Dipartimento di Scienze Economiche, Matematiche e Statistiche

Università degli Studi di Foggia

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Quaderno n. 8/2007

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A Monte Carlo approach to value exchange options using a single stochastic factor.

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Abstract

Exchange options give the holder the right to exchange one risky asset V for another risky asset D. The asset V is referred to as the optioned (underlying) asset, while D is the delivery asset. So, when an exchange option is valued, we generally are exposed to two sources of uncertainity, namely we have two stochastic variables.
Exchange options arise quite naturally in a number of significant financial arrangements including bond futures contracts, investment performance, options whose strike price is an average of the experienced underlying asset price during the life ot the option and so on.
In this paper we propose some algorithms to estimate exchange options by Monte Carlo simulation reducing the bi-dimensionality of valuation problem to single stochastic factor.

Keyword: Exchange Options; Monte Carlo Simulations.
JEL Codes: G13; C15.

1 Introduction

The pricing of options by simulation techniques is an important task especially where analytical solutions are not available. With the aid of ever faster computers coupled with the development of new numerical methods, we are nowadays able to solve numerically an increasing number of important security pricing models. Even where we appear to have analytical solutions it is often desirable to have an alternative implementation that is supposed to give the same answer. Simulation methods for asset pricing were introduced in finance by (Boyle, 1977). Since that time simulation has been successfully applied to a wide range of pricing problems, particularly to value american options as witnessed by the contributions of Tilley (1993), Barraquand & Martineau (1995),

Broadie & Glasserman (1997), Raymar & Zwecher (1997).
The aim of this paper is to propose some algorithms, based on Monte Carlo simulation, for the estimation of exchange options that give its owner the right to exchange one risky asset for another. Exchange options arise quite naturally in a number of significant financial arrangements such as bond futures contracts, investment performance, options whose strike price is an average of the experienced underlying asset price during the life of the option.
The most relevant models that value exchange options are given in Margrabe (1978), McDonald & Siegel (1985), Carr (1988,1995), Armada et al. (2007). We can synthesize the main characteristics of these models.
Margrabe (1978) values an European exchange option which gives the right to realize such exchange only at expiration. Margrabe (1978) also proves that the exercise of American exchange option will only occur at expiration when neither underlying asset pays dividends. McDonald & Siegel (1985) value an European exchange option considering that the assets distribute dividends and Carr (1988) values a compound European exchange option in which the underlying asset is another exchange option. However, when the asset to be received in the exchange pays sufficient large dividends, there is a positive probability that an American exchange option will be exercised strictly prior to expiration. This positive probability induced additional value for an American exchange option as given in Carr (1988,1995) and Armada et al. (2007).
The paper is organized as follows. The section (2) presents the estimation of an European Exchange option, the section (3) introduces the Monte Carlo's valuation of a Compound European Exchange option while the section (4) gives us the estimation of a Pseudo American Exchange option.
In the section (5) we present a numerical study to compare the results obtained applying the theoretical models with those deriving by Monte Carlo simulations. Finally, the section (6) concludes.

2 The Price of an European Exchange Option (EEO)

We begin our discussion by focusing on an EEO to exchange asset D for asset V at time T. Asset D is referred to as the delivery asset, and V the optioned asset. Denoting with $s(V, D, T-t)$ the value of EEO at time t, the final payoff at the option's maturity date T is $s(V, D, 0) = \max(0, V_T - D_T)$, where V_T and D_T are the underlying assets' terminal prices.
Following Margrabe (1978) and McDonald & Siegel (1985) models, we suppose two Browninan processes $(Z_t^v)_{t\in[0,T]}$ and $(Z_t^d)_{t\in[0,T]}$ which are defined on a filtered probability space $(\Omega, \mathcal{A}, \mathcal{F}, \mathbb{P})$. We assume that the risky assets V and D are described by the following stochastic differential equations:

$$\frac{dV}{V} = (\mu_v - \delta_v)dt + \sigma_v dZ_t^v \tag{1}$$

$$\frac{dD}{D} = (\mu_d - \delta_d)dt + \sigma_d dZ_t^d \tag{2}$$

$$cov\left(\frac{dV}{V}, \frac{dD}{D}\right) = \rho_{vd}\sigma_v\sigma_d \, dt \tag{3}$$

where μ_v and μ_d are the expected rates of return on the two assets, δ_v and δ_d are the corresponding dividend yields, σ_v^2 and σ_d^2 are the respective variance rates and ρ_{vd} is the correlation between changes in V and D.
So, under certain assumptions, Margrabe (1978) and McDonald & Siegel (1985) show that the value of an EEO on dividend-paying assets, when the valuation date is $t = 0$, is given by:

$$s(V, D, T) = Ve^{-\delta_v T} N(d_1(P, T)) - De^{-\delta_d T} N(d_2(P, T)) \tag{4}$$

where:

- $P = \frac{V}{D}$;
- $\sigma = \sqrt{\sigma_v^2 - 2\rho_{v,d}\sigma_v\sigma_d + \sigma_d^2}$;
- $\delta = \delta_v - \delta_d$;
- $d_1(P, T) = \frac{\log P + \left(\frac{\sigma^2}{2} - \delta\right) T}{\sigma\sqrt{T}}$;
- $d_2(P, T) = d_1(P, T) - \sigma\sqrt{T}$;
- $N(d)$ is the cumulative standard normal distribution.

The typical simulation approach is to price the EEO as the expectation value of discounted cash-flows:

$$s(V, D, T) = e^{-rT} E_{\mathbb{Q}}[\max(0, V_T - D_T)] \tag{5}$$

where $\max[0, V_T - D_T]$ denotes the payoff at expiration time T and the probability $\mathbb{Q}$ is the risk-neutral probability for the pricing problem. So, for the risk-neutral version of the Eq. (1) and Eq. (2), it's just replace the expected rates of return μ_v and μ_d by the risk-free interest rate r plus the premium-risk, namely $\mu_i = r + \lambda_i\sigma_i$, for $i = V, D$. So, we obtain the risk-neutral stochastic equations:

$$\frac{dV}{V} = (r - \delta_v)dt + \sigma_v(dZ_t^v + \lambda_v dt) = (r - \delta_v)dt + \sigma_v dZ_v^* \tag{6}$$

$$\frac{dD}{D} = (r - \delta_d)dt + \sigma_d(dZ_t^d + \lambda_d dt) = (r - \delta_d)dt + \sigma_d dZ_d^* \quad (7)$$

$$Cov(dZ_v^*, dZ_d^*) = \rho_{vd} dt \quad (8)$$

The Brownian processes $dZ_v^* \equiv dZ_t^v + \lambda_v dt$ and $dZ_d^* \equiv dZ_t^d + \lambda_d dt$ are the new Geometric Brownian Motions under the filtered risk-neutral probability space $(\Omega, \mathcal{A}, \mathcal{F}, \mathbb{Q})$. Applying the Ito's lemma and using a logarithm transformation (see Appendix A.1 and A.2), we can reach the equation for the ratio-price simulation $P = \dfrac{V}{D}$ under the risk-neutral measure $\mathbb{Q}$:

$$\frac{dP}{P} = \left(-\delta + \sigma_d^2 - \sigma_v \sigma_d \rho_{vd}\right) dt + \sigma_v dZ_v^* - \sigma_d dZ_d^* \quad (9)$$

where $-\delta = \delta_d - \delta_v$. Applying the logarithm transformation for D_T, under the risk-neutral probability measure $\mathbb{Q}$, it results that:

$$D_T = D_0 \exp\left\{(r - \delta_d)T\right\} \cdot \exp\left(-\frac{\sigma_d^2}{2}\, T + \sigma_d Z_d^*(T)\right) \quad (10)$$

where D_0 is the value of asset D at initial time.
Since $Z_d^*(T) \sim \mathcal{N}(0, \sqrt{T})$ we have that $U \equiv \left(-\dfrac{\sigma_d^2}{2}\, T + \sigma_d Z_d^*(T)\right) \sim \mathcal{N}\left(-\dfrac{\sigma_d^2}{2} T, \sigma_d \sqrt{T}\right)$ and therefore $\exp(U)$ is a log-normal which expectation value is:

$$E_{\mathbb{Q}}\left[\exp(U)\right] = \exp\left(-\frac{\sigma_d^2}{2} T + \frac{\sigma_d^2}{2} T\right) = 1 \quad (11)$$

So, by Girsanov's theorem, we can define the new probability measure $\widetilde{\mathbb{Q}}$ equivalent to $\mathbb{Q}$ and the Radon-Nikodym derivative is:

$$\frac{d\,\widetilde{\mathbb{Q}}}{d\,\mathbb{Q}} = \exp\left(-\frac{\sigma_d^2}{2}\, t_1 + \sigma_d Z_d^*(t_1)\right) \quad (12)$$

Hence, using the Eq. (10) we can write:

$$D_T = D_0\, e^{(r-\delta_d)T} \cdot \frac{d\,\widetilde{\mathbb{Q}}}{d\,\mathbb{Q}} \quad (13)$$

By the Girsanov theorem, the process:

$$d\hat{Z}_d = dZ_d^* - \sigma_d dt \quad (14)$$

is a Brownian motion under the new risk-neutral probability space $(\Omega, \mathcal{A}, \mathcal{F}, \widetilde{\mathbb{Q}})$. We can write dZ_v^* as:

$$dZ_v^* = \rho_{vd}\, dZ_d^* + \sqrt{1 - \rho_{vd}^2}\, dZ' \quad (15)$$

where Z' is a Brownian motion independent of Z_d^* under measure $\mathbb{Q}$. But, with $\widetilde{\mathbb{Q}}$ defined by Eq. (12), Z' remains a Brownian motion under $\widetilde{\mathbb{Q}}$ independent of $\hat{Z}_d$. Hence $d\hat{Z}_v$ defined by:

$$d\hat{Z}_v = \rho_{vd} d\hat{Z}_d + \sqrt{1-\rho_{vd}^2}\, dZ' \tag{16}$$

is a Brownian motion under $\widetilde{\mathbb{Q}}$. Moreover, using the Eq. (14) for $\hat{Z}_d$, we can rewrite the process $\hat{Z}_v$ under the new risk-neutral probability space $(\Omega, \mathcal{A}, \mathcal{F}, \widetilde{\mathbb{Q}})$ as:

$$d\hat{Z}_v = dZ_v^* - \rho_{vd}\sigma_d\, dt \tag{17}$$

By the Brownian motions defined in the Eq. (14) and Eq. (17), we can rewrite the Eq.(9) for the asset P under the risk-neutral probability $\widetilde{\mathbb{Q}}$. So it results that:

$$\begin{aligned}
\frac{dP}{P} &= (-\delta + \sigma_d^2 - \sigma_v\sigma_d\rho_{vd})\, dt + \sigma_v dZ_v^* - \sigma_d dZ_d^* \\
&= (-\delta + \sigma_d^2 - \sigma_v\sigma_d\rho_{vd} + \sigma_v\sigma_d\rho_{vd} - \sigma_d^2)\, dt + \sigma_v\, d\hat{Z}_v - \sigma_d\, d\hat{Z}_d \\
&= -\delta\, dt + \sigma_v\, d\hat{Z}_v - \sigma_d\, d\hat{Z}_d
\end{aligned} \tag{18}$$

Using the Eq. (16), it results that:

$$\sigma_v d\hat{Z}_v - \sigma_d d\hat{Z}_d = (\sigma_v\rho_{vd} - \sigma_d)\ d\hat{Z}_d + \sigma_v \left(\sqrt{1-\rho_{vd}^2}\right)\ d\hat{Z}' \tag{19}$$

where $\hat{Z}_v$ and Z' are independent under $\widetilde{\mathbb{Q}}$. Moreover, we have that:

$$\begin{aligned}
E_{\widetilde{\mathbb{Q}}}\left[(\sigma_v\rho_{vd} - \sigma_d)\ d\hat{Z}_d + \sigma_v\left(\sqrt{1-\rho_{vd}^2}\right)\ dZ'\right] &= (\sigma_v\rho_{vd} - \sigma_d)\, E_{\widetilde{\mathbb{Q}}}\left[d\,\hat{Z}_d\right] \\
&+ \sigma_v\left(\sqrt{1-\rho_{vd}^2}\right) E_{\widetilde{\mathbb{Q}}}\left[d\hat{Z}'\right] \\
&= 0
\end{aligned} \tag{20}$$

$$
\begin{aligned}
Var\left[(\sigma_v\rho_{vd}-\sigma_d)\ d\hat{Z}_d+\sigma_v\left(\sqrt{1-\rho_{vd}^2}\right)\ dZ'\right] &= (\sigma_v\rho_{vd}-\sigma_d)^2\, Var\left[d\hat{Z}_d\right] \\
&+ \left(\sigma_v\sqrt{1-\rho_{vd}^2}\right)^2 Var\left[dZ'\right] \\
&= \left(\sigma_v^2+\sigma_d^2-2\rho_{vd}\sigma_v\sigma_d\right) dt \\
&= \sigma^2\, dt \qquad (21)
\end{aligned}
$$

Therefore, as $(\sigma_v d\hat{Z}_v - \sigma_d d\hat{Z}_d) \sim \mathcal{N}(0, \sigma\sqrt{dt})$, we can rewrite the Eq. (18):

$$
\frac{dP}{P} = -\delta\, dt + \sigma dZ^p \qquad (22)
$$

where $\sigma = \sqrt{\sigma_v^2+\sigma_d^2-2\sigma_v\sigma_d\rho_{vd}}$ and Z^p is a Geometric Brownian motion under $\widetilde{\mathbb{Q}}$.
Using the logarithm transformation, we obtain the equation for the risk-neutral price simulation P:

$$
P(t) = P_0 \exp\left\{\left(-\delta - \frac{\sigma^2}{2}\right) t + \sigma Z^p(t)\right\} \qquad (23)
$$

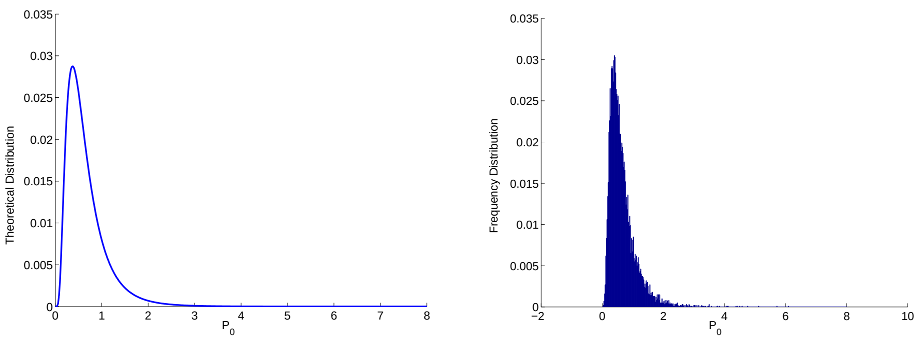
So, using the Eq.(13), we can price an EEO as the expectation value of discounted cash-flows under the risk-neutral probability measure:

$$
\begin{aligned}
s(V,D,T) &= e^{-rT} E_{\mathbb{Q}}[\max(0, V_T - D_T)] \\
&= e^{-rT} E_{\mathbb{Q}}\left[\max\left(0, D_T\left(\frac{V_T}{D_T}-1\right)\right)\right] \\
&= e^{-rT} E_{\mathbb{Q}}\left[\max(0, D_T(P_T-1))\right] \\
&= e^{-rT} D_0 e^{(r-\delta_d)T} E_{\mathbb{Q}}\left[\max\left(0, (P_T-1)\,\frac{d\,\widetilde{\mathbb{Q}}}{d\,\mathbb{Q}}\right)\right] \\
&= D_0 e^{-\delta_d T} E_{\widetilde{\mathbb{Q}}}[g(P_T)] \qquad (24)
\end{aligned}
$$

where $g(P_T) = \max(P_T - 1, 0)$. In addition, the Appendix A.3 shows the valuation of EEO under the risk-neutral probability $\widetilde{\mathbb{Q}}$.
The simulation of the risk-neutral price P (see the Eq.(23)) is performed applying the discretization dt from the continuous-time model:

$$
P(t+dt) = P(t)\exp\left\{(\delta_d - \delta_v - 0.5\cdot\sigma^2)t + \sigma\sqrt{dt}\cdot\epsilon(t)\right\} \qquad (25)
$$

where $\epsilon(t) \sim \mathcal{N}(0,1)$ is a standard normal distribution. Therefore, if we know the value of σ_v, σ_d, ρ_{vd}, δ_v, δ_d and P_0, it's possible to compute, at any time t, the ratio-price P under the risk-neutral probability $\widetilde{\mathbb{Q}}$ simulating the standard Normal distribution $\epsilon(t)$. The figures 1(a) and 1(b) show the comparison between the simulated lognormal distribution of P (using the function "lognpdf"of Matlab Statistics Toolbox) and the theoretical one. We can observe that, when the number of simulations increasing, than the simulated distribution converge to theoretical one.



(a) Theoretical distribution of P

(b) Simulated distribution of P

Figure 1: Distribution of asset P.

Finally, it's possible to implement the Monte Carlo simulation to approximate:

$$E_{\widetilde{\mathbb{Q}}}[g(P_T)] \approx \frac{1}{n}\sum_{i=1}^{n} g(P_i) \tag{26}$$

where n is the number of simulated-paths effected, P_i for $i = 1, 2...n$ are the simulated values and $g(P_i) = \max(0, P_i - 1)$. The section (2.1) shows the Matlab algorithm to derive the simulated value of EEO $s(V, D, T)$. The function "randn (1,1)"generates the stochastic process $\epsilon(t)$ in order to descrive the evolution of ratio-asset P.

2.1 Matlab Algorithm for the EEO

```
function EEO = MCEuroSimple (V0,D0,m,T,sigV,sigD,rhoVD,dV,dD,n)
%Statement of the counter:
SUM =0;
%Statement of the variables:
sig=sqrt(sigV.^2+sigD.^2-2*rhoVD*sigV*sigD);
P0=V0/D0;
d=dV-dD;
%Discretization of timing parameters:
dt=T/m;
```

```
drifts = (-d-0.5*sig*sig).*dt;
stds = sig.*sqrt(dt);
%Computation of simulations:
P=zeros(m+1,1);
for I=1:n
     P(1)=P0;
    for j = 1:m
        P(j+1)=P(j)*exp(drifts +stds*randn(1,1));
    end
    SUM = SUM + max(P(m+1)-1,0);
end
%Computation of European Exchange option:
EEO = D0*exp(-dD*T)*SUM/n
```

3 The price of a Compound European Exchange Option (CEEO)

The CEEO is a derivative in which the underlying asset is another exchange option. Carr (1988) develops a model to value the CEEO assuming that the underlying asset is an EEO $s(V, D, T)$ whose maturity is T, the exercise price is a ratio q of asset D at time t_1 and the expiration date is t_1. So, considering that the valuation date is $t = 0$ and assuming that the evolutions of assets V and D are given by Eq. (1) and Eq. (2) respectively, under certain assumptions, the CEEO value given by Carr (1988) is:

$$\begin{aligned} c(s(V,D,T-t_1),qD,t_1) &= Ve^{-\delta_v T} N_2\left(d_1\left(\frac{P}{P_2^*},t_1\right),d_1\left(P,T\right);\rho\right) \\ &\quad -De^{-\delta_d T} N_2\left(d_2\left(\frac{P}{P_2^*},t_1\right),d_2\left(P,T\right);\rho\right) \\ &\quad -qDe^{-\delta_d t_1} N_1\left(d_2\left(\frac{P}{P_2^*},t_1\right)\right) \end{aligned} \tag{27}$$

where:

- q is the exchange ratio of CEEO;
- t_1 and T are the expiration dates of the CEEO and EEO, respectively, where $T > t_1$;
- $\tau = T - t_1$ is the time to maturity of EEO;

- $d_1\left(\frac{P}{P_2^*}, t_1\right) = \frac{\log\left(\frac{P}{P_2^*}\right) + \left(-\delta + \frac{\sigma^2}{2}\right) t_1}{\sigma\sqrt{t_1}}$;
- $d_2\left(\frac{P}{P_2^*}, t_1\right) = d_1\left(\frac{P}{P_2^*}, t_1\right) - \sigma\sqrt{t_1}$;
- $\rho = \sqrt{\frac{t_1}{T}}$;
- P_2^* is the critical price ratio that solves the following equation:

$$P_2^* e^{-\delta_v \tau} N(d_1(P_2^*, \tau)) - e^{-\delta_d \tau} N(d_2(P_2^*, \tau)) = q. \qquad (28)$$

The critical ratio-price P_2^* makes equal the underlying asset and the exercise price. It's obvious that the CEEO will be exercised at time t_1 if the ratio-price P at time t_1 is higher than P_2^*, namely if $P_{t_1} \geq P_2^*$.
We price the CEEO as the expectation value of discounted cash-flows under the risk-neutral probability $\mathbb{Q}$:

$$c(s, qD, t_1) = e^{-rt_1} E_{\mathbb{Q}}[g(s, qD)] \qquad (29)$$

where $g(s, qD)$ is the CEEO final payoff at the maturity t_1, namely:

$$\begin{aligned} g(s, qD) &= \max[s(V_{t_1}, D_{t_1}, \tau) - qD_{t_1}, 0] \\ &= \max[(V_{t_1} e^{-\delta_v \tau} N(d_1(P_{t_1}, \tau)) - D_{t_1} e^{-\delta_d \tau} N(d_2(P_{t_1}, \tau)) - qD_{t_1}) \mathbf{1}_{(P_{t_1} \geq P_2^*)}] \end{aligned}$$

If we assume D_{t_1} as numeraire and considering that $D_{t_1} = D_0 e^{(r-\delta_d)t_1} \cdot \frac{d\,\widetilde{\mathbb{Q}}}{d\,\mathbb{Q}}$ (see Eq.(13)), we obtain:

$$\begin{aligned} c &= e^{-rt_1} E_{\mathbb{Q}}[D_{t_1}(P_{t_1} e^{-\delta_v \tau} N(d_1(P_{t_1}, \tau)) - e^{-\delta_d \tau} N(d_2(P_{t_1}, \tau)) - q) \mathbf{1}_{(P_{t_1} \geq P_2^*)}] \\ &= e^{-rt_1} D_0 e^{(r-\delta_d)t_1} E_{\widetilde{\mathbb{Q}}}[(P_{t_1} e^{-\delta_v \tau} N(d_1(P_{t_1}, \tau)) - e^{-\delta_d \tau} N(d_2(P_{t_1}, \tau)) - q) \mathbf{1}_{(P_{t_1} \geq P_2^*)}] \\ &= D_0 e^{-\delta_d t_1} E_{\widetilde{\mathbb{Q}}}[g'(s(P_{t_1}), qD)] \qquad (30) \end{aligned}$$

where $c \equiv c(s, qD, t_1)$ is the CEEO and:

$$g'(s(P_{t_1}), qD) = \max[P_{t_1} e^{-\delta_v \tau} N(d_1(P_{t_1}, \tau)) - e^{-\delta_d \tau} N(d_2(P_{t_1}, \tau) - q, 0] \qquad (31)$$

Using Monte Carlo simulation, it's possible to approximate the value of CEEO as:

$$c(s, qD, t_1) \approx D_0 e^{-\delta_d t_1} \left(\frac{\sum_{i=1}^n g'(s(P_{t_1}^i), qD)}{n}\right) \qquad (32)$$

where n is the number of simulated-paths. Furthermore, the section (3.1) shows the Monte Carlo algorithm to simulate the CEEO.

3.1 Matlab Algorithm for the CEEO

```
function CEEO=MCEuroComp(V0,D0,q,dV,dD,m,T1,T2,sigV,sigD,rhoVD,n)
%Statement of the counters:
SUM=0;
%Statement of the variables:
sig=sqrt(sigV.^2+sigD.^2-2*rhoVD.*sigV.*sigD);
P0=V0/D0;
d=dV-dD;
%Discretization of timing parameters:
dT1=T1/m;
drifts=(dD-dV-0.5*sig.*sig).*dT1;
stds=sig.*sqrt(dT1);
P=zeros(m+1,1);
%Computation of simulations:
for i=1:n
    P(1)=P0;
    for j =1:m
        P(j+1)=P(j)*exp((drifts)+stds*randn(1,1));
    end
    d1=(log(P(m+1)*exp(-d*(T2-T1)))+ 0.5*(sig.^2)*(T2-T1))/...
    (sig*sqrt(T2-T1));
    d2=(log(P(m+1)*exp(-d*(T2-T1)))- 0.5*(sig.^2)*(T2-T1))/...
    (sig*sqrt(T2-T1));
    SUM = SUM +max(P(m+1)*exp(-dV*(T2-T1))*normcdf(d1)-...
    exp(-dD*(T2-T1))*normcdf(d2)-q,0);
end
%Computation of Compound European Exchange option:
CEEO=D0*exp(-dD*T1)*SUM/n
```

4 The price of a Pseudo American Exchange Option (PAEO)

Let $t = 0$ the evaluation date and T be the maturity date of the exchange option. Let S_2 the value of a PAEO that can be exercised at time $\frac{T}{2}$ or T. Following Carr (1988,1995), the payoff of PAEO (S_2) can be replicate by a portafolio containing two EEOs and one CEEO. Hence, the value of PAEO is:

$$\begin{aligned} S_2 &= Ve^{-\delta_v T} N_2\left(-d_1^*, d_1; -\rho\right) - De^{-\delta_d T} N_2\left(-d_2^*, d_2; -\rho\right) \\ &\quad + Ve^{-\delta_v \frac{T}{2}} N(d_1^*) - De^{-\delta_d \frac{T}{2}} N(d_2^*) \end{aligned} \tag{33}$$

where:

- $d_1 \equiv d_1(P,T); \quad d_2 \equiv d_2(P,T)\,;$

- $d_1^* \equiv d_1\left(\frac{P}{P_1^*}, \frac{T}{2}\right) = \dfrac{\log(\frac{P}{P_1^*}) + \left(\frac{\sigma^2}{2} - \delta\right)\frac{T}{2}}{\sigma\sqrt{\frac{T}{2}}}\,;$

- $d_2^* \equiv d_2\left(\frac{P}{P_1^*}, \frac{T}{2}\right) = d_1^* - \sigma\sqrt{\frac{T}{2}}\,;$

- $\rho = \sqrt{\frac{T}{2\cdot T}} = \sqrt{0.5}\,;$

- $N_2(x_1, x_2; \rho)$ is the standard bivariate normal distribution function evaluated at x_1 and x_2 with correlation ρ:

  $$N_2(x_1, x_2; \rho) \equiv \int_{-\infty}^{x_1}\int_{-\infty}^{x_2} \frac{\exp\{-\frac{1}{2(1-\rho^2)}[z_1^2 - 2\rho z_1 z_2 + z_2^2]\}}{2\pi\sqrt{1-\rho^2}} dz_2 dz_1$$

- P_1^* is the unique value which makes indifferent the option exercise or not at time $\frac{T}{2}$ and it solves the following equation:

  $$P_1^* e^{-\delta_v \frac{T}{2}} N\left(d_1\left(P_1^*, \tfrac{T}{2}\right)\right) - e^{-\delta_d \frac{T}{2}} N\left(d_2\left(P_1^*, \tfrac{T}{2}\right)\right) = P_1^* - 1$$

The PAEO (S_2) will be exercised at mid-life time $\frac{T}{2}$ if the cash flows $(V_{T/2} - D_{T/2})$ exceeds the opportunity cost of exercise, i.e the value of the option $s(V, D, T/2)$:

$$V_{T/2} - D_{T/2} \geq s(V, D, T/2) \tag{34}$$

It's clear that if the PAEO (S_2) is not exercised at time $\frac{T}{2}$, then it's just the value of an EEO (s) with maturity $\frac{T}{2}$ as given by Eq. (4). However, the exercise condition can be re-expressed in terms of just one random variable by taking the delivery asset as numeraire. Dividing by the delivery asset price $D_{T/2}$ it results:

$$P_{T/2} - 1 \geq P_{T/2}\, e^{-\delta_v \frac{T}{2}} N(d_1(P_{T/2}, T/2)) - e^{-\delta_d \frac{T}{2}} N(d_2(P_{T/2}, T/2)) \tag{35}$$

So, if the condition (35) takes place, namely, if the value of P is higher than P_1^* at moment $\frac{T}{2}$, the PAEO will be exercised at time $\frac{T}{2}$ and the payoff will be $(V_{T/2} - D_{T/2})$ otherwise the PAEO will be exercised at time T and the payoff will be $\max[V_T - D_T, 0]$. So, using Monte Carlo approach, we can value the PAEO (S_2) as the expectation value of discounted cash flows under the risk-neutral probability measure:

$$\begin{aligned} S_2(V,D,T) &= e^{-r\frac{T}{2}}E_{\mathbb{Q}}[(V_{T/2}-D_{T/2})\mathbf{1}_{(P_{T/2}\geq P_1^*)}] \\ &+ e^{-rT}E_{\mathbb{Q}}[\max(0,V_T-D_T)\mathbf{1}_{(P_{T/2}<P_1^*)}] \end{aligned} \quad (36)$$

Using the Eq. (13) for $D_{T/2}$ and D_T, with the same previous method we can write that:

$$\begin{aligned} S_2(V,D,T) &= D_0e^{-\delta_d\frac{T}{2}}E_{\tilde{\mathbb{Q}}}[(P_{T/2}-1)\mathbf{1}_{(P_{T/2}\geq P_1^*)}] \\ &+ D_0e^{-\delta_dT}E_{\tilde{\mathbb{Q}}}[\max(0,P_T-1)\mathbf{1}_{(P_{T/2}<P_1^*)}] \end{aligned} \quad (37)$$

Hence we have that:

$$S_2(V,D,T) = D_0\left(e^{-\delta_d\frac{T}{2}}E_{\tilde{\mathbb{Q}}}[g(P_{T/2})] + e^{-\delta_d T}E_{\tilde{\mathbb{Q}}}[g(P_T)]\right) \quad (38)$$

where $g(P_{T/2}) = (P_{T/2}-1)$ if $P_{T/2} \geq P_1^*$ and $g(P_T) = \max[P_T-1,0]$ if $P_{T/2} < P_1^*$.
So, with the simulation, we can approximate the PAEO as:

$$S_2(V,D,T) \simeq D_0\left(\frac{\sum_{i\in A} g(P^i_{T/2})e^{-\delta_dT/2} + \sum_{i\in B} g(P^i_T)e^{-\delta_dT}}{n}\right) \quad (39)$$

where $A = \{i = 1..n \text{ s.t. } P^i_{T/2} \geq P_1^*\}$ and $B = \{i = 1..n \text{ s.t. } P^i_{T/2} < P_1^*\}$ and n is the number of simulated-paths. The section (4.1) presents the Monte Carlo algorithm for the PAEO simulation.
Finally, the American Exchange option (AEO) is valued using the Richardson extrapolation process. If we denote with S the AEO's price, the extrapolation formula presented by Carr (1988,1995) is:

$$S \simeq s + \frac{S_2 - s}{3} \quad (40)$$

where s and S_2 are the EEO and PAEO values, respectively. Instead, the corrected version of the second order estrapolation presented by Armada et al. (2007) gives:

$$S \simeq S_2 + \frac{S_2 - s}{3} \quad (41)$$

4.1 Matlab Algorithm for the PAEO

```
function PAEO = MCAmerPseudo (V0,D0,dV,dD,m,T,sigV,sigD,rhoVD,n)
```

```
%Statement of the counters:
SUM1=0;
SUM2=0;
%Statement of the variables:
sig=sqrt(sigV.^2+sigD.^2-2*rhoVD.*sigV.*sigD);
P0=V0/D0;
d=dV-dD;
%Discretization of timing parameters:
dT=T/m;
drifts=(-d-0.5*sig.*sig).*dT;
stds=sig.*sqrt(dT);
%Computation of simulations:
P=zeros(m+1,1);
for i = 1:n
    P(1)=P0;
    for j=1:m
        P(j+1)=P(j)*exp(drifts+stds*randn(1,1));
    end
    d1=(log(P(m/2)*exp(-d*0.5*T))+(sig.^2)*0.5*0.5*T)/(sig*sqrt(0.5*T));
    d2=(log(P(m/2)*exp(-d*0.5*T))-(sig.^2)*0.5*0.5*T)/(sig*sqrt(0.5*T));
%Exercise condition at time T/2 and computation of simulations:
    if P(m/2)-1>=P(m/2)*exp(-dV*0.5*T)*normcdf(d1)-...
       exp(-dD*0.5*T)*normcdf(d2)
         SUM1=SUM1+max(P(m/2)-1,0);
    else SUM2=SUM2 + max(P(m+1)-1,0);
    end
end
%Computation of Pseudo American Exchange option:
PAEO=(D0/n)*(exp(-dD*T/2)*SUM1+exp(-dD*T)*SUM2)
```

5 Numerical Examples of Exchange Option Simulations

In this section we report the results of numerical simulations of EEO, CEEO and PAEO. To compute the simulations we have assumed that the number of simulated-paths n is equal to 50 000 and the time-steps $m = 500$. The parameter values are $\sigma_v = 0.93$, $\sigma_d = 0.30$, $\rho_{vd} = 0.20$, $\delta_d = 0$, $\delta_v = 0.15$ and $T = 2$ years. Furthermore, to compute the CEEO we assume that $t_1 = 1$ year and the exchange ratio $q = 0.10$.
The Table (1) summarizes the results of EEO simulations. In the first column and in the second one are indicated the values of optioned asset V and delivery asset D while the third column gives the EEO's prices using Margrabe (1978) and McDonald & Siegel (1985) formula. For each option we have reported four results given by Monte Carlo's simulation and we can observe that the simu-

lated values are very close to true ones. The last column presents the Standard Average Error (SAE) between the four simulated prices and the true value. The SAE is:

$$SAE = \frac{\sum_{i=1}^{k} \frac{|True - Sim_i|}{True}}{k} \tag{42}$$

where $k = 4$ is the number of simulations effected. We can observe that the error ranges from 0.39% up to $1.09.\%$ Moreover, we denote by bold type the simulations that are closer then others to true value.
The Table (2) shows the comparison between the CEEO's prices given by Carr (1988) and the simulated values. In this case, the SAE is included between 0.24% and 1.10%. Instead, the Table (3) summarizes the numerical results of PAEO simulations. Comparing the true values given by Carr (1988,1995) and the simulated ones, we can observe that the minimum SAE is 0.37% while the maximum is 1.02%.
At last, the Table (4) shows the values of AEO given by Armada et al. (2007) and the results by Monte Carlo's simulation. Using the same simulated-paths of ratio-asset P, we compute the simulated prices of EEO and PAEO that we allow to obtain the AEO's value using the two moments Richardson extrapolation as shown in Armada et al. (2007). In this case, the minimum SAE is 0.41% while the maximum one is 1.02%.

6 Concluding Remarks

In this paper we have shown the power of Monte Carlo simulation for the estimations of exchange options in which also the exercise price is stochastic. Using the delivery asset D as numeraire we have reduced the bi-dimensionality of valuing exchange options to one stochastic variable P. After that, our MATLAB simulation procedures have given exchange option values that are very similar to those reported in Margrabe (1978), McDonald & Siegel (1985), Carr (1988,1995) and Armada et al. (2007). In fact, if we examine the numerical simulation examples presented for each option, we can remark that the Standard Average Error (SAE) is in the range $0.24\% - 1.10\%$. This result shows the good approximation obtained with Monte Carlo simulation that validates the methodology presented.
Finally, the Monte Carlo method used here can be very helpful to an increasing literature that use the contingent claim approach to value real investment opportunities. Many times, a real investment valuation requires a complex set of interacting exchange options making them more difficult or impossible to value analitically. Therefore, a numerical approach can be very useful to reach this objective.

V_0	D_0	EEO (true)	1^{st} MC Sim.	2^{nd} MC Sim	3^{rd} MC Sim.	4^{th} MC Sim	SAE
180	180	54.2158	**54.2475**	54.9142	54.4762	54.3627	0.0052
180	200	50.6472	50.0317	**50.6944**	50.7088	51.1733	0.0061
180	220	47.4670	47.7322	46.7522	48.2115	**47.2700**	0.0101
200	180	64.2474	**64.9018**	63.6167	63.7417	65.1252	0.0103
200	200	60.2397	58.9443	59.5512	**60.0234**	59.8030	0.0109
200	220	56.6506	56.0296	57.5219	55.8789	**57.1896**	0.0123
220	180	74.6816	74.5260	**74.5612**	75.0287	73.9450	0.0045
220	200	70.2502	70.9405	69.4675	**69.9030**	70.9966	0.0091
220	220	66.2638	66.3882	**66.2263**	66.3854	67.0260	0.0039

Table 1: Simulation Prices of European Exchange Option (EEO)

V_0	D_0	CEEO (true)	1^{st} MC Sim.	2^{nd} MC Sim	3^{rd} MC Sim.	4^{th} MC Sim	SAE
180	180	43.4257	**43.3429**	43.7377	43.0498	43.5326	0.0050
180	200	39.5055	39.9160	**39.4733**	39.1651	39.3772	0.0057
180	220	36.0654	36.3095	**36.0160**	36.3510	36.2202	0.0051
200	180	52.7139	52.9475	**52.5557**	51.8133	52.1714	0.0087
200	200	48.2508	48.6794	48.7689	49.1844	**48.0151**	0.0109
200	220	44.3050	44.8884	**43.9257**	44.7236	44.8644	0.0110
220	180	62.4989	62.6382	**62.3685**	62.3434	62.6766	0.0024
220	200	57.5128	57.7545	56.9350	**57.6725**	57.7031	0.0040
220	220	53.0759	52.5395	53.5500	52.8895	**53.1271**	0.0056

Table 2: Simulation Prices of Compound European Exchange Option (CEEO)

V_0	D_0	PAEO (true)	1^{st} MC Sim.	2^{nd} MC Sim	3^{rd} MC Sim.	4^{th} MC Sim	SAE
180	180	59.6336	**59.7028**	58.9575	59.2504	60.1984	0.0070
180	200	55.3457	54.6789	**55.5961**	55.0738	54.2569	0.0102
180	220	51.5639	51.7402	**51.6076**	50.8616	52.1739	0.0074
200	180	71.1254	**71.4235**	70.7895	70.8628	71.7302	0.0052
200	200	66.2596	65.9114	**66.2981**	66.1768	66.1609	0.0021
200	220	61.9446	62.1061	**61.9604**	62.0855	62.7222	0.0044
220	180	83.1511	82.5760	82.6444	83.8784	**83.0990**	0.0055
220	200	77.7234	78.1077	**77.9315**	77.3294	78.0138	0.0041
220	220	72.8856	**72.8567**	73.3973	73.0429	73.2667	0.0037

Table 3: Simulation Prices of Pseudo American Exchange Option (PAEO)

V_0	D_0	AEO (true)	1^{st} MC Sim.	2^{nd} MC Sim	3^{rd} MC Sim.	4^{th} MC Sim	SAE
180	180	61.4396	**61.8739**	60.6656	62.1803	60.8769	0.0102
180	200	56.9118	**56.9387**	57.1939	56.4493	57.4565	0.0058
180	220	52.9296	52.6991	52.3327	53.4573	**52.8554**	0.0067
200	180	73.4181	73.1077	72.7118	**73.6618**	73.8786	0.0059
200	200	68.2662	67.8717	67.7270	67.9204	**67.9483**	0.0058
200	220	63.7092	63.1479	63.0386	**63.4105**	64.3292	0.0084
220	180	85.9743	86.9115	**86.2913**	86.4667	85.4575	0.0066
220	200	80.2144	79.9109	79.5847	**80.1922**	79.8635	0.0041
220	220	75.0928	**75.2690**	75.8604	75.7047	75.6117	0.0070

Table 4: Simulation Prices of American Exchange Option (AEO) given by Armada et al. (2007)

A General Computations

A.1 Stochastic differential equation of asset P under the risk-neutral probability $\mathbb{Q}$

To determine $dP = d\left(\frac{V}{D}\right)$ we apply ITO's formula. Computing the derivatives:

$$\frac{\partial P}{\partial t} = 0; \qquad \frac{\partial P}{\partial V} = \frac{1}{D}; \qquad \frac{\partial P}{\partial D} = -\frac{V}{D^2};$$

$$\frac{\partial^2 P}{\partial D^2} = \frac{2V}{D^3}; \quad \frac{\partial^2 P}{\partial V^2} = 0; \quad \frac{\partial^2 P}{\partial V \partial D} = -\frac{1}{D^2};$$

we obtain:

$$\begin{aligned}
dP &= \frac{\partial P}{\partial t}\,dt + \frac{\partial P}{\partial V}\,dV + \frac{\partial P}{\partial D}\,dD + \frac{1}{2}\left[\frac{\partial^2 P}{\partial V^2}(dV)^2 + 2\frac{\partial^2 P}{\partial V \partial D}dD\,dV + \frac{\partial^2 P}{\partial D^2}(dD)^2\right] \\
&= \frac{1}{D}\,[(r-\delta_v)V dt + \sigma_v V dZ_v^*] - \frac{V}{D^2}\,[(r-\delta_d)Ddt + \sigma_d D\,dZ_d^*] \\
&+ \frac{1}{2}\left[-2\cdot\frac{1}{D^2}\,((r-\delta_d)Ddt + \sigma_d DdZ_d^*)\,((r-\delta_v)V dt + \sigma_v V dZ_v^*)\right] \\
&+ \frac{1}{2}\left[\frac{2V}{D^3}((r-\delta_d)Ddt + \sigma_d DdZ_d^*)^2\right] \\
&= P(r-\delta_v)dt + P\sigma_v dZ_v^* - P(r-\delta_d)dt - P\sigma_d dZ_d^* - P(\sigma_v\sigma_d\rho_{vd}) + P\sigma_d^2 dt
\end{aligned}$$

So, under the risk-neutral probability measure $\mathbb{Q}$, we have that:

$$\frac{dP}{P} = (-\delta + \sigma_d^2 - \sigma_v\sigma_d\rho_{vd})\,dt + \sigma_v dZ_v^* - \sigma_d dZ_d^* \tag{43}$$

where $\delta = \delta_v - \delta_d$. We can also compute the variance of $\frac{dP}{P}$:

$$\begin{aligned}
Var\left(\frac{dP}{P}\right) &= [var(\sigma_v dZ_v) + var(-\sigma_d dZ_d) + 2cov(-\sigma_d\sigma_v dZ_v dZ_d)] \\
&= (\sigma_v^2 + \sigma_d^2 - 2\sigma_v\sigma_d\rho_{vd})\,dt \\
&\equiv \sigma^2\,dt
\end{aligned}$$

A.2 Explicit value of P.

Let denote with $Y = \log P$. If we apply ITO's formula we have that:

$$\frac{\partial f}{\partial t} = 0; \quad \frac{\partial f}{\partial P} = \frac{1}{P}; \quad \frac{\partial^2 f}{\partial P^2} = -\frac{1}{P^2};$$

and therefore:

$$\begin{aligned}
dY &= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial P}dP + \frac{1}{2}\frac{\partial^2 f}{\partial P^2}(dP)^2 \\
&= \frac{1}{P}\left[P(-\delta + \sigma_d^2 - \sigma_v\,\sigma_d\,\rho_{vd})\,dt + P\sigma_v\,dZ_v^* - P\sigma_d\,dZ_d^*)\right] \\
& -\frac{1}{2P^2}\left(P^2\sigma_v^2 dt + P^2\sigma_d^2 dt - 2P^2\sigma_v\sigma_d\rho_{vd}dt\right) \\
&= \left(-\delta + \sigma_d^2 - \sigma_v\sigma_d\rho_{vd} - \frac{\sigma_v^2}{2} - \frac{\sigma_d^2}{2} + \sigma_v\sigma_d\rho_{vd}\right)dt + \sigma_v\,dZ_v^* - \sigma_d\,dZ_d^* \\
&= \left(-\delta + \frac{\sigma_d^2}{2} - \frac{\sigma_v^2}{2}\right)dt + \sigma_v\,dZ_v^* - \sigma_d\,dZ_d^*
\end{aligned}$$

Integrating between $[0, t]$ we obtain:

$$\int_0^t dY = \int_0^t \left(-\delta + \frac{\sigma_d^2}{2} - \frac{\sigma_v^2}{2}\right)dt - \int_0^t \sigma_v dZ_v^* - \int_0^t \sigma_d dZ_d^*$$

Hence:

$$Y(t) - Y(0) = \left(-\delta + \frac{\sigma_d^2}{2} - \frac{\sigma_v^2}{2}\right)t + \sigma_v Z_v^*(t) - \sigma_d Z_d^*(t)$$

As $Y(t) = \log P(t)$ we have:

$$\log\frac{P(t)}{P_0} = \left(-\delta + \frac{\sigma_d^2}{2} - \frac{\sigma_v^2}{2}\right)t + \sigma_v Z_v^*(t) - \sigma_d Z_d^*(t)$$

Therefore, the explicit value of P under the risk-neutral probability $\mathbb{Q}$ is:

$$P(t) = P_0 \exp\left\{\left(-\delta + \frac{\sigma_d^2}{2} - \frac{\sigma_v^2}{2}\right)t + \sigma_v Z_v^*(t) - \sigma_d Z_d^*(t)\right\} \tag{44}$$

A.3 Valuation of EEO under the risk-neutral probability measure $\widetilde{\mathbb{Q}}$

In this section we determine the EEO value as the expectation of discounted cash-flows under the risk-neutral probability measure $\widetilde{\mathbb{Q}}$. Developing the Eq.(24)

we obtain that:

$$\begin{aligned} s(V,D,T) &= De^{-\delta_d T} E_{\widetilde{\mathbb{Q}}}[\max(P_T - 1, 0)] \\ &= De^{-\delta_d T} E_{\widetilde{\mathbb{Q}}}[(P_T - 1)\mathbf{1}_{(P_T - 1)}] \\ &= De^{-\delta_d T} E_{\widetilde{\mathbb{Q}}}[P_T \mathbf{1}_{(P_T \geq 1)}] - De^{-\delta_d T} E_{\widetilde{\mathbb{Q}}}[\mathbf{1}_{(P_T \geq 1)}] \end{aligned} \tag{45}$$

Recalling that the evolution of P under the risk-neutral probability $\widetilde{\mathbb{Q}}$ is:

$$P(t) = P \exp\left\{ \left(-\delta - \frac{\sigma^2}{2} \right) t + \sigma Z^p(t) \right\} \tag{46}$$

we can observe that:

$$\begin{aligned} E_{\widetilde{\mathbb{Q}}}[\mathbf{1}_{(P_T \geq 1)}] &= [\widetilde{\mathbb{Q}}\,(P_T \geq 1) \cdot 1 + \widetilde{\mathbb{Q}}\,(P_T < 1) \cdot 0] \\ &= \widetilde{\mathbb{Q}} \left(P \exp\left\{ \left(-\delta - \frac{\sigma^2}{2} \right) T + \sigma Z^p(T) \right\} \geq 1 \right) \\ &= \widetilde{\mathbb{Q}} \left(x \geq \frac{\log(\frac{1}{P}) + (\delta + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \right) \\ &= \widetilde{\mathbb{Q}} \left(x \leq -\frac{\log(\frac{1}{P}) + (\delta + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \right) \\ &= \widetilde{\mathbb{Q}} \left(x \leq \frac{\log P - (\delta + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \right) \\ &= N(d_2(P,T)) \end{aligned} \tag{47}$$

where:

- $x \sim \mathcal{N}(0,1)$ and $Z^p(T) = x \cdot \sqrt{T}$ is a Geometric Brownian motion under $\widetilde{\mathbb{Q}}$;
- N is the cumulative standard normal distribution;
- $d_2(P,T) \equiv \dfrac{\log P - (\delta + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$.

Now we can compute:

$$\begin{aligned}
E_{\widetilde{\mathbb{Q}}}[P_T \mathbf{1}_{(P_T \geq 1)}] &= E_{\widetilde{\mathbb{Q}}}\left[P \exp\left\{\left(-\delta - \frac{\sigma^2}{2}\right)T + \sigma Z^p(t)\right\} \mathbf{1}_{(P_T \geq 1)}\right] \\
&= \int_{-d_2(P,T)}^{+\infty} P \exp\left\{\left(-\delta - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}\cdot x\right\} f(x)dx \\
&= \int_{-d_2(P,T)}^{+\infty} P \exp\left\{\left(-\delta - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}\cdot x\right\} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\
&= Pe^{-\delta T} \int_{-d_2(P,T)}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\sigma^2 T - 2\sigma\sqrt{T} + x^2)} dx \\
&= Pe^{-\delta T} \int_{-d_2(P,T)}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x - \sigma\sqrt{T})^2} dx \qquad (48)
\end{aligned}$$

where $-d_2(P,T)$ is the value that make $P_T \geq 1$. If we make the sostitution $u = x - \sigma\sqrt{T}$ we have that:

$$\begin{aligned}
E_{\widetilde{\mathbb{Q}}}[P_T \mathbf{1}_{(P_T \geq 1)}] &= Pe^{-\delta T} \int_{-d_2(P,T) - \sigma\sqrt{T}}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du \\
&= Pe^{-\delta T} \int_{-\infty}^{d_2(P,T) + \sigma\sqrt{T}} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du \\
&= Pe^{-\delta T} N(d_1(P,T)) \qquad (49)
\end{aligned}$$

where $d_1(P,T) \equiv d_2(P,T) + \sigma\sqrt{T} = \dfrac{\log P + (-\delta + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$.

Hence, recalling that $\delta = \delta_v - \delta_d$ and $P = \dfrac{V}{D}$, the value of EEO is:

$$\begin{aligned}
s(V, D, T) &= De^{-\delta_d T} \cdot Pe^{-\delta T} N(d_1(P,T)) - De^{-\delta_d T} \cdot N(d_2(P,T)) \\
&= Ve^{-\delta_v T} N(d_1(P,T)) - De^{-\delta_d T} \cdot N(d_2(P,T)) \qquad (50)
\end{aligned}$$

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