2, 3\}$. *For all* $R \in \mathscr{R}^{\mathscr{N}}$, *if* $B(R_1) \supseteq B(R_2)$, *then* $\varphi_1(R) \equiv B(R_1)$, $\varphi_2(R) \equiv \emptyset$, *and* $\varphi_3(R) \equiv B(R_3) \backslash B(R_1)$. *Otherwise,* $\varphi_2(R) \equiv B(R_2)$, $\varphi_1(R) \equiv B(R_1) \backslash B(R_2)$, *and* $\varphi_3(R) \equiv B(R_3) \backslash (B(R_1) \cup B(R_2))$. *The rule* φ *satisfies non-bossiness and efficiency, but not strategy-proofness.* ◁

Rules that are similarly defined as sequential dictatorships but the choice of the next dictators depends on the exact preference relation of the first dictator are *efficient* and *strategy-proof*, but violate *non-bossiness*.

4 Resource-monotonicity

In this section we admit variations of the set of objects to be assigned. First we extend our model and notation. We interpret K to be the set of potential objects. We use $\mathscr{K}$ to denote a set of nonempty subsets of K satisfying
(a) $K \in \mathscr{K}$ and
(b) there exists a partition $K^1, \ldots, K^l$, $l \geq 2$, of K such that $\{K^1, \ldots, K^l\} \subseteq \mathscr{K}$ and for all $K' \in \mathscr{K}$, there exists an index set $I \subseteq \{1, \ldots, l\}$ such that $K' = \cup_{h \in I} K^h$.[8]

Sets $K' \in \mathscr{K}$ are called *admissible*. Throughout the remaining part of the paper an *assignment problem* consists of an admissible set of objects $K' \in \mathscr{K}$

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[6] For example, if at a certain profile where all agents announce the same preference relation agent 1 receives the commonly most preferred object, then *neutrality* implies that he does so at all profiles with "maximal conflict".

[7] Let $k \geq n$. Then "hierarchical exchange rules" (Pápai 2000a) are rules that assign at every preference profile to each agent exactly one object. In the multiple assignment model "hierarchical exchange rules" satisfy *strategy-proofness* and *non-bossiness* but not *efficiency*.

[8] The case where objects can come in any possible combination is the special case where $l = k$ and for all $h \in \{1, \ldots, k\}$, $K^h = x_h$.

and a preference profile $R \in \mathscr{R}^{\mathscr{N}}$.[9] A *rule* φ associates with each assignment problem $(R, K') \in \mathscr{R}^{\mathscr{N}} \times \mathscr{K}$ an assignment $\varphi(R, K')$ such that $\cup_{i \in N} \varphi_i(R, K') \subseteq K'$ and for all $i, j \in N$ such that $i \neq j$, $\varphi_i(R, K') \cap \varphi_j(R, K') = \emptyset$. It is straightforward to adjust the properties of rules we introduced in Sects. 2 and 3 to the model at hand.

When the set of objects varies, then a natural requirement is *resource-monotonicity*. Conditions of *resource-monotonicity* have been studied by Chun and Thomson (1988), Moulin and Thomson (1988), and Thomson (1994).

Resource-monotonicity describes the effect of a change in the available resource on the welfare of the agents. A rule satisfies *resource-monotonicity*, if after such a change either all agents (weakly) lose together or all (weakly) gain together.

It is easy to show that in combination with *efficiency*, *resource-monotonicity* implies the following: given some fixed preference profile and some fixed set of objects, if new additional objects are available, then – this being good news – all agents (weakly) gain. We use the following weaker version of *resource-monotonicity*.

Resource-monotonicity. For all $R \in \mathscr{R}^{\mathscr{N}}$ and all $K', K'' \in \mathscr{K}$, if $K' \subseteq K''$, then for all $i \in N$, $\varphi_i(R, K'') R_i \varphi_i(R, K')$.

Our notion of *resource-monotonicity* is weak because we do not require that all subsets of K are admissible. It is possible that $\mathscr{K}$ contains only three subsets of K.

Next, we prove that in combination with *efficiency* and *coalitional strategy-proofness* only serial dictatorships satisfy *resource-monotonicity*. Before stating the result we extend the definition of serial dictatorships.

Serial dictatorship. Let $\pi \in \Pi^N$. The serial dictatorship φ^π with respect to π is defined as follows: for all $(R, K') \in \mathscr{R}^{\mathscr{N}} \times \mathscr{K}$,

$$\begin{aligned}
\varphi^\pi_{\pi(1)}(R, K') &= B(R_{\pi(1)}) \cap K', \\
\varphi^\pi_{\pi(2)}(R, K') &= (B(R_{\pi(2)}) \cap K') \backslash B(R_{\pi(1)}), \\
\varphi^\pi_{\pi(3)}(R, K') &= (B(R_{\pi(3)}) \cap K') \backslash [B(R_{\pi(1)}) \cup B(R_{\pi(2)})], \\
&\vdots \\
\varphi^\pi_{\pi(n)}(R, K') &= (B(R_{\pi(n)}) \cap K') \backslash [\cup_{i=\pi(1)}^{\pi(n-1)} B(R_i)].
\end{aligned}$$

We call agent $\pi(1)$ the first dictator, agent $\pi(2)$ the second dictator, etc. ∘

[9] Note that even though not all objects in K may be available, agents' preferences are defined over 2^K. This assumption mainly simplifies our notation, but all results would remain true if instead we would assume that in an assignment problem preferences are defined over the set of admissible objects.

Theorem 2. *Serial dictatorships are the only rules satisfying efficiency, coalitional strategy-proofness, and resource-monotonicity.*

Proof. It is easy to check that serial dictatorships satisfy the properties listed in Theorem 2.

Conversely, let φ be a rule satisfying *efficiency*, *coalitional strategy-proofness*, and *resource-monotonicity*. Let $K' \in \mathscr{K}$. Then, $\varphi(\cdot, K')$ satisfies *efficiency* and *coalitional strategy-proofness* on $\mathscr{R}^{\mathscr{N}}$. By Theorem 1, $\varphi(\cdot, K')$ is a sequential dictatorship. Hence, for each $R \in \mathscr{R}^{\mathscr{N}}$ we can find $\pi_R^{K'} \in \Pi^N$ as in the definition of a sequential dictatorship such that $\varphi(R, K') = \varphi^{\pi_R^{K'}}(R, K')$. We show in three steps that φ is a serial dictatorship.

Step 1. Let $(R, K') \in \mathscr{R}^{\mathscr{N}} \times \mathscr{K}$ and $\bar{m} \in N$ be such that $\cup_{i=1}^{\bar{m}-1} \varphi_{\pi_R^{K'}(i)}(R, K') \neq K'$. We prove by induction that for all $i \in \{1, \ldots, \bar{m}\}$, $\pi_R^{K'}(i) = \pi_R^K(i)$.

For all $m \in \{1, \ldots, \bar{m}\}$, let $R^m \in \mathscr{R}^{\mathscr{N}}$ be such that

$$R^m_{\{\pi_R^{K'}(1), \ldots, \pi_R^{K'}(m-1)\}} = R_{\{\pi_R^{K'}(1), \ldots, \pi_R^{K'}(m-1)\}}$$

and for all $i \in N \backslash \{\pi_R^{K'}(1), \ldots, \pi_R^{K'}(m-1)\}$, $B(R_i^m) = K$.

Induction basis $m = 1$: Since $\varphi(\cdot, K')$ and $\varphi(\cdot, K)$ are sequential dictatorships, the first dictators are uniquely determined. Hence, $\pi_R^{K'}(1) = \pi_{R^1}^{K'}(1)$ and $\pi_R^K(1) = \pi_{R^1}^K(1)$. Thus, $\varphi_{\pi_R^{K'}(1)}(R^1, K') = K'$ and $\varphi_{\pi_R^K(1)}(R^1, K) = K$. By *resource-monotonicity*, $\pi_R^{K'}(1) = \pi_R^K(1)$, the desired conclusion for the induction basis.

Induction hypothesis. For all $i \in \{1, \ldots, m\}$, let $\pi_R^{K'}(i) = \pi_R^K(i)$.

Induction step $m \to m + 1$ *for* $m < \bar{m}$. Since φ is a sequential dictatorship, (ii) in the definition of sequential dictatorships implies that for all $i \in \{1, \ldots, m+1\}$, $\pi_R^{K'}(i) = \pi_{R^{m+1}}^{K'}(i)$ and $\pi_R^K(i) = \pi_{R^{m+1}}^K(i)$. Since $m + 1 \leq \bar{m}$, $\varphi_{\pi_R^{K'}(m+1)}(R^{m+1}, K') = K' \backslash \cup_{i=1}^m \varphi_{\pi_R^{K'}(i)}(R^{m+1}, K') \neq \emptyset$ and $\varphi_{\pi_R^K(m+1)}(R^{m+1}, K) = K \backslash \cup_{i=1}^m \varphi_{\pi_R^K(i)}(R^{m+1}, K) \neq \emptyset$. By the induction hypothesis and *resource-monotonicity*, $\pi_R^{K'}(m+1) = \pi_R^K(m+1)$, the desired conclusion. This completes the proof of Step 1.

Step 2. Let $(R, K') \in \mathscr{R}^{\mathscr{N}} \times \mathscr{K}$. If for all $i \in N$, $\varphi_i(R, K) \subseteq K'$, then $\varphi(R, K') = \varphi(R, K)$.

By *resource-monotonicity*, for all $i \in N$, $\varphi_i(R, K) R_i \varphi_i(R, K')$. Since for all $i \in N$, $\varphi_i(R, K) \subseteq K'$, *efficiency* implies $\varphi(R, K') = \varphi(R, K)$.

Step 3. We prove that there exists $\pi \in \Pi^N$ such that for all $(R, K') \in \mathscr{R}^{\mathscr{N}} \times \mathscr{K}$, $\varphi(R, K') = \varphi^\pi(R, K')$.

Let $K^1 \in \mathscr{K}$ be such that $K^1 \neq K$. First, we show that $\varphi(\cdot, K^1)$ is a serial dictatorship.

Let $R^\emptyset \in \mathscr{R}^{\mathscr{N}}$ be such that for all $i \in N$, $B(R_i^\emptyset) = \emptyset$. Define $\pi_\emptyset^1 \equiv \pi_{R^\emptyset}^{K^1}$. Since $\varphi(\cdot, K^1)$ is a sequential dictatorship, $\pi_\emptyset^1(1)$ is uniquely defined; i.e., for all $R \in \mathscr{R}^{\mathscr{N}}$, $\pi_R^{K^1}(1) = \pi_\emptyset^1(1)$ and for all $m \in \{1, \ldots, n-1\}$: if for all $i \in \{1, \ldots, m\}$, $\pi_R^{K^1}(i) = \pi_\emptyset^1(i)$ and $\varphi_{\pi_R^{K^1}(i)}(R) = \varphi_{\pi_\emptyset^1(i)}(R^\emptyset) = \emptyset$, then

$\pi_R^{K^1}(m+1) = \pi_\emptyset^1(m+1)$. We show that $\varphi(\cdot, K^1)$ is a serial dictatorship with respect to $\pi_\emptyset^1$.

Let $R \in \mathscr{R}^{\mathscr{N}}$ be such that for all $i \in N$, $B(R_i) \subseteq K \backslash K^1$. Then, $\cup_{i=1}^{n-1} \varphi_{\pi_R^{K^1}(i)}(R, K^1) = \emptyset \neq K^1$. So, by Step 1, $\pi_R^K = \pi_\emptyset^1$. Hence, $\varphi(\cdot, K)$ is a serial dictatorship on those profiles. By the assumptions on $\mathscr{K}$, there exists $K^2 \in \mathscr{K}$ such that $K^2 \neq \emptyset$ and $K^1 \cap K^2 = \emptyset$.

Let $R \in \mathscr{R}^{\mathscr{N}}$ such that for all $i \in N$, $B(R_i) \subseteq K^2$. Thus, for all $i \in N$, $B(R_i) \subseteq K \backslash K^1$. Hence, $\varphi(R, K) = \varphi^{\pi_\emptyset^1}(R, K)$. By Step 2, $\varphi(R, K^2) = \varphi^{\pi_\emptyset^1}(R, K)$. By *strategy-proofness*, if some agent i changes his preference R_i to R_i' such that $B(R_i') \cap K^2 = B(R_i)$, then his allotment will not change. Thus, by *non-bossiness*, $\varphi((R_i', R_{-i}), K^2) = \varphi^{\pi_\emptyset^1}((R_i', R_{-i}), K)$. Since the same conclusion remains valid for all $R \in \mathscr{R}^{\mathscr{N}}$, by *coalitional strategy-proofness*, $\varphi(\cdot, K^2)$ is a serial dictatorship with respect to $\pi_\emptyset^1$; i.e., for all $R \in \mathscr{R}^{\mathscr{N}}$, $\varphi(R, K^2) = \varphi^{\pi_\emptyset^1}(R, K^2)$.

Analogously we can define $\pi_\emptyset^2 \equiv \pi_{R^\emptyset}^{K^2}$ and show that for all $R \in \mathscr{R}^{\mathscr{N}}$ such that for all $i \in N$, $B(R_i) \subseteq K \backslash K^2$, we have $\pi_R^K = \pi_\emptyset^2$. Moreover, the above argument shows that $\varphi(\cdot, K^1)$ is a serial dictatorship with respect to $\pi_\emptyset^2$; i.e., for all $R \in \mathscr{R}^{\mathscr{N}}$, $\varphi(R, K^1) = \varphi^{\pi_\emptyset^2}(R, K^1)$.

Suppose that $\pi_\emptyset^1 \neq \pi_\emptyset^2$. Then there exist agents $i, j \in N$ and $\bar{R} \in \mathscr{R}^{\mathscr{N}}$ such that $B(\bar{R}_i) = B(\bar{R}_j) = K^1 \cup K^2$, $\bar{R}_{-i,j} = R_{-i,j}^\emptyset$, $\varphi_i(\bar{R}, K^1) = K^1$, and $\varphi_j(\bar{R}, K^2) = K^2$. But then, since $\varphi(\cdot, K^1 \cup K^2)$ is also a sequential dictatorship, either $[\varphi_i(\bar{R}, K^1 \cup K^2) = K^1 \cup K^2$ and $\varphi_j(\bar{R}, K^1 \cup K^2) = \emptyset]$ or $[\varphi_i(\bar{R}, K^1 \cup K^2) = \emptyset$ and $\varphi_j(\bar{R}, K^1 \cup K^2) = K^1 \cup K^2]$. Either case is in contradiction to resource-monotonicity. Thus, $\pi_\emptyset^1 = \pi_\emptyset^2$. Let $\pi \equiv \pi_\emptyset^1$.

By our definition of $\mathscr{K}$, there exists a partition $K^1, \ldots, K^l$, $l \geq 2$, of K such that $\{K^1, \ldots, K^l\} \subseteq \mathscr{K}$ and for all $K' \in \mathscr{K}$ there exists an index set $I \subseteq \{1, \ldots, l\}$ such that $K' = \cup_{h \in I} K^h$. So far we have shown that for all $h \in \{1, \ldots, l\}$ and all $R \in \mathscr{R}^{\mathscr{N}}$,

$$\varphi(R, K^h) = \varphi^\pi(R, K^h). \tag{8}$$

Next, we show that $\varphi(\cdot, K)$ is a serial dictatorship with respect to π. Let $R \in \mathscr{R}^{\mathscr{N}}$ and $h \in \{1, \ldots, l\}$. By (8), $\varphi(R, K^h) = \varphi^\pi(R, K^h)$. Let $\bar{m}^h \in N$ be maximal such that $\cup_{i=1}^{\bar{m}^h - 1} \varphi_{\pi(i)}(R, K^h) \neq K^h$. By Step 1, for all $i \in \{1, \ldots, \bar{m}^h\}$, $\pi_R^K(i) = \pi(i)$. Thus, since $\varphi(\cdot, K)$ is a sequential dictatorship, for all $i \in \{1, \ldots, \bar{m}^h\}$,

$$\varphi_{\pi(i)}(R, K^h) \subseteq \varphi_{\pi(i)}(R, K). \tag{9}$$

Since

$$\cup_{h=1}^l \cup_{i=1}^{\bar{m}^h} \varphi_{\pi(i)}(R, K^h) = \cup_{i \in N} \varphi_i(R, K),$$

(9) implies for all $i \in N$,

$$\varphi_{\pi(i)}(R, K) = \cup_{h=1}^l \varphi_{\pi(i)}(R, K^h).$$

Hence, $\varphi(R, K) = \varphi^\pi(R, K)$. Thus, $\varphi(\cdot, K)$ is a serial dictatorship with respect to π.

Finally, we prove that φ is a serial dictatorship. Let $R \in \mathscr{R}^{\mathcal{N}}$ and $K' \in \mathscr{K}$. Let $R' \in \mathscr{R}^{\mathcal{N}}$ be such that for all $i \in N$, $B(R'_i) = B(R_i) \cap K'$. *Coalitional strategy-proofness* implies

$$\varphi(R, K') = \varphi(R', K'). \tag{10}$$

We have proven that $\varphi(R', K) = \varphi^{\pi}(R', K)$. Since for all $i \in N$, $B(R'_i) \subseteq K'$, by *efficiency*, for all $i \in N$, $\varphi_i(R', K) \subseteq K'$. Thus, Step 2 implies $\varphi(R', K') = \varphi(R', K)$. Note that $\varphi(R', K) = \varphi^{\pi}(R', K) = \varphi^{\pi}(R', K')$. Hence, $\varphi(R', K') = \varphi^{\pi}(R', K')$ and by (10), $\varphi(R, K') = \varphi^{\pi}(R, K')$. This completes the proof. ■

Remark 3. *The proof of Theorem 2 remains valid on the preference domain* $\mathscr{A}$. ◁

Similar examples as for Theorem 1 can be used to establish the independence of the properties in Theorem 2. In particular, in Theorem 2 *coalitional strategy-proofness* cannot be weakened to *strategy-proofness* because then the choice of the next dictators may depend on the exact preference relation of the first dictators.

The following example shows that Theorem 2 becomes an incompatibility on the domain of strict preference relations $\mathscr{S}$.

Example 2. *Let* $N = \{1, 2\}$*,* $K = \{x_1, x_2\}$*, and* $\mathscr{K} = \{\{x_1\}, \{x_2\}, \mathscr{K}\}$*. Let* φ *denote the serial dictatorship with agent 1 as first dictator and agent 2 as the second dictator. Let* $R = (R_1, R_2) \in \mathscr{S}^{\{N\}}$ *be such that* $\{x_1, x_2\} P_1 \emptyset P_1 x_1 P_1 x_2$ *and* $R_2 \in L(x_1)$*. Then,* $\varphi(R, \{x_1, x_2\}) = (\{x_1, x_2\}, \emptyset)$ *and* $\varphi(R, \{x_1\}) = (\emptyset, \{x_1\})$*. Hence, in contradiction to resource-monotonicity,* $\varphi_1(R, \{x_1, x_2\}) P_1 \varphi_1(R, \{x_1\})$ *and* $\varphi_2(R, \{x_1\}) P_2 \varphi_2(R, \{x_1, x_2\})$*.* ◁

Remark 4. *The restriction of the structure of* $\mathscr{K}$ *is tight. If* $\mathscr{K}$ *does not satisfy property (b), then it is possible to find rules that are sequential dictatorships on some subsets of* $\mathscr{K}$ *and that satisfy all the properties of Theorem 2.* ◁

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