Approximate Social Nash Equilibria and Applications

Vincenzo Scalzo

Dipartimento di Matematica e Statistica, Università Federico II
via Cinthia, 80126 Napoli, Italy
e-mail: scalzo@unina.it

Abstract. In this paper, a concept of approximate social Nash equilibria is considered and an existence result is given when the strategic spaces of the players are not compact. These have been obtained using an approximate fixed point theorem. As an application of the existence of such approximate social Nash equilibria, sufficient conditions for the existence of a suitable approximate walrasian equilibrium in finite economies are obtained. Among others things, it is shown that the approximate walrasian equilibrium here considered is approximatively weakly efficient.

Key words. Abstract economy, approximate social Nash equilibrium, finite economy, approximate walrasian equilibrium, approximate fixed point theorems.

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1 Introduction

In the framework of finite exchange economies, the closedness of the sets of consumption bundles has played a crucial role in order to obtain sufficient conditions for the existence of walrasian equilibria (see, for example, [6], [7], [10]). When the consumption sets are not closed, even if an exchange economy has continuous and convex preferences, walrasian equilibria may not exist. Moreover, an economy may have not walrasian equilibria also when the preferences are not convex and the consumption sets are closed. So, interesting can be concepts of approximate walrasian equilibria.
Several works have been devoted to concepts of approximate equilibria for exchange economies with a finite number of goods and non necessarily convex preferences. There, existence results have been obtained when the consumption sets coincide with the (closed) positive orthant (see, for example, [16], [9], [1], [2]).
In this paper, in order to investigate free disposal exchange economies with bounded and *non* closed consumption sets and with convex preferences, a concept of approximate walrasian equilibrium is considered. This concept deals with pairs of prices and allocations such that any consumer gets an approximation of his "Utility Maximization Problem" over a "trimmed" budget set and the lost power of purchase of any consumer is "small enough". An existence result is obtained using approximate fixed point arguments (as considered in [5]) and a suitable approximate social Nash equilibrium. Moreover, such an approximate walrasian equilibrium satisfies an approximate weakly efficiency condition.
The paper is planned as follows. In Section 2, a concept of approximate walrasian equilibrium is presented and it is shown that it is approximatively weakly efficient and that a converging sequence of approximate equilibria converges to an exact equilibrium whenever the approximations are converging to zero. Section 3 presents a concept of approximate social Nash equilibrium, together with sufficient conditions for its existence. Using an approximate fixed point theorem ([5]), these are obtained when the strategic spaces are not closed and totally bounded subsets of real Banach spaces. Finally, in Section 4, with the aid of the results of Section 3, sufficient conditions for the existence of an approximate walrasian equilibria for economies with non closed consumption sets are established.

2 An approximate walrasian equilibrium in finite exchange economies

We are interested in a concept of approximate walrasian equilibrium for finite free disposal exchange economies having non closed consumption sets.
More precisely, let $\mathcal{E} = \{X_i, \omega_i, u_i \ / \ i \in \{1, ..., m\}\}$ be an exchange economy (where $m \geq 2$) with ℓ goods and m consumers (we remaind to [6] and [7] for more details on the meaning of an exchange economy). For any consumer i, $X_i \subseteq \mathbb{R}^\ell$ is the set of consumption bundles, u_i is the utility function and ω_i is the bundle of its initial endowment. Set $\omega = \sum_{i=1}^m \omega_i$ (the total resource of the economy), a tuple $(x_1, ..., x_m) \in X = \times_{i=1}^m X_i$ is said to be an *allocation* if $\sum_{i=1}^m x_i \leq \omega$ (where $\leq$ denotes the Pareto order relation). Now, given $(x_1, ..., x_m) \in X$ and a vector $p \in \mathbb{R}_+^\ell$ of prices, the system $(p, (x_1, ..., x_m))$ is said to be a *free-disposal walrasian (or competitive) equilibrium* ([6], [7], [10]) if $(x_1, ..., x_m)$ is an allocation and:

> for all $i \in \{1, ..., m\}$, $p \cdot x_i \leq p \cdot \omega_i$ and $u_i(x_i) \geq u_i(z_i)$ for any $z_i \in X_i$ such that $p \cdot z_i \leq p \cdot \omega_i$.

In order to have sufficient conditions for the existence of walrasian equilibria, a central role has been played by the closedness of the sets $X_1, ..., X_m$ (see, for example, [6], [7], [10]). Now, an economy may not have walrasian equilibria when $X_1, ..., X_m$ are not closed sets. So, when these sets are not closed, useful can be concepts of approximate walrasian equilibrium, as considered in the next definition where the approximation is on both levels of utilities and power of purchase.

Definition 2.1 Let ε be a positive real number. An allocation $(x_1^*, ..., x_m^*)$ is said to be an *ε-walrasian allocation* if there exist a prices vector p^* and m positive real numbers $\nu_1, ..., \nu_m$ (called *associated trimmed budgets*) such that:

i) for all $i \in \{1, ..., m\}$: $\nu_i \leq p^* \cdot \omega_i$, $p^* \cdot x_i^* \leq p^* \cdot \omega_i$ and $u_i(x_i^*) \geq u_i(x_i) - \varepsilon$ for all $x_i \in X_i$ such that $p^* \cdot x_i \leq \nu_i$

ii) $p^* \cdot \sum_{i=1}^m \omega_i - \sum_{i=1}^m \nu_i \leq \varepsilon$

The tuple $(p^*, (x_1^*, ..., x_m^*))$ is called *ε-walrasian equilibrium.*

If $(x_1^*, ..., x_m^*)$ is an *ε-walrasian allocation*, the consumer i asks the bundle x_i^* which is an ε-maximizer of the utility u_i on the "trimmed" budget set $\{x_i \in X_i \ / \ p^* \cdot x_i \leq \nu_i\}$ and he is able to ask x_i^* since $p^* \cdot x_i^* \leq p^* \cdot \omega_i$. Moreover, the part of the value of the total resource of the economy (with

respect to the price vector p^*) which is lost, it is not greater than ε.
In Section4 it will be given sufficient conditions for the existence of such ε-*walrasian equilibria.*

Remark 2.1 Note that if $(p^*, (x_1^*, ..., x_m^*))$ is an ε-walrasian equilibrium (with the associated trimmed budgets $\nu_1, ..., \nu_m$) and if λ is a positive real number, set $p' = \lambda p^*$, the list $(p', (x_1^*, ..., x_m^*))$ is still an ε-walrasian equilibrium (with the associated trimmed budgets $\lambda\nu_1, ..., \lambda\nu_m$) whenever $\lambda < 1$. If $\lambda > 1$, $(p', (x_1^*, ..., x_m^*))$ is an $\lambda\varepsilon$-walrasian equilibrium.

As an exact walrasian allocation in a free disposal economy is weakly efficient (see, for example, [7], [10]), an ε-*walrasian allocation* is *approximatively weakly efficient* (if the value of the total resource of the economy is not "too small") in the following sense, which is an extension of the concept of *approximate efficiency*[1] considered in [14] and in [15].

Definition 2.2 Let $\varepsilon > 0$. An allocation $(x_1, ..., x_m)$ is said to be ε-*weakly efficient* if there are not allocations $(y_1, ..., y_m)$ such that the following holds:

$$\begin{array}{ll} \text{i)} & \sum_{i=1}^m y_i \leq (1-\varepsilon) \sum_{i=1}^m \omega_i \\ \text{ii)} & u_i(y_i) > u_i(x_i) + \varepsilon \ \ \forall\, i \in \{1, ..., m\} \ . \end{array}$$

Proposition 2.1 *Let $(p^*, (x_1^*, ..., x_m^*))$ be an ε-walrasian equilibrium and let $p^* \cdot \sum_{i=1}^m \omega_i \geq 1$. Then, the allocation $(x_1^*, ..., x_m^*)$ is ε-weakly efficient.*

Proof. Assume that $(x_1^*, ..., x_m^*)$ is not ε-*weakly efficient* and let $\nu_1, ..., \nu_m$ be the associated trimmed budgets. So, there exists an allocation $(y_1, ..., y_m)$ such that i) and ii) in Definition 2.2 hold. In light of i) one has $(1-\varepsilon)\, p^* \cdot \sum_{i=1}^m \omega_i \geq p^* \cdot \sum_{i=1}^m y_i$; in light of ii) one has $p^* \cdot y_i > \nu_i$ for all $i \in \{1, ..., m\}$. So, it results:

$$(1-\varepsilon)\, p^* \cdot \sum_{i=1}^m \omega_i \geq p^* \cdot \sum_{i=1}^m y_i > \sum_{i=1}^m \nu_i \geq p^* \cdot \sum_{i=1}^m \omega_i - \varepsilon$$

and

$$\varepsilon\, p^* \cdot \sum_{i=1}^m \omega_i < \varepsilon \ ,$$

which is a contradiction. □

[1]An allocation $(x_1, ..., x_m)$ is said to be ε-*efficient* (see [14] and [15]) if, given a list of bundles $(y_1, ..., y_m)$, the following holds: $u_i(y_i) \geq u_i(x_i)\ \forall\ i \in \{1, ..., m\} \Longrightarrow \sum_{i=1}^m y_i \not\leq (1-\varepsilon) \sum_{i=1}^m \omega_i$.

Finally, as shown by the following proposition, we observe that a converging sequence of approximate walrasian equilibria converges to a walrasian equilibrium if the approximations are converging to zero.

Proposition 2.2 *Assume that the utility functions of the consumers are continuous and, for all* i*,* X_i *is convex and all* $p \in \mathbb{R}^\ell_+$*, there exists* $z_i \in X_i$ *such that* $p \cdot z_i < p \cdot \omega_i$*. Let* $\varepsilon_n \longrightarrow 0^+$*. If* $(p^n, (x_1^n, ..., x_m^n))_n$ *is a sequence converging to* $(p^o, (x_1^o, ..., x_m^o)) \in \mathbb{R}^\ell_+ \times X$ *such that, for all* n*,* $(p^n, (x_1^n, ..., x_m^n))$ *is an* ε_n*-walrasian equilibrium, then* $(p^o, (x_1^o, ..., x_m^o))$ *is a walrasian equilibrium.*

Proof. For any n, there exist positive real numbers $\nu_1^n, ..., \nu_m^n$ such that, for all $i \in \{1, ..., m\}$ and all n:

(i) $p^n \cdot x_i^n \leq p^n \cdot \omega_i$ and $\nu_i^n \leq p^n \cdot \omega_i$

(ii) $u_i(x_i^n) \geq u_i(x_i) - \varepsilon_n$ for all $x_i \in X_i$ such that $p^n \cdot x_i \leq \nu_i^n$

(iii) $p^n \cdot \sum_{i=1}^m \omega_i - \sum_{i=1}^m \nu_i^n \leq \varepsilon_n$

Being $0 < \nu_i^n \leq p^n \cdot \omega_i$ for all n and all i, it follows that $(\nu_i^n)_n$ is bounded and so it admits a converging subsequence. For sake of simplicity, we assume that $(\nu_i^n)_n$ is converging to ν_i^o for all i. By (iii) one obtains $p^o \cdot \sum_{i=1}^m \omega_i = \sum_{i=1}^m \nu_i^o$. So, in light of the second of (i), it results $\nu_i^o = p^o \cdot \omega_i$ for all i. Now, the set-valued function

$$K_i : (p, \nu) \in \mathbb{R}^\ell_+ \times \mathbb{R}_+ \longrightarrow \{z_i \in X_i \ / \ p \cdot z_i \leq \nu\}$$

is lower semicontinuous at (p^o, ν_i^o). So, taken $z_i \in K_i(p^o, \nu_i^o)$, there exists a sequence $(z_i^n)_n$ converging to z_i such that $z_i^n \in K_i(p^n, \nu_i^n)$ for n sufficiently large. Finally, in light of the continuity of utilities, the thesis follows from (ii). □

3 An approximate social Nash equilibrium in abstract economies

Let $m > 1$ be an integer. For any $i \in \{1, ..., m\}$, let Y_i be a non-empty set, f_i be a real valued function defined on $Y = \times_{j=1}^m Y_j$ and K_i be a set-valued function from $Y_{-i} = \times_{j \neq i} Y_j$ to Y_i. The list of data $\Gamma = \{Y_i, K_i, f_i \ / \ i \in \{1, ..., m\}\}$ is said to be an *abstract economy* ([6]), also called *pseudo-game* ([10]). We recall that a profile of strategies $y^* \in Y$ is said to be a *social Nash equilibrium* ([6]) if:

$$y_i^* \in K_i(y_{-i}^*) \quad \text{and} \quad f_i(y_i^*, y_{-i}^*) = \sup_{z_i \in K_i(y_{-i}^*)} f_i(z_i, y_{-i}^*) \qquad \forall\, i \in \{1, ..., m\}.$$

When $K_i(y_{-i}) = Y_i$ for all $y_{-i} \in Y_{-i}$ and all $i \in \{1, ..., m\}$, a social Nash equilibrium is nothing but a Nash equilibrium ([12], [13]).
Suppose that the set-valued functions $K_1, ..., K_m$ are described by inequalities: for any player i there exists a function $g_i : Y \longrightarrow \mathbb{R}$ (called *constraint function*) such that:

$$K_i(y_{-i}) = \{y_i \in Y_i \ / \ g_i(y_i, y_{-i}) \leq 0\} \ \text{ for all } \ y_{-i} \in Y_{-i} \tag{1}$$

Moreover, for any positive real number σ and any $y_{-i} \in Y_{-i}$, we set $K_i^\sigma(y_{-i}) = \{y_i \in Y_i \ / \ g_i(y_i, y_{-i}) \leq \sigma\}$.

Definition 3.1 Let ε and σ be two positive real numbers. A profile of strategies $y^* \in Y$ is said to be an (ε, σ)-*social Nash equilibrium* of Γ if the following holds for any $i \in \{1, ..., m\}$:

$$y_i^* \in K_i^\sigma(y_{-i}^*) \quad \text{and} \quad f_i(y_i^*, y_{-i}^*) \geq \sup_{y_i \in K_i(y_{-i}^*)} f_i(y_i, y_{-i}^*) - \varepsilon \ .$$

Given y^* an (ε, σ)-social Nash equilibrium, y_i^* is a strategy "close" to the set of feasible strategies $K_i(y_{-i}^*)$ of the player i (under a continuity assumption on g_i) and it is such that i cannot improve the payoffs $f_i(y_i^*, y_{-i}^*)$ with more than ε by unilateral deviations on $K_i(y_{-i}^*)$. Note that if $K_i(y_{-i}) = Y_i$ for all $y_{-i} \in Y_{-i}$ and all $i \in \{1, ..., m\}$, the approximate social Nash equilibria by Definition 3.1 coincides with the classical approximate Nash equilibria for games in strategic form[2].

The set of (ε, σ)-social Nash equilibria of Γ coincides with the set of fixed points of the following (ε, σ)-*aggregate best response set-valued function* $B^{(\varepsilon,\sigma)}$:

$$B^{(\varepsilon,\sigma)} : y \in Y \longrightarrow \times_{i=1}^m B_i^{(\varepsilon,\sigma)}(y_{-i}) \in 2^Y$$

where, for any $i \in \{1, ..., m\}$ and any $y_{-i} \in Y_{-i}$, we set

$$B_i^{(\varepsilon,\sigma)}(y_{-i}) = \left\{ y_i \in K_i^\sigma(y_{-i}) \ / \ f_i(y_i, y_{-i}) \geq \sup_{z_i \in K_i(y_{-i})} f_i(z_i, y_{-i}) - \varepsilon \right\} \tag{2}$$

In order to use classical fixed point theorems to obtain the existence of (ε, σ)-social Nash equilibria, among others assumptions, one has to consider compact strategic spaces. Now, using *approximate* fixed point theorems (see [17]

[2]Let $\Gamma = \{Y_1, ..., Y_m, f_1, ..., f_m\}$ be a game in strategic form and $\varepsilon > 0$. A profile of strategies y^* is said to be an ε-*Nash equilibrium* if $f_i(y^*) + \varepsilon \geq f_i(y_i, y_{-i}^*)$ for any $y_i \in Y_i$ and any $i \in \{1, ..., m\}$.

and [5]), it is possible to obtain the existence of fixed points of $B^{(\varepsilon,\sigma)}$ without compactness assumptions on the strategic spaces. In fact, in the same spirit of [5] (see the "Key Proposition"), under extra continuity assumptions on payoff and constraint functions, some δ-fixed point[3] of the set-valued function $B^{(\varepsilon/2,\sigma/2)}$ is also an (ε,σ)-social Nash equilibrium. The next theorem presents sufficient conditions for the existence of (ε,σ)-social Nash equilibria of abstract economies having non closed subsets of real Banach spaces as strategic sets.

Theorem 3.1 *Let $\Gamma = \{Y_i, K_i, f_i \ / \ i \in \{1,...,m\}\}$ be an abstract economy, where $Y_1,...,Y_m$ are convex and totally bounded subsets, with non empty interior, of real Banach spaces. Let the constraints $K_1,...,K_m$ be described by (1) with the functions $g_1,...,g_m$ respectively. Assume that the following is satisfied for all $i \in \{1,...,m\}$:*

(i) the payoff function f_i and the constraint function g_i are uniformly continuous on Y with respect to the norm $\| \cdot \|_Y = \sum_{j=1}^m \| \cdot \|_{Y_j}$;

(ii) the function $f_i(\cdot, y_{-i})$ is quasi-concave on Y_i for any $y_{-i} \in Y_{-i}$;

(iii) for any $y_{-i} \in Y_{-i}$, there exists $y_i \in Y_i$ such that $g_i(y_i, y_{-i}) < 0$;

(iv) the function $g_i(\cdot, y_{-i})$ is strictly quasi-convex[4] on Y_i for all $y_{-i} \in Y_{-i}$.

Then, for any $\varepsilon, \sigma > 0$, Γ has at least an (ε,σ)-social Nash equilibrium.

Proof. Let ε and σ be positive real numbers. First, we prove that $B^{(\varepsilon/2,\sigma/2)}$ is a closed set-valued function with non-empty and convex values.
It is easy to prove that $B^{(\varepsilon/2,\sigma/2)}$ has non-empty and convex values (note that since f_i is uniformly continuous on Y totally bounded, then it is a bounded function). To prove that $B^{(\varepsilon/2,\sigma/2)}$ is a closed set-valued function, we show that the set-valued function $B_i^{(\varepsilon/2,\sigma/2)}$, defined on Y_{-i} by (2), is closed for all i. Let $(y_{-i}^n)_n$ be a sequence converging to y_{-i} and let $(y_i^n)_n$ be a sequence converging to y_i and such that $y_i^n \in B_i^{(\varepsilon/2,\sigma/2)}(y_{-i}^n)$ for n sufficiently large. So we obtain

$$f_i(y_i^n, y_{-i}^n) \geq \sup_{z_i \in K_i(y_{-i}^n)} f_i(z_i, y_{-i}^n) - \varepsilon/2 \quad \text{for } n \text{ sufficiently large.} \qquad (3)$$

[3] If $F : Y \longrightarrow 2^Y$ and (Y,d) is a metric space, taken $\delta > 0$, a point $y \in Y$ is said to be a *δ-fixed point* of F (see, for example, [17] and [5]) if $d(y, F(y)) = \inf\{d(y,z) \ / \ z \in F(y)\} \leq \delta$.

[4] A function $h : C \longrightarrow \mathbb{R}$, where C is a non-empty and convex subset of a vector space, is said to be *strictly quasi-convex* (see, for example, [3]) if for any $c_1, c_2 \in C$ such that $h(c_1) \neq h(c_2)$ and any $t \in]0,1[$, it results $h((1-t)c_1 + tc_2) < \max\{h(c_1), h(c_2)\}$.

In light of [11], Corollary 3.3.1, from *(i)*, *(iii)* and *(iv)*, it follows that the set-valued function K_i is lower semicontinuous. So, in light of [4], Theorem 1 page 121, the marginal function:

$$z_{-i} \mapsto \sup_{z_i \in K_i(z_{-i})} f_i(z_i, z_{-i})$$

is lower semicontinuous and we obtain from (3):

$$f_i(y_i, y_{-i}) \geq \liminf_{n\to\infty} \sup_{z_i \in K_i(y^n_{-i})} f_i(z_i, y^n_{-i}) - \varepsilon/2 \geq \sup_{z_i \in K_i(y_{-i})} f_i(z_i, y_{-i}) - \varepsilon/2 \; .$$

Moreover, since $g_i(y^n_i, y^n_{-i}) \leq \sigma/2$, it follows $g_i(y_i, y_{-i}) \leq \sigma/2$, that is $y_i \in K_i^{\sigma/2}(y_{-i})$. So, $B_i^{(\varepsilon/2,\sigma/2)}$ is a closed set-valued function with non-empty and convex values for all i and in light of [5], Theorem 2.3[5], there exist δ-fixed points of $B^{(\varepsilon/2,\sigma/2)}$ for any $\delta > 0$.
Since the payoff and constraint functions are uniformly continuous on Y, given $\varepsilon > 0$ and $\sigma > 0$, there exists $\delta > 0$ such that for any $y, z \in Y$ with $\| y - z \| < \delta$, it results

$$f_i(y_i, y_{-i}) \geq f_i(z_i, z_{-i}) - \varepsilon/2 \;\text{ and }\; g_i(y_i, y_{-i}) \leq g_i(z_i, z_{-i}) + \sigma/2 \qquad (4)$$

for all $i \in \{1, ..., m\}$.
Taken y^* a $\delta/2$-fixed point of $B^{(\varepsilon/2,\sigma/2)}$, there exists $\hat{y} \in B^{(\varepsilon/2,\sigma/2)}(y^*)$ such that $\| y^* - \hat{y} \| < \delta$ and we have for all i:

$$f_i(\hat{y}_i, y^*_{-i}) \geq \sup_{z_i \in K_i(y^*_{-i})} f_i(z_i, y^*_{-i}) - \varepsilon/2 \quad \text{and} \quad g_i(\hat{y}_i, y^*_{-i}) \leq \sigma/2 \qquad (5)$$

Finally, from (4) and (5), we obtain

$$f_i(y^*_i, y^*_{-i}) \geq \sup_{z_i \in K_i(y^*_{-i})} f_i(z_i, y^*_{-i}) - \varepsilon \quad \text{and} \quad g_i(y^*_i, y^*_{-i}) \leq \sigma$$

for all $i \in \{1, ..., m\}$, which means that y^* is an (ε, σ)-social Nash equilibrium of Γ. □

Remark 3.1 Note that if $K_i(y_{-i}) = Y_i$ for any $y_{-i} \in Y_{-i}$ and any $i \in \{1, ..., m\}$, the hypothesis on the strategic spaces and on the payoffs functions in Theorem 3.1 are sufficient conditions for the existence of approximate Nash equilibria.

[5] **[5], Theorem 2.3**. Let E be a real Banach space and let Y be a convex and totally bounded subset of E with non-empty interior. Assume that $F : Y \longrightarrow 2^Y$ is a closed set-valued function with non-empty and convex values. Then F has δ-fixed points for any $\delta > 0$.

Remark 3.2 Assume that the payoff and constraint functions are continuous and that the hypothesis (iii) and (iv) of Theorem 3.1 are satisfied. Let $\varepsilon_n \longrightarrow 0^+$, $\sigma_n \longrightarrow 0^+$ and $(y^n)_n$ be a sequence such that y^n is an $(\varepsilon_n, \sigma_n)$-social Nash quilibrium of Γ for all n. It is easy to prove that if $y^n \longrightarrow y^o$, then y^o is a social Nash equilibrium of Γ.

Remark 3.3 In order to obtain sufficient conditions for the existence of (ε, σ)-social Nash equilibria, the assumptions of uniform continuity on the payoff and constraint functions in Theorem 3.1 cannot be relaxed in only continuity assumptions. In fact, Example 3.1 considers a game without approximate Nash equilibria but with bounded strategic sets and continuous payoffs.

Example 3.1 Let $\Gamma = \{Y_1, Y_2, f_1, f_2\}$ be the game with $Y_1 = Y_2 =]0,1[$ and f_1, f_2 defined on $]0,1[\times]0,1[$ by

$$f_1(y_1, y_2) = \frac{y_1}{y_2} \qquad \text{and} \qquad f_2(y_1, y_2) = -\frac{y_2}{1 - y_1}$$

The functions f_1 and f_2 are continuous but not uniformly continuous on $]0,1[\times]0,1[$ and Γ has not ε-Nash equilibria for any $\varepsilon \in]0,1[$. In fact, fixed $\varepsilon \in]0,1[$, the set of all ε-Nash equilibria coincides with the set of fixed points of the set-valued function B^ε defined by

$$B^\varepsilon(y_1, y_2) = B_1^\varepsilon(y_2) \times B_2^\varepsilon(y_1)$$

where, for $i, j \in \{1, 2\}$ and $i \neq j$,

$$B_i^\varepsilon(y_j) = \left\{ y_i \in Y_i \ / \ f_i(y_i, y_j) \geq \sup_{z_i \in Y_i} f_i(z_i, y_j) - \varepsilon \right\}.$$

Now, for all $(y_1, y_2) \in]0,1[\times]0,1[$, it results

$$B_1^\varepsilon(y_2) = [1 - y_2\,\varepsilon, 1[\quad \text{and} \quad B_2^\varepsilon(y_1) =]0, (1 - y_1)\,\varepsilon].$$

So, one can verify that B^ε has not fixed points on $]0,1[\times]0,1[$.

The next example considers a game with totally bounded strategic sets (in real Banach spaces), which has *approximate* Nash equilibria but not Nash equilibria.

Example 3.2 Let $\Gamma = \{Y_1, Y_2, f_1, f_2\}$ be the game in which $Y_1 = Y_2 = Y$ is a convex and totally bounded subset, with non-empty interior, of a real

Banach space E. Assume that Y is not closed and $0 \in \overline{Y} \backslash Y$. The payoffs are defined on $Y_1 \times Y_2$ as follows:

$$f_1(y_1, y_2) = - \parallel y_1 - y_2 \parallel$$

and

$$f_2(y_1, y_2) = - \left\| \ y_2 - \tfrac{\|y_1\|}{1+\|y_1\|} y_1 \ \right\| .$$

For any $(y_1, y_2) \in Y_1 \times Y_2$, set B the aggregate best response set-valued function, it results

$$B(y_1, y_2) = \left\{ \left(y_2, \frac{\parallel y_1 \parallel}{1+ \parallel y_1 \parallel} y_1 \right) \right\}.$$

So, Γ does not have Nash equilibria. Since the functions f_1 and f_2 are uniformly continuous on $Y_1 \times Y_2$ and they satisfy the hypothesis of Theorem 3.1, Γ has at least an ε-Nash equilibrium for any $\varepsilon > 0$.

4 Existence of an approximate walrasian equilibrium

In the next theorem, sufficient conditions for the existence of the ε-*walrasian equilibria* considered in Definition 2.1 are given.

Theorem 4.1 *Let $\mathcal{E} = \{X_i, \omega_i, u_i \ / \ i \in \{1, ..., m\}\}$ $(m > 1)$ be an exchange economy, where $X_1, ..., X_m$ are bounded and convex subsets of $\mathbb{R}^\ell$ with non-empty interior. Assume that for all $i \in \{1, ..., m\}$, all $p \in \mathbb{R}^\ell_+$ and all $\delta > 0$, there exists $x_i \in X_i$ such that $p \cdot x_i < p \cdot \omega_i - \delta$. If u_i is a quasi-concave and uniformly continuous function on X_i, for any i, then there exists at least an ε-walrasian equilibrium, for all $\varepsilon > 0$.*

Proof. We proceed as follows. First, we construct an abstract economy Γ whose approximate social Nash equilibria are also approximate competitive equilibria of $\mathcal{E}$. Then the thesis will be obtained applying Theorem 3.1.
Let $\varepsilon > 0$ and $\bar{\omega}_1, ..., \bar{\omega}_m$ such that $\bar{\omega}_i = \omega_i - (1/m)\, \boldsymbol{\varepsilon}$ for all $i \in \{1, ..., m\}$, where $\boldsymbol{\varepsilon}$ is the vector of $\mathbb{R}^\ell$ whose components are equal to ε. Let

$$\Gamma = \big\{ Y_i, K_i, f_i \ / \ i \in \{1, ..., m, m+1\} \big\}$$

be the abstract economy defined as following:

- $Y_i = X_i$ if $i \in \{1, ..., m\}$ and $Y_{m+1} = \{q \in \mathbb{R}^\ell_+ \ / \ \frac{1}{m} \leq q_1 + ... + q_\ell \leq 1\}$;

- $K_i(p, y_{-i}) = \{y_i \in Y_i \ / \ p \cdot y_i \leq p \cdot \bar{\omega}_i\}$ for any $(p, y_{-i}) \in Y_{m+1} \times Y_{-i}$ and for any $i \in \{1, ..., m\}$;
- $K_{m+1}(y) = \{p \in Y_{m+1} \ / \ \sum_{k=1}^{\ell} p_k - 1 \leq 0\}$;
- $f_i(p, y) = u_i(y_i)$ for any $(p, y) \in Y_{m+1} \times Y$ and for any $i \in \{1, ..., m\}$;
- $f_{m+1}(p, y) = p \cdot \sum_{i=1}^{m}(y_i - \bar{\omega}_i)$ for any $(p, y) \in Y_{m+1} \times Y$;

where $Y = \times_{i=1}^{m} Y_i$ and, for any $i \in \{1, ..., m\}$, $Y_{-i} = \times_{j \in \{1,...,m\} \setminus \{i\}} Y_j$. We set $\sigma = \varepsilon/(2m^2)$.
All hypothesis of Theorem 3.1 are satisfied. So, there exists an $(\varepsilon/2, \sigma)$-social Nash equilibrium $(p^*, (x_1^*, ..., x_m^*))$ of Γ. For any $i \in \{1, ..., m\}$, it results:

$$p^* \cdot x_i^* \leq p^* \cdot \bar{\omega}_i + \frac{\varepsilon}{2m^2} < p^* \cdot \omega_i$$

and

$$u_i(x_i^*) \geq u_i(x_i) - \frac{\varepsilon}{2} \ \text{ for all } \ x_i \in X_i \ \text{ such that } \ p^* \cdot x_i \leq p^* \cdot \bar{\omega}_i \ .$$

On the other hand, we have

$$p \cdot \sum_{i=1}^{m}(x_i^* - \bar{\omega}_i) - \frac{\varepsilon}{2} \leq p^* \cdot \sum_{i=1}^{m}(x_i^* - \bar{\omega}_i) \leq \frac{\varepsilon}{2} \quad \forall \, p \in Y_{m+1} \ .$$

So, it results $\sum_{i=1}^{m}(x_i^* - \bar{\omega}_i) \leq \boldsymbol{\varepsilon}$, which implies that $(x_1^*, ..., x_m^*)$ is an allocation of the economy. Now, set $\nu_i = p^* \cdot \bar{\omega}_i$ for all $i \in \{1, ..., m\}$, we obtain:

$$u_i(x_i^*) \geq u_i(x_i) - \varepsilon \ \text{ for all } \ x_i \text{ such that } p^* \cdot x_i \leq \nu_i \ .$$

Finally,

$$p^* \cdot \sum_{i=1}^{m} \omega_i - \sum_{i=1}^{m} \nu_i = p^* \cdot \sum_{i=1}^{m}(\omega_i - \bar{\omega}_i) = p^* \cdot \boldsymbol{\varepsilon} \leq \varepsilon$$

and the proof is concluded. □

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