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Optimal Risk Management, Risk Aversion, and Production Function Properties

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For production risk with identified physical causes, the nature of risk, production characteristics, risk preference, and prices determine optimal input use. Here, a two-way classification for pairs of inputs—each input as being risk increasing or decreasing and pairs as being risk substitutes or complements—provides sufficient conditions to determine how risk aversion should affect input use. Unlike the Sandmo price risk case in which a more risk averse firm produces less output, a more risk averse firm may produce more expected output and use more inputs than a risk neutral firm. Sufficient conditions to determine types for pairs of inputs are also related to properties of the production function.

Key words: production risk, risk aversion, risk management.

Introduction

"The nature of uncertainty in a production function is crucial to determine the effects of production risk on the firm's input decision . . ." (Honda, p. 91). Accordingly, this article examines the case of production risk and risk management when there are identified physical causes of risk, such as insects and weather, with known probability distributions.

The characterization of production decisions under risk is the subject of a growing literature, and conjectures have been made regarding how increasing risk aversion should affect production decisions. Early studies considered exogenous price risk. For example, Sandmo studied price risk and output decisions in the firm. The result obtained with a nonrisky cost function was that the risk averse firm should produce less output than the risk neutral firm. Batra and Ullah considered input decisions in the firm with price risk. They also found that the output of the firm should decline with risk aversion based on effects on input use. Although a production model was used to model input effects, production risk was not considered in Batra and Ullah's work.

Recent research has focused on production risk rather than price risk. In such cases, input use can affect the nature of risk so that risk becomes at least in part endogenous or manageable; for example, insecticide use affects insect risk (Antle; Carlson; Feder). As in classical production theory, economic optimization modeling can provide rules for risk management to determine optimal input use. Here, rules are shown to be related to prices, level of risk aversion, the nature of inputs, and characteristics of the production process.

Previous research in production risk studied special cases. Pope and Kramer used a specific type of stochastic production function with only two inputs and a constant relative risk aversion (power) utility function. MacMinn and Holtman investigated the case of a general risk averse utility function and a general production function, but with only one input. Babcock, Chalfant, and Collender included multiple inputs; however, the moment

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generating function method they used for optimization under risk assumes an exponential utility function.

Comparative static results for production under risk are obtained here for more general conditions. Results are based on a general stochastic production function with multiple inputs, and both exponential and power utility function cases are included. In comparison to Pope and Kramer, who were also concerned with the effect of risk aversion on input use, more complete comparative static results and sufficient conditions are given.

Categorization of pairs of inputs—each as being risk increasing or decreasing and pairs as being risk substitutes and complements—is used to delineate conditions under which input use should increase or decrease with risk aversion. For the exponential and power utility functions, sufficient conditions to identify input types according to the proposed categorization are given in terms of properties of the stochastic production function. In contrast to the Sandmo price risk case, we show that the risk averse firm may even produce more expected output and use more inputs than the risk neutral firm when there are multiple inputs.

This approach differs from that used by Meyer and Ormiston and by Jewitt because they described risk in terms of parameterized changes in a distribution function rather than using a production function to model risk. The advantage of using the production function approach, as demonstrated here, is that sufficient conditions can be based on classical production concepts extended to the risk case.

Another approach to study production under risk is based on approximating first order conditions in mean-variance terms (see Anderson, Dillon, and Hardaker; Robison and Barry). The approach in this article avoids approximation and instead relies heavily on application of a theorem in statistics called Lehmann's Theorem (Barlow and Proschan).

Endogenous Production Risk and Input Choice

Since input choice affects the nature of yield risk, risk is partly endogenous. Optimal management of risk entails the optimal choice of inputs. Inputs are assumed to be chosen before the state of nature is revealed.

Production risk will be described here by a stochastic production function,

$$y = F(x, \theta), \tag{1}$$

which relates random output (y) to a vector of inputs (x) and the state of nature (θ). θ represents physical causes of production risk such as weather and insects with a known cumulative distribution $G(\theta)$. In contrast to Pope and Kramer, the stochastic production function need not be the linear in θ. Specification of this production function should be based on physical relationships between inputs and sources of risk, rather than being a model of measurement error with heteroskedasticity as in Just and Pope (1978, 1979) (see example below). To describe yield risk, a distribution for yield is determined from (1) as a transformation of $G(\theta)$ given input use. Below, only one source of risk (θ) is used.

Optimal input choice for a risk averse producer is assumed to satisfy maximization of expected utility:

$$\operatorname*{Max}_{x} \int_{A} u(W_o + pF(x, \theta) - w \cdot x)\, dg(\theta), \tag{2}$$

where u denotes the utility function for income, W_o is riskless income, x denotes a vector of inputs with input price vector w, p is price of output, and A is the support of θ. Prices p and w are assumed here to be riskless. Profit is defined by:

$$\pi(x, \theta) = pF(x, \theta) - w \cdot x. \tag{3}$$

A risk neutral producer would maximize expected profit rather than (2).

The following lemma gives sufficient conditions to determine the sign of expected marginal profit (as in MacMinn and Holtman) at the optimum of (2) based on production function properties. The lemma applies for a general production function and concave utility function. Note that, in contrast to production in the certainty case with zero marginal profit for each input, here marginal profit may be positive or negative depending on the properties of the production function at optimum input use. F_θ, F_{x_i}, and $F_{x_i\theta}$ denote partial derivatives of the production function with respect to x_i and θ.

Lemma 1: Given a concave utility function, an input (x_i) gives a positive expected marginal profit at the optimum (x^*) if $F_\theta(x^*, \theta)$ and $F_{x_i\theta}(x^*, \theta)$ have the same sign for all values of θ, and a negative expected marginal profit at the optimum if they have opposite signs.

Proof: The expected marginal profit ($E\pi_{x_i}$) evaluated at optimum input use x^* satisfies the following first order conditions for (1):

$$E\pi_{x_i} = E(pF_{x_i}) - w_i = -\text{Cov}(u', pF_{x_i})/Eu' \tag{4}$$

from the definition of covariance (MacMinn and Holtman; Pope and Kramer). By Lehmann's Theorem (see appendix B) applied with a concave utility function, the sign of the covariance $\text{Cov}(u', F_{x_i})$ is positive if F_θ and $F_{x_i\theta}$ have opposite signs and is negative if they have the same signs. Then, a negative expected marginal profit is obtained for a positive covariance term and, conversely, a positive marginal expected profit is obtained for a negative covariance term. QED.

Input Classification

How the solution for optimal input use in (2) would change as the Pratt–Arrow risk aversion coefficient changes is a question of comparative statics. For this purpose, Pope and Kramer proposed two alternative definitions to classify inputs:

Definition 1: An input is said to be "marginally risk reducing" if the expected marginal profit is negative at the optimum input use and "marginally risk increasing" if the expected marginal profit is positive at the optimum input use.

Definition 2: An input is said to be marginally risk reducing (increasing) if the risk averse firm uses a larger (smaller) quantity of the input than the corresponding risk neutral firm.

The equivalence of these two definitions implies that a more risk averse person should use more of an input having a negative expected marginal profit and less of an input having a positive expected marginal profit. Lemma 1, above, is sufficient to determine whether inputs are marginally risk reducing or increasing according to Definition 1.

When the production function has a form linear in θ,

$$F(L, K, \theta) = f(L, K) + g(L, K)\theta, \text{ with } E\theta = 0,$$

assuming a power utility function, Pope and Kramer showed that these two definitions are equivalent if certain sign conditions are imposed on derivatives of the stochastic production function $F(L, K, \theta)$ with respect to two inputs, L and K.

Below, instead of the categorization in Definition 1, pairs of inputs will be categorized in two ways: whether each input is risk increasing or decreasing (Definition 3); and for the pair, whether they are risk substitutes or complements (Definition 4). This categorization is more complex than that in Definition 1. However, this categorization gives rise to definitive comparative static results for a general production function, as stated in the theorem below.

Definitions below are in terms of the certainty equivalent (CE) as derived from expected utility (Eu):

$$u(CE(W_o, x, w, p)) = Eu(W_o + \pi(x, \theta)).$$

Definitions are given in terms of the certainty equivalent rather than in terms of expected utility because then interpretations are in dollar-valued terms (see below). Note that maximization of the certainty equivalent gives the same results as expected utility maximization. The "marginal certainty equivalent" CE_i denotes the partial derivative of the certainty equivalent with respect to an input; it is proportional to the marginal expected utility due to input i:

$$CE_i = \frac{\partial CE}{\partial x_i} = \frac{\partial EU}{\partial x_i}(1/u'(CE))$$

because utility is monotone increasing.

The interpretations of the definitions below are that: for a risk increasing input, increasing risk aversion will reduce the marginal certainty equivalent, whereas the marginal certainty equivalent for a risk reducing input will increase with risk aversion. For a pair of inputs which are risk substitutes, an increase in one input will reduce the marginal certainty equivalent of the other, whereas for complements, an increase in one input will increase the marginal certainty equivalent.

Note also that the definitions are given in terms of the optimum solution. The (*) notation indicates that input use (x^*) and profit (π^*) are evaluated for input use held constant at the optimum value for (2). Since the optimum solution could vary by geographic area (e.g., not the same in high and low rainfall areas), the type of an input may vary depending on the nature of physical risk. The Pratt–Arrow risk aversion coefficient for the exponential utility function is denoted by r.

Definition 3: An input will be termed "risk increasing" if $c_i < 0$ and "risk reducing" if $c_i > 0$, where

$$c_i = \frac{\partial}{\partial r}\frac{\partial CE^*}{\partial x_i}.$$

Definition 4: For a pair of inputs (x_i, x_j), they will be termed "risk substitutes" if q_{ij} is negative and "risk complements" if q_{ij} is positive, where

$$q_{ij} = \frac{\partial^2 CE^*}{\partial x_j x_i}.$$

If $q_{ij} = 0$, there is "no risk substitution" between inputs i and j.

Below, we compare risk increasing (reducing) inputs in Definition 3 to marginally risk increasing (reducing) inputs in Definition 1. Definition 1 can be shown to relate to first order effects of an input on the risk premium. The risk premium (RP) and certainty equivalent (CE) are related by

$$RP = E\pi - CE,$$

where RP is a positive number for a risk averse person. The certainty equivalent satisfies $\frac{\partial CE^*}{\partial x_i} = 0$ at the optimum input use. Therefore, at the optimum,

$$\frac{\partial RP^*}{\partial x_i} = \frac{\partial E\pi^*}{\partial x_i}.$$

From this correspondence at the optimum, a "marginally risk increasing" input in Definition 1 will have a positive marginal risk premium at the optimum; thus, increasing input use from the optimum will increase the risk premium. A "marginally risk reducing" input will have a negative marginal risk premium; thus, increasing input use from the optimum will reduce the risk premium.

Risk increasing (reducing) inputs in Definition 3 are defined in terms of the certainty equivalent, specifically the effect of risk aversion on the marginal certainty equivalent for an input:

$$c_i = \frac{\partial^2 CE^*}{\partial r \partial x_i}.$$

Thus, the sign of c_i shows the change in the marginal certainty equivalent for an input as the Pratt–Arrow risk aversion coefficient is varied. If $c_i < 0$ (risk increasing), an increase in r will reduce the marginal certainty equivalent for an input; for $c_i > 0$ (risk reducing), an increase in r will increase the marginal certainty equivalent for an input.

Comparative Static Effects of Risk Aversion

Here, for a general production function and both power and exponential utility functions, the above categorization of inputs is shown to give rise to comparative static results specifying when a more risk averse producer should use more or less of a given type of input than a less averse producer. In contrast to the Pope and Kramer special case, comparative static results obtained here for general conditions require inputs to be classified by more than just the sign of the expected marginal profit at the optimum. Also, sufficient conditions for categorizing inputs are shown to be based not only on properties of the production function but also on relative prices.

For a pair of inputs, in addition to the "no substitution" case, there are six possible combinations of input types according to Definitions 3 and 4. Separately, both inputs can be risk increasing, both can be risk reducing, or one input can be risk increasing and the other risk reducing. For the pair, they can be either risk complements or substitutes for each of the three cases above.

The comparative statics result of this article is stated in the theorem below. Effects of risk aversion for each of the above six types plus the case of no substitution are described in the theorem. Second order conditions link combinations of input types to effects of risk aversion (see appendix A proving the theorem). In comparison, Pope and Kramer presented proofs for only cases (ii), (v), and (vi) of this theorem and only for the power utility function and production function linear in θ. This theorem extends their results in terms of generality of the production function, allowing both exponential and power utility functions, and completeness of the input combinations.

Theorem: The following comparative static results hold for pairs of inputs, with a general stochastic production function, for either an exponential or a power utility function.

(i) In the case of production with only one input, or no risk substitution for a pair of inputs, a risk increasing input should be used less intensively by a more risk averse person, and a risk decreasing input should be used less intensively by a less risk averse person.

(ii) For a pair of risk substitutes, when both inputs are risk increasing, either both should decrease with increasing risk aversion or one should increase while the other decreases.

(iii) For a pair of risk substitutes when both inputs are risk reducing, either both should increase with increasing risk aversion, or one should increase while the other decreases.

(iv) For a pair of risk substitutes, when one input is risk increasing and the other is risk reducing, then use of the risk reducing input should increase and use of the risk increasing input should decrease with increasing risk aversion.

(v) For a pair of risk complements, when both inputs are risk increasing, then use of both should go down with increasing risk aversion.

(vi) For a pair of risk complements, when both inputs are risk reducing, then use of both should go up with increasing risk aversion.

(vii) For a pair of risk complements, when one is risk increasing and the other is risk reducing, then the risk increasing input should not increase with increasing risk aversion unless the risk reducing input also increases. The risk reducing input should not decrease with risk aversion unless the risk increasing input also decreases. The risk increasing input may decrease if the risk reducing input increases.

Proof: The proof (see appendix A) identifies possible and impossible cases for input use combinations based on comparative static analysis of first and second order conditions for optimization of (2).

These results can be compared to the results of Sandmo, and Batra and Ullah for price risk, assuming that expected output is nondecreasing with increasing input use. As in their results, if there is only one input and one output and if that input is risk increasing, then the risk averse firm will produce less expected output than the risk neutral firm. However, here the more risk averse firm also may produce more expected output than the risk neutral firm if this single input is risk reducing.

The theorem also implies that expected output will decline with increasing risk aversion if pairs of inputs are complements and all inputs are risk increasing. Expected output also could decline under other conditions (e.g., if pairs of inputs are risk substitutes and all are risk increasing). Conversely, expected output will increase with risk aversion if pairs of inputs are risk complements and both are risk reducing. Expected output could also increase in other cases (e.g., if pairs of inputs are risk complements, and one is risk increasing and one is risk reducing).

Sufficient Conditions for Identifying Risk Increasing/Decreasing Inputs

We now give sufficient conditions related only to production function properties to identify if an input is risk increasing or risk reducing for two commonly used utility functions—exponential and power utility functions.

Exponential Function

For the exponential function, with initial wealth W_o

$$u(\pi) = k_0 - k_1 e^{-r(\pi + W_o)},$$

the Pratt–Arrow risk aversion coefficient is a constant ($r \geq 0$). At the optimum, since the marginal certainty equivalent and expected marginal utility are zero at this optimum,

$$c_i = \frac{\partial}{\partial r}\frac{\partial CE^*}{\partial x_i} = \frac{\partial}{\partial r}(k_1/u'(CE^*)) \int_A re^{-r(\pi^* + W_o)}\pi^*_{x_i}\, dG(\theta)$$

$$= -(k_1/u'(CE^*)) \int_A re^{-r(\pi^* + W_o)}(\pi^* + W_o)\pi^*_{x_i}\, dG(\theta).$$

Sufficient conditions to determine the sign of the above integral are obtained from Lehmann's Theorem. First, from the definition of covariance,

$$E(e^{-r(\pi^* + W_o)}(\pi^* + W_o)\pi^*_{x_i}) = E(e^{-r(\pi^* + W_o)}(\pi^* + W_o))E(\pi^*_{x_i}) + \mathrm{Cov}(\pi^*_{x_i}, e^{-r(\pi^* + W_o)}(\pi^* + W_o)).$$

From Lehmann's Theorem, the sign of the covariance above is determined by the signs of the derivatives of its two arguments with respect to θ:

(i) This covariance is positive if $F^*_{x_i\theta}$ and $\frac{\partial}{\partial\theta}[e^{-r(\pi^* + W_o)}(\pi^* + W_o)]$ have the same sign;

and

(ii) this covariance is negative if $F^*_{x_i\theta}$ and $\frac{\partial}{\partial\theta}[e^{-r(\pi^*+W_o)}(\pi^* + W_o)]$ have opposite signs.

The second term above will be positive if $F_\theta > 0$, $(\pi^* + W_o) > 0$, and $[1 - r(\pi^* + W_o)] > 0$ since:

$$\frac{\partial}{\partial\theta}[e^{-r(\pi^*+W_o)}(\pi^* + W_o)] = e^{-r(\pi^*+W_o)}\pi^*_\theta[1 - r(\pi^* + W_o)].$$

Also, $E(e^{-r(\pi^*+W_o)}(\pi^* + W_o))$ is positive if $(\pi^* + W_o)$ is positive. The sign of $E(\pi^*_{x_i})$ also depends on the sign of $F^*_{x_i\theta}$ as in Lemma 1. The sign of c_i is then determined from the combination of the above conditions, as summarized below:

Lemma 2: If $(\pi^* + W_o)$ is positive and $\underset{\theta}{\mathrm{Max}}[\pi^* + W_o] < \frac{1}{r}$: x_i is risk reducing if $F_{x_i\theta}(x^*, \theta)$ and $F_\theta(x^*, \theta)$ have opposite signs; x_i is risk increasing if $F_{x_i\theta}(x^*, \theta)$ and $F_\theta(x^*, \theta)$ have the same sign.

Power Function

With initial wealth W_o, the power function with relative risk aversion s is:

$$u(\pi) = (W_o + \pi)^{1-s}.$$

At the optimum with expected profit $E\pi^*$, the following Pratt–Arrow coefficient corresponds to this power function:

$$r = \frac{s}{W_o + E\pi^*}.$$

To determine risk increasing/risk reducing properties, consider the sign of

$$c_i = \frac{\partial}{\partial r}\frac{\partial CE^*}{\partial x_i} = \left[\frac{\partial}{\partial s}\frac{\partial CE^*}{\partial x_i}\right]\frac{\partial s}{\partial r}.$$

Since $\frac{\partial s}{\partial r} > 0$, the sign of c_i is the same as the sign of

$$\begin{aligned}\frac{\partial^2 Eu}{\partial s\partial x_i} &= -\int_A \ln(W_o + \pi^*)(W_o + \pi^*)^{-s}\pi^*_{x_i}\, dG(\theta)\\ &= -[\mathrm{Cov}(\pi^*_{x_i}, \ln(W_o + \pi^*)(W_o + \pi^*)^{-s})\\ &\quad + E(\ln(W_o + \pi^*)(W_o + \pi^*)^{-s})E(\pi^*_{x_i})].\end{aligned}$$

The same sign conditions as above are shown to determine the sign of c_i from Lehmann's Theorem—with the requirements that $(W_o + \pi^*) > 0$ and $\underset{\theta}{\mathrm{Max}}\ \ln(W_o + \pi^*) < \frac{1}{s}$. The results for the power function are summarized by the following lemma.

Lemma 3: For the power utility function with $(W_o + \pi^*) > 0$ for all θ and $\underset{\theta}{\mathrm{Max}}[\ln(W_o + \pi^*)] < \frac{1}{s}$: x_i is risk reducing if $F_{x_i\theta}(x^*, \theta)$ and $F_\theta(x^*, \theta)$ have opposite signs; x_i is risk increasing if $F_{x_i\theta}(x^*, \theta)$ and $F_\theta(x^*, \theta)$ have the same sign.

Thus, for these two commonly used utility functions, for risk aversion coefficients satisfying the bound conditions, properties of the stochastic production function are sufficient to determine whether an input is risk reducing or increasing.

Because of the bound condition, the magnitude of the Pratt–Arrow risk aversion coefficient should be inversely related to the magnitude of income—similar to the magnitude restriction obtained by Raskin and Cochran but for different reasons. The bound condition for the power function is more restrictive than that for the exponential function. Note that these two lemmas do not require that the sign conditions for the production function derivatives hold for all values of x, merely at the optimum. However, if the sign conditions hold for all input values, then specific information about the optimum may not be required to determine the type of an input (see example below).

Comparing Definitions 1 and 3, note that when utility is of the exponential or power function form, and the level of risk aversion satisfies the appropriate bound condition, then a risk reducing/increasing input is also marginally risk reducing/increasing by Lemma 1. That is, the set of inputs which are risk reducing/increasing intersects the set of inputs which are marginally risk reducing/increasing for a given production function. In particular, Definition 2 and Definition 1 correspond in Pope and Kramer's special case.

Sufficient Conditions for Identifying Risk Complements/Substitutes

Sufficient conditions for identifying risk complements/substitutes are more complex than those above; prices are well as production function properties and bounds on risk aversion determine the types of input pairs.

The coefficient q_{ij} shows the effect of one input on the marginal certainty equivalent for the other input. If $q_{ij} > 0$, then an increase in input i will increase the marginal certainty equivalent for input j; if $q_{ij} < 0$, then the marginal certainty equivalent for input j will be reduced. The general definition of q_{ij}, obtained from the first order condition $\partial CE^*/\partial x_i = 0$, is:

$$q_{ij} = (1/u'(CE)) \int_A u'(\pi^*)\left[\pi_{x_ix_j}(x^*, \theta) + \frac{u''(\pi^*)}{u'(\pi^*)}\pi_{x_i}(x^*, \theta)\pi_{x_j}(x^*, \theta)\right] dG(\theta).$$

One sufficient condition to determine the type for a pair of inputs extends the classical definitions of substitutes and complements based on the cross-product derivatives of the production function. The sign of q_{ij} may be determined from the sign of:

$$\left[\pi_{x_ix_j}(x^*, \theta) + \frac{u''(\pi^*)}{u'(\pi^*)}\pi_{x_i}(x^*, \theta)\pi_{x_j}(x^*, \theta)\right];$$

if this expression has a fixed sign, the sign of q_{ij} is the same as the sign of this term because of monotone utility. With risk aversion, a sufficient (but not necessary) condition for risk substitutes ($q_{ij} < 0$) is when inputs are substitutes in the classical production sense ($F_{x_ix_j} < 0$) for all values of θ, and their marginal profits have the same signs at the optimum. Conversely, a sufficient condition for inputs to be risk complements ($q_{ij} > 0$) is when inputs are complements in the classical production sense ($F_{x_ix_j} > 0$) for all values of θ, and their marginal profits have the opposite sign at the optimum. Note that prices play a role in determining the signs of marginal profits.

A weaker sufficient condition for determining the sign of q_{ij} for the exponential utility function is obtained from Lehmann's Theorem by considering the sign of the following covariance together with the sign of $F_{x_ix_j}$ at the optimum:

$$\mathrm{Cov}(\pi^*_{x_j}, e^{-r(\pi^*+W_0)}\pi^*_{x_i}) = E(e^{-r(\pi^*+W_0)}\pi^*_{x_i}\pi^*_{x_j}) - E(e^{-r(\pi^*+W_0)}\pi^*_{x_i})E(\pi^*_{x_j}).$$

This covariance is positive by Lehmann's Theorem (similar to the analysis for risk increasing/risk reducing inputs) if $\pi^*_{x_j\theta}$ is positive and $[\pi^*_{x_i\theta} - r\pi^*_{x_i}\pi^*_{\theta}]$ is positive; then

$E(e^{-r(\pi^*+W_0)}\pi^*_{x_i}\pi^*_{x_j})$ is positive from first order conditions. Conversely, this covariance is negative if signs of these terms are opposite. The lemma below summarizes sufficient conditions:

Lemma 4: For the exponential utility function satisfying the bound condition $[\pi^*_{x_i\theta} - r\pi^*_{x_i}\pi^*_\theta] > 0$: x_i and x_j are risk complements if $F_{x_ix_j} > 0$ and $\pi^*_{x_j\theta} < 0$; x_i and x_j are risk substitutes if $F_{x_ix_j} < 0$ and $\pi^*_{x_j\theta} > 0$.

Note the above bound condition is based not only on physical properties of the production function but also on relative prices. Similar results could be obtained for the power function.

Production Function Example

The following example illustrates use of the above definitions and lemmas. The nature of inputs is determined in part by physical relationships expressed in terms of production function properties.

Suppose production under risk is described by:

$$y = h(x_1, x_2) + f(x_1)\theta + \lambda(1 + g(x_2))I(\theta),$$

where y is the crop yield, θ represents rainfall, x_1 is fertilizer, and x_2 is insecticide treatment. $I(\theta)$ represents insect population which increases with rainfall. λ is a negative number showing that insects reduce yield. A positive value of $I'(\theta)$ indicates that insect population increases with good rainfall. For $g(x_2)$ negative, insecticide treatment would reduce the negative effect of insects. The derivative of y with respect to rainfall θ should be positive in some range of x_1 and θ:

$$F_\theta = f(x_1) + \lambda(1 + g(x_1))I'(\theta) > 0.$$

Then, if $f(x_1)$ increases with x_1, x_1 will be a risk increasing input:

$$F_{x_i\theta} = f'(x_1) > 0.$$

For x_2 to be a risk reducing input, $g'(x_2)$ should be positive to give:

$$F_{x_2\theta} = \lambda g'(x_2)I'(\theta) < 0.$$

A sufficient condition for expected marginal profit of fertilizer and insecticide to have opposite signs is that $F_{x_1\theta}$ and $F_{x_2\theta}$ have opposite signs, e.g., if $f'(x_1)$, $g'(x_2)$, and $I'(\theta)$ are positive and λ is negative. These are also the same conditions for fertilizer to be risk increasing and insecticide to be risk reducing!

Substitution/complementarity for a pair of inputs relates to the sign of

$$F_{x_1x_2} = \frac{\partial^2 h}{\partial x_1 \partial x_2},$$

which expresses the effect of insecticide on the marginal product of fertilizer. We could expect that insecticide would increase the expected marginal product of fertilizer, i.e., that insecticide and fertilizer potentially would be risk complements. Risk complementarity also requires that the signs of the marginal profits are opposite; i.e., if fertilizer is relatively cheap compared to the value of its marginal product and pesticide is relatively expensive compared to the value of its marginal product.

Case (vii) of the above theorem then applies. If a risk averse person uses more fertilizer than a risk neutral person, then also the risk averse person should use more insecticide; the more risk averse person would then produce more output than the risk neutral person. However, it is also possible that a risk averse person would use less of both fertilizer and insecticide, then producing less output than a risk neutral person. Or, the risk averse person could use less fertilizer and more insecticide than a risk neutral person with the output effect not identifiable.

Conclusions

This article has generalized models of optimal input use in Pope and Kramer, MacMinn and Holtman, and Babcock, Chalfant, and Collender to apply for multiple inputs, a general stochastic production function, and both exponential and power utility functions. Also, our comparative static results expand results obtained by Pope and Kramer. Determination of input use effects of risk aversion was based on a two-way categorization for pairs of inputs—each input separately as being risk increasing or decreasing and the pair of inputs as being risk substitutes or complements.

With production risk, unlike the price risk case in which a more risk averse firm produces less output, a more risk averse firm may produce more expected output and use more inputs than a risk neutral firm. Results in this article outline when this could occur and, conversely, when increased risk aversion would imply less output.

Properties of the stochastic production function were relevant to identify alternative input use cases. Production properties describe the physical interactions of inputs with explicit causes of risk. Production properties were sufficient to determine whether inputs were risk increasing or risk reducing. However, the nature of sufficient conditions for determining risk substitutes/complements implies that production function properties alone are not sufficient to identify inputs of these types. Not surprisingly, the results in this article show that not only production function properties but also the level of risk aversion relative to income and relative prices determine how risk aversion should affect risk management.

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Appendix A: Proof of Theorem

We will prove the result for one pair of inputs at the optimum. For multiple inputs, each pair would be treated similarly. Inputs are denoted by the indices 1 and 2. (All functions are evaluated at the optimum but the * notation is omitted.) Totally differentiating the first order conditions for $i = 1, 2$ with respect to the Pratt–Arrow risk aversion coefficient r in the exponential case,

$$\int_A u'(\pi)\pi_{x_i}\, dG(\theta) = 0,$$

the following simultaneous system is obtained:

$$q_{11}\frac{dx_1}{dr} + q_{12}\frac{dx_2}{dr} = -c_1, \qquad \text{and}$$

$$q_{21}\frac{dx_1}{dr} + q_{22}\frac{dx_2}{dr} = -c_2,$$

where

$$q_{ij} = q_{ji} = (1/u'(CE)) \int_A u'(\pi)\pi_{x_ix_j} + u''(\pi)\pi_{x_i}\pi_{x_j}\, dG(\theta), \qquad \text{and}$$

$$c_i = (1/u'(CE))\frac{\partial}{\partial r}\int_A u'(\pi)\pi_{x_i}\, dG(\theta).$$

Note that q_{ij} corresponds to the definition of risk substitutes/complements, and c_i corresponds to the definition of risk increasing/decreasing inputs. Solving the above system, we obtain

$$\frac{dx_1}{dr} = \det\begin{bmatrix} -c_1 & q_{12} \\ -c_2 & q_{22} \end{bmatrix} \Big/ |D| = (-c_1q_{22} + c_2q_{12})|D|, \qquad \text{and}$$

$$\frac{dx_2}{dr} = \det\begin{bmatrix} q_{11} & -c_1 \\ q_{21} & -c_2 \end{bmatrix} \Big/ |D| = (-c_2q_{11} + c_1q_{12})|D|,$$

where

$$|D| = \det\begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}.$$

Second order sufficient conditions for a pair of inputs are that: $q_{11} < 0$, $q_{22} < 0$, and $D = q_{11}q_{22} - q_{12}^2 > 0$.

For the one input case or the case of no substitution with $q_{12} = 0$, $\frac{dx_i}{dr} = -c_i/q_{ii}$; thus, $c_i < 0$ implies $\frac{dx_i}{dr} < 0$, and the reverse sign for $c_i > 0$.

In the six cases for $q_{12} \neq 0$, there are eight possible sign combinations for q_{12}, c_1, c_2, as shown in appendix table A1; these eight cases reduce to the six cases of the theorem as indicated. Table A1 shows four cases ($E1$, $E2$, $E3$, and $E4$) for sign combinations of $\frac{dx_i}{dr}$. Note that $E3$ corresponds to a decrease in expected output and $E4$ gives an increase in expected output; $E1$ and $E2$ produce indeterminate output effects. For these combinations, "S" denotes sufficient conditions, "NP" corresponds to cases which are not possible, and "P" indicates possible cases.

The S, P, and NP results are obtained from $\frac{dx_i}{dr}$ and second order conditions. For the sufficient (S) cases, the signs of $\frac{dx_i}{dr}$ are determined directly by the stated conditions and second order conditions. As an example of a sufficient (S) case, if $q_{12} < 0$, $c_1 < 0$, and $c_2 > 0$, then $\frac{dx_1}{dr} < 0$ and $\frac{dx_2}{dr} > 0$ are obtained directly from the expressions for $\frac{dx_i}{dr}$. When there is a sufficient condition in a row of table A1, the other cases in the row are not possible (NP).

Table A1. Sign Combinations

Combination			Input Sign Cases				Theorem Case
q_{12}	c_1	c_2	$E1$	$E2$	$E3$	$E4$	
+	−	−	NP	NP	S	NP	(v)
+	−	+	NP	P	P	P	(vii)
+	+	−	P	NP	P	P	(vii)
+	+	+	NP	NP	NP	S	(vi)
−	−	−	P	P	P	NP	(ii)
−	−	+	NP	S	NP	NP	(iv)
−	+	−	S	NP	NP	NP	(iv)
−	+	+	P	P	NP	P	(iii)

Notation:

$E1$: $dx_1/dr > 0$, $dx_2/dr < 0$.
$E2$: $dx_1/dr < 0$, $dx_2/dr > 0$.
$E3$: $dx_1/dr < 0$, $dx_2/dr < 0$.
$E4$: $dx_1/dr > 0$, $dx_2/dr > 0$.

For cases (ii), (iii), and (vii) of the theorem, proofs of the not possible (NP) cases are obtained from second order conditions.

Case (ii): Suppose $q_{12} < 0$, $c_2 < 0$, and $\frac{dx_1}{dr} > 0$. Then

$$c_1 q_{22} < c_2 q_{12}.$$

From second order conditions,

$$c_1 q_{12} q_{22} > c_2 q_{12}^2 > c_2 q_{11} q_{22}.$$

So, since $q_{22} < 0$,

$$c_1 q_{12} < c_2 q_{11}$$

implies $\frac{dx_2}{dr} < 0$; thus, $\frac{dx_2}{dr} > 0$ is not possible.

Case (iii): Suppose $q_{12} < 0$, $c_2 > 0$, and $\frac{dx_1}{dr} < 0$. Then

$$c_1 q_{22} > c_2 q_{12};$$

also,

$$c_1 q_{22} q_{12} < c_2 q_{12}^2 < c_2 q_{11} q_{22}.$$

This implies (since $q_{22} < 0$)

$$c_1 q_{12} > c_2 q_{11},$$

or $\frac{dx_2}{dr} > 0$. Thus, $\frac{dx_2}{dr} < 0$ is not possible.

Case (vii): Suppose $q_{12} > 0$, $c_2 > 0$, and $\frac{dx_1}{dr} > 0$. Then

$$c_1 q_{22} < c_2 q_{12}.$$

From second order conditions,

$$c_1 q_{12} q_{22} < c_2 q_{12}^2 < c_2 q_{11} q_{22}.$$

Since $q_{22} < 0$,

$$c_1 q_{12} > c_2 q_{11}$$

implies $\frac{dx_2}{dr} > 0$. Thus, $\frac{dx_2}{dr} < 0$ is not possible.

Possible (P) outcomes occur when the numerator for $\frac{dx_i}{dr}$ contains terms of opposite sign; then signs depend on the relative sizes of c_1, c_2, q_{12}, q_{22}, and q_{11}. For example, when $q_{12} < 0$ and c_1 and c_2 have the same sign, terms in the numerator of $\frac{dx_i}{dr}$ have opposite signs. The proof for the power function is similar because of a positive relationship between s and r. QED.

Appendix B: Summary of Lehmann's Theorem (from Barlow and Proschan) and Application to Production Under Risk

Definition of Quadrant Dependence

Given a pair (U, V) of random variables with a joint distribution, the pair is *positively* quadrant dependent if

$$P(U \leq u, V \leq v) \geq P(U \leq u)P(V \leq v) \text{ for all } u, v,$$

and *negatively* quadrant dependent if the reverse inequality holds.

Lehmann's Theorems

The following is a special case of Lehmann's Theorem 1:

> *Theorem 1:* Let U, V be positively quadrant dependent and $r(U)$, $s(V)$ be monotone transformations of U, V. Then the pair $(r(U), s(V))$ is positively quadrant dependent if r and s are monotone in the same direction and negatively quadrant dependent if r and s are monotone in opposite directions.

The following is a restatement of Lehmann's Theorem 2:

> *Theorem 2:* If (U, V) are positively quadrant dependent, then $E(UV) \geq E(U)E(V)$; the reverse holds for negative quadrant dependence. Let $r(U)$, $s(V)$ be monotone transformations of U, V. Then, if r and s are monotone in the same direction,
>
> $$E(r(U) \cdot s(V)) \geq E(r(U)) \cdot E(s(V)),$$
>
> and if r and s are monotone in the opposite directions, then
>
> $$E(r(U) \cdot s(V)) \leq E(r(U)) \cdot E(s(V)).$$

Application of Lehmann's Theorems for Production Under Risk

We apply the above theorems in this article by

(*a*) taking $U = V = \theta$ [then (θ, θ) is positively quadrant dependent] and
(*b*) considering $F(x, \theta)$, $F_x(x, \theta)$, $u'(pF(x, \theta) - wx)$ to be transformations of θ.

The following results are obtained by direct application of the above theorems:

(*a*) F_θ, $F_{x\theta}$ have the same sign implies $\text{Cov}(F, F_x) > 0$;
(*b*) F_θ, $F_{x\theta}$ have opposite signs implies $\text{Cov}(F, F_x) < 0$;
(*c*) $u''F_\theta$, $F_{x\theta}$ have the same sign implies $\text{Cov}(u', F_x) > 0$; for $u'' < 0$, this is when F_θ and $F_{x\theta}$ have opposite signs;
(*d*) $u''F_\theta$, $F_{x\theta}$ have opposite signs implies $\text{Cov}(u', F_x) < 0$; for $u'' < 0$, this is when F_θ and $F_{x\theta}$ have the same sign.

Even if F_θ and $F_{x\theta}$ are not monotone, the following weaker results are obtained since $u'(\cdot)$ and its inverse $(u')^{-1}$ are monotone transformations when $u'(\cdot)$ is strictly decreasing:

(*e*) If (F, F_x) is positively quadrant dependent and if u is strictly concave, then (u', F_x) is negatively quadrant dependent and $\text{Cov}(u', F_x) < 0$. The reverse holds if (F, F_x) is negatively quadrant dependent.
(*f*) If (u', F_x) is positively quadrant dependent [so that $\text{Cov}(u', F_x) > 0$] and $u(\cdot)$ is strictly concave, then $\text{Cov}(F, F_x) < 0$; the reverse holds if (u', F_x) is negatively quadrant dependent.