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A Theory of the Allocation of Time and Goods Over the Life Cycle

1.1 ASSUMPTIONS AND EQUILIBRIUM CONDITIONS

The three main building blocks of our analysis are: (i) the now traditional Fisherian theory of consumption planning over time; (ii) the recent approach to the allocation of time that treats it on equal footing with the allocation of goods; (iii) the household production function approach that considers time and goods not as objects of choice in utility functions but as inputs into the production of household outputs that are these objects. This marrying of the old and the new permits us to obtain novel results while preserving much of the Fisherian format.

In order to simplify the presentation and bring out the main points we make several assumptions that are relaxed later on. Each

NOTE: We consider ourselves equally important contributors to this chapter. Becker's primary contribution is his unpublished paper, "The Allocation of Time and Goods Over Time" (June 1967), and Ghez's is a series of papers starting in 1966 and culminating in "A Theory of Life Cycle Consumption" (Ph.D. diss., Columbia University, 1970). Ghez is solely responsible for Chapter 2 and Becker for Chapter 3. We are equally responsible for Chapter 4.

decision unit is assumed to be a single person rather than the more common multiperson families. Each unit is assumed to know with perfect certainty its life expectancy, utility function, production functions, flows of goods and time, and all other relevant magnitudes. Calendar time is divided into T periods of equal length, called years, and a single output is assumed to be produced in each period with a household production function that is the same in each period. The arguments in the production function are the service flows of goods and time. We assume that all goods are nondurable; this assumption is relaxed in Chapter 2. Time can be allocated to only two sectors: the market sector, where command over goods is received in return, or the nonmarket sector, where it is used directly to produce household outputs. In particular, we rule out any allocation of time (or goods) to the production of human capital.

Symbolically, these assumptions are expressed in a series of relations for each decision unit between the input of goods and time and the output of what we shall henceforth call commodities:

$$C_t = F(X_t, L_t), \qquad t = 1, 2, \ldots, T. \quad (1.1)$$

where X_t is the aggregate input of the services of goods in the tth period, L_t is the input of the individual's own time, and C_t is his output of commodities.[1] The C_t in principle can be measured and observed, but they are not marketable; instead they enter directly into the utility function:

$$U = U(C_1, C_2, \ldots, C_T). \quad (1.2)$$

This function depends on the stream of present and future commodity flows.

By substituting the relations given by equation (1.1) into equation (1.2), we get the "derived" utility function of goods and time:

$$\begin{aligned} U &= U[F(X_1, L_1), F(X_2, L_2), \ldots, F(X_T, L_T)] \\ &= V(X_1, X_2, \ldots, X_T; L_1, L_2, \ldots, L_T). \quad (1.3) \end{aligned}$$

A full justification of our decision to restrict the presentation to the seemingly more complicated two-stage formulation given by equations (1.1) and (1.2), rather than the simple utility function of equation

1. C_t is to be thought of as a quantity index over all nonmarket outputs. For a more disaggregated analysis, see Ghez, "Life Cycle Consumption," App. A.

(1.3) is presented elsewhere.[2] Here we only point out that the two-stage formulation emphasizes the special relation between goods and time in the same period compared to the relation between goods and time from randomly selected periods. Put more technically, the two-stage formulation implies that goods and time of the same period can be separated in the derived utility function; that is, the ratio of their marginal utilities does not depend on the goods and time of other periods.[3]

Let N_t denote the time an individual spends at market activities, usually called "work," during his tth year of age, and θ the length of each time period. Since we have assumed that time can be used only at work or in producing consumption,[4] we have the following T time constraints: [5]

$$L_t + N_t = \theta, \qquad t = 1, 2, \ldots, T. \quad (1.4)$$

where L_t, $N_t \geq 0$. By its very nature time cannot be transferred directly from one period to another, but we show later that it can be transferred indirectly.

If goods can be transferred between periods, consumption of goods at age t, unlike consumption of time, will not be limited by the

2. See Robert T. Michael and Gary S. Becker, "On the New Theory of Consumer Demand," *Journal of Swedish Economics*, vol. 75 (1973), pp. 378–396; and Kelvin Lancaster, *Consumer Demand—A New Approach* (New York: Columbia University Press, 1971).

3. Since

$$\frac{\partial U}{\partial X_t} = \frac{\partial U}{\partial C_t}\frac{\partial F}{\partial X_t}$$

and

$$\frac{\partial U}{\partial L_t} = \frac{\partial U}{\partial C_t}\frac{\partial F}{\partial L_t},$$

then

$$\frac{\partial U}{\partial X_t} \Big/ \frac{\partial U}{\partial L_t} = \frac{\partial F}{\partial X_t} \Big/ \frac{\partial F}{\partial L_t} = H(X_t, L_t).$$

4. In particular, we rule out the use of time in savings or in asset management. The analysis can easily be extended to cover these cases.

5. C_t, X_t, L_t, and N_t have the dimensions of total quantities produced and consumed during period t of length θ. Alternatively, and with no change in substance, the analysis could proceed with all variables defined as within-period rates (say $c_t = C_t/\theta$, $x_t = X_t/\theta$, $l_t = L_t/\theta$); accordingly, the sum of the proportions of time spent on each activity would equal unity in each period.

flow of resources at t but by the discounted value of the whole lifetime flow. Let R_t be the value at the beginning of period zero of one dollar received at age t:

$$R_t = 1/(1 + r_0)(1 + r_1) \ldots (1 + r_{t-1}), \tag{1.5}$$

where r_t is the rate of interest in period t. Barring bequests, the budget constraint for goods may be written as

$$\sum_{t=1}^{T} R_t p_t X_t = \sum_{t=1}^{T} R_t w_t N_t + A_0, \tag{1.6}$$

where p_t is the price of a unit of services of market goods at age t, w_t is the wage rate at t, and A_0 is the discounted value of property income, i.e., initial assets. Substituting the T time constraints of equation (1.4) into equation (1.6), we obtain: [6]

$$\sum_{t=1}^{T} R_t(p_t X_t + w_t L_t) = \sum_{t=1}^{T} R_t w_t \theta + A_0. \tag{1.7}$$

If both wage rates and interest rates at each year of age were given and were independent of an individual's behavior and if all his time were spent at work, the right-hand side of equation (1.7) would be the discounted value of money income, which we call "full wealth." [7] It is the sum of "full human wealth" and nonhuman wealth. The left side of equation (1.7) shows how full wealth is spent: in part directly on

6. Equation (1.7) may be rewritten in terms of real prices alone. The left-hand side is

$$\sum_t R_t(p_t X_t + w_t L_t) = \sum_t R_t p_t (X_t + \frac{w_t}{p_t} L_t) = p_0 \sum_t R_t^* (X_t + w_t^* L_t),$$

where $w_t^* = (w_t/p_t)$ and

$$R_t^* = \frac{(1 + \tilde{p}_0)(1 + \tilde{p}_1) \ldots (1 + \tilde{p}_{t-1})}{(1 + r_0)(1 + r_1) \ldots (1 + r_{t-1})}; \tilde{p}_t = \frac{p_{t+1} - p_t}{p_t}.$$

The right-hand side of equation (1.7) is

$$\sum_t R_t w_t \theta + A_0 = \sum_t R_t p_t \frac{w_t}{p_t} \theta + A_0 = p_0 \left(\sum_t R_t^* w_t^* \theta + A_0^*\right),$$

where $A_0^* = (A_0/p_0)$. Hence we have the full-wealth constraint with prices expressed in terms of goods in period 0:

$$\sum_t R_t^* (X_t + w_t^* L_t) = \sum_t R_t^* w_t^* \theta + A_0^*.$$

7. By analogy with the "full-income" concept developed by Becker in "A Theory of the Allocation of Time," *Economic Journal* (September 1965), pp. 493–517.

goods and in part indirectly by using time for consumption rather than at work.

We assume a person maximizes his utility subject to the constraints given by the production functions and full wealth. If the utility and production functions are twice differentiable, necessary conditions for an interior maximum include:[8]

$$\frac{\partial U}{\partial C_t} = \lambda R_t \pi_t; \qquad t = 1, 2, \ldots, T. \quad (1.8)$$

$$\pi_t = \frac{w_t}{\partial F_t/\partial L_t} = \frac{p_t}{\partial F_t/\partial X_t}; \ t = 1, 2, \ldots, T. \quad (1.9)$$

where $\partial U/\partial C_t$ is the marginal utility (at the beginning of the initial period) of commodity consumption at age t, λ is the marginal utility of wealth, π_t is the marginal cost of producing commodities at age t, and $\partial F_t/\partial L_t$ and $\partial F_t/\partial X_t$ are the marginal products of consumption time and market goods respectively at age t.

Conditions (1.8) state that the marginal utility of commodity consumption at any age should be proportional to the discounted value of the marginal cost of producing commodities at that age. Put differently, the marginal rate of substitution between commodity consumption at any two ages should equal the ratio of their discounted marginal costs. If C_t is decreased by a small amount, π_t dollars of resources are released in the form of X_t or L_t, or a combination of the two. The goods released may be lent at the market rate r_t.[9] Although time is not transferable between periods, its yield is: a reduction in L_t means a rise in work in t and hence a rise in income, which also may be lent at rate r_t. The increment in income next period of $\pi_t(1 + r_t)$ buys $\pi_t(1 + r_t)/\pi_{t+1}$ units of C_{t+1}. In equilibrium, the willingness to substitute commodity consumption at time $t + 1$ for commodity consumption at t should equal the cost of increasing commodity consumption at $t+1$ in lieu of commodity consumption at t.

Conditions (1.9) are the familiar cost minimization conditions. At each age, the increment in output from an additional dollar "spent" on time should equal the increment in output from an additional

8. The corner solution obtained when no time is spent at work is discussed briefly in section 6.

9. If money prices of goods are changing over time, the net return from lending one unit of goods in period t is equal to the real rate of interest, $r_t - \dot{p}_t$.

dollar spent on goods. If factor proportions were fixed, conditions (1.9) would be discarded and marginal cost would equal the increase in total cost when both factors are increased in fixed proportions.

In the remainder of this study, we assume that the production functions are homogeneous of the first degree: a 1 per cent increase in goods combined with a 1 per cent increase in time in period t increases commodity output by 1 per cent. This assumption appears to be rather innocuous, especially at the level of abstraction we deal with. Taken together with the assumption that wage rates are independent of hours of work, the assumption of constant returns to scale ensures that household production will be subject to constant unit costs. Hence, the marginal cost at age t, denoted by π_t, is independent of the level of commodity output at t.

1.2 MARKET PRODUCTIVITY EFFECTS OVER THE LIFE CYCLE

In this and the next several sections we analyze some implications of the model just set out. Our interest in this study is centered on the life cycle: we seek to explain the allocation of goods and time over the life cycle.[10] Our primary focus is on the demand for market goods and time, because these data are used to test the model. The pattern of consumption of commodities is described only enough to make the pattern of derived demand for goods and time understandable.

The basic method of analysis is to decompose the changes in the demand for goods and consumption time into substitution between goods and time in production, and substitution between commodities in consumption. Equation (1.9) can be written as

$$\frac{\partial F/\partial L_t}{\partial F/\partial X_t}=\frac{w_t}{p_t}. \tag{1.10}$$

This states that, in equilibrium, the marginal rate of substitution in production is equal to the ratio of factor prices, which is the opportunity cost of time expressed in terms of goods, or, for short, the real wage rate. Taking equations (1.10) and (1.11) together, we can ex-

10. Although the comparative statistics of the model are not discussed, the fundamental principles of demand analysis apply here. In particular, a fall in the marginal cost of C_t compensated for so as to hold real wealth constant increases the amount of C_t consumed.

press the demand for goods and consumption time at age t as functions of the real wage rate and household output at t

$$X_t = X(w_t/p_t, C_t); \tag{1.11}$$

$$L_t = L(w_t/p_t, C_t). \tag{1.12}$$

An increase in the real wage rate induces substitution away from the relatively more expensive factor of production. If the real wage rate rose over time and household output were held constant, the demand for goods would increase while the demand for consumption time would fall. Therefore, substitution of factors in production makes the demand for goods positively related and the demand for consumption time negatively related to the wage rate over the life cycle.

Even if output varied systematically with the real wage rate, the *ratio* of goods to time must rise with the real wage rate, for if the production function is homogeneous, the ratio of the marginal product of goods to the marginal product of time depends only on the quantity of goods relative to time. Consequently, the demand for goods relative to time would be independent of household output,[11] and would be positively related to the wage rate over time: it would rise as the real wage rate rose, and fall as the real wage rate fell.

The percentage change in the demand for goods relative to time due to a 1 per cent change in the real wage rate is given by the elasticity of substitution in production. Let σ_f denote this elasticity, and let $\tilde{x}_t = (x_{t+1} - x_t)/x_t$ for any variable x. Then the change in goods intensity at time t is described by[12]

$$\tilde{X}_t - \tilde{L}_t = \sigma_f(\tilde{w}_t - \tilde{p}_t),$$

$$\sigma_f \geqslant 0;\ t = 1, 2, \ldots, T-1. \tag{1.13}$$

where $\tilde{w}_t - \tilde{p}_t$ is the percentage change in the real wage rate at time t. For any given change in the real wage rate, the change in the demand

11. Given homogeneity of the production function, a 1 per cent increase in output raises the demand for all inputs by the same proportion when factor prices are held constant. In particular, with a homogeneous production function of the first degree we have $X(w/p, C) = x(w/p)C$ and $L(w/p, C) = l(w/p)C$.

12. This equation holds as an approximation if σ_f is interpreted as the point elasticity of substitution evaluated at, say, the point w_t/p_t.

for goods relative to time is larger, the larger the elasticity of substitution in production.

Consider now the effects of substitution between commodities in consumption on the demand for goods and time. At constant factor prices, an increase in commodity output increases the demand for both goods and consumption time.[13] How does this output vary with age? Since perfect foresight and constant tastes have been assumed, there would be no unanticipated changes in real wealth with age. Therefore, variations in commodity consumption with age would not be due to wealth effects; they would be entirely due to time preference and to substitution effects generated by anticipated variations in prices with age.

The relevant prices for commodity consumption decisions are discounted commodity prices. Indeed, equilibrium conditions (1.8) state that the marginal utility of commodity consumption in period t should be proportional to the marginal cost of commodities in period t discounted to the initial period. Put differently, the marginal rate of substitution between commodity consumption in any periods t and $t+1$ should equal the ratio of their discounted prices:

$$\frac{\partial U/\partial C_t}{\partial U/\partial C_{t+1}} = \frac{R_t\pi_t}{R_{t+1}\pi_{t+1}}, \qquad t = 1, 2, \ldots, T-1. \quad (1.14)$$

From these conditions, we get the demand function for commodity consumption at any age t,

$$C_t = C(R_1\pi_1, R_2\pi_2, \ldots, R_T\pi_T, t, U), \quad (1.15)$$

where U is the utility index.

It is intuitively plausible that in the absence of time preference, commodity consumption would be relatively high during periods when the discounted cost of producing commodities was relatively low. Preference for the present makes early consumption relatively more attractive, whereas preference for the future makes later consumption relatively more attractive.

These implications can be derived more formally by imposing certain restrictions on the utility function. We assume that the

13. Inferior factors are ruled out by the assumption that the production function is homogeneous.

marginal rate of substitution between commodity consumption at time t and $t+1$ depends only on the commodities consumed at those two dates; it is independent of consumption at all other times. Second, we assume that all indifference curves between consecutively dated commodities are symmetric.[14] These two assumptions combined imply that the marginal rate of substitution between t and $t+1$ can be written as

$$\frac{\partial U/\partial C_t}{\partial U/\partial C_{t+1}} = \beta_{t,t+1} \frac{g(C_t)}{g(C_{t+1})}, \qquad t = 1, 2, \ldots, T-1. \quad (1.16)$$

with $g' < 0$.

Neutral time preference is said to exist if the marginal utilities of C_t and C_{t+1} are the same when $C_t = C_{t+1}$. There is preference for the present or for the future as the marginal utility of C_t is greater than or smaller than the marginal utility of C_{t+1} when $C_t = C_{t+1}$. In other words

$$\left(\frac{\partial U/\partial C_t}{\partial U/\partial C_{t+1}}\right)_{C_t = C_{t+1}} \left.\begin{matrix} > 1 \\ = 1 \\ < 1 \end{matrix}\right\} \text{defines} \left\{\begin{matrix} \text{preference for the present} \\ \text{neutral time preference} \\ \text{preference for the future} \end{matrix}\right.$$

In terms of equations (1.16), preference is for the present, for the future, or for neither as $\beta_{t,t+1}$ is greater than, smaller than, or equal to unity. By substituting equations (1.16) into the intertemporal equilibrium conditions (1.14), we obtain:

$$\beta_{t,t+1} \frac{g(C_t)}{g(C_{t+1})} = \frac{R_t \pi_t}{R_{t+1} \pi_{t+1}}. \qquad t = 1, 2, \ldots, T-1. \quad (1.17)$$

14. The first assumption implies that the lifetime utility function is additively separable. For a proof, see William Gorman, "Conditions for Additive Separability," *Econometrica* (July–October 1968), pp. 605–609. The second assumption implies that the rate of time preference is independent of wealth. Together, these assumptions imply that the utility function may be written as

$$U = \sum_{t=1}^{T} \beta_t \, G(C_t),$$

where in equation (1.16)

$$g(G_t) \equiv \frac{\partial G}{\partial C_t}$$

$$\beta_{t,t+1} \equiv \beta_t / \beta_{t+1}.$$

It will be convenient to transform all prices into real prices. Let R_t^* denote the value in terms of goods in the initial period of one unit of goods received in t, and let π_t^* denote the marginal cost of commodities in period t in terms of goods in period t: $\pi_t^* = \pi_t/p_t$. Then the equilibrium conditions (1.17) can be written equivalently as

$$\beta_{t,t+1} \frac{g(C_t)}{g(C_{t+1})} = \frac{R_t^* \pi_t^*}{R_{t+1}^* \pi_{t+1}^*}, \qquad t = 1, 2, \ldots, T-1. \tag{1.18}$$

$$= (1 + r_t^*) \frac{\pi_t^*}{\pi_{t+1}^*}, \qquad t = 1, 2, \ldots, T-1.$$

where r_t^* is the real rate of interest in period t.

Assume for the moment that the real rate of interest is equal to zero, and that time preference is neutral ($\beta_{t,t+1} = 1$ for all t). Then conditions (1.18) become

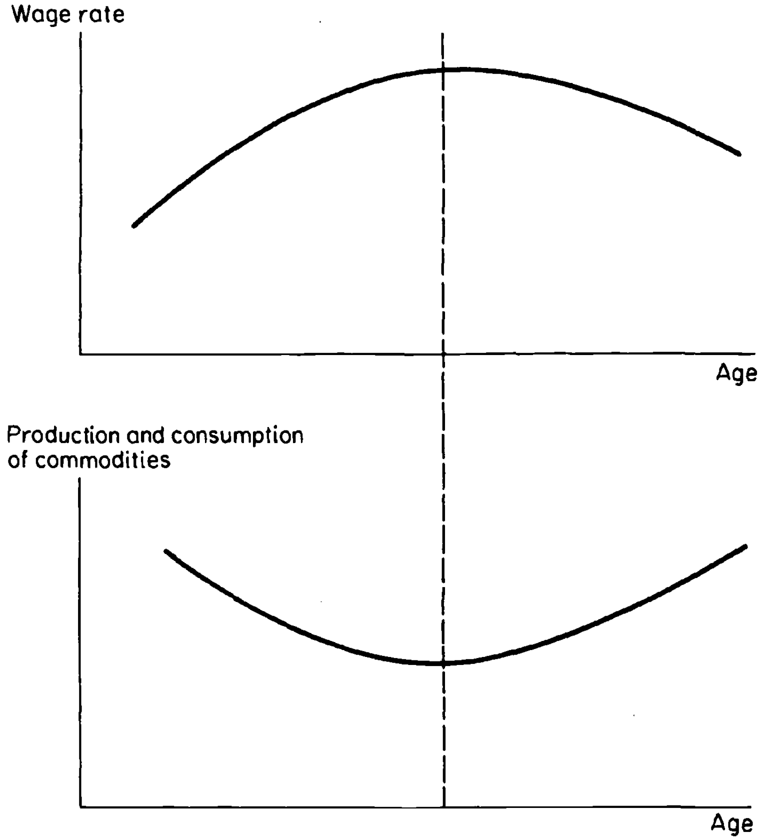
$$\frac{g(C_t)}{g(C_{t+1})} = \frac{\pi_t^*}{\pi_{t+1}^*}. \qquad t = 1, 2, \ldots, T-1. \tag{1.19}$$

Since $g' < 0$, commodity consumption would rise or fall over time as the real marginal cost of producing commodities falls or rises. Put differently, the household shifts its consumption toward periods when the real cost of consumption is relatively low, for in so doing it achieves the maximum possible lifetime utility consistent with its resources.[15]

Real marginal costs depend only on the real wage rate (the opportunity cost of time used in household activities), because we have assumed constant returns to scale and constant household technology. Since time is important in home production, marginal costs would be relatively high when the wage rate was relatively high: they would rise together, peak at the same age, and fall together. It follows from equations (1.19) that commodity consumption would be falling

15. In other words, if $G(C_t)$, in note 14, is identified as the household's per period level of utility (= real income), then it will be low when the cost of consumption is relatively high.

FIGURE 1.1
CONSUMPTION OF COMMODITIES OVER THE LIFE CYCLE



when the wage rate was rising and rising when the wage rate was falling;[16] in Figure 1.1 these patterns are portrayed over the life cycle.

Variations in the wage rate with age set in motion two effects: substitution between goods and time of the same period and substitution between commodities of different periods. While the wage rate is rising, a household substitutes goods for time and present commodities for future ones. Therefore, substitution in production

16. The conclusion that commodity consumption is inversely related to the wage rate (at a given level of real income) does not depend on the assumption of constant costs (or constant returns to scale). If costs were a rising function of output, the same rise in costs over time would simply make the decline in home consumption somewhat smaller than if costs were independent of output.

and substitution in consumption both reduce the demand for consumption time: hours spent in the nonmarket sector fall as the wage rate rises because less time is used per unit of output and because the level of output falls. Since wage rates typically rise rapidly initially with age, taper off, and then often fall at older ages, hours spent in household activities would fall rapidly initially, taper off, and reach a trough at the peak wage rate age, and rise later on when the wage rate fell (see Figure 1.2).

Since we assume that time can be allocated only to market or consumption activities, hours spent in the market, i.e., hours at "work," would be positively related to the wage rate over the life cycle. They would rise as long as the wage rate rose and fall when the wage rate fell. In the standard analysis of the supply of labor, a rise in the wage rate generates a substitution effect in favor of working time and an income effect away from it. The income effect is often supposed to dominate and cause a "backward-bending" supply curve of labor. In our analysis there is no income or wealth effect because all changes in wealth are perfectly foreseen. Hence a rise in wage rates with age generates only substitution effects, and the supply curve of labor would be positively sloped.[17]

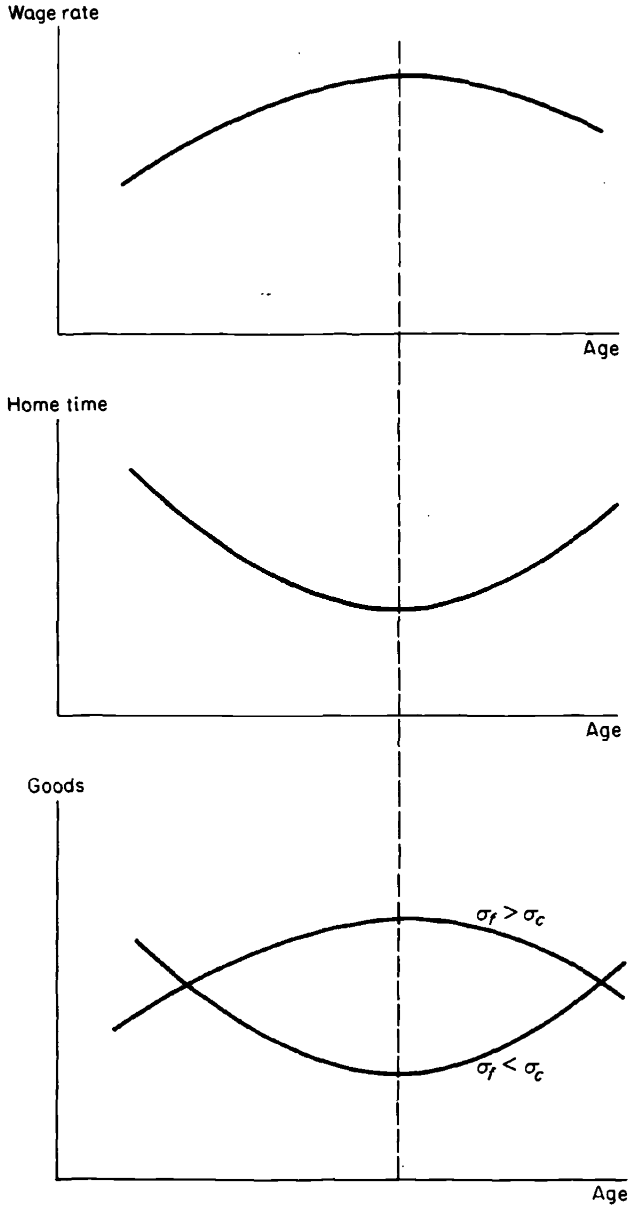
The life cycle pattern of the demand for market goods is not as clearly defined as that for home time. As the wage rate rises with age, the demand for goods increases relative to home time. If output were stationary, the absolute demand for goods would also rise. However, as the wage rate rises, commodity consumption falls, and this reduces the demand for both goods and time, that is, substitutions in production and in consumption have opposite effects on the demand for goods: to predict the direction of change in the demand for goods as the wage rate varies, it is essential to know the relative strengths of these two substitution effects. If substitution in production is easier than in consumption, a household will increase its consumption of goods as wage rates rise and decrease it when wage rates fall. The opposite will be true if substitution in consumption is easier. These two types of paths of goods consumption are displayed in the bottom panel of Figure 1.2.

More formally, it can be shown that with neutral time preference

17. This conclusion is not a negation of the observation that a parametric shift in the wage profile generates both income and substitution effects.

FIGURE 1.2

ALLOCATION OF TIME AND GOODS OVER THE LIFE CYCLE



and zero interest rates, the change in the demand for consumption time is related to the change in the wage rate as follows:[18]

$$\tilde{L}_t = -[\sigma_f(1 - s_t) + \sigma_c s_t](\tilde{w}_t - \tilde{p}_t). \qquad t = 1, 2, \ldots, T-1. \quad (1.20)$$

Similarly, the change in the demand for market goods is

$$\tilde{X}_t = (\sigma_f - \sigma_c)s_t(\tilde{w}_t - \tilde{p}_t), \qquad t = 1, 2, \ldots, T-1. \quad (1.21)$$

where

$\tilde{w}_t - \tilde{p}_t$ = percentage change in the real wage rate during period t;

σ_f = elasticity of substitution between goods and time in production, $\sigma_f \geqslant 0$;

σ_c = elasticity of substitution in consumption, $\sigma_c \geqslant 0$;

s_t = proportion of household production costs accounted for by time during period t; $s_t = w_t L_t/(p_t X_t + w_t L_t)$.

The elasticity of substitution in consumption, σ_c, measures the percentage change in commodity demand due to a 1 per cent change in its price, whereas the proportion of forgone earnings in total costs, s, measures the percentage change in the marginal cost of commodities due to a 1 per cent change in the wage rate. Therefore, $-\sigma_c s$ measures the percentage change in commodity demand due to a 1 per cent change in the wage rate. Since the production function is assumed to be homogeneous of the first degree, substitution in consumption is the same for goods and time as it is for commodities. This explains why $-\sigma_c s$ enters both equations (1.20) and (1.21).

Substitution in production, however, has different effects on goods and time. On goods the effect is measured by $\sigma_f s$, whereas on consumption time it is measured by $-\sigma_f(1 - s)$. A 1 per cent increase in the wage rate raises goods intensity, or the demand for goods *relative* to time, by σ_f per cent. A 1 per cent increase in goods intensity at constant output raises the absolute demand for goods by s per cent and reduces the absolute demand for time by $(1 - s)$ per cent.

Since σ_f, σ_c, and s are nonnegative numbers (s has a maximum

18. Equations (1.20)–(1.23), (1.25), and (1.26) are approximations. Abstracts of proofs are given in the appendix. Complete proofs are contained in Ghez, "Life Cycle Consumption."

value of unity), the implication of equation (1.20) is that a change in the demand for consumption time is negatively related to a change in the wage rate. The change in demand is a weighted average of the elasticity of substitution in consumption and in production. The direction of the change in the demand for market goods, on the other hand, is not clear a priori and depends on the difference between these elasticities. Changes in the demand for goods will be positively or negatively related to changes in the wage rate as the elasticity of substitution in production is greater than or less than the elasticity of substitution in consumption.[19]

1.3 EFFECTS OF THE INTEREST RATE AND OF TIME PREFERENCE

We have been assuming neutral time preference and a zero interest rate. A positive interest rate reduces the value of discounted future commodity prices lelative to present ones, and thus induces a substitution toward future commodities, consumption time, and goods relative to present ones. By contrast with variations in the price of time, however, positive interest rates do not affect the optimal combination of factors. The ratio of goods to time is independent of the rate of interest,[20] and depends only on the concurrent real wage rate [see equation (1.10)].

Given a zero rate of interest and neutral time preference, the consumption of commodities and time will fall as the wage rate rises, reach a trough at the peak wage age, and then rise. Therefore, a positive interest rate will push the trough to an earlier age than the peak wage age. Put differently, the peak in hours of work will come earlier than the peak wage rate. If goods rise with the wage rate ($\sigma_f > \sigma_c$), the peak in goods consumption will be pushed to a later age than the peak wage rate age.[21] If the rate of interest is sufficiently high, the

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19. Since σ_f can depend on the real wage rate and σ_c on the level of commodity consumption, the difference $(\sigma_f - \sigma_c)$ could be positive for some values and negative for other values of w_t^* and C_t along a given life cycle path.

20. This implication is no longer true in a model incorporating durable consumer goods. On this point, see Chapter 2.

21. By the same reasoning, if goods fell when the wage rate rose ($\sigma_f < \sigma_c$), the trough in goods consumption would occur earlier than the peak wage rate age.

trough in commodity and time consumption will come at the initial age and the peak in goods at the terminal age; hence, all three will rise continuously over the life cycle.[22]

Time preference for the future has the same kind of effect as a positive interest rate: it encourages future consumption relative to present consumption, but does not affect the optimal combination of factors. The troughs in the consumption of commodities and time will fall earlier and the peak in goods later than the peak wage rate. Preference for the present has precisely the opposite effects.

Equations (1.20) and (1.21) are easily modified to accommodate the effects of the interest rate and of time preference. Changes in consumption time are now described by

$$\tilde{L}_t = -[\sigma_f(1-s) + \sigma_c s](\tilde{w}_t - \tilde{p}_t) + \sigma_c(r_t - \tilde{p}_t - \rho_t); \qquad t = 1, 2, \ldots, T-1. \quad (1.22)$$

and changes in goods, by

$$\tilde{X}_t = (\sigma_f - \sigma_c)s(\tilde{w}_t - \tilde{p}_t) + \sigma_c(r_t - \tilde{p}_t - \rho_t); \qquad t = 1, 2, \ldots, T-1. \quad (1.23)$$

where $r_t - \tilde{p}_t$ is the real rate of interest at time t, and ρ is an index of time preference[23] that is positive, negative, or zero depending on whether preference is for the present, the future, or neither. Only the difference between the rates of interest and time preference enters these equations.[24] They affect time and goods in exactly the same way because we have assumed that the production functions are homogeneous.

In equations (1.22) and (1.23) the changes in time and goods are decomposed into more fundamental determinants: changes in wage rates and in interest rates net of time preference. The Fisherian model of lifetime planning as developed by Modigliani and associates

22. Note that if the rate of interest net of time preference were positive, multiple extremes could occur in the consumption paths for commodities, time, and goods (even if the path of the wage rate had only a single peak) if $\tilde{w}$ did not decline monotonically, if the share of time, s, were variable, or if the rate of interest were variable. However, the last trough in time would precede, and the first peak in goods (assuming $\sigma_f > \sigma_c$) would occur after, the peak wage rate.

23. ρ_t and $\beta_{t,t+1}$ are related by: $\beta_{t,t+1} = 1 + \rho_t$.

24. This is borne out by the intertemporal equilibrium conditions (1.17).

neglects the first determinant and concentrates exclusively on the second.[25]

1.4 NONMARKET PRODUCTIVITY EFFECTS

In this section we examine the effect of variations in nonmarket productivity over time on the life cycle demand for goods and time. We need not at this point detail the sources of change in nonmarket efficiency; in the next section changes in both market and nonmarket efficiency are related to changes in the stock of human capital.

Changes in nonmarket efficiency are reflected in shifts in the productivity of goods and time in the household production functions. Formally, the production function at age t can be written as:

$$C_t = F(X_t, L_t; t). \tag{1.24}$$

To begin with, notice that the utility-maximizing conditions set out in equation (1.8) and (1.9) or in equations (1.10) and (1.17) still hold, but that the marginal products of goods and time now depend not only on input proportions but also on age itself (t). Changes in age result in either an increase or decrease in the output producible with given inputs. For simplicity, in the following discussion we talk only about improvements with age.

Technological improvement with age raises the marginal product of goods, consumption time, or both, in future periods relative to present ones, for given levels of these inputs. It thus lowers the marginal cost of commodities in future periods compared to present ones, and induces substitution toward future commodities.

The effect on the derived demand for goods and time depends on the magnitude of the output response relative to the saving in inputs generated by the technological improvement. The output response is measured by the elasticity of substitution in consumption: a 1 per cent fall in marginal costs generates a σ_c per cent rise in the consumption of commodities. On the other hand, there would be a 1 per

25. See Franco Modigliani and Richard Brumberg, "Utility Analysis and the Consumption Function: An Interpretation of Cross-Section Data," in K. Kurihara, ed., *Post-Keynesian Economics* (New Brunswick: Rutgers University Press, 1954), pp. 383–436. Also Franco Modigliani and Albert Ando, "The 'Permanent Income' and the 'Life Cycle' Hypothesis of Saving Behavior: Comparison and Tests," in Irving Friend and Robert Jones, eds., *Proceedings of Conference on Consumption and Saving*, vol. 2 (Philadelphia: University of Pennsylvania Press, 1960), pp. 49–174.

cent reduction in the inputs needed to produce a given output. Hence, consumption of goods and time would rise along with technology if the elasticity of substitution in consumption were greater than unity, and would fall if the elasticity of substitution in consumption were smaller than unity.

The changes in consumption time and goods described by equations (1.22) and (1.23) can be expanded to include changes in nonmarket efficiency with age,

$$\tilde{L}_t = -[\sigma_f(1-s) + \sigma_c s](\tilde{w}_t - \tilde{p}_t) + (\sigma_c - 1)\hat{F}_t + \sigma_c(r_t - \tilde{p}_t - \rho_t); \quad t = 1, 2, \ldots, T-1. \quad (1.25)$$

$$\tilde{X}_t = (\sigma_f - \sigma_c)s(\tilde{w}_t - \tilde{p}_t) + (\sigma_c - 1)\hat{F}_t + \sigma_c(r_t - \tilde{p}_t - \rho_t); \quad t = 1, 2, \ldots, T-1. \quad (1.26)$$

where $\hat{F}_t$ denotes the rate of change in nonmarket efficiency at time t.[26]

1.5 THE PRODUCTION OF HUMAN CAPITAL

We have been explaining the allocation of time and goods over a lifetime by life cycle variations in wage rates, nonmarket efficiency, interest rates, and preferences. The thrust of the substantial research during the last fifteen years on investment in human capital, however, has been precisely to show that variations in wage rates and even in nonmarket efficiency are not simply given: they are largely determined by investments in schooling, on-the-job training, health, preschooling, and other kinds of human capital.[27]

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26. These equations assume Hicksian factor-neutral technological change. If the change is factor biased it is necessary to add $-\sigma_f(1-s)B_t$ to equation (1.25) and $\sigma_f s B_t$ to (1.26), where $B_t = \widehat{MPX}_t - \widehat{MPL}_t$, with $\widehat{MPX}_t$ and $\widehat{MPL}_t$ measuring the percentage changes in the marginal products of goods and time at time t. Then $\hat{F}_t$ is a weighted average of these changes: $\hat{F}_t = (1-s)\widehat{MPX}_t + s\widehat{MPL}_t$.

27. For an outstanding discussion of the effects of schooling and post-school investments, see Jacob Mincer, *Schooling, Experience, and Earnings* (New York: NBER, 1974). For interesting discussions of the effect of human capital on nonmarket efficiency, see Robert T. Michael, *The Effect of Education on Efficiency in Consumption*, NBER Occasional Paper 116 (New York: NBER, 1972) and Michael Grossman, *The Demand for Health: A Theoretical and Empirical Investigation*, NBER Occasional Paper 119 (New York: NBER, 1972).

The essence of the approach is to define a stock of human capital owned by each person; he can produce more, but since the capital is embodied in his own person and since even voluntary slavery is considered illegal, he cannot sell or buy any capital. Let H_t denote the stock of human capital he holds at the beginning of time t. The wage rate at t is assumed to be proportional to H_t

$$w_t = e_t H_t. \qquad t = 1, 2, \ldots, T. \quad (1.27)$$

Expressing this relationship in real prices, i.e., in terms of units of consumer goods, we have

$$w_t^* = e_t^* H_t, \qquad t = 1, 2, \ldots, T. \quad (1.28)$$

where $e_t^* = e_t/p_t$. The factor of proportionality e_t^* measures the service yield per unit of human capital for each hour spent at work at time t. The service yield can vary over time because of changes in the economy at large.

Human capital is produced by using own time and a bundle of market goods and services, which we call educational goods. Let h_t denote the amount of human capital produced at time t, and N_t' and X_t' the time and educational goods used in the production of h_t. The production function that relates these inputs and outputs is

$$h_t = h(N_t', X_t'). \qquad t = 1, 2, \ldots, T. \quad (1.29)$$

We are assuming for the present that H_t, the stock of human capital at t, does not affect the productivity of N_t' and X_t'. If human capital never depreciated, the change in its stock at any point would simply equal the amount produced. More generally, if δ_t denotes the rate of depreciation on this stock at age t, we have [28]

$$H_{t+1} = H_t(1 - \delta_t) + h_t.$$
$$t = 1, 2, \ldots, T-1. \quad (1.30)$$

Gross investment at time t (h_t) equals net investment ($H_{t+1} - H_t$) plus depreciation ($\delta_t H_t$).

The amount of human capital held at any age can be expressed in terms of the undepreciated component of the initial inherited

28. Since the stock of human capital is durable, $\delta_t < 1$ for all t; and since nature by itself does not create human capital, $\delta_t \geqslant 0$ for all t.

stock[29] and the amounts previously produced:

$$H_t = H_1 D_{1,t} + \sum_{v=1}^{t-1} h_v D_{v+1,t}, \qquad t = 1, 2, \ldots, T. \quad (1.31)$$

where $D_{v,t}$ is the fraction of human capital held at age v and remaining at age t, i.e.,

$$D_{v,t} = (1 - \delta_v)(1 - \delta_{v+1}) \ldots (1 - \delta_{t-1}). \qquad v = 1, 2, \ldots, t-1. \quad (1.32)$$

Equation (1.31) clearly shows that human capital held at any age depends on the past resources devoted to its production. Indeed, this may be confirmed by substituting the production function (1.29) into equation (1.31):

$$\begin{aligned} H_t &= H_1 D_{1,t} + \sum_{v=1}^{t-1} h(N'_v, X'_v) D_{v+1,t} \\ &= H(N'_1, N'_2, \ldots, N'_{t-1}; X'_1, X'_2, \ldots, X'_{t-1}; t; H_1). \end{aligned} \qquad t = 1, 2, \ldots, T. \quad (1.33)$$

We start with the seemingly more complicated two-stage formulation given by equations (1.29) and (1.31) rather than with the direct formulation given by equation (1.33)[30] because in the former the emphasis is on the special relationship that exists between investment time and educational goods used at the same time as compared to use of these inputs at different times.[31]

29. Since the initial age is arbitrary in our analysis, the initial stock depends on past accumulation. It also depends on the individual's native ability and on his environment.

30. In fact, once the two-stage formulation is dropped, the concept of a stock of human capital can also be dropped and replaced by a generalized wage rate function

$$w_t^* = w^*(N'_1, N'_2, \ldots, N'_{t-1}; X'_1, X'_2, \ldots, X'_{t-1}; t). \qquad t = 1, 2, \ldots, T.$$

31. Put more technically, the two-stage formulation implies that

$$\partial H_t / \partial N'_v = \psi_{N'}(N'_v, X'_v, t, v);$$
$$\partial H_t / \partial X'_v = \psi_{X'}(N'_v, X'_v, t, v). \qquad t = 1, 2, \ldots, T; \quad v = 1, 2, \ldots, t-1.$$

Any time used in the production of human capital must be diverted from other possible uses, namely, from working time and consumption time; hence, the time constraints become

$$L_t + N_t + N_t' = \theta. \quad t = 1, 2, \ldots, T. \quad (1.34)$$

Similarly, expenditures on educational goods compete with expenditures on consumption goods. If p_t' is the price of educational goods at time t, the budget constraint (1.6) is modified to

$$\sum_{t=1}^{T} R_t(p_t X_t + p_t' X_t') = \sum_{t=1}^{T} R_t w_t N_t + A_0. \quad (1.35)$$

If the time constraints (1.34) are substituted into the budget constraint (1.35), we obtain[32]

$$\sum_{t=1}^{T} R_t(p_t X_t + w_t L_t) + \sum_{t=1}^{T} R_t(p_t' X_t' + w_t N_t') = \sum_{t=1}^{T} R_t w_t \theta + A_0. \quad (1.36)$$

For the moment the production functions (1.1) are assumed to be unaffected by the accumulation of human capital.

32. The right-hand side of equation (1.36) measures the amount of wealth attainable if all time is spent at work, but no longer measures "full wealth," for it excludes the production of human capital. An increase in the production of human capital at time t would raise all future wage rates, and would therefore increase both the right-hand side of equation (1.36) and the cost of all household production beyond time t.

Full wealth can, however, still be meaningfully defined. Let $V(L_1, L_2, \ldots, L_T)$ denote the maximum value of consumable wealth when consumption time in each period is held at fixed levels $L_1, L_2, \ldots, L_T$. That is,

$$V(L_1, L_2, \ldots, L_T) = \max\left(\sum_t R_t w_t N_t - \sum_t R_t p_t' X_t' + A_0\right),$$

given $L_1, L_2, \ldots, L_T$; hence

$$\sum_t R_t p_t X_t = V(L_1, L_2, \ldots, L_T).$$

Full wealth, W, is defined as the value of $V(L_1, L_2, \ldots, L_T)$ when no time is spent in consumption: $W = V(0, 0, \ldots, 0)$. If Ψ denotes the wealth forgone by using time to produce commodities, we have the full-wealth constraint:

$$\sum_t R_t p_t X_t + \Psi(L_1, L_2, \ldots, L_T) = W.$$

This is a development of the general definition of full income given in Becker, "Theory of Allocation." We recognize that the concept of full wealth does not add any new information to the analysis when prices are not parametric.

If utility is maximized subject to the budget constraint given by equation (1.35), the production functions for human capital [equation (1.29)] and household commodities [equation (1.24)], and the human capital constraints given by equation (1.31), the necessary conditions for an interior maximum include the following:

$$\frac{\partial U}{\partial C_t} = \lambda R_t \pi_t. \qquad t = 1, 2, \ldots, T. \quad (1.37)$$

$$\pi_t = \frac{w_t}{\partial F/\partial L_t} = \frac{p_t}{\partial F/\partial X_t}. \quad t = 1, 2, \ldots, T. \quad (1.38)$$

$$\mu_t = \frac{w_t}{\partial h/\partial N_t'} = \frac{p_t'}{\partial h/\partial X_t'}. \quad (1.39)$$

$$R_t \mu_t = \sum_{v=t+1}^{T} R_v e_v D_{t,v-1} N_v. \quad (1.40)$$

The equilibrium conditions (1.37) and (1.38) are identical to equations (1.8) and (1.9), the equilibrium conditions derived when the production of human capital was excluded. (For a modification, see page 28.) Consequently, our analysis of the paths of commodities, consumption time, and goods in the previous sections is not affected by allowing human capital to be endogenous. Put differently, the question of whether these paths are rising or falling as the wage rate is rising or falling is completely independent of the reasons for changes in the wage rate. This fundamental proposition is at the heart of all the empirical work reported in this volume.

The new equilibrium conditions due to the accumulation of human capital are given by equations (1.39) and (1.40). According to equation (1.39), at each point in time the increment in human capital (h) from an additional dollar of expenditure on time equals the increment from an additional dollar spent on educational goods. These define the marginal cost of producing human capital, which is μ_t at time t.

The left-hand side of equation (1.40), namely $R_t\mu_t$, yields the discounted value of the marginal cost of producing human capital, while the right-hand side yields the discounted value of returns from an additional unit of such capital. The term $e_v D_{t,v-1} = e_v(\partial H_v/\partial h_t)$ is the increase in the wage rate, and $e_v D_{t,v-1} N_v$ is the increase in earnings at time v attributable to an additional unit of human capital produced at time t. The discounted value of the increase in earnings measures the benefit from additional production at time t.[33] Of course, in equilibrium, the marginal cost of production should equal the marginal benefit.

It will be convenient for further analysis to express costs and returns from investment at time t in terms of consumer goods at time t, rather than in terms of dollars in the initial period. Dividing both sides of equation (1.40) by $R_t p_t$, we obtain:

$$\mu_t^* = B_t^w, \tag{1.41}$$

where $\mu_t^* = \mu_t / p_t$; and

$$B_t^w = \sum_{v=t+1}^{T} R_{v,t}^* e_v^* D_{t,v-1} N_v; \tag{1.42}$$

with $e_v^* = e_v / p_v$; and

$$R_{v,t}^* = \frac{R_v^*}{R_t^*} = 1/(1 + r_t^*)(1 + r_{t+1}^*) \ldots (1 + r_{v-1}^*).$$

Correspondingly, equation (1.39) becomes

$$\mu_t^* = \frac{w_t^*}{\partial h / \partial N_t'} = \frac{p_t'^*}{\partial h / \partial X_t'}, \tag{1.43}$$

where $p_t'^* = p_t' / p_t$. From this condition and the production function for human capital, we get the marginal cost function for human capital

$$\mu_t^* = \mu^*(w_t^*, p_t'^*, h_t). \tag{1.44}$$

Henceforth it is assumed that the price of educational goods relative to consumption goods remains unchanged over time; therefore, $p_t'^* = p_t' / p_t$ is a constant.

Whether real wage rates rise or fall over time depends on whether the stock of human capital and the index e_t^* are rising or falling.

33. Note that actual hours worked are used in the evaluation of benefits no matter how they vary by age or level of human capital. In rate-of-return calculations, if the difference in the amounts invested is small, it will not matter whether hours of work of the unskilled or of the skilled group are used, since the effect on real income of any differences in hours worked induced by the different amounts invested is of second-order smallness. On the other hand, for large differences in amounts invested, using hours of work of the unskilled group will understate returns, while using hours of work of the skilled group will overstate returns.

Indeed, from equation (1.28),

$$\tilde{w}_t^* = \tilde{e}_t^* + \tilde{H}_t. \quad t = 1, 2, \ldots, T-1. \quad (1.45)$$

The stock of human capital rises or falls depending on whether the output of human capital is larger or smaller than depreciation. If human capital never depreciated ($\delta_t = 0$ for all t), then its stock could never decrease with age, and would increase so long as some human capital were being produced. Human capital is being produced in period t if and only if the equality (1.40) or (1.41) holds for period t.

The time path of the output of human capital is implicit in the set of equilibrium conditions. Suppose for the moment that human capital never depreciated (hence $D_{t,v} = 1$ for all v, t), that the index e_v^* was constant, and that the real rate of interest was always equal to zero. Then, if human capital is being produced at time t, the stock of human capital will be higher in $t+1$ than in t (since $\delta_t = 0$), and, therefore, the real wage rate will be higher in $t+1$ than in t (since $\tilde{e}_t^* = 0$). Hence the marginal cost function of producing human capital will be higher in $t+1$ than in t. This provides one incentive for producing human capital early in life rather than later.

How do benefits from producing human capital at time t compare with benefits at later ages? Since the depreciation rate, interest rates, and the growth in efficiency of a unit of human capital are all assumed to equal zero, marginal benefits cannot increase. The change in marginal benefits would equal $-eN_{t+1}$, i.e., the loss in the service yield provided by an additional output of human capital when this addition is produced in $t+1$ rather than in t. Therefore, as long as some time is spent at work, marginal benefits must fall with age.

Since marginal benefits fall and marginal costs rise with age, the optimal production of human capital necessarily falls with age. Hence our analysis predicts the well-known finding that the real wage rate rises with age at a decreasing absolute rate.[34]

Time spent in the production of human capital must fall with age both because output falls and because of the inducement to substitute away from time and in favor of educational goods as the price of time rises.[35] Since investment time falls as the wage rate rises, and since we showed earlier that consumption time also falls as the wage

34. See, for example, Mincer, *Schooling.*

35. The demand for educational goods would fall or rise over time depending on whether the effect of output expansion was greater or smaller than the effect of factor substitution.

rate rises (barring time preference effects), hours of work would rise as the wage rate rises. The effect of the decline in consumption time on working time is reinforced by the decline in investment time.

If human capital depreciates, the stock of human capital and hence the real wage rate will decline toward the end of life, when the incentive to invest becomes small. On the other hand, growth in the efficiency of a unit of human capital causes the wage rate to rise beyond the age where net investment ceased. Indeed, depreciation and efficiency have symmetrical and opposite effects on the wage rate:[36]

$$w^*_{t+1} - w^*_t = e^*_t(H_{t+1} - H_t) + H_t(e^*_{t+1} - e^*_t)$$
$$= e^*_t[h_t - (\delta_t - \tilde{e}^*_t)H_t].$$
$$t = 1, 2, \ldots, t-1. \quad (1.46)$$

Once we drop the assumptions of zero depreciation and interest rates and constant efficiency, marginal benefits may rise for a while with age (they must ultimately decline of course). If they rose faster than marginal costs, the output of human capital would also rise with age for a while.[37]

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36. In the equation, $-\tilde{e}^*_t$ might be defined as the rate of obsolescence on human capital at time t.

37. If equilibrium condition (1.40) holds for periods t and $t+1$, by taking first differences we obtain $\mu^*_{t+1} - \mu^*_t = B^w_{t+1} - B^w_t$. But since

$$B^w_{t+1} - B^w_t = -e^*_{t+1}N_{t+1} + (r^*_t + \delta_t)B^w_t$$

and

$$\mu^*_{t+1} - \mu^*_t = \frac{\partial \mu^*}{\partial w^*_t}(w^*_{t+1} - w^*_t) + \frac{\partial \mu^*}{\partial h_t}(h_{t+1} - h_t),$$

then solving for $h_{t+1} - h_t$, we obtain

$$h_{t+1} - h_t = \frac{-\dfrac{\partial \mu^*}{\partial w^*_t}(w^*_{t+1} - w^*_t) - e^*_{t+1}N_{t+1} + (r^*_t + \delta_t)B^w_t}{\dfrac{\partial \mu^*}{\partial h_t}}.$$

Converting this expression into percentage changes, and using the relation $w^*_t = e^*_t H_t$, we have

$$\tilde{h}_t = \frac{-s'_t \dfrac{h_t}{H_t} - \dfrac{e^*_{t+1}N_{t+1}}{B^w_t} + s'_t(\delta_t - \tilde{e}^*_t) + r^*_t + \delta_t}{\dfrac{h_t}{\mu^*_t}\dfrac{\partial \mu^*}{\partial h_t}},$$

where

$$s'_t = \frac{w^*_t}{\mu^*_t}\frac{\partial \mu^*}{\partial w^*_t}.$$

If output of human capital falls monotonically with age, so will the time spent in its production.[38] Therefore, hours of work will rise as the wage rate rises because both consumption time and training time will fall.[39] If the difference between the rates of interest and time preference is zero, consumption time will reach a trough at the peak wage age, whereas the trough in training time will come later if human capital depreciates and its efficiency does not change with age. Hence, working time will peak later than the trough in consumption time, but will tend to decline eventually because the rise in consumption time will more than offset any fall in training time.

Marginal benefits at any age are positively related to the rate of output of human capital at that age, as shown by the curves B_t and B_t' in Figure 1.3.[40] Finite nonzero investment could occur only if marginal cost rose faster than marginal benefit as output of human capital rose.

If the production function for human capital given in equation (1.29) is homogeneous of the first degree, the marginal cost of producing human capital at any age will be independent of output as long as the value of time is given by the (assumed) fixed market wage rate at that age. If so much human capital is being produced that all of working time is drawn into its production, the value of time can no longer be measured by the market wage rate, since no more time will be available at that price. Additional time will have to be drawn from consumption, and the value of time in producing additional human capital will then be measured by the money equivalent of the marginal productivity of time in consumption. As more and more time is drawn out of consumption, this marginal productivity will rise, and so will the shadow price of time and the marginal cost of producing human

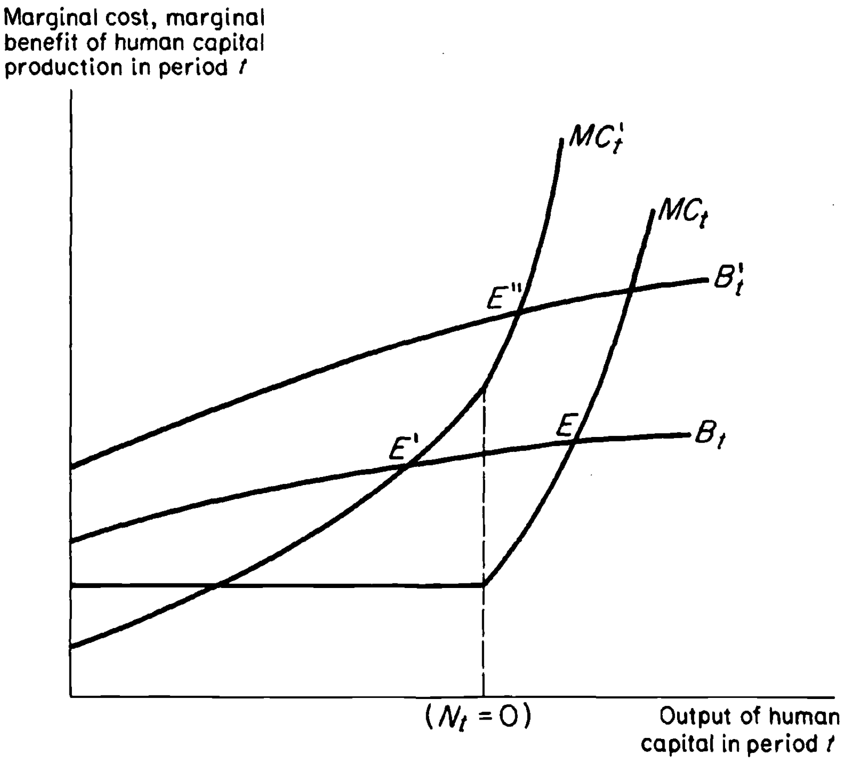
38. An exception might occur if wages fell eventually while gross output continued to be positive, because the substitution toward time induced by the declining wages could increase the time spent in training.

39. Measured working time often includes training time. For a further discussion, see Chapter 3.

40. A proof of this statement can be developed along the following lines: An increase in output of human capital at age t, with future levels of output held constant, raises the future stock of human capital. The resulting higher future wages induce substitution away from training and consumption times. Future hours of work, and thus marginal benefits at age t, rise as output of human capital rises at age t.

This point is further developed in Gilbert R. Ghez, "A Note on the Earnings Function When Human Capital Is Biased Toward Earnings" (unpublished, February 1973).

FIGURE 1.3
PRODUCTION OF HUMAN CAPITAL AT A GIVEN YEAR OF AGE



capital. The marginal cost curve, *MC* in Figure 1.3, is infinitely elastic until all working time is exhausted ($N_t = 0$) and then rises as consumption time is reduced.

Since benefits are a nondecreasing function of output, an equilibrium with positive investment is attained only if no time is spent at work. With marginal benefit schedule B_t and marginal cost schedule MC_t, equilibrium output is *E* in Figure 1.3, where no time is spent at work. The value of time is then measured by the consumption shadow price. That is, the following equality holds:

$$w_t^s = B_t^w \frac{\partial h}{\partial N_t'}, \tag{1.47}$$

where w_t^s is the value of the marginal hour spent in consumption.

If the production function for human capital exhibited diminishing returns to scale, say because new knowledge could only be absorbed at a decreasing rate, the marginal cost function would rise throughout (see MC_t' in Figure 1.3) and the equilibrium hours of work could readily be positive (see point E'). Of course, with sufficiently large benefits, the equilibrium could be at a "corner" even if marginal costs were rising throughout (point E'').

Specifying diminishing returns to scale is an ad hoc way of ensuring that positive investment will occur even when some time is spent at work. A more appealing alternative is to suppose that the production of human capital is homogeneous of the first degree not only in goods and training time, but also in working time. This, after all, is the rationale for on-the-job training: productivity is enhanced by combining work and training. Human capital will be produced with increasing cost even if working time is positive because working time and training time (and goods) can be increased in the same proportion only if consumption time is reduced. The reduction in consumption time, not diminishing returns, causes marginal costs to rise.[41]

If, for whatever reason, the equilibrium hours of work are zero for several ages—"corner equilibriums"—the change over time in benefits during these ages will be

$$\Delta B_t^w = -e_{t+1}^* N_{t+1} + (r_t^* + \delta_t) B_t^w = (r_t^* + \delta_t) B_t^w, \tag{1.48}$$

41. In this formulation, the production function of human capital would be $h_t = h(N_t, N_t', X_t')$, rather than (1.29), and equilibrium conditions (1.38) and (1.39) would be replaced by

$$\pi_t = \frac{w_t^s}{\partial F/\partial L_t} = \frac{p_t}{\partial F/\partial X_t};$$

$$\mu_t = \frac{w_t^s}{\partial h/\partial N_t'} = \frac{p_t'}{\partial h/\partial X_t'};$$

where w_t^s is the shadow wage rate;

$$w_t^s = eH_t + \sum_{v=t+1}^{n} R_{v,t} e_v D_{t,v-1} N_v \frac{\partial h_t}{\partial N_t}.$$

An alternative formulation, which would also generate increasing costs, is to suppose that the depreciation rate on human capital is negatively related to its rate of utilization, say hours of work. For a general discussion of the relationship between depreciation and utilization, see Gilbert R. Ghez, "Life Cycle Demand for Durable Consumer Goods" (unpublished, February 1968); and Robert T. Michael and Edward P. Lazear, "On the Shadow Price of Children" (unpublished, December 1971).

since hours of work, N_{t+1}, are assumed to equal zero in this interval. Marginal benefits from investment and with them the equilibrium shadow wage rate will *rise* during this interval.[42] The potential market wage rate, eH, will also be rising during this interval, and must eventually overtake the shadow wage rate; that is, the marginal benefit and cost curves must eventually intersect at a point where $N_t > 0$. The value of the individual's time would then be measured by his observed wage rate.

We have been assuming that investment in human capital at any age raises only market wage rates at later ages. Yet it presumably also raises the efficiency of later production of human capital and commodities and of asset management; that is, the stock of human capital enters the production functions given by equations (1.1) and (1.31) and influences the rate of return on nonhuman wealth, r, as follows:

$$C_t = F(X_t, L_t; H_t, t); \tag{1.49}$$

$$h_t = h(N'_t, X'_t, H_t, t); \tag{1.50}$$

and

$$r^*_t = r(H_t, t); \tag{1.51}$$

with

$$\frac{\partial C_t}{\partial H_t} \geqslant 0; \frac{\partial h_t}{\partial H_t} \geqslant 0; \frac{\partial r}{\partial H_t} \geqslant 0.$$

The more general set of equilibrium conditions that replaces equations (1.43) is:

$$\mu^*_t = B^w_t + B^c_t + B^h_t + B^r_t; \tag{1.52}$$

where

$$B^c_t = \sum_{v=t+1}^{T} R^*_{v,t}\pi^*_v C_v \left(\frac{1}{C_v}\frac{\partial F_v}{\partial H_v}\right) D_{t,v-1};$$

$$B^h_t = \sum_{v=t+1}^{S} R^*_{v,t} e^*_v H_v \frac{\partial N'_v}{\partial H_v} D_{t,v-1};$$

$S =$ number of periods during which human capital is produced ($S < T$); and

42. Therefore, once again, the consumption of commodities and time would tend to fall during this interval, and the change in goods would depend on the relative importance of substitution in production and consumption.

$$B_t^T = \sum_{v=t+1}^{T} R^*_{v+1,t} A^*_v \frac{\partial r}{\partial H_v} D_{t,v-1};$$

with

$(1/C_v)(\partial F_v/\partial H_v)$ = percentage reduction in the total cost of producing a given amount of C_v.

$\partial N'_v/\partial H_v$ = the reduction in the amount of training time in period v required to achieve a given amount of h_v (the level of education goods held constant).

$\partial r_v/\partial H_v$ = the rise in the rate of return on nonhuman wealth.

We do not discuss here the full implications of this widening of the benefits flowing from investments in human capital since they are not incorporated in the empirical discussion. We simply note that these increases in efficiency are benefits of investing in human capital that not only add to the total benefit of such investment and thus increase the amount invested at any age, but also affect its time profile. For example, an increase in the efficiency of producing human capital would reduce and perhaps reverse the tendency for the marginal cost of producing human capital to rise over time as its stock increases. This in turn would reduce the decline over time in the output of human capital, and could even result in a rise for a time.[43]

Second, the nonmarket benefits to investment in human capital depend on the percentage reduction in the total cost of producing commodities and on the planned stream of expenditures on commodities. In the particular case where human capital is time augmenting in the home, with the result that human capital carries neutral efficiency effects between work and home activities, the nonmarket returns would be simply the discounted value of the increase in cost of home time due to a small increment in human capital produced today.[44] Consequently, benefits would be positively related to

43. In note 36 we show that a rise over time in market benefits could also cause a temporary rise in the output of human capital.

44. Suppose the household production function were $C_t = F(H_tL_t, X_t)$. Then

$$\pi_t = \frac{eH_t}{\frac{\partial F}{\partial H_tL_t} H_t} = \frac{e}{\frac{\partial F}{\partial H_tL_t}};$$

and

home time and negatively related to market time. If human capital carried neutral efficiency effects across all sectors, including its own production, marginal benefits would depend on total time and would therefore decrease monotonically with age.

Third, the benefits in the form of increased efficiency in portfolio management are weighted by the planned asset holdings; therefore, the larger one's planned portfolio the larger the benefits to current investment.[45]

Our model of capital accumulation over a lifetime is related to the models developed by others in the last decade.[46] We have extended these models, however, by integrating consumption and investment decisions; in particular, the size of stock of human capital is assumed to affect efficiency in consumption as well as in market production, and the optimal allocation of time between work and "leisure" is determined simultaneously with the optimal accumulation of human capital. Efficiency in consumption is an added incentive to investment in human capital. By incorporating the effect of efficiency into the model, we are better able to explain why, say, men and women invest in different ways.[47] One consequence of making hours of work endogenous is that since they should rise with age until about the peak wage age and then decline, B^w, the benefit in higher wages resulting from investment in human capital, would decline

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$$\frac{\partial F}{\partial H_t} = \frac{\partial F}{\partial H_t L_t} L_t.$$

Therefore, B_t^c in equation (1.52) would become

$$B_t^c = \sum_{v=t+1}^{T} R_{v,t}^* e^* L_t D_{t,v-1}.$$

45. For a further development, see Uri Ben-Zion and Isaac Ehrlich, "A Model of Productive Saving" (unpublished, October 1972).

46. See, for example, Gary S. Becker, *Human Capital,* 2nd ed. (New York: NBER, forthcoming) and *Human Capital and the Personal Distribution of Income,* Woytinsky Lecture 1 (Ann Arbor: University of Michigan Press, 1967); Yoram Ben-Porath, "The Production of Human Capital and the Life Cycle of Earnings," *Journal of Political Economy* (August 1967); and Assaf Razin, "Investment in Human Capital and Economic Growth: A Theoretical Study" (Ph.D. diss., University of Chicago, 1969).

47. The reason is that women spend relatively more time in consumption than men do. If, for equal amounts of time spent at home, the marginal product of women's home time is at least as large as that of men, and if men earn more than women per unit of time, then the family's optimal allocation of resources is to have men working more hours than their wives. This point was first made by Haim Ofek in "The Allocation of Goods and Time in a Family Context" (Ph.D diss., Columbia University, 1971).

more slowly initially and more rapidly ultimately than in models in which constant hours of work are assumed.

1.6 MULTIPLE EARNERS

We now relax the assumption that the household is composed of only one person, and assume instead that it is composed of a husband and wife. The allocation of time of both members is determined simultaneously. The production function for commodities is still assumed to be homogeneous of the first degree, and can be written as

$$C_t = F_t(X_t, L_{1t}, L_{2t}), \tag{1.53}$$

where L_{1t} and L_{2t} denote the consumption time of husband and wife at age t of the household head.

If N_{1t} and N_{2t} are the time at work and w_{1t} and w_{2t} the wage rates of husband and wife at age t of the head, the budget constraint becomes

$$\sum_{t=1}^{T} R_t p_t X_t = \sum_{t=1}^{T} R_t(w_{1t}N_{1t} + w_{2t}N_{2t}) + A_0. \tag{1.54}$$

If all production of human capital is ignored for the present, the time constraints are

$$\begin{aligned} L_{1t} + N_{1t} &= \theta; \\ L_{2t} + N_{2t} &= \theta; \end{aligned} \tag{1.55}$$

with $L_{it} \geqslant 0$, $N_{it} \geqslant 0$; $i = 1, 2$. If the constraint on goods given by equation (1.54) is combined with the time constraints of equations (1.55), we get the family full-wealth constraint:

$$\sum_{t=1}^{T} R_t(p_t X_t + w_{1t}L_{1t} + w_{2t}L_{2t}) = \sum_{t=1}^{T} R_t(w_{1t}\theta + w_{2t}\theta) + A_0. \tag{1.56}$$

To maximize the utility function given by (1.2), subject to the full-wealth constraint of equation (1.56), necessary conditions (for an interior solution) must include

$$\frac{\partial U}{\partial C_t} = \lambda R_t \pi_t; \tag{1.57}$$

$$\pi_t = \frac{p_t}{\partial F_t/\partial X_t} = \frac{w_{1t}}{\partial F_t/\partial L_{1t}} = \frac{w_{2t}}{\partial F_t/\partial L_{2t}}. \tag{1.58}$$

The marginal cost of commodities at age t of the head, π_t, depends now on the wage rate at that age of both members. The equilibrium conditions given by equations (1.57), which determine how commodity output is distributed over time, continue to hold.

If the difference between the rate of interest and time preference equals zero, the consumption of commodities will rise or fall with age as their marginal cost rises or falls. If the household production function is the same at all ages, consumption will rise or fall as the real wage rates of the husband and wife rise or fall. If both their wage rates rise, consumption will fall, whereas if the wage rate of one member rises while that of the other falls, the change in marginal cost and hence in consumption will depend on the relative magnitudes of the changes in wage rates and on the importance of each member's time in the production of commodities.

Changes in real wage rates induce substitution effects between factors of production as well. If the husband's real wage rate rises while the wife's remains stationary, the demand for husband's time will fall relative to the demands for goods and wife's time as long as goods and wife's time are substitutes for husband's time in the production of commodities.

Therefore, if the real wage rate of the husband rises while that of the wife remains constant, substitution in production and in consumption will both reduce the demand for his time. Her time will also fall only if the elasticity of substitution between the two time inputs is less than the elasticity of substitution in consumption; a similar conclusion holds for goods.

The changes in demand for goods and time given by equations (1.25) and (1.26) are replaced by the following: [48]

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48. These equations assume factor-neutral technological change. If they were biased, one would add to equation (1.60) the terms

$$-(s_1\sigma_{11}+1)B_1 - s_2\,\sigma_{12}B_2 - s_x\,\sigma_{1x}B_x;$$

to equation (1.61), the terms

$$-s_1\sigma_{21}B_1 - (s_2\sigma_{22}+1)B_2 - s_x\sigma_{2x}B_x;$$

and to equation (1.62), the terms

$$-s_1\sigma_{x1}B_1 - s_2\sigma_{x2}B_2 - (s_x\sigma_{xx}+1)B_x;$$

with $B_1 = \widehat{MPL_1} - \hat{F}$; $B_2 = \widehat{MPL_2} - \hat{F}$; $B_x = \widehat{MPX} - \hat{F}$; where $\widehat{MPL_1}$, $\widehat{MPL_2}$, and $\widehat{MPX}$ measure the percentage increases in the marginal products of L_1, L_2, and X, and $\hat{F}$ measures the percentage reduction in the marginal cost of commodities: $\hat{F} = s_1\,\widehat{MPL_1} + s_2\,\widehat{MPL_2} + s_x\,\widehat{MPX}$.

$$\tilde{L}_{1t} = s_1(\sigma_{11} - \sigma_c)(\tilde{w}_{1t} - \tilde{p}_t) + s_2(\sigma_{12} - \sigma_c)(\tilde{w}_{2t} - \tilde{p}_t) + (\sigma_c - 1)\hat{F}_t + \sigma_c(r_t - \tilde{p}_t - \rho_t); \quad (1.59)$$

$$\tilde{L}_{2t} = s_1(\sigma_{21} - \sigma_c)(\tilde{w}_{1t} - \tilde{p}_t) + s_2(\sigma_{22} - \sigma_c)(\tilde{w}_{2t} - \tilde{p}_t) + (\sigma_c - 1)\hat{F}_t + \sigma_c(r_t - \tilde{p}_t - \rho_t); \quad (1.60)$$

$$\tilde{X}_t = s_1(\sigma_{x1} - \sigma_c)(\tilde{w}_{1t} - \tilde{p}_t) + s_2(\sigma_{x2} - \sigma_c)(\tilde{w}_{2t} - \tilde{p}_t) + (\sigma_c - 1)\hat{F}_t + \sigma_c(r_t - \tilde{p}_t - \rho_t); \quad (1.61)$$

where

s_1 and s_2 = proportions of total costs of commodities accounted for by husband's and wife's time;

σ_{ij} = partial elasticity of substitution between factors i and j ($i, j = L_1, L_2, X$), with $\sigma_{ii} < 0$ and $\sigma_{ij}(j \neq i) >$ or < 0 as i and j are substitutes or complements.[49]

Husbands and wives are not in the labor force at all ages. Both retire eventually as their market earnings are reduced due to failing health, reductions in their human capital, restrictions of social security legislation, etc. Wives often remain out of the labor force at younger ages as well, partly because their wage rates are low relative to their husband's, and partly because their household productivity is relatively high—primarily because of the presence of young children in the home.

1.7 FAMILY SIZE

We have been assuming that the number of persons in a family is exogenously given and is constant over time. Considerable research

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49. There are certain restrictions on these partial elasticities of substitution. Let $\boldsymbol{\sigma}$ denote the matrix

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} & \sigma_{x1} & \sigma_{x2} \\ \sigma_{1x} & \sigma_{11} & \sigma_{12} \\ \sigma_{2x} & \sigma_{21} & \sigma_{22} \end{bmatrix}$$

$\boldsymbol{\sigma}$ is negative semidefinite. In particular, it is symmetric; all diagonal elements are negative (nonpositive); and $\boldsymbol{\sigma}\mathbf{s} = \mathbf{0}$, where $\mathbf{s}$ is the column vector $(s_x s_1 s_2)$ of factor shares and $\mathbf{0}$ is the zero vector.

is now in progress by economists on the formation, growth, and dissolution of families.[50] The core of these studies is the assumption that family size and composition are decision variables, basically no different than the usual ones considered by economists. We do not seek in this chapter to integrate these decisions fully into those dealing with the allocation of lifetime resources, although ultimately that must be done.

We assume that the marriage decision is made exogenously, i.e., independently of decisions about the lifetime allocation of resources. The parent's utility function is assumed to depend not only on the commodities previously defined, but also on commodities measuring child services: the number and "quality" of children of given years of age at each age of the household head. The raising of children requires time, especially wife's time,[51] and goods. Thus, time and goods must be allocated between child services and other commodities.

We note here those implications of this model that are most relevant to the empirical work reported in the next three chapters. If only the real wage rate of, say, the husband rises with age, other inputs will be substituted for his time in the production of all commodities, including child services; and present commodities, again including child services[52] will be substituted for future ones.

If husband's and wife's time are substitutes in the production of all commodities, the demand for her time relative to his will increase over time, and will increase absolutely if the total substitution effect in production is stronger than the total substitution effect in consumption. Thus, once we allow for changes in family size, the substitutions in production and consumption incorporated in equations (1.59), (1.60), and (1.61) must be interpreted as reflecting the combined effects of all commodities, including child services.

50. An early study is Gary S. Becker, "An Economic Analysis of Fertility," in *Demographic and Economic Change in Developed Countries*, Universities–National Bureau Conference 11 (Princeton, N.J.: Princeton University Press for NBER, 1960). For more recent studies, see for instance, T. W. Schultz, ed., "New Economic Approaches to Fertility," *Journal of Political Economy*, March–April 1973, Part II; and Schultz, ed., "Marriage, Human Capital, and Fertility," ibid., March–April 1974, Part II.

51. The importance of wife's time has been demonstrated in several empirical studies. See, for instance, Jacob Mincer, "Market Prices, Opportunity Costs, and Income Effects," in C. Christ et al., eds., *Measurement in Economics* (Stanford: Stanford University Press, 1963). See also the references listed in the preceding note.

52. Unless an increase in present child services greatly reduced the marginal utility of other commodities in the future.

1.8 SUMMARY

In this chapter a model of a family's consumption, work time, and investment in human capital was constructed under three basic assumptions: First, the primal objects of choice entering the utility function are nonmarket activities, called commodities, and these commodities are produced with market goods and own time. Second, the household can allocate its time between nonmarket and market activities, including the production of own human capital, at prices governed by its productivity in each of these sectors. Third, the household is endowed with perfect foresight; hence it predicts accurately its life-span and all its future income, wages, and interest rates. Under these assumptions, the following principal implications were drawn:

i. At a zero rate of interest and with neutral time preference, consumption time will be inversely related to the wage rate over the life cycle.

ii. Again at a zero rate of interest and with neutral time preference, the consumption of goods will be positively related to the wage rate over the life cycle if substitution between goods and time is easier than intertemporal substitution between nonmarket activities produced at different points in time.

iii. With a positive rate of interest (or preference for the present), nonmarket time will reach a trough before the peak-wage-rate age, and consumption of goods will reach a peak after the peak-wage-rate age (if $\sigma_f > \sigma_c$).

iv. Changes in nonmarket productivity over the life cycle can modify these patterns. In particular, if improvements in nonmarket efficiency are neutral between goods and home time, they will lessen the rise in the demand for goods and increase the incentive to contract home time during periods of rising wages, provided the intertemporal elasticity of substitution between commodities produced at different points in time is less than unity.

v. The incentive to engage in the production of human capital is shown to depend on an individual's planned future working time, since market returns from current investment are larger the larger his attachment to the labor force.

vi. Nonmarket returns to human capital increase future efficiency in the home. These returns will be positively related to home time if human capital is time-augmenting in the home.

vii. Barring nonmarket returns to human capital, the production of human capital will rise during the early years of life, when the incentive to produce is so large that the household specializes by spending no time at work. Eventually the rate of production falls, since, to the extent that human capital is not transferred to one's children, the benefit from additional production must fall to zero at the end of life.

viii. Time spent investing in human capital will also rise initially if the production of human capital rises, and will eventually fall along with the output of human capital, unless the substitution effect between time and goods in its production is larger than the effect of the reduced scale of investments.

ix. Hours of work will rise initially and reach a peak later than home time reaches its trough. If the rate of interest net of time preference is zero, home time will reach a trough at the peak-wage-rate age, while working time will reach a peak later, essentially because training time is declining in the neighborhood of the peak-wage-rate age (regardless of the degree of substitutability between time and goods in human capital production).

APPENDIX

1 WAGE RATE AND INTEREST RATE EFFECTS

Notation:

t = age of household head.
C_t = consumption of commodities at age t.
L_{1t}, L_{2t} = time spent in consumption by husband and wife at age t.
N_{1t}, N_{2t} = time spent at work by husband and wife at age t.
w_{1t}, w_{2t} = money wage rate of husband and wife at age t.
X_t = consumption of market goods at age t.
p_t = price index of market goods at age t.
r_t = rate of interest at age t.
R_t = value in period zero of \$1.00 received at age t, i.e., $R_t = 1/(1 + r_0)(1 + r_1) \ldots (1 + r_{t-1})$.
A_t = assets at age t after consumption decisions have been made at that age.

Each variable carries only one time subscript because all plans are assumed to be consistent and realized.[53]

We have the following four sets of relations:

i. A production function for commodities:

$$C_t = F(X_t, L_{1t}, L_{2t}); \qquad t = 1, 2, \ldots, T. \quad \text{(A1.1)}$$

which is assumed to be twice differentiable and homogeneous of the first degree. For the present the production function is also assumed to be the same at all ages.

ii. A utility function which is assumed to be twice differentiable and the same at all years of age of the head:

$$U = U(C_1, C_2, \ldots, C_T), \quad \text{(A1.2)}$$

where T is the lifetime horizon measured in years. Later on, the utility function is specialized to the following additive form:

$$U = \sum_{t=1}^{T} \beta_t G(C_t). \quad \text{(A1.3)}$$

iii. A budget constraint:

$$\sum_{t=1}^{T} R_t p_t X_t = \sum_{t=1}^{T} R_t(w_{1t}N_{1t} + w_{2t}N_{2t}) + A_0. \quad \text{(A1.4)}$$

iv. A set of time constraints:

$$L_{it} + N_{it} = \theta.$$

$$i = 1, 2; \; t = 1, 2, \ldots, T. \quad \text{(A1.5)}$$

Substitute the time constraints of equations (A1.5) into the budget constraint of equation (A1.4) to obtain

$$\sum_{t=1}^{T} R_t(p_t X_t + w_{1t}L_{1t} + w_{2t}L_{2t}) = W_0, \quad \text{(A1.6)}$$

where W_0 is full wealth:

$$W_0 = \sum_{t=1}^{T} R_t(w_{1t}\theta + w_{2t}\theta) + A_0.$$

Finally, the non-negativity constraints are

$$\left.\begin{array}{l} L_{it}, N_{it} \geqslant 0; \; i = 1, 2; \\ X_t \geqslant 0; \\ C_t \geqslant 0. \end{array}\right\} t = 1, 2, \ldots, T. \quad \text{(A1.7)}$$

53. We assume that all expectations are fulfilled and that the utility function given below is consistent. Conditions for consistency have been examined by Robert H. Strotz, "Myopia and Inconsistency in Dynamic Utility Maximization," *Review of Economic Studies*, vol. 23 (1955–56), pp. 165–180.

The household is assumed to maximize its utility subject to the full-wealth constraint given by equation (A1.6). For the present the non-negativity constraints given by (A1.7) are assumed to be ineffective. We construct the Lagrangean function

$$\mathscr{L} = U(C_1, C_2, \ldots, C_t) - \lambda \left[\sum_{t=1}^{T} R_t(p_t X_t + w_{1t} L_{1t} + w_{2t} L_{2t}) - W_0 \right]; \quad \text{(A1.8)}$$

and set its derivatives equal to zero:

$$\frac{\partial \mathscr{L}}{\partial X_t} = \frac{\partial U}{\partial C_t} \frac{\partial F}{\partial X_t} - \lambda R_t p_t = 0; \qquad t = 1, 2, \ldots, T. \quad \text{(A1.9)}$$

$$\frac{\partial \mathscr{L}}{\partial L_{it}} = \frac{\partial U}{\partial C_t} \frac{\partial F}{\partial L_{it}} - \lambda R_t w_{it} = 0; \qquad i = 1, 2;\ t = 1, 2, \ldots, T. \quad \text{(A1.10)}$$

$$\frac{\partial \mathscr{L}}{\partial \lambda} = \sum_{t=1}^{T} R_t(p_t X_t + w_{1t} L_{1t} + w_{2t} L_{2t}) - W_0 = 0. \quad \text{(A1.11)}$$

Since the marginal products of goods and times are positive, the equilibrium conditions given by equations (A1.9) and (A1.10) can be written equivalently as:

$$\frac{\partial U}{\partial C_t} = \lambda R_t \pi_t, \qquad t = 1, 2, \ldots, T. \quad \text{(A1.12)}$$

where

$$\pi_t = \frac{p_t}{\partial F/\partial X_t} = \frac{w_{1t}}{\partial F/\partial L_{1t}} = \frac{w_{2t}}{\partial F/\partial L_{2t}}. \qquad t = 1, 2, \ldots, T. \quad \text{(A1.13)}$$

The conditions given by (A1.13) could be obtained by minimizing the total cost of producing C_t for a given level of C_t. π_t is, therefore, the marginal cost of commodities at age t. Correspondingly, the equilibrium conditions of (A1.12) could be obtained by minimizing the lifetime expenditures on commodities to attain a given level of utility.

From these cost-minimization conditions and the production function, we get the derived demand functions for goods and time:

$$X_t = X(w_{1t}, w_{2t}, p_t, C_t); \quad \text{(A1.14)}$$

$$L_{it} = L_i(w_{1t}, w_{2t}, p_t, C_t). \qquad i = 1, 2. \quad \text{(A1.15)}$$

Since these demand functions are homogeneous of degree zero in prices and homogeneous of the first degree in output, then

$$X_t = x(w_{1t}^*, w_{2t}^*)C_t; \tag{A1.16}$$

$$L_{it} = l_i(w_{it}^*, w_{2t}^*)C_t; \tag{A1.17}$$

where $w_{it}^* = w_{it}/p_t$, and $i = 1, 2$.

Changes in the derived demands for goods and times would depend on changes in the real wage rates and in the production of commodities.

Although none of the variables in a discrete-time model are differentiable functions of time, we can express the changes in the demand for goods and time in a simple form by using a linear expansion of a function around a point. Let $\tilde{x}_t = (x_{t+1} - x_t)/x_t$ for any variable x. Then

$$\tilde{X}_t = \eta_{x1}\tilde{w}_{1t}^* + \eta_{x2}\tilde{w}_{2t}^* + C_t; \tag{A1.18}$$

$$\tilde{L}_{it} = \eta_{i1}\tilde{w}_{1t}^* + \eta_{i2}\tilde{w}_{2t}^* + \tilde{C}_t; \qquad i = 1, 2. \tag{A1.19}$$

where η_{xj}, η_{ij} are the elasticities of demand for goods and for ith time at the jth wage rate, evaluated at the point (w_{1t}^*, w_{2t}^*), and holding output constant.

Since own substitution effects are negative, $\eta_{ii} < 0$, for $i = 1, 2$. Compensated cross-price elasticities between any two factors are positive or negative as these factors are substitutes or complements. The symmetry property of cross-substitution effects is more explicit with partial elasticities of substitution (PES). The PES σ_{ij} between the ith and jth time is defined by $\eta_{ij} = s_j\sigma_{ij}$, and the PES σ_{xj} between goods and the jth home time by $\eta_{xj} = s_j\sigma_{xj}$, where s_j is the proportion of jth time in the total cost of commodities. By symmetry, $\sigma_{ij} = \sigma_{ji}$ and $\sigma_{xj} = \sigma_{jx}$.

Substituting these expressions into (A1.18) and (A1.19) we get:

$$\tilde{X}_t = s_1\sigma_{x1}\tilde{w}_{1t}^* + s_2\sigma_{x2}\tilde{w}_{2t}^* + \tilde{C}_t; \tag{A1.20}$$

$$\tilde{L}_{it} = s_1\sigma_{i1}\tilde{w}_{1t}^* + s_2\sigma_{i2}\tilde{w}_{2t}^* + \tilde{C}_t. \qquad i = 1, 2. \tag{A1.21}$$

With the additive utility function given by (A1.3), the equilibrium conditions (A1.12) specialize to

$$\beta_t G'(C_t) = \lambda R_t \pi_t, \tag{A1.22}$$

or $G'(C_t) = \lambda\gamma_t$, with $\gamma_t = R_t\pi_t/\beta_t$. Solving for C_t, we obtain $C_t = C(\lambda\gamma_t)$, where $C(.)$ is the inverse of the function $G'(.)$, and λ, the marginal utility of wealth, is a constant over a lifetime in the absence of unexpected changes in prices and incomes. The change in consumption of commodities may be approximated up to first-order terms by

$$C_{t+1} - C_t \doteq \frac{1}{G''}\lambda(\gamma_{t+1} - \gamma_t).$$

Substituting for λ from (A1.22) and dividing by C_t, we obtain

$$\tilde{C}_t = \frac{G'}{G''C_t}\tilde{\gamma}_t.$$

But $\tilde{\gamma}_t$ can be approximated by $\tilde{\pi}_t - r_t + \rho_t$, where $\rho_t = (\beta_{t+1} - \beta_t)/\beta_t$. Moreover, it can be shown that $-G'/G''C_t$ is equal to the direct (McFadden) elasticity of substitution σ_c between C_t and C_{t+1} evaluated at the point $C_{t+1} = C_t$. Hence,

$$\tilde{C}_t = -\sigma_c(\tilde{\pi}_t - r_t + \rho_t). \tag{A1.23}$$

To linear approximation, the change in marginal (= average) cost is given by

$$\tilde{\pi}_t = s_{1t}\tilde{w}_{1t} + s_{2t}\tilde{w}_{2t} + (1 - s_1 - s_2)\tilde{p}_t.$$

Hence,

$$\tilde{\pi}^*_t = s_{1t}\tilde{w}^*_{1t} + s_{2t}\tilde{w}^*_{2t}, \tag{A1.24}$$

where $\pi^*_t = \pi_t/p_t$. Therefore, by substituting (A1.24) into (A1.23),

$$\tilde{C}_t = -\sigma_c(s_{1t}\tilde{w}^*_{1t} + s_{2t}\tilde{w}^*_{2t} - r^*_t + \rho_t). \tag{A1.25}$$

Equation (A1.25) enables us to write the derived demands for goods and time as functions only of the interest rate net of time preference and of changes in real wage rates:

$$\tilde{X}_t = s_1(\sigma_{x1} - \sigma_c)\tilde{w}^*_{1t} + s_2(\sigma_{x2} - \sigma_c)\tilde{w}^*_{2t} + \sigma_c(r^*_t - \rho_t); \tag{A1.26}$$

and

$$\tilde{L}_{it} = s_1(\sigma_{i1} - \sigma_c)\tilde{w}^*_{1t} + s_2(\sigma_{i2} - \sigma_c)\tilde{w}^*_{2t} + \sigma_c(r^*_t - \rho_t).$$

$$i = 1, 2. \tag{A1.27}$$

Factor shares and elasticities of substitution need not be constant. Shares would be independent of wage rates if, and only if, all cross-partial elasticities of substitution in production equaled unity.[54]

2 Changes in Nonmarket Productivity

We represent changes in productivity with age in the factor-augmenting form:

$$C_t = F(X_t, L_{1t}, L_{2t}; t) = F(a_{xt}X_t, a_{1t}L_{1t}, a_{2t}L_{2t}), \tag{A1.28}$$

and construct the same Lagrangean function as in (A1.8). If its derivatives are set equal to zero, the following result is obtained:

$$\frac{\partial U}{\partial C_t} = \lambda R_t \pi_t = 0; \tag{A1.29}$$

$$\pi_t = \frac{p_t/a_{xt}}{\partial F/\partial a_{xt}X_t} = \frac{w_{1t}/a_{1t}}{\partial F/\partial a_{1t}L_{1t}} = \frac{w_{2t}/a_{2t}}{\partial F/\partial a_{2t}L_{2t}}. \tag{A1.30}$$

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54. Indeed,

$$\tilde{s}_{it} = s_{xt}(1 - \sigma_{ix})\tilde{w}^*_{it} + s_{jt}(1 - \sigma_{ij})(\tilde{w}^*_{it} - \tilde{w}^*_{jt}). \qquad i, j = L_1, L_2; j \neq i.$$

The derived demand for goods and time are now:

$$a_{xt}X_t = X(w_{1t}/a_{1t},\ w_{2t}/a_{2t},\ p_t/a_{xt})C_t; \qquad \text{(A1.31)}$$

$$a_{it}L_{it} = L_i(w_{1t}/a_{1t},\ w_{2t}/a_{2t},\ p_t/a_{xt})C_t. \qquad i = 1, 2. \qquad \text{(A1.32)}$$

Therefore, the percentage changes in X and L are

$$\tilde{X}_t = -\tilde{a}_{xt} + s_1\sigma_{x1}(\tilde{w}_{1t} - \tilde{a}_{1t}) + s_2\sigma_{xt}(\tilde{w}_{2t} - \tilde{a}_{2t}) + s_x\sigma_{xx}(\tilde{p}_t - \tilde{a}_{xt}) + \tilde{C}_t; \qquad \text{(A1.33)}$$

$$\tilde{L}_{it} = -\tilde{a}_{it} + s_1\sigma_{i1}(\tilde{w}_{1t} - \tilde{a}_{1t}) + s_2\sigma_{i2}(\tilde{w}_{2t} - \tilde{a}_{2t}) + s_x\sigma_{ix}(\tilde{p}_t - \tilde{a}_{xt}) + \tilde{C}_t.$$

$$i = 1, 2. \qquad \text{(A1.34)}$$

Equation (A1.23) still holds, since a change in productivity affects the consumption of commodities only through variations in their prices. However, the price changes are now given by

$$\tilde{\pi}_t = s_1(\tilde{w}_{1t} - \tilde{a}_{1t}) + s_2(\tilde{w}_{2t} - \tilde{a}_{2t}) + s_x(\tilde{p}_t - \tilde{a}_{xt}). \qquad \text{(A1.35)}$$

A substitution of (A1.35) and (A1.23) into (A1.33) and (A1.34) yields the following results:

$$\tilde{X}_t = -\tilde{a}_{xt} + s_1\sigma_{x1}(\tilde{w}_{1t} - \tilde{a}_{1t}) + s_2\sigma_{x2}(\tilde{w}_{2t} - \tilde{a}_{2t}) + s_x\sigma_{xx}(\tilde{p}_t - \tilde{a}_{2t})$$

$$- \sigma_c[s_1(\tilde{w}_{1t} - \tilde{a}_{1t}) + s_2(\tilde{w}_{2t} - \tilde{a}_{2t}) + s_x(\tilde{p}_t - \tilde{a}_{xt}) - r_t^* + \rho_t]; \qquad \text{(A1.36)}$$

$$L_{it} = -\tilde{a}_{it} + s_1\sigma_{i1}(\tilde{w}_{1t} - \tilde{a}_{1t}) + s\sigma_{i2}(\tilde{w}_{2t} - \tilde{a}_{2t}) + s_x\sigma_{ix}(\tilde{p}_t - \tilde{a}_{xt})$$

$$- \sigma_c[s_1(\tilde{w}_{1t} - \tilde{a}_{1t}) + s_2(\tilde{w}_{2t} - \tilde{a}_{2t}) + s_x(\tilde{p}_t - \tilde{a}_{xt}) - r_t^* + \rho_t];$$

$$i = 1, 2. \qquad \text{(A1.37)}$$

or, by regrouping terms,

$$\tilde{X}_t = (\sigma_{x1} - \sigma_c)s_1(\tilde{w}_{1t} - \tilde{p}_t) + (\sigma_{x2} - \sigma_c)s_2(\tilde{w}_{2t} - \tilde{p}_t) + (\sigma_c - 1)\hat{F}_t$$

$$+ (1 - \sigma_{x1})s_1(\tilde{a}_{1t} - \tilde{a}_{xt}) + (1 - \sigma_{x2})s_2(\tilde{a}_{2t} - \tilde{a}_{xt}) + \sigma_c(r_t - \tilde{p}_t - \rho_t); \qquad \text{(A1.38)}$$

$$L_{it} = (\sigma_{i1} - \sigma_c)s_1(\tilde{w}_{1t} - \tilde{p}_t) + (\sigma_{i2} - \sigma_c)s_2(\tilde{w}_{2t} - \tilde{p}_t) + (\sigma_c - 1)\hat{F}_t$$

$$+ (1 - \sigma_{ij})s_j(\tilde{a}_{jt} - \tilde{a}_{it}) + (1 - \sigma_{ix})s_x(\tilde{a}_{xt} - \tilde{a}_{it}) + \sigma_i - \tilde{p}_t - \rho_t);$$

$$i, j = 1, 2;\ j \neq i. \qquad \text{(A1.39)}$$

where $\hat{F}_t = s_1\tilde{a}_{1t} + s_2\tilde{a}_{2t} + s_x\tilde{a}_{xt}$ is the percentage increase in the output of commodities at constant factor inputs, or the percentage reduction in their marginal cost, due to the increase in productivity.

3 PRODUCTION OF HUMAN CAPITAL

I now permit the accumulation of human capital, and introduce the following additional notation:

H_{it} = stock of human capital held by the ith family member at age t of the household head;

h_{it} = rate of production of human capital by the ith member at age t;
N'_{it} = amount of time spent by the ith member in producing human capital at age t;
X'_{it} = amount of goods used by the ith member in producing human capital at age t;
δ_{it} = rate of depreciation of human capital of the ith member at age t;
$D_{iv,t}$ = undepreciated portion at age t of one unit of the ith member's human capital held at v:

$$D_{iv,t} = (1 - \delta_{iv})(1 - \delta_{iv+1}) \ldots (1 - \delta_{it-1}).$$

$$i = 1, 2; v = 1, 2, \ldots, t-1$$

It is assumed here that only wage rates depend on the stock of human capital, as in

$$w_{it} = e_{it}H_{it}. \qquad i = 1, 2. \quad \text{(A1.40)}$$

In the text I also consider briefly the effects of human capital on the productivity of goods and time in the production of commodities and human capital itself. The initial stocks H_{i1} are given, but later stocks depend on the amounts produced and not depreciated:

$$H_{it} = H_i D_{i1,t} + \sum_{v=1}^{t-1} h_{iv} D_{i,v,t}; \qquad i = 1, 2. \quad \text{(A1.41)}$$

with [55]

$$h_{it} = h_i(X'_{it}, N'_{it}).$$

$$i = 1, 2; t = 1, 2, \ldots, T. \quad \text{(A1.42)}$$

The time constraints are

$$L_{it} + N_{it} + N'_{it} = \theta;$$

$$i = 1, 2; t = 1, 2, \ldots, T. \quad \text{(A1.43)}$$

and the budget constraint is

$$\sum_{t=1}^{T} R_t(p_t X_t + p'_{1t}X'_{1t} + p'_{2t}X_{2t}) = \sum_{t=1}^{T} R_t(e_{1t}H_{1t}N_{1t} + e_{2t}H_{2t}N_{2t}) + A_0. \quad \text{(A1.44)}$$

If the derivatives of the Lagrangean function,

$$\mathscr{L} = U(C_1, C_2, \ldots, C_t) - \lambda\left[\sum_{t=1}^{T} R_t(p_t X_t + p'_t X'_t - e_{1t}H_{1t}N_{1t} - e_{2t}H_{2t}N_{2t}) - A_0\right]$$

$$- \sum_{t=1}^{T}\sum_{i=1}^{2} \kappa_{it}(L_{it} + N_{it} + N'_{it} - \theta),$$

55. Interactions between husband and wife in the production function for human capital are ignored here, but could easily be introduced.

are set equal to zero, then

$$\frac{\partial \mathcal{L}}{\partial X_t} = \frac{\partial U}{\partial C_t}\frac{\partial F}{\partial X_t} - \lambda R_t p_t = 0; \tag{A1.45}$$

$$\frac{\partial \mathcal{L}}{\partial L_{it}} = \frac{\partial U}{\partial C_t}\frac{\partial F}{\partial L_{it}} - \kappa_{it} = 0; \qquad i = 1, 2. \tag{A1.46}$$

$$\frac{\partial \mathcal{L}}{\partial N_{it}} = \lambda R_t e_{it} H_{it} - \kappa_{it} = 0; \qquad i = 1, 2. \tag{A1.47}$$

$$\frac{\partial \mathcal{L}}{\partial N'_{it}} = \lambda\left(\sum_{v=t+1}^{T} R_v e_{iv} N_{iv} \frac{\partial H_{iv}}{\partial h_{it}}\frac{\partial h_{it}}{\partial N'_{it}}\right) - \kappa_{it} = 0; \qquad i = 1, 2. \tag{A1.48}$$

$$\frac{\partial \mathcal{L}}{\partial X'_{it}} = \lambda\left(\sum_{v=t+1}^{T} R_v e_{iv} N_{iv} \frac{\partial H_{iv}}{\partial h_{it}}\frac{\partial h_{it}}{\partial X'_{it}} - R_t p'_t\right) = 0; \qquad i = 1, 2. \tag{A1.49}$$

where λ can be interpreted as the marginal utility of wealth, and κ_{it} as the marginal utility of the ith member's time at age t of the household head.

From (A1.47), it is seen that the monetary equivalent of the marginal utility of time is equal to the wage rate. Therefore, (A1.45) and (A1.46) may be rewritten as

$$\frac{\partial U}{\partial C_t} = \lambda R_t \pi_t; \tag{A1.50}$$

$$\pi_t = \frac{p_t}{\partial F/\partial X_t} = \frac{e_{1t}H_{1t}}{\partial F/\partial L_{1t}} = \frac{e_{2t}H_{2t}}{\partial F/\partial L_{2t}}. \tag{A1.51}$$

Moreover, the equilibrium conditions (A1.48) and (A1.49) may be written as

$$\sum_{v=t+1}^{T} R_v e_{iv} N_{iv} \frac{\partial H_{iv}}{\partial h_{it}} = R_t \mu_{it}; \tag{A1.52}$$

$$R_t \mu_{it} = \frac{\kappa_{it}/\lambda}{\partial h/\partial N'_{it}} = \frac{R_t p'_t}{\partial h/\partial X'_{it}}; \tag{A1.53}$$

where μ_{it} is the "current" marginal cost of producing the ith member's human capital at age t of the household head. The "full" marginal cost includes the increase in future total costs attributable to the effects on future wage rates of an additional unit produced at age t.

Let $K_{it} = K_i(e_{it}H_{it}, p'_t, h_{it})$ denote total costs of producing the ith member's human capital at age t. Then the optimality conditions (A1.52) may be expressed as

$$\sum_{v=t+1}^{T} R_v e_{iv} \theta'_{iv} \frac{\partial H_{iv}}{\partial h_{iv}} = R_t \frac{\partial K_{it}}{\partial h_{it}} + \sum_{v=t+1}^{T} R_v e_{iv} \frac{\partial K_{iv}}{\partial e_{iv} H_{iv}}\frac{\partial H_{iv}}{\partial h_{it}}; \tag{A1.54}$$

where the right-hand side is full marginal cost, with $\partial K_{iv}/\partial e_{iv}H_{iv} = N'_{iv}$. The first term on the right-hand side measures current marginal cost, whereas the second measures the discounted value of the increment in future costs of producing human capital due to a small rise in the rate of output in period t. The left-hand side is "full" benefits, since θ'_{iv} is the market time of the ith household member at time v: $\theta'_{iv} = N_{iv} + N'_{iv}$. If full benefits were independent of the amount produced, second-order conditions would require only that full marginal costs be an increasing function of h_t:

$$R_t \frac{\partial^2 K_{it}}{\partial h_{it}^2} + \sum_{v=t+1}^{T} R_v \frac{\partial^2 K_{iv}}{\partial H_{iv}^2} \frac{\partial H_{iv}}{\partial h_{it}} > 0.$$

A fortiori, full marginal cost must be increasing in h_t if full marginal benefits are positively related to the rate of output of human capital. Since consumption and investment decisions are determined simultaneously, consumption time will fall as the cost of time increases through increased production of human capital. Market time, θ', and thus full marginal benefits will be positively related to the output of human capital in previous periods.