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International Environmental Agreement: a Dynamic Model of Emissions Reduction

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Abstract. We model an International Environmental Agreement as a two stages game: during the first stage each country decides whether or not to join the agreement while, in the second stage, the quantity of emissions reduction is choosen.
Players determine their abatement levels in a dynamic setting, given the dynamics of pollution stock and the strategies of other countries.
Players may act cooperatively, building coalitions and acting according to the interest of the coalition, or they make their choices taking care of their individual interest only. Countries can behave myopically or in a farsighted way. As a consequence, the size of stable coalition can completely change.
A continuous time framework is choosen in the present paper and consequently the problem is studied by a differential game.

Keywords IEA, Differential games, Coalition stability.

1 Introduction

Over the last two decades, the interest in international environmental problems such as climate change, ozone depletion, marine pollution has grown immensely and it has drived an increased sense of interdipendence between countries.
Cooperation among different countries appears necessary and this results in International Environmental Agreements (IEA) such as Helsinky and Olso Protocol signed in 1985 and 1994; Montreal Protocol on the reduction of CFCs that deplete the ozone layer signed in 1987; Kyoto Protocol on the reduction of greenhouse gases causing global warming signed in 1997. It is important to observe that, in these IEAs, the number of signatories varies considerably and this justifies the increasing interest of many authors to explain why IEAs are ratified only by a fraction of the countries and to suggest strategies to increase

the number of signatories.
Economists have emphasized two important aspects: agreements must be profitable (there must be gains to all signatory countries), agreements must be self-enforcing (in the absence of any international authority, there must be incentives for countries to join and to remain in an agreement).
Literature has focused on stability concepts for IEA's in order to obtain some conclusions on the size that can be expected and to explain why some IEAs are large and others are not. Stable IEA means that no individual signatory country has any incentive to leave the IEA and no non-signatory country has an incentive to join the IEA.
Both Cooperative and Non-Cooperative game theory have been used to study coalition formation.
In the Cooperative Game framework, using the γ-core concept and implementing transfers to solve the heterogeneity of the countries, Chander and Tulkens (1995) reach the conclusion that the grand coalition is stable. These results lead to an optimistic view on the size of the stable coalition.
In the Non-Cooperative Game framework the concept of Internal and External stability has been applied to obtain the size of a coalition. The idea is to check for which size of a coalition an individual country is indifferent between remaining in the coalition or leaving it. Carraro and Siniscalco (1991,1993), Hoel (1992), de Zeeuw(2008) show that if signatories act in a Cournot fashion with respect to non-signatories then the size of a stable coalition is very small. If countries act in a Stackelberg fashion, where signatories are the leaders and non-signatories are the followers, a stable IEA can have any number of signatories between two and the grand coalition (see Barret (1994), Diamantoudi and Sartzetakis (2001), Rubio and Ulph (2004)).
Recent developments in game theory advocate the concept of *farsighted* stable coalitions against previous notions of stability which are myopic and don't reflect the complexity and foresight of countries' decisions about agreements.
When an agent contemplates exiting a coalition, it compares the welfare it enjoys as a member of the coalition with the welfare it will enjoy once it exits. The agent implicitly assumes that once it deviates, no one else will want to deviate. But this is not always the case. In fact, it is possible that another country may wish to exit coalition and so on. Thus, the agent must compare the starting situation with the outcome at the end of the process, after a number of deviations. The final outcome can be characterized as such only if no more countries wish to exit and no more countries wish to join.
The concept of farsightedness and its applications to IEAs inspired a series of papers such as e.g. Chwe (1994), Diamantoudi and Sartzetakis (2002), Eyckmans (2003), de Zeeuw (2008).
This literature shows that farsightedness allows both large and small stable coalitions and so this concept reconciles the cooperative and non-cooperative approaches.
All papers quoted above study the stability of an IEA in a static context while dynamic aspect of the problem are ignored, but, abatements processes are usually dynamic as well as stock pollutant. In most of models, it is assumed that

countries reduce emissions in one step, but it is not realistic and also not rational. For this reason we think that the analysis of the stability of an IEA must be proposed in a dynamic setting. Other authors as, for example, Rubio and Casino (2005) and de Zeeuw (2008) have applied differential games and optimal control methodologies to analyse environmental problems. In particular, Rubio and Casino (2005) analyse the internal and the external stability of environmental agreements in a dynamic framework, when environmental damages are associated with a stock externality. Coalition formation has been designed as a two stages game in which, in the first stage each country decides to join or not the coalition and in the second stage signatories and non signatories play an emissions differential game. Authors calculate open loop equilibrium and show, by a numerical simulation, that a bilateral coalition is the unique self enforcing IEA. In de Zeeuw (2008) a model of abatement is proposed as a difference game, because the state transition is given as a difference equation. The feedback Nash equilibrium is found and, in order to study the stability of an IEA, the concept of dynamic farsighted stability is introduced showing as large and small stable coalitions can occur.

In this paper we propose an optimal control model with the objective to reduce pollution at the lowest costs. Players determine their abatement levels in a dynamic setting defined in continuous time. In the differential game proposed, open loop Nash equilibria and Feedback Nash equilibria are calculated in order to determine the optimal paths of the abatement levels and of the stock pollutant. The results obtained are the same and depend on the parameter p which can be seen as a measure of the environmental awareness of countries. Stability conditions, such as internal and external stability or farsighted stability, are applyed showing that different answers about the size of a stable IEA can be given.

The paper is organized as follows. In section 2 we describe the model; in section 3 the open loop Nash equilibria of the differential game are calculated and in section 4 the analysis of the stability is proposed. In section 5 Feedback Nash equilibria are obtained showing that they agree with the ones obtained using open loop strategies. Some concluding remarks are given in section 6.

2 The Model

Let us assume that n identical countries decide to abate emissions in order to reduce the environmental pollution.

Initially the accumulated emissions are at a level s_0 and each country i chooses to abate the quantity of emissions $a_i(t)$ (for $i = 1, 2, ..., n$) where $a_i(t) \geq 0$.

The dynamic of accumulated emissions is given by the following differential equation

$$\dot{s}(t) = L - \sum_{i=1}^{n} a_i(t) - k\, s(t) \quad s(0) = s_0 \tag{1}$$

where L represents a constant source of pollutant and k a positive rate of natural decay.
Since $s(t) \geq 0$ the following constraint must be satisfies

$$0 \leq a_i(t) \leq \frac{L}{n} \;\; \forall\, i = 1, 2, ..., n \tag{2}$$

so, we suppose, by the simmetry, that a single country is allowed to abate only a fraction of the emissions produced by itself.
We assume that players minimize a cost function $c_i(a_i(t))$ which is the sum of two terms: the abatement costs and the costs due to remaining pollution. It is very common in literature to consider this kind of cost functions in which the two terms can be linear or quadratic function. In this model we consider the first term quadratic and the second one linear. So, the cost function for each country is

$$c_i(a_i(t)) = \frac{1}{2}a_i(t)^2 + \frac{1}{2}p\, s(t) \tag{3}$$

A major role is played by the parameter $p > 0$; it can be seen as a measure of the environmental awareness of the country, i.e. it denotes the relative weight attached to the damage costs as compared to the abatement costs. By symmetry, p is the same for every country.
To analyze the stability of an IEA with a stock pollutant we consider a two stages game, in which in the first stage each country decides whether to join or not the agreement while in the second stage each country chooses his abatement level.
The game is solved in a backward order. Let assume that, as the outcome of the first stage game, there are m signatory countries ($i = 1, ..., m$) and $n - m$ non signatory countries ($j = m + 1, ..., n$). So, we consider a simple structure in which there is only one coalition while the other countries play as individual outsiders.
In the second stage, as usual, non signatory countries choose their abatement levels acting noncooperatively in order to minimize the present value of their costs taking as given the strategy of the other countries; signatory countries choose their abatement levels acting noncooperatively against non signatories in order to minimize the present value of the aggregate costs of the m signatories. Signatories also take as given the strategies of non signatories.
The optimal abatement levels and accumulated emission paths are given by the Nash equilibria of a differential game. Consequently it is possible to obtain the equilibrium present value of the costs of a signatory country $C_i(m)$ and of a non signatory country $C_j(m)$.
In the first stage countries play a simultaneous open membership game; in a game of this kind, the strategies for each countries are to sign or not an agreement and any player is free to join it. The choice between the two different kinds of behaviour is simultaneous and the agreement is formed by all players that have choosen to cooperate, the others are non signatories. The usual approach to the self enforcing IEA is based on the use of internal and external stability conditions. They have been proposed for static emission games, but

can be extended to dynamic games.
A coalition of size m is internally stable if no member has an incentive to leave the coalition because the costs for an outsider to a coalition of size $m-1$ are larger than the costs for a member of an $m-sized$ coalition. External stability means that no country has an incentive to join a coalition of size m because the cost for a member of a coalition of size $m+1$ is higher than the ones for an outsider to a coalition of size m. A coalition is called *stable* if it is both internally and externally stable, that is if the following inequalities hold:

$$C_i(m) \leq C_j(m-1) \qquad C_i(m+1) \geq C_i(m).$$

A deeper investigation shows that the stability definition assumes, in some sense, a myopic behaviour of players. To overcome this drawback, the concept of *farsightedness* has been introduced in literature, see e.g. Chwe (1994).
A country belonging to a coalition of size m decides to abandon the coalition if its current cost $C_i(m)$ is higher than the cost he should pay going outside the coalition. Neverthless, by the farsighted approach, he will not simply compare its actual cost with the outsider cost $C_j(m-1)$, but he will take into account the possibility that if he leaves the coalition then other coalition members may find convenient to abandon the coalition, too. So, a disgregation process of the coalition can arise and then a country which decides to abandon a coalition of size m must compare its cost as a member of the coalition with the cost that it should pay as an outsider of the remaining coalition at the end of this disgregation process. If no country has an incentive to leave a coalition of size m, behaving in a farsighted way, then the coalition is said to be internally farsighted stable.
A similar definition is given for the external farsighted stability.
It is clear that the concept of internal/external farsighted stability is a recursive one. For a precise formal definition see Chwe (1994).
Now, we look for a self enforcing IEA calculating the open loop Nash equilibria of the abatements differential game.

3 The open loop Nash equilibria of the differential game

Let us assume that $\delta > 0$ is the discount rate. Taken as given the abatement levels of outsiders, signatories commit to a level of abatement that minimize the sum of the costs present value of the countries in the agreement

$$\min_{a_i} \sum_{h=1}^{m} \int_0^{+\infty} e^{-\delta t} \left(\frac{1}{2} a_h(t)^2 + \frac{1}{2} p\, s(t) \right) dt \tag{4}$$

which is equivalent to

$$\max_{a_i} \sum_{h=1}^{m} \int_0^{+\infty} - e^{-\delta t} \left(\frac{1}{2} a_h(t)^2 + \frac{1}{2} p\, s(t) \right) dt \tag{5}$$

Taken as given the abatement levels of cooperators, non signatories commit to a level of abatement that minimize the discount present value of the costs which is equivalent to

$$\max_{a_j} \int_0^{+\infty} -e^{-\delta t}\left(\frac{1}{2}a_j(t)^2 + \frac{1}{2}p\, s(t)\right) dt \tag{6}$$

In both cases, countries face the same dynamics

$$\dot{s}(t) = L - \sum_{i=1}^{m} a_i(t) - \sum_{j=m+1}^{n} a_j(t) - k\, s(t) \quad s(0) = s_0 \tag{7}$$

with the constraint on the control variables given by (2).
Let us define the current value of the Hamiltonian in the standard way

$$H_i = \left[-\sum_{h=1}^{m}\left(\frac{1}{2}a_h^2 + \frac{1}{2}p\, s\right) + \lambda_i\left(L - \sum_{h=1}^{m} a_h - \sum_{j=m+1}^{n} a_j - ks\right)\right],\ i = 1, ..., m$$

$$H_j = -\left(\frac{1}{2}a_j^2 + \frac{1}{2}p\, s\right) + \lambda_j\left(L - \sum_{i=1}^{m} a_i - \sum_{j=m+1}^{n} a_j - ks\right),\ j = m+1, ..., n$$

where λ_i and λ_j are the adjoint variables. We obtain the following set of necessary conditions for an interior open-loop equilibrium

$$-a_i - \lambda_i = 0,\ i = 1, ..., m \tag{8}$$

$$-a_j - \lambda_j = 0,\ j = m+1, ..., n \tag{9}$$

and the adjoint equations are

$$\dot{\lambda}_i = (\delta + k)\lambda_i + \frac{1}{2}mp,\ i = 1, ..., m \tag{10}$$

$$\dot{\lambda}_j = (\delta + k)\lambda_j + \frac{1}{2}p,\ j = m+1, ..., n \tag{11}$$

The trasversality conditions are

$$\lim_{t\to+\infty} e^{-\delta t}\lambda_i = 0\ i = 1, ..., m \tag{12}$$

$$\lim_{t\to+\infty} e^{-\delta t}\lambda_j = 0\ j = m+1, ..., n \tag{13}$$

Because of the symmetry assumption, the n equations given by (10) and (11) reduce to two, solving it and using the trasversality conditions (12) and (13) which are sufficient because the Hamiltonian functions are strictly concave in s, we obtain

$$\lambda_i = -\frac{mp}{2(\delta + k)} \quad \text{and} \quad \lambda_j = -\frac{p}{2(\delta + k)}$$

The constraint on the control variables given by (2); (8) and (9) lead to the following optimal abatement levels

$$a_i = \begin{cases} 0 & if \quad \frac{mp}{2(\delta+k)} < 0 \\ \frac{mp}{2(\delta+k)} & if \quad 0 \le \frac{mp}{2(\delta+k)} \le \frac{L}{n} \\ \frac{L}{n} & if \quad \frac{mp}{2(\delta+k)} > \frac{L}{n} \end{cases} \qquad i = 1, ..., m$$

for a signatory country;

$$a_j = \begin{cases} 0 & if \quad \frac{p}{2(\delta+k)} < 0 \\ \frac{p}{2(\delta+k)} & if \quad 0 \le \frac{p}{2(\delta+k)} \le \frac{L}{n} \\ \frac{L}{n} & if \quad \frac{p}{2(\delta+k)} > \frac{L}{n} \end{cases} \qquad j = m+1, ..., n$$

for a non signatory country.

Let

$$r = \frac{2L(\delta+k)}{np}.$$

We distinguish three different cases depending on the value of the parameter r.

CASE I

If $r \ge m$ then

$$a_i = \frac{mp}{2(\delta+k)} \; i = 1, ..., m \quad \text{and} \quad a_j = \frac{p}{2(\delta+k)} \; j = m+1, ..., n. \tag{14}$$

Signatories abate m times more than the non signatories. The optimal path for the state variable s is

$$s(t) = s_0 e^{-kt} + \frac{1}{k}\left[L - \frac{m^2 p}{2(\delta+k)} - \frac{(n-m)p}{2(\delta+k)}\right](1 - e^{-kt}) \tag{15}$$

which is a positive, increasing and concave function if $s_0 < \frac{1}{k}\left[L - \frac{m^2 p}{2(\delta+k)} - \frac{(n-m)p}{2(\delta+k)}\right]$ otherwise it is a decreasing and convex one. Moreover, in both cases, for

$t \to +\infty$, $s(t)$ approaches the value $\frac{1}{k}\left[L - \frac{m^2 p}{2(\delta+k)} - \frac{(n-m)p}{2(\delta+k)}\right]$.

CASE II

If $1 \leq r < m$ then

$$a_i = \frac{L}{n} \; i = 1, ..., m \quad \text{and} \quad a_j = \frac{p}{2(\delta+k)} \; j = m+1, ..., n; \tag{16}$$

that is cooperators stop to produce emissions at all. The optimal path for the state variable s is

$$s(t) = s_0 e^{-kt} + \frac{(n-m)}{k}\left[\frac{L}{n} - \frac{p}{2(\delta+k)}\right](1 - e^{-kt}) \tag{17}$$

which is a positive, increasing and concave function if $s_0 < \frac{(n-m)}{k}\left[\frac{L}{n} - \frac{p}{2(\delta+k)}\right]$ otherwise it is a decreasing and convex one. Moreover, in both cases, for $t \to +\infty$, $s(t)$ approaches the value $\frac{(n-m)}{k}\left[\frac{L}{n} - \frac{p}{2(\delta+k)}\right]$.

CASE III

If $r < 1$ then

$$a_i = \frac{L}{n} \; i = 1, ..., m \quad \text{and} \quad a_j = \frac{L}{n} \; j = m+1, ..., n \tag{18}$$

and so both cooperators cease to produce emissions. In this case the role of a signatory and of a non signatory is the same. The optimal path for the state variable s is

$$s(t) = s_0 e^{-kt} \tag{19}$$

which is a positive, decreasing and convex function and for $t \to +\infty$, it approaches zero.

4 Nash equilibria and Stability

In order to apply the stability conditions proposed in the above sections, we need to calculate $C_i(m)$ and $C_j(m)$.
Substituting the obtained optimal control paths of the pollution stock and of the abatement levels in (4) and in (6), we obtain the following results

CASE I

If $r \geq m$ then

$$C_i(m) = \frac{m^2 p^2}{8\delta(\delta+k)^2} + \frac{p}{2\delta(\delta+k)}\left[L - \frac{m^2 p}{2(\delta+k)} - \frac{(n-m)p}{2(\delta+k)}\right] + \frac{p s_0}{2(\delta+k)} \tag{20}$$

$$C_j(m) = \frac{p^2}{8\delta(\delta+k)^2} + \frac{p}{2\delta(\delta+k)}\left[L - \frac{m^2 p}{2(\delta+k)} - \frac{(n-m)p}{2(\delta+k)}\right] + \frac{ps_0}{2(\delta+k)} \quad (21)$$

CASE II

If $1 \leq r < m$ then

$$C_i(m) = \frac{L^2}{2\delta n^2} + \frac{p(n-m)}{2\delta(\delta+k)}\left[\frac{L}{n} - \frac{p}{2(\delta+k)}\right] + \frac{ps_0}{2(\delta+k)} \quad (22)$$

$$C_j(m) = \frac{p^2}{8\delta(\delta+k)^2} + \frac{p(n-m)}{2\delta(\delta+k)}\left[\frac{L}{n} - \frac{p}{2(\delta+k)}\right] + \frac{ps_0}{2(\delta+k)} \quad (23)$$

CASE III

If $r < 1$ then

$$C_i(m) = C_j(m) = \frac{L^2}{2\delta n^2} + \frac{ps_0}{2(\delta+k)} \quad (24)$$

In this case the costs are the same no matter of participation to a coalition. Therefore in the following we definitively assume that $r \geq 1$. We want to determine the size of farsighted stable coalitions. We use a recursive argument. Let us assume that a farsighted stable coalition of size $m \geq 1$ exists and we start supposing that m satisfies the constraint given by the first case, i.e. $r \geq m$. In order to have the smallest farsighted stable coalition larger than the coalition of size m, we need to find the smallest integer h such that $1 \leq h \leq n-m$ and

$$C_i(m+h) \leq C_j(m) \quad (25)$$

Before studying the conditions for which (25) is satisfied, we have to characterize the costs of cooperators and of defectors which depend on the relative positions of $m+h$ and r. Infact, if $m+h \leq r$, then C_i and C_j are given, respectively by (20) and (21); if $m+h > r$ then C_i is given by (22) while C_j by (21), again.
If we suppose that $m+h \leq r$, then (25) becomes:

$$\frac{(m+h)^2p^2}{8\delta(\delta+k)^2} + \frac{p}{2\delta(\delta+k)}\left[L - \frac{(m+h)^2 p}{2(\delta+k)} - \frac{(n-m-h)p}{2(\delta+k)}\right] + \frac{ps_0}{2(\delta+k)} \leq$$

$$\frac{p^2}{8\delta(\delta+k)^2} + \frac{p}{2\delta(\delta+k)}\left[L - \frac{m^2 p}{2(\delta+k)} - \frac{(n-m)p}{2(\delta+k)}\right] + \frac{ps_0}{2(\delta+k)}$$

which is satisfied if

$$h \geq \sqrt{2m(m-1)} - (m-1)\,,$$

that is

$$m+h \geq \sqrt{2m(m-1)} + 1$$

Let $g(m)$ defined as the smallest integer greater than or equal to $\sqrt{2m(m-1)}+1$, i.e.

$$g(m) = [\sqrt{2m(m-1)}+1].$$

If $g(m) \leq \min\{[r], n\}$ then the size of the smallest farsighted stable coalition larger than the coalition of size m is $g(m)$.
If we suppose that $m+h > r$, then (25) becomes:

$$\frac{L^2}{2\delta n^2} + \frac{p(n-m-h)}{2\delta(\delta+k)}\left[\frac{L}{n} - \frac{p}{2(\delta+k)}\right] + \frac{ps_0}{2(\delta+k)} \leq$$

$$\frac{p^2}{8\delta(\delta+k)^2} + \frac{p}{2\delta(\delta+k)}\left[L - \frac{m^2 p}{2(\delta+k)} - \frac{(n-m)p}{2(\delta+k)}\right] + \frac{ps_0}{2(\delta+k)}$$

which is satisfied if

$$h \leq \frac{1}{2}(r+1) + \frac{m(m-r)}{r-1},$$

and so

$$m+h \leq \frac{2m^2 - 2m - 1 + r^2}{2(r-1)} \equiv \lambda.$$

Let $w(m)$ defined as the smallest integer greater than or equal to λ, i.e.

$$w(m) = [\lambda].$$

If $w(m) \geq [r]+1$ and $[r]+1 \leq n$ then the size of the smallest farsighted stable coalition larger than the coalition of size m is $[r]+1$.

Now, we suppose that $1 \leq r < m$, then C_i and C_j, in (25), are given by (22) and (23) and we have:

$$\frac{L^2}{2\delta n^2} + \frac{p(n-m-h)}{2\delta(\delta+k)}\left[\frac{L}{n} - \frac{p}{2(\delta+k)}\right] + \frac{ps_0}{2(\delta+k)} \leq$$

$$\frac{p^2}{8\delta(\delta+k)^2} + \frac{p(n-m)}{2\delta(\delta+k)}\left[\frac{L}{n} - \frac{p}{2(\delta+k)}\right] + \frac{ps_0}{2(\delta+k)}$$

which is satisfied if

$$h \geq \frac{r+1}{2}$$

and so

$$m+h \geq m + \frac{r+1}{2}.$$

Let $z(m)$ defined as the smallest integer greater than or equal to $m + \frac{r+1}{2}$, i.e.

$$z(m) = \left[m + \frac{r+1}{2}\right]$$

then the size of the smallest farsighted stable coalition larger than the coalition of size m is $z(m)$, provided that $z(m) \leq n$.

We propose some numerical examples which show how the size of the farsighted

stable coalitions changes, as the value of p varies.
We fix the following values:

$$n = 100 \ , \ L = 100 \ , \ k = 1 \ , \ \delta = 1 \ , \ s_0 = 0$$

Let $p = 0.01$, then the following coalitions are farsighted stable

$m = 2$
$m = 3$
$m = 5$
$m = 8$
$m = 12$
$m = 18$
$m = 26$
$m = 38$
$m = 55$
$m = 79$

Let $p = 0.1$, then the following coalitions are farsighted stable

$m = 2$
$m = 3$
$m = 5$
$m = 8$
$m = 12$
$m = 18$
$m = 26$
$m = 38$
$m = 41$
$m = 62$
$m = 83$

Let $p = 1$, then the coalitions of size $m = 2$ and $m = 3$ are farsighted stable. Moreover any coalition of size $m = 3t - 1$ with $t = 2, 3, ..., 33$ is farsighted stable. The largest farsighted stable coalition is $m = 98$.

Let $p \geq 4$, then any coalition is farsighted stable.

5 Feedback Nash equilibrium

Open loop strategies imply that each player commits himself to his entire course of action at the beginning of the game and will not revise it at any subsequent moment. In this section we abandon this assumption assuming that players use feedback strategies. A feedback strategy consists of a contingency plan that indicates the optimal value of the control variable for each value of the state

variable at each point in time. It has the property of being subgame perfect, because after each player's actions have caused the state of the system to evolve from its initial state to a new state, the continuation of the game may be regarded as a subgame of the original game. We can say that in this case each player has committed to a rule which yields the optimal value of the control variable in each moment as a function of the state of the system at that moment.
A feedback strategy must satisfy the principle of optimality of dynamic programming.
The Hamilton-Jacobi-Bellman equation for signatories is

$$\delta V_i = \max_{\{a_i\}} \left\{ -\sum_{h=1}^{m} \left(\frac{1}{2} a_h^2 + \frac{1}{2} p\, s \right) + V_i' \left(L - \sum_{h=1}^{m} a_h - \sum_{j=m+1}^{n} a_j - ks \right) \right\} \quad (26)$$

The Hamilton-Jacobi-Bellman equation for non signatories is

$$\delta V_j = \max_{\{a_j\}} \left\{ -\left(\frac{1}{2} a_j^2 + \frac{1}{2} p\, s \right) + V_j' \left(L - \sum_{i=1}^{m} a_i - \sum_{j=m+1}^{n} a_j - ks \right) \right\} \quad (27)$$

where $V_i(s)$ and $V_j(s)$ represent the optimal control value functions of the coalition and of a non signatory associated with the optimization problem (5) and (6), i.e. they denote the minimum present value of the cost flow subject to the dynamic constraint of the accumulated emissions; V_i' and V_j' are the first derivative with respect to the state variable s.

The optimal value of the control variables must satisfy the necessary conditions for an interior feedback Nash equilibrium, that is

$$-a_i - V_i' = 0 \qquad i = 1, ..., m \quad (28)$$

$$-a_j - V_j' = 0 \qquad j = m+1, ..., n \quad (29)$$

These conditions define the optimal strategies for abatements as functions of accumulated emissions; so, the constraint on the control variables given by (2); (28) and (29) lead to the following conditions on the abatement levels

$$a_i = \begin{cases} 0 & if \quad -V_i' < 0 \\ -V_i' & if \quad 0 \le -V_i' \le \frac{L}{n} \quad i = 1, ..., m \\ \frac{L}{n} & if \quad -V_i' > \frac{L}{n} \end{cases}$$

for a signatory country;

$$a_j = \begin{cases} 0 & if \quad -V_j' < 0 \\ -V_j' & if \quad 0 \leq -V_j' \leq \dfrac{L}{n} \quad j = m+1, ..., n \\ \dfrac{L}{n} & if \quad -V_j' > \dfrac{L}{n} \end{cases}$$

for a non signatory country.
We have analysed all possible combinations between interior and boundary a_i and a_j values.

If we suppose that $0 \leq -V_i' \leq \dfrac{L}{n}$ and $0 \leq -V_j' \leq \dfrac{L}{n}$, then $a_i = -V_i'$ and $a_j = -V_j'$. Substituting these abatement level expressions in (26) and in (27), we obtain the following nonlinear differential equations

$$\delta V_i = \frac{m}{2}(V_i')^2 + V_i'(L + (n-m)V_j' - ks) - \frac{1}{2}mps \tag{30}$$

$$\delta V_j = \left(\frac{2n - 2m - 1}{2}\right)(V_j')^2 + V_j'(L + mV_i' - ks) - \frac{1}{2}ps \tag{31}$$

In order to compute the solution of these equations, given the linear quadratic structure of the game, we guess that the optimal value functions are quadratic and consequently the equilibrium strategies are linear respect to the state variable. Precisely, we postulate quadratic value functions of this form

$$V_i = \frac{1}{2}\alpha_i s^2 + \beta_i s + \mu_i \tag{32}$$

$$V_j = \frac{1}{2}\alpha_j s^2 + \beta_j s + \mu_j \tag{33}$$

where α, β, μ are constant parameters of the unknown value functions which are to be determined. Using (32) and (33) to eliminate V_i, V_j, V_i' and V_j' from (30) and from (31), and equating we yield the following system of algebraic Riccati equations for the coefficients of the value functions

$$\begin{cases}
\frac{1}{2}\alpha_i\delta = \dfrac{m}{2}\alpha_i^2 + (n-m)\alpha_i\alpha_j - k\alpha_i \\
\beta_i\delta = m\alpha_i\beta_i + L\alpha_i + (n-m)\alpha_i\beta_j + (n-m)\beta_i\alpha_j - k\beta_i - \frac{1}{2}mp \\
\mu_i\delta = \beta_i\left[\dfrac{m}{2}\beta_i + L + (n-m)\beta_j\right] \\
\frac{1}{2}\alpha_j\delta = \left(\dfrac{2n-2m-1}{2}\right)\alpha_j^2 + m\alpha_i\alpha_j - k\alpha_j \\
\beta_j\delta = (2n-2m-1)\alpha_j\beta_j + L\alpha_j - k\beta_j + m\alpha_j\beta_i + m\alpha_i\beta_j - \frac{1}{2}p \\
\mu_j\delta = \beta_j\left[\left(\dfrac{2n-2m-1}{2}\right)\beta_j + L + m\beta_i\right]
\end{cases}$$

This system has four solutions, but only one produces value functions satisfying the stability condition. To obtain this condition we substitute the linear strategies

$$a_i = -\alpha_i s - \beta_i \ , \ a_j = -\alpha_j s - \beta_j \tag{34}$$

in the dynamical constraint of accumulated emissions. We obtain the following differential equation

$$\dot{s} = [m\alpha_i + (n-m)\alpha_j - k]s + L + m\beta_i + (n-m)\beta_j \tag{35}$$

The stability condition is

$$\frac{d\dot{s}}{ds} = m\alpha_i + (n-m)\alpha_j - k < 0$$

which is satisfied only by the following solution of the system

$$\alpha_i = \alpha_j = 0 \ , \ \beta_i = -\frac{mp}{2(k+\delta)} \ , \ \beta_j = -\frac{p}{2(k+\delta)}$$

$$\mu_i = -\frac{mp(4kL + 4L\delta - p(m^2 - 2m + 2n))}{8\delta(k+\delta)^2}$$

$$\mu_j = -\frac{p(4kL + 4L\delta - p(2m^2 - 2m + 2n - 1))}{8\delta(k+\delta)^2}$$

This solution, combined with the constraints $0 \leq -V_i' \leq \dfrac{L}{n}$ and $0 \leq -V_j' \leq \dfrac{L}{n}$, gives us the optimal abatement levels:

$$a_i = \frac{mp}{2(\delta+k)} \quad \text{and} \quad a_j = \frac{p}{2(\delta+k)}$$

when the following condition on p is satisfied

$$p \leq \frac{2L(\delta+k)}{mn}$$

which is equivalent to

$$m \leq r\,.$$

It is possible to conclude that the Feedback Nash equilibrium obtained, coincide with the open loop Nash equilibrium given by case I.

If we suppose that $-V_i' > \frac{L}{n}$ and $0 \leq -V_j' \leq \frac{L}{n}$, then $a_i = \frac{L}{n}$ and $a_j = -V_j'$. Argumenting as above we obtain

$$a_i = \frac{L}{n} \quad \text{and} \quad a_j = \frac{p}{2(\delta+k)}$$

when the following condition on p is satisfied

$$\frac{2L(\delta+k)}{mn} < p \leq \frac{2L(\delta+k)}{n}$$

which is equivalent to

$$1 \leq r < m\,.$$

Again we conclude that the Feedback Nash equilibrium obtained coincides with the corresponding open loop one.
If we consider the remaining combinations between a_i and a_j values, it is possible to prove that solutions of the Riccati system don't satisfy the constraints and so they don't represent feedback Nash equilibria.
So, we conclude this analysis, claiming that, in the model proposed, Feedback and Open Loop Nash equilibria are the same.

6 Concluding remarks

The present paper studies the problem of computing the size of a stable coalition in an International Environmental Agreement.
We studied a differential game in which abatement levels are associated with a stock pollutant. Coalition formation has been designed as a two stages game in which in the first stage each country decides if to join or not a coalition, instead, in the second stage, non signatories and signatories determine the optimal paths of the abatements and so , also, the path of the global emissions.
The model is characterized by the presence of a parameter p which gives us the measure of the environmental awareness of countries. The results obtained show that if p is quite small, only coalitions of size 2 or 3 are internally and externally stable, but if we consider the farsighted stability, then large stable coalitions can occur.
If p increases it is possible to obtain value of it for which no coalition is internally

and externally stable. Moreover, also in this case it it possible to observe that if we consider farsighted stability both large and small coalitions can be stable. Only if cooperators and defectors abate the same quantities then myopic stability conditions and so farsighted conditions are satisfied, whatever is coalitions' size.
Open loop Nash equilibria and Feedback Nash equilibria have been analysed showing that they carry out to the same solution of the differential game.
A possible step for a forthcoming paper could be the study of stable coalitions modifying the costs function or relaxing some assumptions of the game's rules.

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