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Expected Utility in Models with Chaos

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Abstract

In this paper, we provide a framework for calculating expected utility in models with chaotic equilibria and consequently a framework for ranking chaos. Suppose that a dynamic economic model's equilibria correspond to orbits generated by a chaotic dynamical system $f : X \rightarrow X$ where X is a compact metric space and f is continuous. The map f could represent the forward dynamics $x_{t+1} = f(x_t)$ or the backward dynamics $x_t = f(x_{t+1})$. If f represents the forward/backward dynamics, the set of equilibria forms a direct/inverse limit space. We use a *natural* f -invariant measure on X to induce a measure on the direct/inverse limit space and show that this induced measure is a *natural* σ -invariant measure where σ is the shift operator. We utilize this framework in the cash-in-advance model of money where f is the backward map to calculate expected utility when equilibria are chaotic.

Keywords: chaos, inverse limits, direct limits, natural invariant measure, cash-in-advance.

JEL: C6, E3, and E4.

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1 Introduction

Consider a dynamic general equilibrium model and two fiscal policies, say A and B , both of which result in chaotic equilibria. Which policy is preferred? What are the welfare consequences of switching from policy A to policy B ? Note that in this model, there is *not* a unique mapping from policies to equilibria (outcomes). Not only are there an infinite number of equilibria associated with each policy, but there is an enormous “variety” of equilibria as well. In this paper, we provide a framework for calculating expected utility in models with chaotic equilibria and consequently a framework for ranking chaos.

Suppose that a dynamic economic model’s equilibria correspond to orbits generated by a chaotic dynamical system $f : X \rightarrow X$ where X is a compact metric space and f is continuous. The map f could represent the forward dynamics $x_{t+1} = f(x_t)$ or the backward dynamics $x_t = f(x_{t+1})$. If f represents the forward/backward dynamics, the set of equilibria forms a direct/inverse limit space. The direct/inverse limit space is a subset of X^∞ : the direct limit space consists of the forward orbits of f and the inverse limit space consists of all the backward orbits of f .¹

When integrating real-valued functions like a utility function, we would like to use a probability measure that respects the dynamics of the model by providing the actual probability (in a frequency sense) of seeing certain Borel sets of X and the direct/inverse limit space. These types of measures are called *natural invariant measures*. An f -invariant measure μ has the property that $\mu[A] = \mu[f^{-1}(A)]$ for every measurable set A . The natural invariant measure μ is an f -invariant measure that respects the dynamics of $f : X \rightarrow X$ in the following sense. If $S \subset X$ is a measurable set and almost all points in X have 40% of their respective orbits in S , then μ assigns S measure 0.4. We use an f -invariant measure on X to induce a measure on the direct/inverse limit space. We show that this induced measure is σ -invariant, where σ is the shift map. Moreover, we show that if the f -invariant measure is a *natural*

¹If f is a non-invertible backward map, we say that the model (or dynamical system) has *backward dynamics*, i.e, the relationship describing the equilibrium dynamics is multi-valued going forward in time, but is single-valued going backward in time. Two such models that may have backward dynamics include the overlapping generations (OLG) model (see Grandmont (1985)) and the cash-in-advance (CIA) model (see Michener and Ravikumar (1998)). Inverse limits is a relatively new approach to analyzing dynamic economic models with backward dynamics. Medio and Raines (2006, 2007) use inverse limits to analyze the long-run behavior of an OLG model with backward dynamics. Kennedy et al. (2007, 2005) investigate the topological structure of the inverse limit space associated with the CIA model of Lucas and Stokey (1987). Kennedy and Stockman (2007) utilize the inverse limit space to show that a multi-valued dynamical system with backward dynamics is chaotic going forward in time if and only if it is chaotic going backward in time.

invariant measure, then the induced measure on the direct/inverse limit space will also be a *natural invariant measure*.

One of the uses of the induced measure on the direct/inverse limit space is to perform integration over this space in a way that is dynamically meaningful. To be more concrete, suppose that Y is the direct/inverse limit space from a DGE model with a representative agent and f has a natural invariant measure μ . Then the utility function of the representative agent can be viewed as a function $W : Y \rightarrow \mathbb{R}$ given by

$$W(\mathbf{x}) := \sum_{t=1}^{\infty} \beta^{t-1} U(x_t).$$

Let m be the induced measure on Y by μ , then expected utility is given by:

$$E[W(\mathbf{x})] := \int_Y W(\mathbf{x}) dm(\mathbf{x}).$$

Note that our utility function is essentially inducing a ranking on direct/inverse limit spaces and consequently providing a means for ranking chaos. In the context of the cash-in-advance model, this could be useful in ranking monetary policies (money growth rates) all of which lead to chaotic equilibria and assessing the welfare consequences of changing the money growth rate.

The paper is organized as follows. In section 2, we discuss some preliminary background from dynamics, direct/inverse limit spaces and natural invariant measures. In section 3 we construct our induced measure on the inverse limit space and show that this induced measure is a natural invariant measure. In section 4 we carry out this analysis for the much simpler direct limit case. The formidable problem associated with numerically calculating these integrals is discussed in Section 5 along with a solution. An application of these tools to the cash-in-advance model is in Section 6 illustrating how this framework can be used for policy analysis. We conclude in Section 7.

2 Preliminaries

In this section we cover some preliminaries on dynamical systems, natural invariant measures and direct/inverse limit spaces.

2.1 Dynamical Systems

Suppose that a dynamic economic model's equilibria correspond to orbits generated by a chaotic dynamical system $f : X \rightarrow X$ where X is a compact metric space with

metric d and f is continuous (we assume throughout that f is also a surjection).

Definition 1. We say f has *sensitive dependence on initial conditions* if there exists a sensitivity constant $\delta > 0$ such that, for any $x \in X$ and any neighborhood N of x , there exists $y \in N$ and an integer $n \geq 0$ such that $d(f^n(x), f^n(y)) > \delta$.

Definition 2. Suppose X is a compact metric space and $f : X \rightarrow X$ is a continuous function. We will say that f is *transitive* if whenever U and V are open sets (non-empty), there exists a positive integer n such that $f^n(U) \cap V \neq \emptyset$.

Definition 3. A point $x \in X$ is a *periodic point of period n* if $f^n(x) = x$ and n is the smallest positive integer with $f^n(x) = x$. For $x \in X$, the *orbit of x under the action of f* is defined by $O_+^f(x) := \{x, f(x), f^2(x), \dots\}$.

One of the more commonly used definitions of chaos is that given by Devaney (2003).

Definition 4. Suppose X is a metric space and $f : X \rightarrow X$ is a map. Then f is *chaotic on X* if (1) f has sensitive dependence on initial conditions; (2) f is transitive; and (3) the periodic points of f are dense in X .

2.2 Limit Spaces

Let $f : X \rightarrow X$ where X is a compact metric space with metric d and f is continuous (we assume throughout that f is also a surjection). The *direct limit space* consists of the forward orbits of f :

$$\varinjlim(X, f) := \{(x_1, x_2, \dots) \in X^\infty \mid x_{i+1} = f(x_i), i \in \mathbb{N}\}. \quad (1)$$

The *inverse limit space* consists of all the backward orbits of f :

$$\varprojlim(X, f) := \{(x_1, x_2, \dots) \in X^\infty \mid x_i = f(x_{i+1}), i \in \mathbb{N}\}. \quad (2)$$

In this context, the space X is called the *factor space* and the function f is called the *bonding map*. We use boldface letters, i.e., \mathbf{x} to denote a point of D or Y , with x_n denoting the n th coordinate of \mathbf{x} . We let π_n denote the n th projection on D or Y , so $\pi_n(\mathbf{x}) = x_n$.

We can use the metric d on X to induce a metric on X^∞ (and D and Y) according to

$$\rho(\mathbf{x}, \mathbf{y}) := \sum_{i=1}^{\infty} \frac{d(x_i, y_i)}{2^{i-1}}. \quad (3)$$

Let $Y := \varprojlim(X, f)$ and $D := \varinjlim(X, f)$. Then the bonding map f induces a natural map $F : Y \rightarrow Y$ given by

$$F((x_1, x_2, \dots)) := (f(x_1), f(x_2), \dots). \quad (4)$$

Note that F is a homeomorphism with inverse given by the shift map:

$$\sigma((x_1, x_2, x_3, \dots)) := (x_2, x_3, x_4, \dots). \quad (5)$$

Note that the induced map on D by f given by $(x_1, x_2, \dots) \rightarrow (f(x_1), f(x_2), \dots)$ is simply the shift map σ . This map is onto (provided f is onto), but it in general is *not* a homeomorphism (unless f is a homeomorphism on X).

Here are some well-known results for these spaces: (1) if X is compact, then the direct/inverse limit space will be compact as well, (2) σ is continuous and (3) the topology on the direct/inverse limit space generated by the metric ρ is equivalent to the product topology on these spaces.

We see that if every forward (backward) orbit of $f : X \rightarrow X$ is an equilibrium in the model, then the set of equilibria is a direct (inverse) limit space. Furthermore, the dynamics of f (or f^{-1}) on X are being captured by the shift map σ on the D (or Y). Examples: (1) The point $x \in X$ has period n orbit under f on X if and only if $(x, f(x), f^2(x), \dots) \in D$ is a period n point under the shift map σ . (2) f (or f^{-1}) is chaotic on X if and only if σ is chaotic on D (or Y).

2.3 Natural Invariant Measures

Suppose X is a compact metric space, $f : X \rightarrow X$ is continuous. Let $\mathcal{B}(X)$ be the σ -algebra of Borel sets. If μ is a measure on $(X, \mathcal{B}(X))$ such that $\mu[f^{-1}(S)] = \mu[S]$ for every set $S \in \mathcal{B}(X)$, then μ is called an *invariant* measure for f .

Let x_0 is a point in X , and $S \in \mathcal{B}(X)$. Define the *fraction of the orbit of x_0 lying in S* by

$$G(x_0, S) = \lim_{n \rightarrow \infty} \frac{\#\{f^i(x_0) \in S : 1 \leq i \leq n\}}{n}, \quad (6)$$

provided this limit exists. We would like for this fraction of orbits to be the same for almost every $x_0 \in X$. However, this method of measuring S may be not be appropriate for certain “borderline” set and may imply a type of discontinuity as the following example illustrates.

Example 1. Let $f : [-1, 1] \rightarrow [-1, 1]$ given by $f(x) = \alpha x$ where $|\alpha| < 1$.

Note that for all $x \in [-1, 1]$, we have $f^n(x) \rightarrow 0$. The invariant measure that captures the dynamics of f is the Dirac measure δ_0 . Note the measure is putting all the measure on the attractor $\{0\}$. However we see that for $S := \{0\}$, we have $G(x_0, S) = 0$ for all x_0 except $x_0 = 0$. Note however that for any open set A with $0 \in A$, we have $G(x_0, A) = 1$ for all $x_0 \in [-1, 1]$. So instead of assigning measure to a set by the fraction of orbit lying in the set S , we will use the fraction of the orbit of x_0 lying in a sequence of open sets *shrinking down* to S .

For $A \subset X$ and r is a positive number, define $D_r(A) := \{x \in X : d(x, y) < r \text{ for some } y \in A\}$. Note that $D_r(A)$ is an open set containing A , and as $r \rightarrow 0$, $D_r(A)$ shrinks down to A . We are now ready to define a natural invariant measure.

Definition 5. Suppose X is compact metric with a regular nonatomic Borel measure ν with full support.² Let $f : X \rightarrow X$ be continuous, x_0 a point in X , and S be a closed subset of X . The *natural measure generated by the map f* is defined by

$$\mu_f(S) := \lim_{r \rightarrow 0} G(x_0, D_r(S)), \quad (7)$$

provided that for ν -a.e. $x_0 \in X$ this limit exists and is the same.

In this context, we call ν the *reference measure*. When the space X has Lebesgue measure λ , typically ν is taken to be λ . However, when f is chaotic, the inverse limit space is topologically complicated and does not have Lebesgue measure even if the factor space does. However, we will see that if μ is nonatomic with full support, then the induced measure on the inverse limit space will also be nonatomic with full support. Consequently, when showing that the induced measure on the inverse limit space is natural, the induced measure itself can be used as the reference measure.

3 Measures on Inverse Limit Spaces

Given a compact metric space X , a continuous map $f : X \rightarrow X$, and an f -invariant measure μ defined on X , we wish to define a measure m on $Y := \varprojlim(X, f)$ invariant relative to the induced shift homeomorphisms F and σ on Y . If the measure μ is, in addition, a natural measure on X , we would like the induced measure m on Y to be natural relative to F and σ .

²A measure ν is *nonatomic* if $\nu(\{x\}) = 0$ for every $x \in X$. We call a measure *strictly positive* if it assigns every (non-empty) open set positive measure. The *support* of ν is the set of $x \in X$ such that every open set containing x has positive measure. A measure ν has *full support* if the support of ν is all of X . Being a strictly positive measure is equivalent to having full support.

3.1 Invariant Measures

If X is a locally compact metric space, denote by \mathcal{C} the collection of all compact subsets of X , by $\mathcal{B}(X)$ the σ -algebra generated by \mathcal{C} . A member of the collection $\mathcal{B}(X)$ is called a *Borel set*, and $\mathcal{B}(X)$ is the *collection of Borel sets* in X . A *content* is a nonnegative, finite, monotone, additive, and subadditive set function defined on the class \mathcal{C} of all compact sets of a locally compact metric space X . A *Borel measure* is a measure μ defined on the collection $\mathcal{B}(X)$ of all Borel sets such that $\mu(C) < \infty$ for every $C \in \mathcal{C}$.

It is straightforward to generate a regular Borel measure from a content on the compact subsets of a locally compact space X , see (Halmos, 1974, Section 53, p. 231). Our goal now is to define a content on the compact subsets of $Y := \varprojlim(X, f)$, where X is a compact metric space and $f : X \rightarrow X$ is continuous, and then use that content to generate a measure on Y . The measure we obtain on Y is not new, see Bochner (1955) or Choksi (1958), but establishing that it is σ -invariant and a natural invariant measure when it is induced by a natural invariant measure is new. We include the construction of this measure for completeness and because in this section we give much of the notation we will use in later proofs. Here is an outline of the construction:

- I. Use μ to define a function Γ on the compact subsets of $Y := \varprojlim(X, f)$. Show Γ is a *content* on the compact subsets of Y .
- II. Show that Γ is a *regular content* on the compact subsets of Y .
- III. Use the regular content Γ to induce an outer measure m^* on the σ -bounded sets of Y .
- IV. Finally, this outer measure m^* is used to induce a regular Borel measure m on $\mathcal{B}(Y)$ that agrees with the regular content Γ on the compact subsets of Y .

Suppose then that X is a compact metric space and $f : X \rightarrow X$ is continuous. Let $Y := \varprojlim(X, f)$. Let n be a nonnegative integer, and let B be a compact subset of Y . Define the *tower sets* B_n for B as follows: define $B_n := \{\mathbf{x} \in Y : \pi_n(\mathbf{x}) := x_n \in \pi_n(B)\}$. Note that $B \subset B_n$ with $\pi_i(B) \equiv \pi_i(B_n)$ for $i = 1, 2, \dots, n$. However, for $j > n$ we may have $\pi_j(B) \subset \pi_j(B_n)$. Note also that the compact subsets of Y are the closed subsets of Y .

Now on to the construction of our induced measure. Suppose that μ is an invariant measure on X . Now define the function Γ on the compact subsets of Y by first

declaring that

$$\Gamma[B_n] := \mu[\pi_n(B)], \quad (8)$$

where B_n is a tower set for the compact set B as defined above. Then define

$$\Gamma[B] := \lim_{n \rightarrow \infty} \Gamma[B_n]. \quad (9)$$

The function Γ is a content on the compact sets of Y as we prove below: p

Lemma 1. *The set function Γ is a content on the compact sets of $Y := \varprojlim(X, f)$.*

Proof. It is immediate from the definition that Γ is nonnegative, finite and monotone as long as μ is. To see that Γ is additive, let K and R be disjoint compact subsets of Y . Then there is an integer N such that for all $m > N$, $\pi_m(K) \cap \pi_m(R) = \emptyset$. Thus,

$$\begin{aligned} \Gamma[K_m \cup R_m] &= \mu[\pi_m(K) \cup \pi_m(R)] = \mu[\pi_m(K)] + \mu[\pi_m(R)] \\ &= \Gamma[K_m] + \Gamma[R_m] \end{aligned}$$

for all $m > N$. Hence,

$$\Gamma[K \cup R] = \lim_{n \rightarrow \infty} \Gamma[K_n \cup R_n] = \lim_{n \rightarrow \infty} \Gamma[K_n] + \lim_{n \rightarrow \infty} \Gamma[R_n] = \Gamma[K] + \Gamma[R],$$

and Γ is additive. The proof that Γ is subadditive is similar. \square

A measure ν is *outer regular* provided that $\nu[E] = \inf\{\nu[U] : E \subset U \text{ and } U \text{ is open}\}$. A measure ν is *inner regular* provided that $\nu[E] = \sup\{\nu[C] : C \subset E \text{ and } C \text{ is compact}\}$. A measure ν on a space Z is *regular* provided that it is both inner regular and outer regular.

A content Γ on the compact sets \mathcal{C} is *regular* provided

$$\Gamma[C] = \inf\{\Gamma[D] : C \subset D^\circ \subset D \in \mathcal{C}\}. \quad (10)$$

Let U be an open set and define the *inner content* Γ_* induced by the content Γ by

$$\Gamma_*[U] = \sup\{\Gamma[C] : U \supseteq C \text{ and } C \in \mathcal{C}\}. \quad (11)$$

Suppose X is a metric space and $E \subset X$. We say that E is *σ -compact* provided there is a collection $\{C_i\}_{i=1}^\infty$ of compact sets such that $E = \cup_{i=1}^\infty C_i$. We say that E is *σ -bounded* provided there is a collection $\{C_i\}_{i=1}^\infty$ of compact sets such that $E \subset \cup_{i=1}^\infty C_i$. Given a σ -bounded set E we define

$$m^*[E] = \inf\{\Gamma_*[U] : E \subset U \text{ and } U \text{ is open}\}, \quad (12)$$

and we call m^* the *outer measure induced by the content* Γ .

Lemma 2. *With the assumptions as above, Γ is a regular content on Y provided μ is a regular, invariant measure on X .*

Proof. Let $C \in \mathcal{C}$, the collection of all compact subsets of Y . Let $\epsilon > 0$ and choose a positive integer n such that $|\Gamma[C_n] - \Gamma[C]| < \epsilon/2$, where the C_n 's are the tower sets for C . Since μ is a regular measure on X , there is a compact subset D_n of X such that $D_n^\circ \supseteq \pi_n(C_n)$ and $|\mu[D_n] - \mu[\pi_n(C_n)]| < \epsilon/2$. Let $D = \pi_n^{-1}(D_n)$. Since π_n is continuous and D is compact, $C \subset C_n \subset \pi_n^{-1}(D_n^\circ) \subset D^\circ \subset D$. Also, $|\Gamma[C] - \Gamma[D]| \leq |\Gamma[C] - \Gamma[C_n]| + |\Gamma[C_n] - \Gamma[D]| \leq \epsilon/2 + |\mu[\pi_n(C_n) - \mu[D_n]| = \epsilon$. Thus, Γ is regular. \square

Theorem 1 (Halmos (1974), Theorem E, p. 234). *If m^* is the outer measure induced by a content Γ , then the set function defined for every Borel set E by $m[E] := m^*[E]$ is a regular Borel measure.*

Using this measure m from these definitions to calculate $m[E]$ might be difficult. However, since Γ is a regular content we can use the following result:

Theorem 2 (Halmos (1974), Theorem A, p. 235). *If m is the Borel measure induced by a regular content Γ , then $m[C] = \Gamma[C]$ for each compact set C .*

Thus, these definitions and theorems lead us from the f -invariant measure μ on X to a content Γ on Y , and finally to a regular Borel measure m on Y , and from this last theorem we see that m behaves exactly like Γ on the compact sets. We primarily work with compact subsets K of Y because of the fact that $m[K] = \Gamma[K]$. However, as the next few lemmas show, if $K \subseteq Y$ is Borel and it projects to a Borel subset of X then we can approximate $m[K]$.

Lemma 3. *Let $K \subset Y$ be Borel. Then $m[K] \leq \mu[\pi_n(K)]$ for all nonnegative integers n such that $\pi_n[K]$ is Borel.*

Proof. If K is compact, this follows immediately from the definition of Γ and the fact that $\Gamma[K] = m[K]$.

Next assume that K is open. Then

$$m[K] = \sup\{m[C] : C \subseteq K, C \text{ is compact}\}.$$

Since each $m[C] \leq \mu[\pi_n(C)]$, it follows that

$$\begin{aligned} m[K] &= \sup\{m[C] : C \subseteq K, C \text{ is compact}\} \\ &\leq \sup\{\mu[\pi_n(C)] : C \subseteq K, C \text{ is compact}\}. \end{aligned}$$

Since π_n is continuous, $\pi_n(C) \subset \pi_n(K)$. Thus,

$$\{\pi_n(C) : C \subseteq K, C \text{ is compact}\} \subseteq \{R : R \subseteq \pi_n(K), R \text{ is compact}\},$$

and since $m[C] \leq \mu[\pi_n(C)]$, we have

$$\begin{aligned} m[K] &\leq \sup\{\mu[\pi_n(C)] : C \subseteq K, C \text{ is compact}\} \\ &\leq \sup\{\mu[R] : R \subseteq \pi_n(K), R \text{ is compact}\}. \end{aligned}$$

But since μ is also a regular Borel measure and $\pi_n[K]$ is Borel,

$$\mu[\pi_n(K)] = \sup\{\mu[R] : R \subseteq \pi_n(K), R \text{ is compact}\}.$$

Hence, $m[K] \leq \mu[\pi_n(K)]$ for each n .

Finally let K be any Borel set and recall that

$$m[K] = \inf\{m[U] : U \supseteq K, U \text{ is open in } Y\}.$$

Let $V \supseteq \pi_n(K)$ be open in X . Then $\pi_n^{-1}(V) \supseteq K$ is also open in Y . Thus,

$$\begin{aligned} \{\pi_n^{-1}(V) : V \supseteq \pi_n(K), V \text{ is open in } X\} &\subseteq \\ \{U : U \supseteq K, U \text{ is open in } Y\}, & \end{aligned}$$

and

$$\begin{aligned} \inf\{m[\pi_n^{-1}(V)] : V \supseteq \pi_n(K), V \text{ is open in } X\} \\ \geq \inf\{m[U] : U \supseteq K, U \text{ is open in } Y\} = m[K]. \end{aligned}$$

But by the previous case

$$\mu[V] = \mu[\pi_n \circ \pi_n^{-1}(V)] \geq m[\pi_n^{-1}(V)].$$

Hence,

$$\begin{aligned} \mu[\pi_n(K)] &= \inf\{\mu[V] : V \supseteq \pi_n(K), V \text{ is open in } X\} \\ &\geq \inf\{m[\pi_n^{-1}(V)] : V \supseteq \pi_n(K), V \text{ is open in } X\} \\ &\geq m[K]. \end{aligned}$$

□

Lemma 4. *Let $K \subset X$ be a Borel set. Then, for any nonnegative integer n , $\mu[K] = m[\pi_n^{-1}(K)]$.*

Proof. Let n be a nonnegative integer. Since Γ and m agree on the compact sets of Y , we need only consider open sets and then Borel sets in the proof of this lemma.

Let U be an open subset of X , and denote $\pi_n^{-1}(U)$ by \widehat{U} . By the previous lemma, $\mu[U] \geq m[\widehat{U}]$. Let $\{L_i\}_{i=1}^{\infty}$ be a collection of compact subsets of X such that $L_i \subset U$ for each i and $\mu[U] = \sup\{\mu[L_i] : i \in \mathbb{N}\}$. Then $m[\widehat{U}] = \sup\{m[K] : K \subseteq \widehat{U}, K \text{ is compact}\} \geq \sup\{m[\pi_n^{-1}(L_i)] : i \in \mathbb{N}\}$, because $\{K : K \subseteq \widehat{U}, K \text{ is compact}\} \supseteq \{\pi_n^{-1}(L_i) : i \in \mathbb{N}\}$. Since

$$\sup\{m[\pi_n^{-1}(L_i)] : i \in \mathbb{N}\} = \sup\{\Gamma[\pi_n^{-1}(L_i)] : i \in \mathbb{N}\} = \sup\{\mu[L_i] : i \in \mathbb{N}\} = \mu[U],$$

we see that $m[\widehat{U}] \geq \mu[U]$. Thus, $m[\widehat{U}] = \mu[U]$.

Let $K \subseteq X$ be a Borel set. Let $\widehat{K} = \pi_n^{-1}(K)$. By the previous lemma, $m[\widehat{K}] \leq \mu[K]$. Let $\{Z_i\}_{i=1}^{\infty}$ be a collection of compact subsets of X with $Z_i \subseteq K$ for all $i \in \mathbb{N}$ and with $\mu[K] = \sup\{\mu[Z_i] : i \in \mathbb{N}\}$. For each i , let $\widehat{Z}_i = \pi_n^{-1}(Z_i)$. By the previous paragraph, $\mu[Z_i] = m[\widehat{Z}_i]$ for all $i \in \mathbb{N}$, and since $\{R : R \subseteq \widehat{K}, R \text{ is compact in } Y\} \supseteq \{\widehat{Z}_i : i \in \mathbb{N}\}$,

$$m[K] = \sup\{m[R] : R \subseteq \widehat{K}, R \text{ is compact in } Y\} \geq \sup\{m[\widehat{Z}_i] : i \in \mathbb{N}\}.$$

But $\mu[K] = \sup\{\mu[Z_i] : i \in \mathbb{N}\} = \sup\{m[\widehat{Z}_i] : i \in \mathbb{N}\}$. Thus, $m[\widehat{K}] = \mu[K]$. \square

3.2 Natural Invariant Measures

For the next few propositions, lemmas, and theorems we assume that X is a compact metric space, $f : X \rightarrow X$ is continuous, and μ is an invariant measure on X with respect to f such that μ is regular and nonatomic with $\mu(O) > 0$ for each nonempty open set O in X . We suppose further that $Y = \varinjlim(X, f)$, and m denotes the measure induced by μ .

Theorem 3. *The induced measure, m , is F -invariant.*

Proof. Let $K \subseteq Y$ be closed. Then since Y is compact, K is compact. By definition, if $\mathbf{x} = (x_0, x_1, \dots) \in Y$ then $F^{-1}(\mathbf{x}) = (x_1, x_2, \dots)$. Thus $\pi_{n+1}[K] = \pi_n \circ F^{-1}[K]$. So if $K_n = \pi_n^{-1} \circ \pi_n[K]$ is the n th tower set for K , it is the $n-1$ 'st tower set for $F^{-1}[K]$. By definition, $m[K] = \lim_{n \rightarrow \infty} m[K_n] = \lim_{n-1 \rightarrow \infty} m[K_{n-1}] = m[F^{-1}(K)]$. Thus m is F -invariant. \square

Note that since F is a homeomorphism, m is also σ -invariant as well. The next two propositions say that if the measure μ is strictly positive (or equivalently has

full support) or nonatomic then the induced measure m has these properties as well. However, first we will need a lemma from Ingram and Mahavier (2004). The lemma essentially says that one can guarantee points in the inverse limit space are close to each other if in some factor space their projections are close to each other.

Lemma 5 (Ingram and Mahavier (2004)). *Suppose X is a compact metric space, $f : X \rightarrow X$ is continuous, and $Y := \varprojlim(X, f)$. Let $\epsilon > 0$. Then there is a positive number δ and a positive integer n such that for every $x \in X$, $\pi_n^{-1}(D_\delta(x))$ has diameter less than ϵ .*

Proposition 1. *If U is an open nonempty subset of Y , $m(U) > 0$.*

Proof. Suppose not, i.e., suppose U is a nonempty open set in Y and $m(U) = 0$. Applying Lemma 5, for each $\epsilon > 0$, there are a positive number δ and a positive integer n such that for each x in X , $\text{diam}(\pi_n^{-1}(D_\delta(x))) < \epsilon$. Therefore we can find $x \in X$, an integer n , and a positive number δ such that $\pi_n^{-1}(\overline{D_\delta(x)}) \subset U$. Then $m[\pi_n^{-1}(\overline{D_\delta(x)})] = 0$. However, $m[\pi_n^{-1}(\overline{D_\delta(x)})] = \mu[\overline{D_\delta(x)}]$, by Lemma 4 and this means that $m[\pi_n^{-1}(\overline{D_\delta(x)})] = \mu[\overline{D_\delta(x)}] > 0$. This is a contradiction to μ being regular. \square

Proposition 2. *If μ is nonatomic, then m is nonatomic.*

Proof. Suppose not, i.e., m is atomic. Let $\hat{x} = (x_0, x_1, \dots) \in \varprojlim(X, f)$ be such that $m[\{\hat{x}\}] > 0$. Since $\{\hat{x}\}$ is a compact set we have that $\Gamma[\{\hat{x}\}] = m[\{\hat{x}\}] > 0$. Let A_n be the associated tower sets for $\{\hat{x}\}$, i.e. $A_n = \{\hat{z} \in \varprojlim(X, f) : z_n = x_n\}$. By definition, $\Gamma[A_n] \rightarrow \Gamma[\{\hat{x}\}]$, and since $\Gamma[\{\hat{x}\}] > 0$, there is some $N \in \mathbb{N}$ such that $\Gamma[A_n] > 0$ for all $n \geq N$. This implies that $\mu[\{x_n\}] = \Gamma[A_n] > 0$ for all $n \geq N$, and hence μ is atomic – a contradiction. \square

Proposition 3. *If Z is a measurable subset of X such that $\mu(Z) = 0$, then $\pi_n^{-1}(Z)$ has empty interior for each nonnegative integer n and $m[\pi_n^{-1}(Z)] = 0$.*

Proof. That $m[\pi_n^{-1}(Z)] = 0$ follows from Lemma 4. That $\pi_n^{-1}(Z)$ has empty interior follows from Proposition 1. \square

Proposition 4. *Suppose that $\mathbf{x} \in Y$, O is open in X , and k is a nonnegative integer. Then*

$$\lim_{n \rightarrow \infty} \frac{\#\{F^i(\mathbf{x}) \in \pi_k^{-1}(O) : 0 \leq i \leq n\}}{n} = \lim_{n \rightarrow \infty} \frac{\#\{f^i(x_k) \in O : 0 \leq i \leq n\}}{n},$$

provided one of these limits exists.

Proof. Note that for $\mathbf{x} = (x_0, x_1, \dots) \in Y$, $F^i(\mathbf{x}) = (f^i(x_0), f^i(x_1), \dots)$. Thus, $F^i(\mathbf{x}) \in \pi_k^{-1}(O)$ if and only if $f^i(x_k) \in O$. It follows that $\#\{F^i(\mathbf{x}) \in \pi_k^{-1}(O) : 0 \leq i \leq n\} = \#\{f^i(x_k) \in O : 0 \leq i \leq n\}$ for each n , and the result follows. \square

Notation: For $\mathbf{x} \in Y$, A a measurable subset of Y , let

$$\tilde{G}(\mathbf{x}, A) = \lim_{n \rightarrow \infty} \frac{\#\{F^i(\mathbf{x}) \in A : 0 \leq i \leq n\}}{n},$$

provided this limit exists. For $x \in X$, A a measurable subset of X , let

$$G(x, A) = \lim_{n \rightarrow \infty} \frac{\#\{f^i(x) \in A : 0 \leq i \leq n\}}{n},$$

provided this limit exists.

Recall the Birkhoff Ergodic Theorem [see, for example Katok and Hasselblatt (1995)]:

Theorem 4 (Birkhoff Ergodic Theorem). *Let $T : (X, \mu) \rightarrow (X, \mu)$ be a measure-preserving transformation of a probability space, $\phi \in L^1(X)$. Then for μ -a.e. $x \in X$ the following time-average exists*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi(T^k(x)).$$

By construction (Y, m) is a measure space and since the induced homeomorphism F is continuous it is a measure-preserving transformation. Also the characteristic function χ_A of a measurable subset A of Y is in $L^1(Y)$ we see that the limit $\tilde{G}(\mathbf{x}, A)$ exists for m -a.e. $\mathbf{x} \in Y$.

Lemma 6. *Assume that X is a compact space with Lebesgue measure λ . Also assume that μ is a natural invariant measure on X . If A is a closed tower set in Y , then there is some nonnegative integer r such that $A = \{\mathbf{y} \in Y : y_r \in \pi_r(A)\}$ and $\mu[\pi_r(A)] = m[A]$. Furthermore, for m -a.e. $\mathbf{x} \in Y$, $\mu[\pi_r(A)] = \lim_{\epsilon \rightarrow 0} \tilde{G}(\mathbf{x}, \pi_r^{-1}(D_\epsilon(\pi_r(A)))) = \lim_{\epsilon \rightarrow 0} G(x_r, D_\epsilon(\pi_r(A))) = m[A]$.*

Proof. Suppose $\epsilon > 0$. Since μ is a natural measure, there is a measurable set Z of measure 0 in X such that if $x \notin Z$, then for every closed set S in X , $\mu[S] = \lim_{\epsilon \rightarrow 0} G(x, D_\epsilon(S))$. By Proposition 4, $Z_Y := \cup_{k=0}^{\infty} \pi_k^{-1}(Z)$ is a measurable set of measure 0 in Y .

Suppose then that $\mathbf{x} \in Y \setminus Z_Y$. Then for each k , $\pi_k(\mathbf{x}) = x_k \notin Z$. Thus, $x_m \notin Z$. Then, by the previous proposition, for each $\varepsilon > 0$ and nonnegative integer n ,

$$\#\{f^i(x_r) \in D_\varepsilon(\pi_r(A)) : 0 \leq i \leq n\} = \#\{F^i(\mathbf{x}) \in \pi_r^{-1}(D_\varepsilon(\pi_r(A))) : 0 \leq i \leq n\}.$$

Thus,

$$\begin{aligned} G(x_r, D_\varepsilon(\pi_r(A))) &= \lim_{n \rightarrow \infty} \frac{\#\{f^i(x_r) \in D_\varepsilon(\pi_r(A)) : 0 \leq i \leq n\}}{n} \\ &= \tilde{G}(\mathbf{x}, \pi_r^{-1}(D_\varepsilon(\pi_r(A)))) \\ &= \lim_{n \rightarrow \infty} \frac{\#\{F^i(\mathbf{x}) \in \pi_r^{-1}(D_\varepsilon(\pi_r(A))) : 0 \leq i \leq n\}}{n}. \end{aligned}$$

Since $\mu[\pi_r(A)] = \lim_{\varepsilon \rightarrow 0} G(x_r, D_\varepsilon(\pi_r(A)))$ and $\tilde{G}(\mathbf{x}, \pi_r^{-1}(D_\varepsilon(\pi_r(A)))) = G(x_r, D_\varepsilon(\pi_r(A)))$ for each $\varepsilon > 0$. Because A is a tower set, the result follows. \square

Lemma 7. *Assume that X is a compact space with Lebesgue measure, λ . Also assume that μ is a natural invariant measure on X . Let $A \subseteq Y$ be closed. Then $\lim_{\varepsilon \rightarrow 0} \tilde{G}(\mathbf{x}, D_\varepsilon(A)) \leq m(A)$ for m -a.e. point in Y .*

Proof. For each $r \in \mathbb{N}$ let A_r be the r th tower set for A , i.e. $A_r = \pi_r^{-1} \circ \pi_r(A) = \{\mathbf{x} \in Y : x_r \in \pi_r(A)\}$. Let $Z_r \subseteq X$ be a set of Lebesgue measure zero such that if $x \notin Z_r$ then $\lim_{\delta \rightarrow 0} G(x, D_\delta(\pi_r(A_r))) = \mu[\pi_r(A_r)]$. By Lemma 6, for all $\mathbf{x} \notin \pi_r^{-1}(Z_r)$, $\lim_{\delta \rightarrow 0} \tilde{G}(\mathbf{x}, \pi_r^{-1}(D_\delta(\pi_r(A_r)))) = m[A_r]$.

Let $\mathbf{x} \notin \bigcup_{t=1}^{\infty} \pi_t^{-1}(Z_t) = Z$, and let $r \in \mathbb{N}$. For all sufficiently small $\varepsilon > 0$ there is a $\delta_\varepsilon > 0$ such that $D_\varepsilon(A) \subseteq \pi_r^{-1}(D_{\delta_\varepsilon}(\pi_r(A)))$. Fix $\varepsilon > 0$ small enough and $\delta_\varepsilon > 0$ as above.

Define

$$L_n = \#\{F^i(\mathbf{x}) \in D_\varepsilon(A) : 0 \leq i \leq n\}$$

and

$$M_n = \#\{F^i(\mathbf{x}) \in \pi_r^{-1}(D_{\delta_\varepsilon}(\pi_r(A))) : 0 \leq i \leq n\}$$

Since $D_\varepsilon(A) \subseteq \pi_r^{-1}(D_{\delta_\varepsilon}(\pi_r(A)))$ we see that $0 \leq L_n \leq M_n$. Obviously, $0 \leq \frac{L_n}{n} \leq \frac{M_n}{n}$, and by Proposition 4 $\lim_{n \rightarrow \infty} \frac{M_n}{n} = \tilde{G}(\mathbf{x}, \pi_r^{-1}(D_{\delta_\varepsilon}(\pi_r(A))))$ exists. By the Birkhoff Ergodic Theorem we see that $\lim_{n \rightarrow \infty} \frac{L_n}{n}$ exists. So $0 \leq \lim_{n \rightarrow \infty} \frac{L_n}{n} \leq \lim_{n \rightarrow \infty} \frac{M_n}{n}$, and thus $0 \leq \lim_{n \rightarrow \infty} \frac{L_n}{n} = \tilde{G}(\mathbf{x}, D_\varepsilon(A))$ exists and is less than or equal to $\tilde{G}(\mathbf{x}, \pi_r^{-1}(D_{\delta_\varepsilon}(\pi_r(A))))$.
Let

$$N_\varepsilon = \tilde{G}(\mathbf{x}, D_\varepsilon(A)) \text{ and } P_\delta = \tilde{G}(\mathbf{x}, \pi_r^{-1}(D_\delta(\pi_r(A)))).$$

By above, for all sufficiently small ϵ there is a δ_ϵ with $N_\epsilon \leq P_{\delta_\epsilon}$. As $\epsilon \rightarrow 0$ we can choose δ_ϵ so that $\delta_\epsilon \rightarrow 0$. By Lemma 6, $m[A_r] = \lim_{\delta \rightarrow 0} P_\delta$, and thus we can pass to a subset of the δ 's $m[A_r] = \lim_{\epsilon \rightarrow 0} P_{\delta_\epsilon} \geq \lim_{\epsilon \rightarrow 0} N_\epsilon$. Thus, for each $r \in \mathbb{N}$,

$$m[A_r] \geq \lim_{\epsilon \rightarrow 0} \tilde{G}(\mathbf{x}, D_\epsilon(A))$$

This implies that $m[A] \geq \lim_{\epsilon \rightarrow 0} \tilde{G}(\mathbf{x}, D_\epsilon(A))$ for all $\mathbf{x} \notin Z$. □

Theorem 5. *Assume that X is a compact space with a Lebesgue measure λ . Also assume that μ is a natural invariant measure on X that is nonatomic with full support. Then m is a natural invariant measure on Y .*

Proof. We showed in Theorem 3 that m is F -invariant. By Propositions 1 and 2, m is nonatomic with full support.

Suppose A is a closed subset of Y and, for each nonnegative integer n , A_n denotes the n th tower set for A . There is a measurable set $Z_n \subseteq X$ of Lebesgue measure zero such that if $\mathbf{x} \in Y \setminus \pi_n^{-1}[Z_n]$, then $m[A_n] = \lim_{\epsilon \rightarrow 0} \tilde{G}(\mathbf{x}, \pi_n^{-1}(D_\epsilon(\pi_n(A))))$. Let $\mathbf{x} \notin \bigcup_{n=1}^{\infty} \pi_n^{-1}[Z_n] = Z$. Let $n, r, k \in \mathbb{N}$ and define

$$Q_n = \tilde{G}(\mathbf{x}, D_{1/n}(A))$$

and

$$P_{r,k} = \tilde{G}(\mathbf{x}, \pi_r^{-1}(D_{1/k}(\pi_r(A))))$$

Notice that if we fix $r \in \mathbb{N}$ then $m(A_r) = \lim_{k \rightarrow \infty} P_{r,k}$ and $m(A) = \lim_{r \rightarrow \infty} m(A_r) = \lim_{r \rightarrow \infty} \lim_{k \rightarrow \infty} P_{r,k}$. By Lemma 5, for each $n \in \mathbb{N}$, there is some $r_n, k_n \in \mathbb{N}$ so that $\pi_{r_n}^{-1}(D_{1/k_n}(\pi_{r_n}(A))) \subseteq D_{1/n}(A)$. For this $n, r_n, k_n \in \mathbb{N}$ we have

$$P_{r_n, k_n} = \tilde{G}(\mathbf{x}, \pi_{r_n}^{-1}(D_{1/k_n}(\pi_{r_n}(A)))) \leq \tilde{G}(\mathbf{x}, D_{1/n}(A)) = Q_n$$

As $n \rightarrow \infty$, $r_n, k_n \rightarrow \infty$. So

$$m(A) = \lim_{n \rightarrow \infty} P_{r_n, k_n} \leq \lim_{n \rightarrow \infty} Q_n$$

Since $\lim_{n \rightarrow \infty} Q_n = \lim_{\epsilon \rightarrow 0} \tilde{G}(\mathbf{x}, D_\epsilon(A))$ we see that

$$m(A) \leq \lim_{\epsilon \rightarrow 0} \tilde{G}(\mathbf{x}, D_\epsilon(A))$$

Combining this with the previous lemma,

$$m(A) = \lim_{\epsilon \rightarrow 0} \tilde{G}(\mathbf{x}, D_\epsilon(A))$$

for every $\mathbf{x} \in Y \setminus Z$. □

Corollary 1. *Since $F : Y \rightarrow Y$ is a homeomorphism, m is also a natural invariant measure for σ .*

4 Measures on Direct Limit Spaces

Suppose (X, d) be a compact metric space and $f : X \rightarrow X$ continuous. Let $\mathcal{B}(X)$ denote the Borel sets of X . Suppose μ is an f -invariant measure defined on the $\mathcal{B}(X)$. Let $D := \varinjlim(X, f)$. We wish to define a measure m on $\mathcal{B}(D)$ that is σ -invariant where σ is the shift map on D . Furthermore, if μ is a natural invariant measure, we would like the induced measure m to be a natural invariant measure as well.

Carrying out these tasks is fairly straightforward in the direct limit case since f and σ are conjugate as the next theorem establishes.

Theorem 6. *Suppose $f : X \rightarrow X$ is continuous and X is a compact metric space. Let $D := \varinjlim(X, f)$ with induced metric ρ . Then X and D are homeomorphic with a homeomorphism given by $H : X \rightarrow D$ defined by $H(x) := (x, f(x), f^2(x), \dots)$ with $H^{-1} \equiv \pi_1 : D \rightarrow X$ defined by $\pi_1((x_1, x_2, \dots)) := x_1$. Furthermore, $f : X \rightarrow X$ and the shift map $\sigma : D \rightarrow D$ are conjugate with $f = H^{-1} \circ \sigma \circ H$.*

Corollary 2. *Suppose $f : X \rightarrow X$ is continuous and X is a compact metric space. Let $D := \varinjlim(X, f)$ with induced metric ρ . Then (D, ρ) is a compact metric space.*

4.1 Invariant Measures

Note that since H is a homeomorphism, we can use H to induce a measure m on $\mathcal{B}(D)$ given by

$$m[B] := \mu[H^{-1}(B)] \equiv \mu[\pi_0(B)]. \quad (13)$$

The next theorem establishes that an invariant measures induces an invariant measure for a conjugate dynamical system.

Theorem 7. *Suppose that X and Y are compact metric spaces, $f : X \rightarrow X$ and $g : Y \rightarrow Y$ are continuous and conjugate with conjugacy given by $h : X \rightarrow Y$. Then if μ_f is an f -invariant measure on the Borel sets $\mathcal{B}(X)$ then the induced measure on the Borel sets $\mathcal{B}(Y)$ given by $\mu_g[B] := \mu_f[h^{-1}(B)]$ is g -invariant.*

Proof. Let B be a closed set in Y . Note that since h is a homeomorphism we have $\mu_g[h(A)] = \mu_f[A]$ for $A \in \mathcal{B}(X)$ as well. Since $g^{-1}(B) \equiv h \circ f^{-1} \circ h^{-1}(B)$ we have

$$\mu_g[B] = \mu_f[h^{-1}(B)] = \mu_f[f^{-1} \circ h^{-1}(B)] = \mu_g[h \circ f^{-1} \circ h^{-1}(B)] = \mu_g[g^{-1}(B)].$$

□

Corollary 3. *Suppose $f : X \rightarrow X$ is continuous, X is compact, μ is a Borel f -invariant measure, D is the direct limit space and $\sigma : D \rightarrow D$ the shift map. Then the induced measure on $\mathcal{B}(D)$ given by $m[B] := \mu[\pi_0(B)]$ is σ -invariant.*

4.2 Natural Invariant Measures

Theorem 8. *Suppose we have $(X, f, \mathcal{B}(X), \mu)$ and $(Y, g, \mathcal{B}(Y), m)$ where X and Y are compact metric spaces, $f : X \rightarrow X$ and $g : Y \rightarrow Y$ are continuous, f and g are conjugate with conjugacy $h : X \rightarrow Y$, μ is an f -invariant Borel measure and m is the Borel g -invariant measure induced on $\mathcal{B}(Y)$ by h , i.e., $m[B] = \mu[h^{-1}(B)]$. If μ is a natural invariant measure, then m is a natural invariant measure.*

Proof. Let λ be Lebesgue measure on $\mathcal{B}(X)$ and $\tilde{\lambda}$ be Lebesgue measure on $\mathcal{B}(Y)$ given by $\tilde{\lambda}[B] := \lambda[h^{-1}(B)]$. Since μ is a natural invariant measure, there exists a measurable set P_X with Lebesgue measure zero where $\lim_{r \rightarrow 0} G(x, S^r)$ exists and is the same for $x \in X \setminus P_X$. Let $S \subset Y$ be closed. We define

$$m_g[S] := \lim_{r \rightarrow 0} G(y, S^r).$$

We will show that this limit exists $\tilde{\lambda}$ -almost every y in D and equals $m[S]$. Let $P_Y := h(P_X)$. Then $h^{-1}(P_Y) = P_X$ and $\tilde{\lambda}(P_Y) = \lambda[h^{-1}(P_Y)] = \lambda[P_X] = 0$. Let $y \in Y \setminus P_Y$ and $x = h^{-1}(y)$. N.B. $y \in Y \setminus P_Y$ iff $x = h^{-1}(y) \in X \setminus P_X$. We also have $x \in h^{-1}(S^r)$ iff $h(x) \in S^r$ which implies $f^j(x) \in h^{-1}(S^r)$ iff $h \circ f^j(x) \in S^r$ which is equivalent to $h \circ f^j \circ h^{-1}(y) \in S^r$ or $g^j(y) \in S^r$. This gives

$$\#\{f^j(x) \in h^{-1}(S^r) : 1 \leq j \leq n\} = \#\{g^j(y) \in S^r : 1 \leq j \leq n\}.$$

This implies that $G(y, S^r)$ exists and is the same for all $y \in Y \setminus P_Y$. Furthermore, $G(y, S^r) = \mu[h^{-1}(S^r)]$. So we have

$$m_g[S] := \lim_{r \rightarrow 0} G(y, S^r) = \lim_{r \rightarrow 0} \mu[h^{-1}(S^r)] = \mu[h^{-1}(S)] = m[S].$$

□

Corollary 4. *Suppose $f : X \rightarrow X$ is continuous, X is compact, μ is a Borel f -invariant measure, D is the direct limit space and $\sigma : D \rightarrow D$ the shift map. If μ is a natural invariant measure on $\mathcal{B}(X)$, then the induced measure m is a natural invariant measure on $\mathcal{B}(D)$.*

5 Computational Issues

When approximating the integral over the direct/inverse limit space, one essentially wants to do what one does with a Riemann integral:

1. Form a partition: $\{E_1, E_2, \dots, E_N\}$.
2. Pick a representative point: $x_i \in E_i$, $i = 1, 2, \dots, N$.
3. Add up the “area of the rectangles”:

$$\int_X G(x)m(dx) \approx \sum_{i=1}^N G(x_i)m(E_i).$$

When the bonding map f is chaotic, the property of sensitive dependence on initial conditions may make this approach computationally difficult. The problem is that even if two points in a factor space are close together, as one moves through the direct/inverse limit space, these points may be pulled far apart. As an illustration, take the tent map $f : [0, 1] \rightarrow [0, 1]$ given by $f(x) := 2x$ for $x \in [0, 1/2]$ and $f(x) := 2(1-x)$ for $x \in [1/2, 1]$. Consider a fixed partition $\{E_1, E_2, \dots, E_N\}$ of $[0, 1]$ given by $E_i := [(i-1)/N, i/N]$ for $i = 1, 2, \dots, N-1$ and $E_N := [(N-1)/N, 1]$. Then there exists a k such that $f^k(E_i) = [0, 1]$ for $i = 1, 2, \dots, N$. Heuristically, f^k is becoming very “erratic” for large k . Holding the partition fixed, consider for $k \in \mathbb{N}$:

$$\int_X f^k(x)\mu(dx) \approx \sum_{i=1}^N f^k(x_i)\mu(E_i).$$

For some k , there will always be two different points $x_i, y_i \in E_i$ with $|f^k(x_i) - f^k(y_i)| = 1$. This implies that the area of the rectangle may be very sensitive to the representative point chosen:

$$\left| \sum_{i=1}^N f^k(x_i)\mu(E_i) - \sum_{i=1}^N f^k(y_i)\mu(E_i) \right| = 1.$$

However, for a given k and $\epsilon > 0$, there will always exist a partition $\{E_1, E_2, \dots, E_{N_k}\}$ (sufficiently fine) so that

$$\left| \sum_{i=1}^N f^k(x_i)\mu(E_i) - \sum_{i=1}^N f^k(y_i)\mu(E_i) \right| < \epsilon,$$

for any $x_i, y_i \in E_i$. However, N_k may need to be so large to make this computationally very demanding.

5.1 Inverse Limit Case

Let $f : X \rightarrow X$ be a continuous map on a compact metric space X . Let μ be an invariant measure of f and m be the induced invariant measure on the inverse limit space $Y := \varprojlim(X, f)$. Let $W : Y \rightarrow \mathbb{R}$ be a continuous real-valued function on Y defined by

$$W(\mathbf{x}) := \sum_{t=1}^{\infty} \beta^{t-1} U(x_t),$$

where $U : X \rightarrow \mathbb{R}$ is continuous. The proof used to construct the measure on the inverse limit space in Section 3 suggests the following algorithm for approximating the integral

$$\int_Y W(\mathbf{x}) m(d\mathbf{x}).$$

First, we truncate the infinite sum at some value T . Next, we grid the state space (factor space) at time T into a partition with N pieces $\{I_j\}_{j=1}^N$. Each of these pieces can be used to partition the inverse limit space according to

$$A_j := \{\mathbf{x} \in Y \mid \pi_T(\mathbf{x}) \in I_j\}$$

Each of these “tunnels” in the inverse limit space has measure $\mu(I_j)$. Let H_j denote a truncated tunnel, i.e., the embedding of A_j into \mathbb{R}^T . Then our integral is approximated by

$$\int_Y W(\mathbf{x}) m(d\mathbf{x}) \approx \sum_{t=1}^T \sum_{j=1}^N \beta^{t-1} U(x_t^j) \mu(I_j),$$

where $\{x_0^j, x_1^j, \dots, x_T^j\} \in H_j$. One issue that needs to be addressed is how sensitive is the value of the approximation to the length T , the number (and size) of subintervals and the selections of the point $\{x_0^j, x_1^j, \dots, x_T^j\} \in H_j$. As mentioned earlier, the selection point is potentially very problematic due to the sensitive dependence on initial conditions inherent in the map f . We can have two different points $x_0, y_0 \in I_j$ (close), but $f^{T-1}(x_0)$ and $f^{T-1}(y_0)$ might be far apart. Ideally, one would want the projection of the tunnel $\pi_i(H_j)$ to be “small” for each i , but the sensitive dependence on initial conditions of f is “stretching” I_j apart. The problem of sensitive dependence on initial conditions puts our two objectives at odds with each other: as we increase T to make the tail of the utility function small we must also increase N to ensure $f^j(I_k)$ is small for all $j = 0, 1, \dots, T-1$. Our initial results using this algorithm were very discouraging. Fortunately, the fact that μ is f -invariant can be used to speed things up immensely.

Since μ is an f -invariant measure, we have for $A \in \mathcal{B}(X)$ and $k \in \mathbb{N}$,

$$\int_X \chi_{f^k(A)} \mu(dx) = \mu[f^{-k}(A)] = \mu[A] = \int_X \chi_A \mu(dx).$$

This implies for any continuous real-valued function g we have

$$\int_X g \circ f^k(x) \mu(dx) = \int_X g(x) \mu(dx).$$

In fact we have the following stronger theorem.

Theorem 9 (Walters (1982), Theorem 6.8). *If $f : X \rightarrow X$ is continuous and X is a compact metric space, then μ is an f -invariant measure if and only if $\int g \circ f(x) \mu(dx) = \int g(x) \mu(dx)$ for all $g \in C(X)$.*

The integral of the truncated sum is given by

$$\sum_{t=1}^T \int_X \beta^{t-1} U(f^{T-t}(x)) \mu(dx).$$

However, since μ is an f -invariant measure, we have

$$\int_X U(f^{T-t}(x)) \mu(dx) = \int_X U(x) \mu(dx), \quad t = 1, \dots, T-1.$$

Consequently, our integral is given by

$$\int_Y W(\mathbf{x}) m(d\mathbf{x}) = \frac{1}{1-\beta} \int_X U(x) \mu(dx).$$

5.2 Direct Limit Case

Let $f : X \rightarrow X$ be a continuous map on a compact metric space X . Let μ be an invariant measure of f and m be the induced invariant measure on the direct limit space $D := \varinjlim (X, f)$. Let $W : D \rightarrow \mathbb{R}$ be a continuous real-valued function on D defined by

$$W(\mathbf{x}) := \sum_{t=1}^{\infty} \beta^{t-1} U(x_t),$$

where $U : X \rightarrow \mathbb{R}$ is continuous. The proof used to construct the measure on the inverse limit space in Section 4 suggests the following algorithm for approximating the integral

$$\int_D W(\mathbf{x}) m(d\mathbf{x}).$$

First, we truncate the infinite sum at some value T . Next, we grid the state space (factor space) at time $t = 1$ into a partition with N pieces $\{I_j\}_{j=1}^N$. Each of these intervals can be used to partition the direct limit space according to

$$H_j := \{\mathbf{x} \in D \mid \pi_1(\mathbf{x}) \in I_j\}$$

Each of these “tunnels” in the inverse limit space has measure $m(H_j) = \mu(I_j)$. Let $\mathbf{x}_j := (x_j, f(x_j), f^2(x_j), \dots) \in H_j$. Then our integral is approximated by

$$\int_D W(\mathbf{x})m(d\mathbf{x}) \approx \sum_{j=1}^N W(\mathbf{x}^j)m(H_j) = \sum_{j=1}^N \sum_{t=1}^T \beta^{t-1}U(f^t(x_j))\mu(I_j)$$

This implies that

$$\int_Z W(\mathbf{x})m(d\mathbf{x}) = \sum_{t=1}^T \int_X \beta^{t-1}U(f^t(x))\mu(dx).$$

Again, since μ is an f -invariant measure, we have

$$\int_X U(f^n(x))\mu(dx) = \int_X U(x)\mu(dx), \quad n \in \mathbb{N}.$$

Consequently, our integral is given by

$$\int_Z W(\mathbf{x})m(d\mathbf{x}) = \frac{1}{1-\beta} \int_X U(x)\mu(dx).$$

Note that in the direct limit case, the “Riemann strategy” for approximating the integral might work even though we still have sensitive dependence on initial conditions. Our partition of tunnels is done with a grid that is always on X at time $t = 1$. As n gets large f^n may be very irregular so the choice of $x_i \in E_i$ matters a lot for $f^n(x_i)$, but β^n substantially discounts this problem.

6 Ramsey Meets Chaos in a Cash-in-Advance Model

6.1 Optimal Policy with Multiple Equilibria

The framework in this paper for calculating expected utility can be used to bridge two important literatures in macroeconomic theory: multiple equilibria and optimal policy. Dynamic general equilibrium (DGE) models have become a standard framework for both the positive and normative evaluation of policy. In the optimal

monetary/fiscal policy literature one considers a mapping from a policy space (e.g., money growth rate or set of taxes) to outcomes (e.g. allocations from a competitive equilibrium). If the mapping from policies to outcomes in the DGE model is single-valued, then one can induce a *ranking* on the policy space in a very natural way. For instance, suppose Θ is the policy space and for each $\theta \in \Theta$, there is a unique competitive equilibrium E given by $E = M(\theta)$. If U the utility function of the household defined over the space of competitive equilibria, then one can use the function $W(\theta) := U(M(\theta))$ to define a ranking on Θ . In addition to perhaps locating the most preferred or optimal policy θ^* , such a ranking can be used to measure the welfare gains of switching from some policy θ to another policy θ' . There is a large literature that takes this approach to evaluating policies starting with the work of Ramsey (1927).³

However, when H is not single-valued this method of ranking policies will not work, and it is not clear what one should do since there is more than one equilibrium associated with a particular policy. There are many ways in which M may be multi-valued. For example, the model may exhibit local indeterminacy in which for a given policy θ there exists a continuum of equilibria all converging to the steady state equilibrium. However, one may also have a multi-valued H due to global properties of the model as well. Our framework can be applied to the class of economic models with equilibria that correspond to orbits generated by a chaotic dynamical system $f : X \rightarrow X$ where X is a compact metric infinite space and f is continuous. Thus there is both a large number of equilibria and a large and complicated variety as well. Our framework is designed for this type of multi-valued H . Note that if f represents the backward map, the indeterminacy in the model is greater in the following sense. If f represents the forward map, there is a unique equilibrium associated with each $x \in X$. However, if f is the backward map, there is at least one equilibrium (and perhaps an infinite number of equilibria) associated with each $x \in X$.

6.2 Cash-in-Advance Model

The model is the standard endowment CIA model of Lucas and Stokey (1987). We closely follow the exposition of Michener and Ravikumar (1998), hereafter [MR]. Since our intent is only to apply our techniques to calculate expected utility in a model with backward dynamics and chaos, we will focus on a particular family of utility functions and parameterizations used in [MR].⁴ It is an endowment economy with both cash

³See Chari and Kehoe (1999) for a literature survey.

⁴See [MR] for more details and a more general framework.

and credit goods. There is a representative agent and a government. The government consumes nothing and sets monetary policy using a money growth rule.

The household has preferences over sequences of the cash good (c_{1t}) and credit good (c_{2t}) represented by a utility function of the form

$$\sum_{t=1}^{\infty} \beta^{t-1} U(c_{1t}, c_{2t}), \quad (14)$$

with the discount factor $0 < \beta < 1$. The utility function is assumed to take the following form:

$$U(c_1, c_2) := \frac{c_1^{1-\sigma}}{1-\sigma} + \frac{c_2^{1-\gamma}}{1-\gamma},$$

with $\sigma > 0$ and $\gamma > 0$. To purchase the cash good c_{1t} at time t the household must have cash m_t carried forward from $t-1$. The credit good c_{2t} does not require cash, but can be bought on credit. The household has an endowment y each period that can be transformed into the cash and credit goods according to $c_{1t} + c_{2t} = y$. Since this technology allows the cash good to be substituted for the credit good one-for-one, both goods must sell for the same price p_t in equilibrium and the endowment must be worth this price per unit as well.

The household seeks to maximize (14) by choice of $\{c_{1t}, c_{2t}, m_{t+1}\}_{t=1}^{\infty}$ subject to the constraints $c_{1t}, c_{2t}, m_{t+1} \geq 0$,

$$p_t c_{1t} \leq m_t, \quad (15)$$

$$m_{t+1} \leq p_t y + (m_t - p_t c_{1t}) + \theta M_t - p_t c_{2t}, \quad (16)$$

taking as given m_1 and $\{p_t, M_t\}_{t=1}^{\infty}$. The money supply $\{M_t\}$ is controlled by the government and follows a constant growth path $M_{t+1} = (1 + \theta)M_t$ where θ is the growth rate and $M_1 > 0$ given. Each period the household receives a transfer of cash from the government in the amount θM_t .

A *perfect foresight equilibrium* is defined in the usual way as a collection of sequences $\{c_{1t}, c_{2t}, m_t\}_{t=1}^{\infty}$ and $\{M_t, p_t\}_{t=1}^{\infty}$ satisfying the following. (1) The money supply follows the stated policy rule: $M_{t+1} = (1 + \theta)M_t$. (2) Markets clear: $m_t = M_t$ and $c_{1t} + c_{2t} = y$. (3) The solution to the household optimization problem is given by $\{c_{1t}, c_{2t}, m_{t+1}\}_{t=0}^{\infty}$.

The necessary first-order conditions from the household's problem imply that

$$U_2(c_{1t}, c_{2t})/p_t = \beta U_1(c_{1t+1}, c_{2t+1})/p_{t+1}, \quad (17)$$

where U_i is the partial derivative of U with respect to the i th argument. This condition reflects that at the optimum, the household must be indifferent between spending a

little more on the credit good (giving a marginal benefit $U_2(c_{1t}, c_{2t})/p_t$) versus savings the money and purchasing the cash good in the next period (giving a marginal benefit $\beta U_2(c_{1t+1}, c_{2t+1})/p_{t+1}$).

Let $x_t := m_t/p_t$ denote the level of real money balances. Using the equilibrium conditions that $M_t = m_t$ and $c_{2t} = y - c_{1t}$, equation (17) implies

$$x_t U_2(c_{1t}, y - c_{1t}) = \frac{\beta}{1 + \theta} x_{t+1} U_1(c_{1t+1}, y - c_{1t+1}). \quad (18)$$

Let c be the unique solution to $U_1(x, y - x) = U_2(x, y - x)$. If the cash-in-advance constraint (15) binds, then $c_{1t} = x_t$. If not, then the Lagrange multiplier $\mu_t = 0$ and $c_{1t} = c$. It then follows that $c_{1t} = \min[x_t, c]$ for all t . Using this relationship we can eliminate c_{1t} and c_{1t+1} from (18) to get a difference equation in x alone:

$$x_t U_2(\min[x_t, c], y - \min[x_t, c]) = \frac{\beta}{1 + \theta} x_{t+1} U_1(\min[x_{t+1}, c], y - \min[x_{t+1}, c])$$

or

$$B(x_t) = A(x_{t+1}), \quad (19)$$

where

$$\begin{aligned} B(x) &:= x U_2(\min[x, c], y - \min[x, c]), \\ A(x) &:= \frac{\beta}{1 + \theta} x U_1(\min[x, c], y - \min[x, c]). \end{aligned}$$

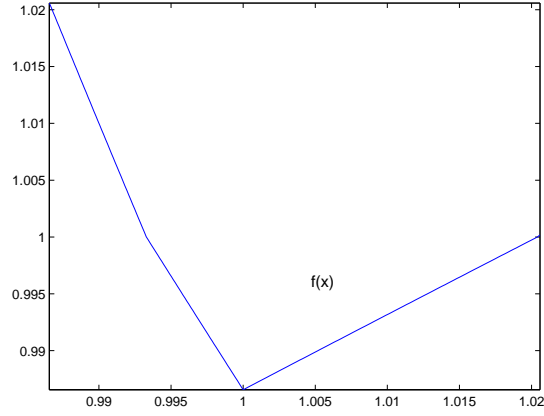
Whether or not the model has backward dynamics depends on whether or not $A(\cdot)$ is invertible. In one parameterization, [MR] set $\beta = 0.98$, $\sigma = 4$, $\gamma = 0.5$, $y = 2$ and consider θ equal to 0, 0.5 and 1.0. In this case the function A is not invertible and there exists an invariant set $[x_l, x_h]$ such that the the backward map has a three cycle. The backward map for this parameterization (with $\theta = 0$) is in Figure 1. We see that for this parameterization, the CIA model has backward dynamics.

Our function $W : \varprojlim(I, f) \rightarrow \mathbb{R}$ is given by

$$W(\mathbf{x}) := \sum_{t=1}^{\infty} \beta^{t-1} U(x_t, y - x_t).$$

To construct the natural f -invariant measure, we approximate μ via a histogram using a sample trajectory of f for some $x \in [\underline{x}, \bar{x}] : \{x, f(x), f^2(x), \dots\}$. This mimics the ‘‘rain gauge’’ description of the natural invariant measure described in Alligood et al. (1996). Figure 2 contains an approximation of μ . This histogram uses 10^4 bins and

Figure 1: Backward map $f : [x_l, x_h] \rightarrow [x_l, x_h]$ from the cash-in-advance model.



a sample trajectory of length 10^8 . Given this approximation to the natural invariant measure, the utility function U and the discount factor β it is now straight-forward to approximate our integral

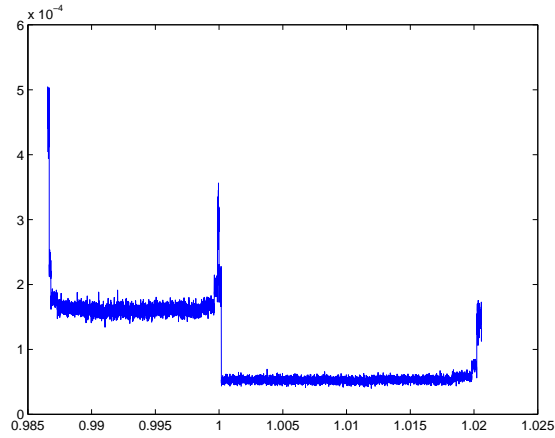
$$\int_{\mathbf{X}} W(\mathbf{x})\mu(d\mathbf{x}) = \frac{1}{1-\beta} \int_I U(x)\mu(dx) \approx 83.3285573.$$

As mentioned in the introduction, our integral allows us to rank direct/inverse limit spaces according to expected utility (a very natural ranking from the model). To give some sense of how this might be used to evaluate different monetary policies, imagine that for money growth rates $\theta \in \Theta := [\underline{\theta}, \bar{\theta}]$, the backward map f is chaotic. However, not all chaotic maps are the same in terms of utility. One way of framing the question through a Ramsey lens, is within this subclass of possible monetary policies Θ , which money growth rate gives the greatest expected utility? We see that for $\theta \in \Theta$, we have a different backward map f_θ , natural invariant measure μ_θ , invariant state space I_θ , inverse limit space $Z_\theta := \varprojlim (I_\theta, f_\theta)$ and induced measure m_θ . We then have an indirect utility function given by

$$V(\theta) := \int_{Z_\theta} W(\mathbf{x})m_\theta(d\mathbf{x}) \equiv \frac{1}{1-\beta} \int_{I_\theta} U(x)\mu_\theta(dx).$$

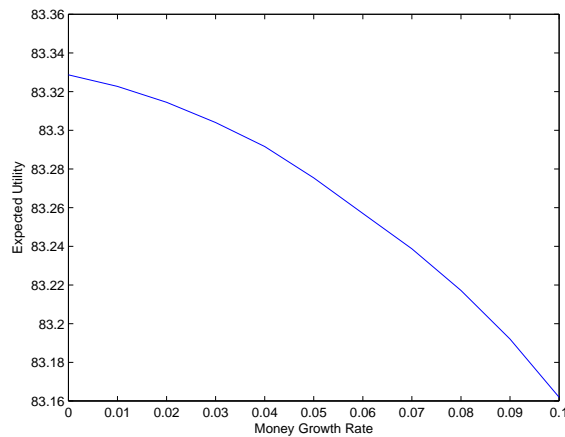
To be more concrete, suppose that the monetary authority is only considering money growth rates in $\Theta := [0, 0.1]$. Which $\theta \in \Theta$ should the monetary authority choose to maximize expected utility? Figure 3 contains a plot of the indirect utility function

Figure 2: Histogram with 10^4 bins using a sample trajectory of length 10^8



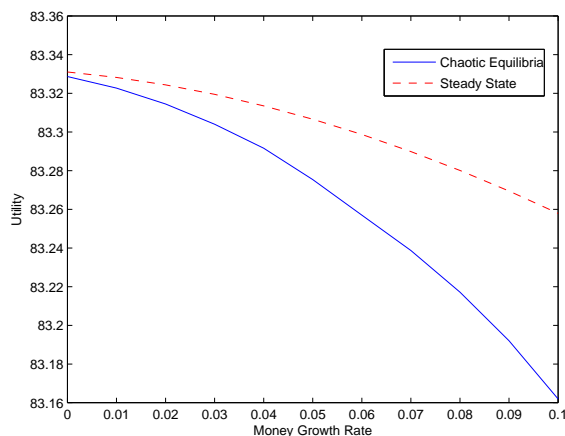
$V : \Theta \rightarrow \mathbb{R}$. We see that a lower money growth rate is preferred to higher money growth rate ($\theta = 0$ is the most preferred). This ranking is qualitatively similar to the ranking when considering only steady state equilibria.⁵ However, Figure 4 illustrates that considering only the steady state equilibria would underestimate the welfare costs of higher money growth rates.

Figure 3: Indirect utility function $V : \Theta \rightarrow \mathbb{R}$.



⁵A priori, this need not have been the case. What is driving the utility results in this example is the distribution (and its support) for different money growth rates θ .

Figure 4: Comparison of expected utility under chaos versus utility in the steady-state equilibrium.



7 Conclusion

In this paper, we developed a framework for calculating expected utility in models with chaotic equilibria and consequently a framework for ranking chaos. Our framework is quite general and applies to any DGE model where the set of equilibria correspond to the orbits generated by a chaotic dynamical system $f : X \rightarrow X$ where X is compact and f is continuous with a natural invariant measure. We have illustrated how this framework can be used to bring together two important literatures in macroeconomic theory: multiple equilibria and optimal policy.

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