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4 Inflation and the Role of Bonds in Investor Portfolios

Zvi Bodie, Alex Kane, and Robert McDonald

4.1 Introduction

The inflation of the past decade and a half has dispelled the notion that default-free nominal bonds are a riskless investment. Conventional wisdom used to be that the conservative investor invested principally in bonds and the aggressive or speculative investor invested principally in stocks. Short-term bills were considered to be only a temporary "parking place" for funds awaiting investment in either bonds or stocks. Today many academics and practitioners in the field of finance have come to the view that for an investor who is concerned about his real rate of return, long-term nominal bonds are a risky investment even when held to maturity.

The alternative view that a policy of rolling-over short-term bills might be a sound long-term investment strategy for the conservative investor has recently gained credibility. The rationale behind this view is the observation that for the past few decades, bills have yielded the least variable real rate of return of all the major investment instruments traded in U.S. financial markets. Stated a bit differently, the nominal rate of return on bills has tended to mirror changes in the rate of inflation so that their real rate of return has remained relatively stable as compared to stocks or longer-term fixed-interest bonds.

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This is not a coincidence, of course. All market-determined interest rates contain an "inflation premium," which reflects expectations about the declining purchasing power of the money borrowed over the life of the loan. As the rate of inflation has increased in recent years, so too has the inflation premium built into interest rates. While long-term as well as short-term interest rates contain such a premium, conventional long-term bonds lock the investor into the current interest rate for the life of the bond. If long-term interest rates on new bonds subsequently rise as a result of unexpected inflation, the funds already locked in can be released only by selling the bonds on the secondary market at a price well below their face value. But if an investor buys only short-term bonds with an average maturity of about 30 days, then the interest rate he earns will lag behind changes in the inflation rate by at most one month. For the investor who is concerned about his real rate of return, bills may therefore be less risky than bonds, even in the long run.

The main purpose of this paper is to explore both theoretically and empirically the role of nominal bonds of various maturities in investor portfolios. How important is it for the investor to diversify his bond holdings fully across the range of bond maturities? We provide a way to measure the importance of diversification, and this enables us to determine the value of holding stocks and a variety of bonds, for example, as opposed to following a less cumbersome investment strategy, such as concentrating in stocks and bills alone.

One of our principal goals is to determine whether an investor who is constrained to limit his investment in bonds to a single portfolio of money-fixed debt instruments will suffer a serious welfare loss. In part, our interest in this question stems from the observation that many employer-sponsored tax-deferred savings plans limit a participant's investment choices to two types, a common stock fund and a money-fixed bond fund of a particular maturity.[1]

A second goal is to study the desirability of introducing a market for indexed bonds (i.e., an asset offering a riskless real rate of return). There is a substantial literature on this subject,[2] but to our knowkedge no one has attempted to measure the magnitude of the welfare gain to an individual investor from the introduction of trading in such securities in the U.S. capital market.

In the first part of the paper we develop a mean-variance model for measuring the value to an investor of a particular set of investment instruments as a function of his degree of risk aversion, rate of time preference, and investment time horizon. We then take monthly data on real rates of return on stocks, bills, and U.S. government bonds of eight different durations, their covariance structure, and combine these estimates with reasonable assumptions about net asset supplies and aggregate risk aversion in order to derive a set of equilibrium risk premia. This procedure allows us to circumvent the formidable problems of deriving reliable estimates of these risk premia from the historical means, which are negative during many subperiods. We then employ these parameter values in our model of optimal consumption and portfolio selection in order to address the two empirical issues of principal concern to us. The paper concludes with a section summarizing the main results and pointing out possible implications for private and public policy.

4.2 Theoretical Model

4.2.1 Model Structure and Assumptions

Our basic model of portfolio selection is that of Markowitz (1952) as extended by Merton (1969, 1971). Merton has shown that when asset prices follow a geometric Brownian motion in continuous time and portfolios can be continuously revised, then as in the original Markowitz model, only the means, variances, and covariances of the joint distribution of returns need to be considered in the portfolio selection process.

In more formal terms, we assume that the real return dynamics on all n assets are described by stochastic differential equations of the form:

$$\frac{dQ_i}{Q_i} = R_i dt + \sigma_i dz_i, \qquad i = 1, \ldots, n,$$

where R_i is the mean real rate of return per unit time on asset i and σ_i^2 is the variance per unit of time. For notational convenience we will let R represent the n-vector of means and Ω the $n \times n$ covariance matrix, whose diagonal elements are the variances σ_i^2 and whose off-diagonal elements are the covariances σ_{ij}.

Investors are assumed to have homogeneous expectations about the values of these parameters. Furthermore, we assume that all n assets are continuously and costlessly traded and that there are no taxes.[3]

The change in the individual's real wealth in any instant is given by

$$(1) \qquad dW = W\sum_1^n w_i R_i dt - Cdt + W\sum_1^n w_i \sigma_i dz_i,$$

where W is real wealth, C is the rate of consumption, and w_i is the proportion of his real wealth invested in asset i.

The individual's optimal consumption and portfolio rules are derived by finding

$$(2) \qquad \max_{\{C, w\}} E_0 \int_0^H e^{-\rho t} U(C_t) dt,$$

where E is the expectation operator, ρ is the rate of time preference, $U(C_t)$ is the utility from consumption at time t, and H is the end of the investor's planning horizon.

The individual's derived utility of wealth function is defined as

$$J(W_t) = \max E_t \int_t^H e^{-\rho s} U(C_s) ds. \tag{3}$$

J is interpreted as the discounted expected value of lifetime utility, conditional on the investor's following the rules for optimal consumption and portfolio behavior. This value can be computed as a function of current wealth. The specific utility function with which we have chosen to work is the well-known constant relative risk aversion form,

$$U(C) = \frac{C^\gamma}{\gamma}, \text{ for } \gamma < 1 \text{ and } \gamma \neq 0,$$
$$\log C, \text{ for } \gamma = 0,$$

with $\delta \equiv 1 - \gamma$ representing Pratt's measure of relative risk aversion. This functional form has several desirable properties for our purposes. First, the investor's degree of relative risk aversion is independent of his wealth, which in turn implies that the optimal portfolio proportions are also independent of wealth. Second, actually solving the problem in (2) allows us to find an explicit solution for the derived utility of wealth function (Merton 1971), which takes the relatively simple form

$$J(W) = q\frac{W^\gamma}{\gamma}, \tag{4}$$

where

$$q = \left[\frac{1 - e^{-H\left(\frac{\rho - \gamma\nu}{\delta}\right)}}{\frac{\rho - \gamma\nu}{\delta}}\right]^\delta$$

and ν is a number which reflects the parameters of the investor's investment opportunity set and his degree of risk aversion.[4] Specifically, when there is no risk-free asset, ν is defined by:

$$\nu = \frac{A}{G} + \frac{D}{2G\delta} - \frac{\delta}{2G} \tag{5}$$

where $A \equiv i'\Omega^{-1}R, B \equiv R'\Omega^{-1}R, G \equiv i'\Omega^{-1}i, D \equiv BG - A^2$, where i is a vector of dimension n all of whose elements are one.

The degree of relative risk aversion plays an important role in the specific numerical results which follow, so we interpret this parameter by means of a simple example. Suppose an individual faces a situation in which there is a .5 probability of losing a proportion x of his current wealth and a .5 probability of gaining the same proportion. What proportion of current wealth would the individual be willing to pay as an insurance premium in order to eliminate this risk?[5]

Table 4.1 displays the value of this insurance premium for various values of x and δ. The second row, for example, shows that for a risk

Table 4.1 Proportion of Current Wealth an Investor Would Be Willing to Pay to Avoid a Risky Prospect with a Payoff of $=xW$ (%of Wealth)

Proportion of Wealth at Risk	Coefficient of Relative Risk Aversion (δ)					
x	1	2	3	4	5	6	10
1%	.005%	.010%	.015%	.020%	.025%	.030%	.050%
10%	.50%	1.00%	1.49%	1.97%	2.43%	2.88%	4.42%
20%	2.02%	4.00%	5.86%	7.56%	9.06%	10.35%	13.84%
50%	13.40%	25.00%	32.92%	37.76%	40.72%	42.61%	46.00%

which involves a gain or loss of 10% of current wealth an investor with a coefficient of relative risk aversion of one would only pay ½ of 1% of his wealth (or 5% of the magnitude of the possible loss) to insure against it, while an investor with a δ of 10 would pay 4.42% of his wealth (which is fully 44.2% of the magnitude of the possible loss). If the investor with a δ of 10 faces a risky prospect involving a possible gain or loss of 50% of his wealth, he would be willing to pay 92% of the possible loss to avoid the risk.

4.2.2 Optimal Portfolio Proportions and Equilibrium Risk Premia

The vector of optimal portfolio weights derived from the optimization model described above is given by

$$w^* = \frac{1}{\delta}\Omega^{-1}\left(R - \frac{A}{G}i\right) + \frac{\Omega^{-1}i}{G}. \tag{6}$$

Note that these weights are independent of the investor's rate of time preference and his investment horizon. Merton (1972) has shown that A/G is the mean rate of return on the minimum variance portfolio and that $(\Omega^{-1}i)/G$ is the vector of portfolio weights of the n assets in the minimum variance portfolio. Denoting these by $R_{\min}$ and $w_{\min}$, respectively, we can rewrite equation (6) as

$$w^* = \frac{1}{\delta}\Omega^{-1}(R - R_{\min}i) + w_{\min}. \tag{6'}$$

The demand for any individual asset can thus be decomposed into two parts represented by the two terms on the right-hand side of equation (7):

$$w_i^* = \frac{1}{\delta}\sum_{j=1}^{n} v_{ij}(R_j - R_{\min}) + w_{i,\min}, \tag{7}$$

where v_{ij} is the ij_{th} element of Ω^{-1}, the inverse of the covariance matrix. The first of these two parts is a "speculative demand" for asset i, which depends inversely on the investor's degree of risk aversion and directly on a weighted sum of the risk premia on the n assets. The second component is a "hedging demand" for asset i which is that asset's weight in the minimum-variance portfolio.[6]

Under our assumption of homogeneous expectations the equilibrium risk premia on the n assets are found by aggregating the individual demands for each asset (eq. [6']) and setting them equal to the supplies. The resulting equilibrium yield relationships can be expressed in vector form as

$$R - R_{\min}i = \bar{\delta}(\Omega w_M - \sigma^2_{\min}i), \tag{8}$$

where $\bar{\delta}$ is a harmonic mean of the individual investors' measures of risk aversion weighted by their shares of total wealth, w_M is the vector of net supplies of the n assets each expressed as a proportion of the total value of all assets, and $\sigma^2_{\min}$ is the variance of the minimum variance portfolio.

The portfolio whose weights are given by w_M has come to be known in the literature on asset pricing as the "market" portfolio, and we will adopt that same terminology here. Equation (8) implies that

$$R_i - R_{\min} = \overline{\delta}(\sigma_{iM} - \sigma^2_{\min}), \qquad i = 1, \ldots, n, \tag{9}$$

where σ_{iM} is the covariance between the real rate of return on asset i and the rate of return on the market portfolio.

This relationship holds for any individual asset and for any portfolio of assets. Thus for the market portfolio we get

$$R_M - R_{\min} = \overline{\delta}(\sigma^2_M - \sigma^2_{\min}). \tag{10}$$

It is interesting to compare this with the traditional form of the capital asset pricing model which assumes the existence of a riskless asset. In that special case $R_{\min}$ is simply the riskless rate and $\sigma^2_{\min}$ is zero.

By substituting the equilibrium values of $R_i - R_{\min}$ from equation (8) into equation (6′), we get for investor k

$$w_k^* = \frac{\overline{\delta}}{\delta_k} w_M + \left(1 - \frac{\overline{\delta}}{\delta_k}\right) w_{\min}. \tag{11}$$

This implies that in equilibrium every investor will hold some combination of the market and the minimum variance portfolios. If the investor is more risk averse than the average he will divide his portfolio into positive positions in both the market portfolio and the minimum variance portfolio, with a higher proportion in the latter the greater his degree of risk aversion. If he is less risk averse than the average he will sell the minimum variance portfolio short in order to invest more than 100% of his funds in the market portfolio.

4.2.3 The Welfare Loss from Incomplete Diversification

Suppose the investor faces an investment opportunity set consisting of less than the full set of n assets. How much additional current wealth would he have to be given in order to make him as well off as he was with the full set of n assets?

Let $J(W|n)$ be the lifetime utility of an investor who chooses from among n assets, and let $J(W|n - m)$ be the lifetime utility of an investor choosing from among a restricted set of assets. Let W represent the investor's actual level of current wealth and $\hat{W}$ the level at which his welfare would be the same under the restricted opportunity set. $\hat{W}$ is defined by $J(W|n) = J(\hat{W}|n - m)$.

Thus $\hat{W} - W$ is the extra wealth necessary to compensate the investor for having a restricted opportunity set and is greater than or equal to zero. From equation (4) we get

$$\hat{W} = W\left\{\frac{\left[1 - e^{-H\left(\frac{\rho - \gamma\nu}{\delta}\right)}\right]}{(\rho - \gamma\nu)} \cdot \frac{(\rho - \gamma\hat{\nu})}{\left[1 - e^{-H\left(\frac{\rho - \gamma\hat{\nu}}{\delta}\right)}\right]}\right\}^{\frac{\delta}{\gamma}}, \tag{12}$$

where $\hat{\nu}$ is calculated according to equation (5) and corresponds to the restricted opportunity set.[7]

Equation (12) implies that the magnitude of the welfare loss will in general depend on the investor's risk aversion, δ, rate of time preference, ρ, and investment horizon, H. Since $\hat{W}$ is proportional to W, a convenient measure of this loss is $\hat{W}/W - 1$, the loss per dollar of current wealth, which is independent of the investor's wealth level. Since $\hat{W} \geq W$, this number is always greater than or equal to zero.

Of course, certain restrictions on the investment opportunity set need not decrease investor welfare. We know from equation (11) that even if the investor had only two mutual funds to choose from, there would be no loss in welfare, provided they were the market portfolio and the minimum variance portfolio. Merton (1972) has shown that any two portfolios along the mean-variance portfolio frontier would serve as well. But, in general, restricting the number of assets in the opportunity set does lead to a loss in investor welfare.

4.2.4 The Shadow Riskless Rate and the Gain from Introducing a Riskless Asset

We define the shadow riskless real rate of interest as that rate at which an investor would have no change in welfare if his opportunity set were expanded to include a riskless asset. When the investment opportunity set includes a riskless asset, Merton (1971) shows that the lifetime utility of wealth function is the same as (4), except that ν is replaced by λ, where

$$\lambda = R_F + \frac{(R - R_F i)' \, \Omega^{-1}(R - R_F i)}{2\delta}. \tag{13}$$

We find the expression for the shadow riskless rate by setting ν equal to λ and solving for R_F. This gives

$$R_F = R_{\min} - \delta\sigma^2_{\min}. \tag{14}$$

This implies that a risk-averse investor will always have a shadow riskless real rate which is less than the mean real return on the minimum variance portfolio. The return differential is equal to his degree of relative risk aversion times the variance of the minimum variance portfolio.

If there is a zero net supply of this riskless asset in the economy, the equilibrium value of R_F will just be $R_{\min} - \bar{\delta}\sigma^2_{\min}$. Therefore, by assumption, an investor with average risk aversion will not gain from the introduction of a market for index bonds. For an investor whose risk aversion is different from the average there will be a welfare gain, ignoring the costs of establishing and operating such a market. We measure this gain analogously to the way we measured the welfare cost of incomplete diversification in the previous section.

As before, let W be the investor's actual level of wealth and $\hat{W}$ the level at which his welfare would be the same under an opportunity set expanded to include a riskless asset offering a real rate of $R_{\min} - \bar{\delta}\sigma^2_{\min}$. Since in this case $\hat{W} \leq W$, we take as our measure of the welfare gain from indexation $1 - (\hat{W}/W)$, or the amount the investor would be willing to give up per dollar of current wealth for the opportunity to trade index bonds.

4.3 The Data and Parameter Estimates

In this section we will describe our data and how we used them to estimate the parameters needed to evaluate the welfare loss from restricting an investor's opportunity set and the gain from introducing a real riskless asset. It must be borne in mind that we were not trying to test the model of capital market equilibrium presented in section 4.2 empirically but rather to derive its implications for the specific questions being addressed in this paper. It was therefore important to maintain consistency between the underlying theoretical model and the parameter estimates derived from the historical data, even if that meant ignoring some of the descriptive statistics yielded by those data.

Our raw data were monthly real rates of return on stocks, one-month U.S. government Treasury bills, and eight different U.S. bond portfolios. We used monthly data in order to best approximate the continuous trading assumption of Merton's model, and because one month is the shortest interval for which information about the rate of inflation is available. The measure of the price level that we used in computing real rates of return was the Bureau of Labor Statistics' Consumer Price Index, excluding the cost-of-shelter component. We excluded the cost-of-shelter component because it gives rise to well-known distortions in the measured rate of inflation.

The bill data are from Ibbotson and Sinquefield (1982), while the bond data are from the U.S. Government Bond File of the Center for Research in Security Prices (CRSP) at the University of Chicago. The stock data are from the CRSP monthly NYSE file. We divided the bonds into eight different portfolios based on duration. We felt that duration was superior to maturity as a criterion for grouping the bonds since it takes into account a bond's coupon as well as its maturity.[8] The durations of the bond portfolios range from 1 to 8 years.

Table 4.2 presents the means, variances, and correlation coefficients of the monthly real rates of return on the 10 asset categories for three subperiods between January 1953 and December 1981. The first is the 12 years from January 1953 to December 1964, a period of relative price stability; the second is the 8 years from 1965 to 1972, a period of moderate inflation; and the third is the 9 years from 1973 to 1981, a period of relatively rapid inflation.

The measure of the real rate of return used in all cases was the natural logarithm of the monthly real wealth relatives $Q_i(t)/Q_i(t - 1)$. On the assumption that these returns follow a geometric Brownian motion in continuous time, $dQ_i | Q_i = R_i dt + \sigma_i dz_i$, the log of the wealth relative over a discrete time interval is normally distributed with mean μ_i and variance σ_i^2, where $\mu_i = R_i - (\sigma_i^2/2)$. The means reported in table 2 were converted to annual rates by multiplying them by 12 and the standard deviations by multiplying them by $\sqrt{12}$. This makes them comparable to the means and standard deviations one would obtain using a 1-year holding period.

A most striking aspect of these descriptive statistics can be seen in part C of the table: all assets have negative mean returns over the last subperiod. This presents a dilemma for anyone requiring estimates of the risk premia called for in models of capital market equilibrium, since their recent historical pattern is grossly inconsistent with the pattern implied by the variance-covariance matrix estimated from the same data.

As Merton (1980) has shown, in order to get a reliable estimate of the mean of a continuous-time stochastic process, it is necessary to observe the process over a long span of time. Variances and covariances, however, can be measured fairly accurately over much shorter observation periods. We therefore chose to ignore the historical means reported in table 4.2, while using the estimated covariance matrix.

The standard deviations of all 10 assets reported in table 4.2 increased significantly over the 3 periods. Since we were interested in computing welfare losses and gains for investors in today's U.S. capital markets, we used in our calculations the variance and correlation coefficients estimated for the most recent period, 1973–81.

The standard deviations for this last subperiod fall into a clear pattern. The lowest is for bills, .0126, which is well below that on 1-year bonds, the next lowest reported in the table. The standard deviation on bonds rises continuously with duration, reaching a maximum of .1095 on duration 8. Stocks have a standard deviation of .1735, which is 1.6 times that of duration 8 bonds and about 14 times that of bills. In the previous two subperiods, while all the standard deviations are lower than in the 1973–

Table 4.2 **Distribution of Monthly Real Rates of Return (Annualized)**

	Common Stocks	1-Month Bills	Bonds (by Duration in Years) 1	2	3	4	5	6	7	8
A. 1953–64:										
Mean	.1202	.0113	.0183	.0213	.0218	.0151	.0188	.0122	.0074	.0243
S.D.	.1179	.0081	.0105	.0170	.0235	.0268	.0311	.0305	.0361	.0155
Observations	144	144	144	144	144	144	138	110	84	42
	Correlation Coefficient									
Stocks		.04	−.02	−.09	−.10	−.06	−.13	−.15	−.12	.15
Bills			.70	.40	.25	.23	.22	.24	.25	.42
Bonds 1				.81	.70	.68	.65	.65	.56	.63
2					.85	.82	.78	.80	.74	.59
3						.90	.83	.82	.80	.60
4							.86	.86	.83	.65
5								.88	.86	.79
6									.91	.75
7										.95

Table 4.2 (continued)

	Common Stocks	1-Month Bills	Bonds (by Duration in Years) 1	2	3	4	5	6	7	8
	Common Stocks	1-Month Bills	Bonds (by Duration in Years) 1	2	3	4	5	6	7	8
B. 1965–72:										
Mean	.0413	.0122	.0162	.0179	.0101	.0007	.0032	.0078	.0339	.0113
S.D.	.1345	.0064	.0155	.0298	.0370	.0427	.0519	.0427	.0453	.0377
Observations	96	96	96	96	96	96	96	52	30	6
	Correlation Coefficients									
Stocks		.11	.25	.33	.38	.33	.33	.08	.03	.23
Bills			.64	.52	.37	.38	.27	.32	.38	0
Bonds 1				.86	.78	.74	.67	.69	.60	.11
2					.83	.80	.73	.76	.77	.98
3						.82	.84	.85	.86	.57
4							.83	.86	.49	.41
5								.94	.77	.61
6									.66	.72
7										.88

	Common Stocks	1-Month Bills	Bonds (by Duration in Years) 1	2	3	4	5	6	7	8
C. 1973–81:										
Mean	−.0269	−.0050	−.0044	−.0141	−.0186	−.0284	−.0320	−.0549	−.0298	−.0485
S.D.	.1735	.0126	.0316	.0529	.0693	.0812	.0922	.1034	.1049	.1095
Observations	108	108	108	108	108	108	108	95	99	106
					Correlation Coefficients					
Stocks		.20	.32	.32	.27	.31	.22	.22	.30	.33
Bills			.54	.39	.35	.35	.26	.22	.27	.22
Bonds 1				.88	.85	.82	.80	.73	.78	.77
2					.94	.92	.87	.81	.85	.86
3						.95	.93	.85	.88	.88
4							.91	.83	.89	.87
5								.89	.92	.89
6									.91	.88
7										.91

Notes: The measure of the real rate of return used is the natural logarithm of the monthly real wealth relative. The reported means were converted to annual rates by multiplying them by 12 and the standard deviations by multiplying them by $\sqrt{12}$. This makes them comparable to the means and standard deviations of the continuously compounded rates of return one would obtain using a one-year holding period.

81 subperiod, they fall into approximately the same pattern of relative magnitudes.

Turning to the matrix of correlation coefficients, we see that in the last subperiod all of the correlations are positive. Stocks had correlations ranging from .20 (with bills) to .33 (with duration 8 bonds), and they do not rise uniformly with the duration of the bonds. The pattern for bonds and bills is that correlations are highest among bonds of adjacent durations and fall off more or less uniformly as one moves to more distant durations. In the 1965–72 subperiod the pattern of correlations is quite similar to 1973–81 for all assets, but in the noninflationary 1953–64 subperiod the correlations among bills and bonds follow the same pattern, while the real returns on stocks appear to be essentially uncorrelated with the real returns on bills and bonds.

In addition to the variance-covariance matrix, the next input we need for equation (8) in order to generate numerical results is the vector of weights for the market portfolio. Here we face some problems of both a theoretical and an empirical sort.

At the theoretical level, one issue is whether to treat U.S. government bonds as net wealth. There is considerable controversy among monetary theorists on this issue, and a substantial literature on it exists.[9] We decided to treat U.S. government debt as net wealth of the private sector.

We also ignore the default risk premium on corporate bonds by lumping them together with Treasury bonds. This amounts to assuming that they have the same variance-covariance structure.

Another problem is our exclusion of some important categories of assets in our computation of the market portfolio. Most notable among these are residential real estate, consumer durables, human capital, and social security wealth.[10] While we do not include these in the present paper, our plan for future extensions of this research is to seek appropriate data on these other asset classes and redo our calculations to include them.

There remains the empirical problem of determining the relative weights of those assets which we do include in the market portfolio in the present study. The ratio of the market value of corporate equity to the book value of total government debt was approximately 1.5 in 1980. Thus, 60% was the equity weight in the market portfolio. The relative supplies of government debt by duration were approximated from a table in the *Treasury Bulletin* which breaks down the quantities of government debt by maturity: issues maturing in less than 1 year, in 1–5 years, and so forth. We arbitrarily spread the weights evenly among the years within each of these groupings.

This procedure obviously omits corporate debt. However, using flow-of-funds data we computed the percentage of equity by treating both corporate equity and the net worth of unincorporated businesses as equity. Debt then consisted of federal, corporate, and unincorporated business credit market liabilities. This procedure also yielded a 60% equity-to-wealth ratio. By lumping corporate debt together with U.S. government debt we are ignoring any default risk premia.

The foregoing ignores financial intermediaries, in effect supposing that households hold the securities of nonfinancial businesses and the government directly. A different procedure would be to net out securities held by intermediaries, and consider the public's holding of bank liabilities as debt. (Deposits could be treated as Treasury bills, for example.) We plan to experiment with this alternative in future research.

The ultimate set of weights we used for the market portfolio was:

		Bonds by Duration in Years							
Stocks	Bills	1	2	3	4	5	6	7	8
.60	.05	.15	.033	.033	.033	.033	.022	.022	.022

Finally, in order to determine the equilibrium risk premia we need to set a value for $\bar{\delta}$, the economy-wide average degree of relative risk aversion. In a recent paper, Grossman and Shiller (1981) concluded that a value of 4 is most consistent with the observed movements of the value of the stock market over the past 90 years. Friend and Blume (1975) had estimated it to be 2, while Friend and Hasbrouck (1982) found 6 to be more appropriate. As we show below, a value of 4 produces an imputed risk premium on stocks which is in line with direct time-series estimates of this premium obtained by other researchers using a variety of estimation techniques. We therefore choose 4 as our value for $\bar{\delta}$ in the calculation of the equilibrium risk premia which we use in the remainder of the paper. To a large extent the particular value of $\bar{\delta}$ is unimportant, since the deviation of δ_k from $\bar{\delta}$, and not the level, is what matters most for our results.

Table 4.3 presents the full set of imputed real risk premia $(R_i - R_{\min})$ which we calculated using the formula embodied in equation (8), the variance-covariance matrix of monthly real returns estimated over the period 1973–81, and the vector of market weights and value of $\bar{\delta}$ presented above. The table also shows the individual asset variances, their covariances with the market portfolio, and their betas on the market portfolio. The last two columns give the values corresponding to the minimum variance and market portfolios, respectively.

The table shows that the real risk premium on the market portfolio is approximately 5% per year, which is almost 4 times its variance of 1.26% per year. Since we have set $\bar{\delta}$ at 4, the risk premium on the market portfolio would be exactly 4 times its variance if the variance of the minimum variance portfolio were zero rather than .0144% per year. The

Table 4.3 **Imputed Risk Premia, Variances, and Covariances with the Market Portfolio (Annualized)**

			Bonds (by Duration in Years)								Portfolios	
	Stocks	1-Month Bills	1	2	3	4	5	6	7	8	Minimum Variance	Market
Risk premium	.0760	.0009	.0061	.0108	.0126	.0161	.0151	.0165	.0205	.0227	0	.0497
Variance	.0301	.00016	.0010	.0028	.0048	.0066	.0085	.0107	.0110	.0120	.00014	.0126
Covariance with market	.0191	.0003	.0017	.0028	.0033	.0042	.0039	.0043	.0053	.0058	.00014	.0126
Beta coefficient	1.52	.02	.13	.22	.26	.33	.31	.34	.42	.46	.01	1.00

Notes: 1. The risk premia were computed according to the formula $R_i - R_{min} = \bar{\bar{\delta}}(\sigma_{iM} - \sigma^2_{min})$, with $\bar{\bar{\delta}}$ the economy-wide average coefficient of relative risk aversion set equal to 4; the σ_{iM} are the covariances with the market portfolio reported in the third row of the table.

2. The variances and covariances reported above were computed from the distribution of the natural logs of the monthly real wealth relatives over the period 1973–81. They were annualized by multiplying them by 12.

3. The reported beta coefficients are the covariance with the market divided by the variance of the market portfolio.

risk premium on bills is only 9 basis points, and the variance is only slightly higher than the minimum, which is not surprising since as we shall see in the next section the minimum variance portfolio is essentially bills.

With the sole exception of duration 5, the risk premia on bonds rise uniformly with duration reaching a maximum of 2.27% per year. Finally, the risk premium on stocks is 7.60% per year or approximately 1.5 times the risk premium on the market portfolio. Since the beta of stocks is approximately 1.5, this result should not be surprising to readers familiar with the capital asset pricing model.[11] It is also in line with the long-run time-series estimates derived by Ibbotson and Sinquifield (1982) and Merton (1980).

4.4 The Welfare Loss from Incomplete Diversification

In this section we address the question of how much welfare an investor loses by having his choice of assets limited. The main conclusion of the theoretical discussion in section 4.2 was that even if an investor's opportunity set is limited to only two assets, there will be no loss in welfare provided that these two assets are the market portfolio and the minimum variance portfolio (or any other set of two frontier portfolios). But we are interested in the actual menu of asset choices offered in practice by many employer-sponsored tax-sheltered savings plans in the United States. These plans usually offer participants a choice of two or three funds: a stock fund, an intermediate-term fixed interest bond fund, and sometimes as a third option a money market fund.

Table 4.4 presents the risk premia, variances, and asset compositions of the optimal portfolios chosen from the full set of 10 assets for investors with coefficients of relative risk aversion ranging from 2 to 10. Figure 4.1, which is the familiar efficient portfolio frontier, displays graphically the mean-variance combinations tabulated in the second and third columns of table 4.4.

The middle row of table 4.4 corresponds to the market portfolio and the last row to the minimum variance portfolio, which consists essentially of bills, hedged with small offsetting short and long positions in bonds of the various durations. Table 4.4 shows that a very risk-averse investor, with a coefficient of risk aversion of 6, would hold 40% of his portfolio in stocks, 40% in bills, and the remaining 20% in bonds of various durations. He would thereby attain a risk premium of about 3.3% per year with a variance of 0.57% per year. Even an extremely risk-averse investor, one whose δ value is 10, would still invest roughly 24% of his funds in stocks, 67% in bills, and the remainder in bonds of various durations, in order to attain a mean risk premium of 1.95% per year with a variance of only 0.21% per year.

Note that for coefficients of relative risk aversion smaller than the

Table 4.4 **Risk Premia, Variances, and Asset Composition of Optimal Portfolios**

Coefficient of Relative Risk Aversion	Risk Premium (% per Year)	Variance (% per Year)	Portfolio Proportions (%)									
					Bonds (by Duration in Years)							
			Common Stocks	1-Month Bills	1	2	3	4	5	6	7	8
2	9.94	4.98	120.6	−98.4	37.0	1.2	13.9	12.6	1.3	3.4	8.4	−.1
3	6.62	2.22	80.2	−29.5	22.3	2.6	6.9	6.4	2.7	2.6	4.3	1.4
4 (market portfolio)	4.97	1.26	60.0	5.0	15.0	3.3	3.3	3.3	3.3	2.2	2.2	2.2
5	3.97	.81	47.9	25.7	10.6	3.8	1.2	1.5	3.7	2.0	1.0	2.7
6	3.29	.57	39.8	39.5	7.7	4.0	−.2	.3	4.0	1.8	.2	3.0
10	1.95	.21	23.6	67.0	1.8	4.6	−3.0	−2.2	4.6	1.5	−1.5	3.6
Minimum variance portfolio	0	.014	−.6	108.4	−7.0	5.5	−7.2	−5.9	5.4	1.0	−4.0	4.5

Notes: The covariance matrix used was estimated over the 1973–81 period and reported in table 5.2, part C. The risk premia of stocks, bills, and bonds used are the ones reported in table 5.3.

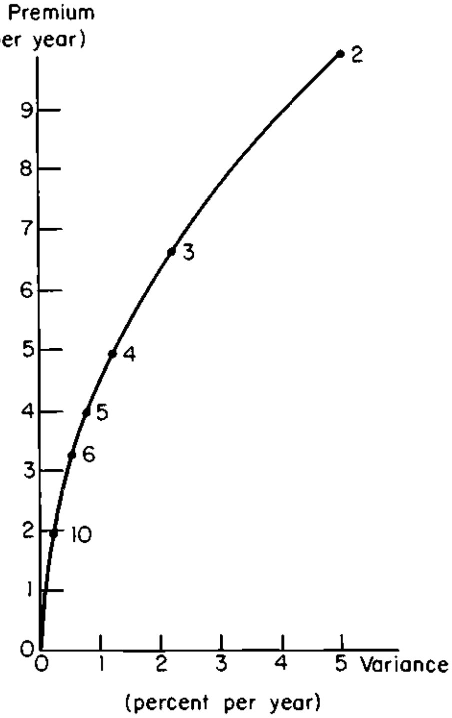

Fig. 4.1 Efficient portfolio frontier. Source: table 4.4. Note: The numbers on the frontier are coefficients of relative risk aversion and indicate the point which would be optimal for an investor having the corresponding degree of risk aversion.

economy-wide average of 4, the investor takes larger short positions in bills and long positions in stocks and bonds of most durations. In the first row, for example, we see that an investor with a risk aversion coefficient of 2 nearly doubles the mean risk premium on his portfolio relative to the average investor but also increases the variance by a factor of 4.

Short-selling Treasury bills is difficult in practice. This difficulty can be overcome in two ways. First, a large investment house or pension fund could allow its less risk-averse investors to sell short to the more risk-averse investors, as a purely internal transaction. Second, and more likely, a less risk-averse investor can simply take a long position in stock market futures as a way to hold a levered position in stocks.

Table 4.5 and figure 4.2 present our estimates of the welfare loss to an investor from having his opportunity set restricted to various subsets of the 10 asset classes. The numbers in this table represent the amount of money the investor would need to be given per $10,000 of his current wealth to make him as well off with the restricted choice set as he would

Table 4.5 **Welfare Loss from Incomplete Diversification (Dollars per $10,000 of Wealth)**

	2 Assets					3 Assets:
		Stocks and bonds of duration:				
Coefficient of Relative Risk Aversion	Stocks and Bills	1-Year	2-Years	4-Years	8-Years	3 Assets: Stocks, Bills, and Bonds of 2 Years Duration
2	$325	$442	$551	$671	$756	$40
3	246	174	113	58	39	29
4	202	63	26	119	372	24
5	170	34	137	596	1,445	21
6	146	61	389	1,426	3,301	19

Notes: These estimates correspond to an assumed rate of time preference of 4% per year and an infinite horizon. In setting the mean rates of return we assumed that the mean on bills is zero and calibrated the others accordingly. The risk premia and covariances used were the ones estimated over the 1973–81 period, reported in table 4.2, part C, and table 4.3.

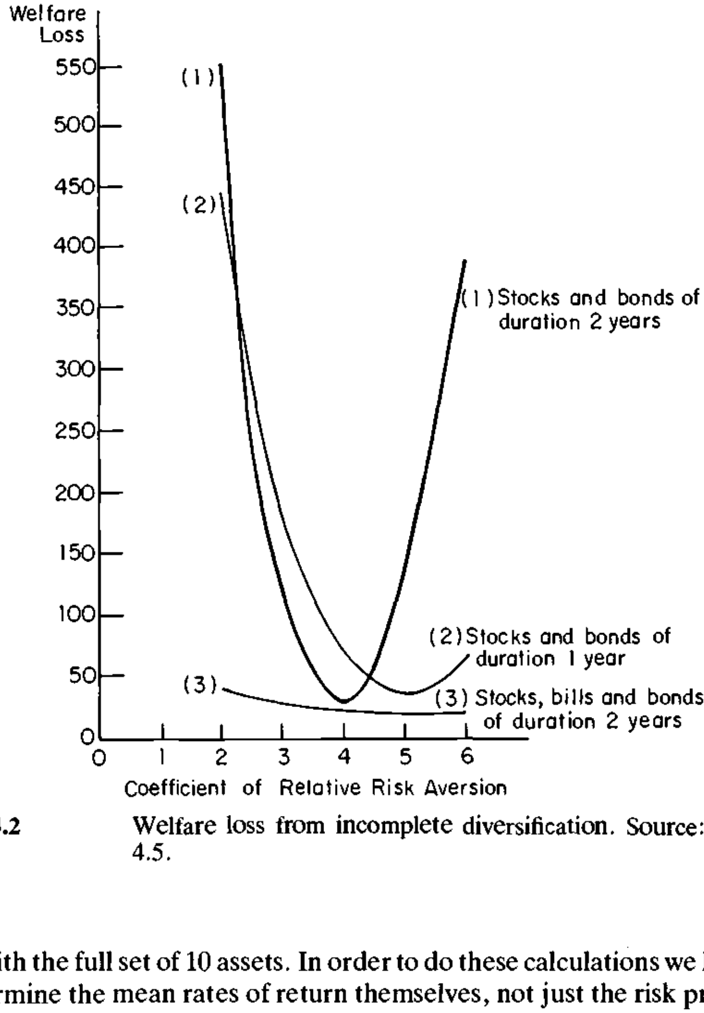

Fig. 4.2 Welfare loss from incomplete diversification. Source: table 4.5.

be with the full set of 10 assets. In order to do these calculations we had to determine the mean rates of return themselves, not just the risk premia. We did this by assuming that the mean on bills is zero and calibrating all other rates accordingly. This assumption was based on the actual mean real return on bills observed over the past 30 years.

We also had to assume a rate of time preference and a specific planning horizon. We arbitrarily set these at 4% per year and infinity, respectively, but did a sensitivity analysis which we report below in table 4.6. It should be noted that the infinite horizon assumption is really meant to represent the case where time of death is uncertain and the parameter ρ in (2) incorporates the rate of mortality as in Merton (1971). Note also that table 4.5 shows the welfare loss from restricting the investor's portfolio choice forever, not just for a limited period.

Table 4.5 and figure 4.2 show that when the investor is restricted to only two assets, the welfare impact of the restriction can be quite sensitive

Table 4.6 **Effect of Rate of Time Preference and Time Horizon on the Welfare Loss from Incomplete Diversification (Dollars per $10,000 of Wealth)**

A. Rate of Time Preference	2 Assets: Stocks and Bills
0	$394
2% per year	305
4% per year	249

B. Time horizon	2 Assets: Stocks and Bills
1 month	$0.28
5 years	17
Infinite	249

Notes: Part A assumes a coefficient of risk aversion of 4 and an infinite time horizon. The risk premia and covariances used were the ones estimated over the 1973–81 period and reported in table 4.2, part C, and table 4.3.

Part B assumes a rate of time preference of 4% per year. The risk premia and covariances used were the ones estimated over the 1973–81 period and reported in table 4.2, part C, and table 4.3.

to his coefficient of risk aversion. If the two assets are stocks and bonds of 2 years duration (curve 1), we see that the welfare loss is small for an investor with a risk aversion coefficient equal to the average, 4, but increases sharply on either side of this value. If, on the other hand, the two assets are stocks and bonds of 1 year duration (curve 2), then the welfare loss is greatest for the least risk-averse investor but is not extreme for any investor. Investors with coefficients of risk aversion equal to 3 or 4 are better off with stocks and bonds of 2 years duration, whereas investors who are either more or less risk averse than that would prefer stocks and bonds of 1 year duration.

A comparison of the first two columns in table 4.5 reveals that stocks and bonds of duration one year are preferable to stocks and bills for all investors except those with risk aversion of 2. Moving across table 4.5 we see that as the duration of the bond fund increases the welfare loss becomes more sensitive to the coefficient of risk aversion. Thus for bonds of 8 years duration the smallest welfare loss relative to the full 10-asset opportunity set occurs at a coefficient of risk aversion of 3, rising quite sharply on either side of that value and becoming particularly severe for very risk-averse investors.

The last column in table 4.5 shows that when the choice set is expanded from two to three assets, stocks, bills, and bonds of 2 years duration, the magnitude of the welfare loss falls dramatically for all investors, regard-less of their degree of risk aversion. Having these three assets to choose from is thus almost as good as having all 10.

The effects of changing our assumptions about the rate of time preference and the horizon are shown in table 4.6. The magnitude of the welfare loss from restricting the choice set to stocks and bills is greater the lower the rate of time preference and the longer the horizon.

These numerical results suggest that if an employer-sponsored savings plan is going to restrict its participants to a choice of only two funds, then since the sponsor does not know the exact degree of risk aversion of the participants, it would make sense to let the two funds be stocks and bonds of 1 or 2 years duration. If, however, the sponsor is willing to expand the number of funds to three, then stocks, bills, and bonds of 2 years duration will eliminate almost all of the welfare loss relative to the full 10-asset opportunity set.

The applicability of our analysis to employer-sponsored tax-deferred savings plans is limited by two factors: assets held outside the plan, and taxes. Without taxes it is trivially obvious that the omission of bills from a savings plan is of no consequence if investors can hold a money market fund on their own account. When there are tax advantages to investing in a savings plan, however, on the margin the investor prefers to hold assets inside the plan. If the plan fails to offer a full menu of assets, the investor will suffer a welfare loss. Our numerical calculations can be viewed as applying to a world in which all assets are invested in a tax-deferred savings plan with a restricted menu of assets. In general, however, our numerical calculations still provide an upper bound on the possible welfare loss, for the following reason: if the investor could in principle invest all wealth in the plan, and *chooses* not to do so in order to diversify, then the welfare loss must be less than for an investor who is (as in our calculations) constrained to hold only those assets offered by the plan.

In practice, of course, additional complications reduce the importance of tax-deferred savings plans. The IRS imposes a limit on the contributions to these plans, and frequently there are penalties or delays associated with the premature withdrawal of funds. These considerations will reduce the percentage of an investor's wealth which is held in such savings plans. Therefore the failure of the plan to offer certain assets is less important, since freely chosen assets held outside the plan will undo the effect of restrictions imposed within the plan. Our numerical estimates again provide an upper bound on the welfare loss.

4.5 Shadow Riskless Rates and the Welfare Gain from Introduction of a Riskless Real Asset

In section 4.2 we defined the shadow riskless rate of interest as that rate at which an investor would have a zero gain in welfare from having his choice set expanded to include an asset which was riskless in real terms. Equation (14) showed that this rate is below the mean real rate of return on the minimum variance portfolio by an amount equal to the investor's degree of relative risk aversion times the variance of the minimum variance portfolio. Given that our estimate of this variance is a mere .0144% per year, it follows that even a very risk averse investor ($\delta = 6$) would be willing to give up less than 9 basis points.

Since the average degree of risk aversion is 4, if a market for riskless real bonds could be established costlessly, the market clearing real interest rate would be about 6 basis points below the mean rate on the minimum variance portfolio. Table 4.7 shows what the welfare gain would be to investors with varying degrees of risk aversion.

The magnitude of the welfare gain to investors does not appear to be large. The numbers in the first column of table 4.7 show the results obtained using the actual covariance matrix estimated for the 1973–81 subperiod. The second column shows the results of an experiment in which we made all nominal debt securities twice as risky by doubling their variances and covariances, leaving the variance of stocks unchanged. While the effect is to approximately double the welfare gain to investors at any degree of risk aversion, the magnitude of the gain still appears small.

These results suggest one possible reason for the nonexistence of index bonds in the U.S. capital market. Since there would probably be some costs associated with creating a new market for such bonds, the benefits would have to exceed those costs. Given the assumptions of our model, in particular the assumption of homogeneous expectations, the benefit from trading in index bonds would have to arise from differences in the degree of risk aversion among investors. If as table 4.7 suggests, the welfare gain does not appear to be large over a fairly broad range of risk aversion coefficients, then one should not be surprised at the failure of a market for index bonds to appear.

Table 4.7 **Welfare Gain from Introduction of a Real Riskless Asset (Dollars per $10,000 of Wealth)**

	Welfare Gain	
Coefficient Relative Risk Aversion	Actual Covariance Matrix	Double All Variances and Covariances but Stocks
2	$32	$63
3	7	13
4	0	0
5	6	12
6	25	49

Note: Assumptions are the same as for table 5.5.

One should bear in mind that table 4.7 is derived assuming a zero net aggregate supply of index bonds. Thus it does not answer the question of whether the welfare gain from indexing government debt would be significant.

4.6 Summary and Discussion of Findings

We undertook this research with two main policy questions in mind: (1) Is there a significant welfare loss stemming from the practice on the part of many employer-sponsored savings plans of restricting a participant's choice of investments to two or three asset classes? (2) What is the potential welfare gain from the introduction of trading in privately issued index bonds? In this section we summarize and discuss the implications of our findings for each.

With regard to the first of these, we have shown that there is no necessary loss of welfare from restricting an investor's choice set to only two funds, provided these two are properly chosen. If they are the market portfolio and the minimum variance portfolio, then there will be no loss at all. In practice, however, many plans offer a diversified common stock fund and an intermediate-term fixed-interest bond fund as the only two assets, and in such cases there can be a substantial welfare loss to participants whose degree of risk aversion differs appreciably from the average. Most of this loss can be eliminated for risk-averse participants by introducing as a third option a money market fund.

With regard to the second question, our results indicate that the potential welfare gain from the introduction of index bonds in the current U.S. capital market is probably not large. The major reason for the small gain we calculate is the fact that one-month T-bills with their small variance of real returns are an effective substitute for index bonds.

There are some important factors bearing on these two policy questions that we either excluded or ignored in our analysis, and we must consider their potential effect on our conclusions. The first is the fact that we limited ourselves to only a subset of the assets which individuals in the United States hold in their portfolios. Specifically, we excluded residential real estate, consumer durables, and nontradeable assets like human capital and social security wealth.

Undoubtedly the inclusion of these other assets would affect the magnitude of the welfare effects we calculated. Thus the welfare loss to an individual whose employer-sponsored savings plan offers only a stock fund and a bond fund would almost surely be smaller. The loss would appear smaller still were we to take into account the fact that individuals have access to other assets outside of the plan. Nonetheless, it is probably still true that not having a money market option lowers the welfare of investors who are more risk averse than the average. Similarly, the small welfare gain from index bonds, which we calculated, would probably become even smaller, in the context of the broader spectrum of assets, especially when one considers that social security is indexed.

Our agenda for future research starts with a more detailed quantitative analysis of the impact of these additional assets.

Notes

1. An example of particular relevance to academics is the plan managed by the Teachers Insurance and Annuity Association and offered by many private educational institutions in the United States. Under this plan the participant can choose between a common stock fund, the College Retirement Equities Fund (CREF), and a second fund which is essentially a portfolio of intermediate term nominal bonds.

2. See, for example, the paper by Fischer (1975) and the references cited therein.

3. All of these simplifying assumptions are, of course, counterfactual, and there is a considerable literature on the effect of relaxing each of them. The only one we think would materially affect the main results in this paper is the no-taxes assumption. We discuss its likely effects in sect. 4.4.

4. A necessary condition for (4) to be correct is $\rho > \gamma\nu$. See Merton (1969).

5. Pratt (1964) shows that for small changes in wealth this insurance premium is approximately $1/2\delta x^2$. Note that x^2 is the variance of the proportional change in wealth caused by the risky prospect.

6. See Bodie (1982) for a discussion in terms of nominal rates of return and unanticipated inflation.

7. Note that if $(\rho - \gamma\nu)/\delta > 0$ then as $H \to \infty$ eq. (12) reduces to

$$\hat{W} = W\left(\frac{\rho - \gamma\hat{\nu}}{\rho - \gamma\nu}\right)^{\frac{\delta}{\gamma}}. \tag{12'}$$

8. Duration, as defined by Macaulay (1938), is a weighted average of the years to maturity of each of the cash flows from a security. The weights are the present value of each year's cash flow as a proportion of the total present value of the security. Duration equals final maturity only in the case of pure discount bonds. For coupon bonds and mortgages, duration is always less than maturity. The difference between maturity and duration for ordinary coupon bonds and mortgages is greater the longer the final maturity and the higher the level of interest rates. In our sample of bonds this difference rose steadily over the 1953–81 period due to the rising trend in interest rates. The most pronounced differences were in the 8-year duration category. In 1953 the average maturity of the bonds in our 8-year duration portfolio was just under 9 years, whereas in 1981 the average maturity of the 8-year duration portfolio was 23 years. This variation over the last 30 years calls into question the appropriateness of a bond return series with a constant maturity of 20 years, such as the one tabulated by Ibbotson and Sinquefield (1982).

9. For the arguments on both sides of this debate, see Barro (1974) and Tobin (1980).

10. Including residential real estate would raise another theoretical issue. Individual holdings of residential real estate serve both to diversify the portfolio and to hedge against changes in the relative price of housing services. This hedging demand is ignored in our model, and including it would substantially increase the difficulty of solving for the J function.

11. Equation (9) in our model implies that

$$R_i - R_{\min} = \frac{(\sigma_{iM} - \sigma^2_{\min})}{(\sigma^2_M - \sigma^2_{\min})} (R_M - R_{\min}) .$$

Since $\sigma^2_{\min}$ is very small relative to the covariance of stocks with the market and to the variance of the market, we get $R_{\text{stocks}} - R_{\min} \cong \beta_{\text{stocks}} (R_M - R_{\min})$.

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Comment Martin J. Gruber

The paper by Bodie, Kane, and McDonald, "Inflation and the Role of Bonds in Investor Portfolios," is both interesting and innovative. To the best of my knowledge, it is the first attempt to employ a model of multiperiod equilibrium to examine the welfare loss of restricting the set of assets from which investors can choose. This is a huge undertaking, and the paper shows a great deal of intelligent effort. Like all first papers in an area, there are some things which could be done differently and perhaps improved.

My major concern with this paper is the policy implications which the authors suggest, based on their analysis. The analysis shows that investors can gain most of the advantages of holding the 10 assets classes examined from choosing among three asset classes: a money market fund, stocks, and bonds of 2 years duration. If two funds are offered, they should be a stock fund and a bond fund of 1 or 2 years duration. While these conclusions do describe the investment process facing an investor, one must be very careful in applying these results to the type of investment vehicles which should be offered by private retirement savings plans. This is the stated objective of this paper. The authors' results depend heavily on the investors' ability to sell short as well as to buy any of the portfolios offered. For example, in examining the optimal investment for an individual with a relative risk aversion coefficient of 2 with $10,000 to invest, the authors advocate the following (table 4.4, line 1):

Common stock	Buy	12,060
One-month bill	Sell short	9,840
All other bonds	Buy	7,780

Note that the investment in common stocks is larger than the $10,000 which the investor placed in the pension fund. The authors' results depend on the ability of the investor to short sell any type of portfolio offered by the pension fund and to use the proceeds of this short sale to buy other portfolios. The implications the authors draw for the type of investment portfolios a pension fund should offer individuals is based on

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this ability. But in choosing from among the portfolios offered by a pension fund, one cannot short sell one or more portfolios. To draw more meaningful conclusions about the appropriate portfolios for pension funds to offer, the authors must reexamine their problem with short sales restrictions.[1]

Also, in drawing policy implications from this paper, the reader should be aware that the model used assumes that the holder of the pension funds (*a*) owns no other assets; (*b*) has no other sources of income; and (*c*) finances all consumption over his lifetime from the pension fund. While this might be a reasonable description of a subclass of retired individuals, it certainly does not fit the average participant in a pension fund. Care must be taken in drawing policy implications from this paper without a more careful examination of the assets and income stream of pension participants.

I would like briefly to raise a question about the authors' empirical results and then to make a comment about their model. It is always difficult to choose a time frame from which to draw parameters on a model such as that presented by the authors.

Bodie et al. employ an equilibrium model of the form

$$R_i = R_{\min} = \bar{\delta}(\sigma_{im} - \sigma^2_{\min})\,.$$

They have chosen to use the variance of returns and correlation matrix of returns for the period 1973–81 together with an estimate of $\bar{\delta}$ (the coefficient of relative risk aversion) for the period 1889–1979. They have rejected using the returns for the period 1973–81, but rather generate them from the equation presented above. The authors' results are affected by the fact that the 1973–81 period was a time of rapid and changing inflation. This resulted in a different absolute and relative risk of securities than in earlier periods. For example, long-term bonds have over twice the risk (standard deviation) relative to stocks in this period that they had in earlier periods. One might reasonably ask if this change in absolute and relative risk was accompanied by a change in $\bar{\delta}$. While Bodie et al. accepted the Grossman and Shiller estimate of $\bar{\delta}$ found as 4 over the period 1889–1979, they ignored the comments made by Grossman and Shiller in the same article that the estimates of $\bar{\delta}$ for recent subperiods were much higher.

I would like to discuss briefly an alternate way of viewing the equilibrium model employed in this paper. Solving the authors' equation (10) for $\bar{\delta}$ and substituting into equation (9), we see that

$$(1) \qquad R_i - R_{\min} = \frac{(R_m - R_{\min})}{\sigma^2_m - \sigma^2_{\min}}(\sigma_{im} - \sigma^2_{\min})\,.$$

While this form of the model is correct, there is an alternative way of writing it which, I think, is simpler and will be more familiar to readers.

Since this model must hold for all assets and portfolios, it holds for an asset which has a zero beta with the market portfolio, or

$$R_z - R_{\min} = \frac{R_m - R_{\min}}{\sigma_m^2 - \sigma_{\min}^2}(-\sigma_{\min}^2).$$

Solving for $R_{\min}$, substituting into equation (1) above, and simplifying, we find

$$R_i = R_z + \beta_i(R_m - R_z). \qquad (2)$$

The two-factor (Black) CAPM model holds in real terms with the continuous formulation of variables.

In fact, we could have derived this quite simply without resorting to Merton's work. Under a log or power utility function and i.i.d. returns, we know that myopic decisions are optimal.[2] Since investors are maximizing a utility function in terms of means and variances of real returns, equation (2) follows directly from Roll's work.[3]

In summary, I believe the authors have made an interesting start at examining an important and complex problem. They indicate at several points in their paper that this is the first step in a continuing research project. I look forward to following their continuing research.

Notes

1. In the revised version of their paper, the authors advocate the use of futures as the more likely way to alleviate the short sales restriction for the less risk-averse investor. One should be aware that the incorporation of futures into the analysis could significantly modify the authors' conclusions. The expected return, variance, and covariance with other assets of a portfolio of futures is significantly different from a leveraged portfolio of stocks because the purchase of futures involves no cash outflow; dividends are not received by the holder of stock futures; and marking to the market involves intermediate cash flows.

2. See Edwin J. Elton and Martin J. Gruber, "The Multiperiod Consumption Investment Problem and Single Period Analysis," *Oxford Economic Papers* 26 (July, 1974):289–301.

3. See Richard Roll, "A Critique of the Asset Pricing Theory's Tests; Part 1: On Past and Potential Testability of the Theory," *Journal of Financial Economics* 4 (March, 1977):129–76.