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Convexity on Nash Equilibria without Linear Structure *

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Abstract

To give sufficient conditions for Nash Equilibrium existence in a continuous game is a central problem in Game Theory. In this paper, we present two games in which we show how the continuity and quasi-concavity hypotheses are unconnected one to each other. Then, we relax the quasiconcavity assumption by exploiting the multiconnected convexity's concept (Mechaiekh & Others, 1998) in spaces without any linear structure. These results will be applied to two non-zero-sum games lacking the classical assumptions and more recent improvements (Ziad, 1997), (Abalo & Kostreva, 2004). As a minor result, some counterexamples about relationship between some continuity conditions due to Lignola (1997), Reny (1999) and Simon (1995) for Nash equilibria existence are obtained.

Keywords: Nash Equilibria Existence; Fixed Point Theorem; Generalized Convexity; 2 Person Game; 3 Person Game; Symmetric Game; Generalized Continuity.

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1 Introduction

In mathematical economics, showing existence of an equilibrium is the main problem of investigating various kinds of economic models and, till now, a number of equilibrium existence results in general economic models have been investigated by several authors.

The quasiconcavity assumption is central one in the existence of Nash equilibria. Some attempts to relax this hypothesis can be found in works due to Park [15] (Acyclic uplevels); Abalo & Kostreva [1]; Nishimura & Friedman [14] (Monotonic Best Reply's Mapping); Baye & Oth. [3] (Diagonally Transfer Quasi Concavity); Ricceri [18] (Connected uplevels).

We introduce some useful notations and definitions. Let X_i be a nonempty subset of an Hausdorff topological space for all $i \in I = \{1, \dots n\}$; and $X := \prod_{i=1}^{n} X_i$ the joint strategy space; 2^X the set of all subset included in X; and $F(X) \subset 2^X$ the set of all finite subsets included in X. We shall say $x = (x_1, x_2, \dots, x_n) \in X$ a multistrategy. We denote by $X_{-i} := \prod_{j \in I \setminus \{i\}} X_j$ and $u_i : X \longrightarrow \mathbb{R}$ the i^{th} player's utility function that evaluates the i^{th} player's gain $u_i(x)$ by each multistrategy x. A decision rule for the i^{th} player is a correspondence C_i from X_{-i} to X_i which associates the multistrategies $x_{-i} \in X_{-i}$, determined by other players, with a strategy subset $C_i(x_{-i}) \subset X_i$. The classical concept of equilibrium for a game (X_i, u_i) and for its generalization (X_i, u_i, C_i) with constraints, is given in seminal papers [11],[12] and [5]. Moreover, let $BR_i : X_{-i} \longrightarrow X_i$

$$BR_i(x_{-i}) = \arg \max_{x_{-i} \in X_i} u_i(x_i, x_{-i})$$

be the *Best Reply multifunction* for the player i. For any subset $A \subseteq X$, we denote by $\bar{A}$ the closure of A in X and, respectively, by $\overset{\circ}{A}$ the interior of A in X. Let

$$co_{t \in A}\left[P_1, P_2, \dots, P_n\right] = \sum_{j=1}^{n-1} co_{t \in \left[\frac{j-1}{n-1}, \frac{j}{n-1}\right] \cap A}[P_j, P_{j+1}] =$$

$$\sum_{j=1}^{n-1} \left\{ \left[(n-1) \left(\frac{j}{n-1} - t \right) \right] \; P_j + \left[(1-j) + (n-1)t \right] \; P_{j+1} \right\} \chi_{\left[\frac{j-1}{n-1}, \frac{j}{n-1}\right] \cap A}(t)$$

be a subset in $[0,1]$ for every $A \subseteq [0,1]$, $P_i \in \mathbb{R}^2$ with χ_A the characteristic function of A. Given a function f of one variable, we denote its first derivative by $\dot{f}$. Let lim sup and lim inf be the superior limit and the infimum limit of real valued functions but, also, the outer limit and the inner limit of real multivalued functions according to the Painvelevé-Kuratowsky convergence's meaning.

A game G is Better-Reply Secure, in [16][pp.1033], if whenever (x^*, u^*) is in the closure of the graph of its vector payoff function and x^* is not an equilibrium and other players deviate slightly from x^*_{-i}, some player i can secure a payoff strictly above u^*_i at x^*. This hypothesis generalized the *Complementary Discontinuities* (*Reciprocally Upper Semicontinuity*) assumption introduced by Simon in [19]; and the *Payoff Security* assumption introduced by Reny in [17]. In particular, payoff security, in [16][pp. 1032], requires that for every strategy $x \in X$, each player has a strategy $\bar{x}_i \in X_i$ that, virtually, guarantees the payoff he receives at x even if the others deviate slightly from x. In mathematical words, for every strategy $x \in X$ and $\epsilon > 0$, there exists $\bar{x}_i \in X_i$ such that $u_i(\bar{x}_i, x'_{-i}) > u_i(x) - \epsilon$ for all x'_{-i} in a neighborhood of x_{-i} and for all $i = 1, \ldots, n$. Reciprocal upper semicontinuity, in [16][pp. 1034], requires that some player's payoff jumps up whenever some other player's payoff jumps down. In mathematical words, if whenever (x, u) is in the closure of the graph of its vector payoff function and $u_i(x) \leq u_i$ for every player i, then $u_i(x) = u_i$ for some player i. Moreover, the function $\phi : (x, y) \in X \times X \to \sum_{i=1}^{n} u_i(x_i, y_{-i})$ is the equilibrium bifunction for the game G. Such a function ϕ is *diagonal transfer continuous on $A \subseteq X$ in $y \in Z \subseteq X$*, in [3][Definition 1], if, by assuming that for every point $(x, y) \in A \times Z$ such that $\phi(x, y) > \phi(y, y)$, there exists $\bar{x} \in A$ and $U \subset Z$ a neighborhood of y in Z such that $\phi(\bar{x}, y') > \phi(y', y')$ for all $y' \in U$. We shall simply say that ϕ is diagonal transfer continuous in y when $A = X$ and $Z = X$.

The paper is organized as follows: In Section 2, two examples, in which the failure of equilibrium is due to lack of quasiconcavity in spite of some discontinuities, are presented; in Section 3, two New Results relaxing the aforesaid assumption in the setting of constrained and not constrained games, are presented; in Section 4, two applications to non zero-sum games are introduced; in Appendix we show some missing proofs in the paper.

Now, some further notations used in Appendix are introduced. Given x, $x_0, x_1 \in \mathbb{R}$, we denote by $x \to (x_0 \neq x_1)^{+(-)}$ the convergence of x towards x_0 from the right hand side (from the left hand side) by assuming that $x_0 \neq x_1$; by co_{x_0,x_1} the convex hull generated by x_0, x_1. Besides, given a subset $A \subset \prod_{s=1}^{m} A_s$, we denote by $Pr_j(A)$ the projection of A on the subset A_j. Let $B_\delta(x)$ the ball of radius δ centered at $x \in \mathbb{R}^m$; $I_{[0,1]}$ the identity function on $[0,1]$. For not making heavier notations, we can identify $x_2(m_x)$, defined at page 15, with $x_2(m)$ as the point having m as coordinate in the subset $co_{\frac{4}{5},1}$. Finally, References are also referred to the Appendix.

2 How much are the Quasi Concavity Hypothesis and the Continuity one unconnected?

In this section, the question that we propose is the following: How much is decisive the quasi concavity assumption for Nash Equilibrium existence? We show two simple but meaningful discontinuous games without a pure Nash equilibrium in which the continuity conditions established by Reny in [16][pp.1033], hold, but the quasi concavity assumption fails.

2.1 A 3-person symmetric game

Let $G_1 = ([0,1]^3, u_i, u_j, u_k)$ a symmetric game and $(x_i, x_j, x_k) \in [0,1]^3$ as their strategies. Suppose that x_j and x_k are fixed and $x_j \leq x_k$. The payoff for the player i is defined in the following way:

1. **Case 1** $[x_j < x_k]$.

$$u_i(x_i, x_j, x_k) = \begin{cases} \dfrac{x_i + x_j}{2} & 0 \leq x_i < x_j \neq 0 \\ \dfrac{x_j + x_k}{4} & x_i = x_j \\ \dfrac{x_k - x_j}{2} & x_j < x_i < x_k \\ \dfrac{1}{2} - \dfrac{x_k + x_j}{4} & x_i = x_k \\ 1 - \dfrac{x_i + x_k}{2} & 1 \neq x_k < x_i \leq 1 \end{cases}$$

2. **Case 2** $[x_j = x_k]$.

$$u_i(x_i, x_j, x_j) = \begin{cases} \dfrac{x_i + x_j}{2} & 0 \leq x_i < x_j \neq 0 \\ \dfrac{1}{3} & x_i = x_j = x_k \\ 1 - \dfrac{x_i + x_k}{2} & 1 \neq x_j < x_i \leq 1 \end{cases}$$

.

The same rules hold if $x_j \geq x_k$. The main result in [5] doesn't hold since the i^{th} player's payoff is not continuous on X. The main existence result in [13] and [4][Theorem 2], [14][Theorem 1] don't hold since the i^{th} player's payoff is not continuous at variable $x_{-i} \in X_{-i}$ and G_1 is a symmetric game. Moreover, by fixing $(\bar{x}_i, \bar{x}_i, \bar{x}_k) \in [0,1]^3$ with $\bar{x}_k < 3\,\bar{x}_j$, we obtain

$$\sup_{x_i \in [0,1]} \liminf_{(x_j, x_k) \to (\bar{x}_j, \bar{x}_k)} u_i(x_i, x_j, x_k) = \sup_{x_i \leq \min\{\bar{x}_k - 2\bar{x}_i, 1\}} \left\{ \frac{\bar{x}_k - \bar{x}_i}{2}, \frac{\bar{x}_i + x_i}{2} \right\} =$$

$$\frac{\bar{x}_k - \bar{x}_i}{2} < \frac{\bar{x}_k + \bar{x}_i}{4} = u_i(\bar{x}_i, \bar{x}_i, \bar{x}_k).$$

Therefore, the conditions $c)$ or $d)$ in [9][Theorem 3.1] fails, notwithstanding $\sum_{j=1}^{3} u_j = cost$ is upper semicontinuous. Now, the following two results are introduced.

Proposition 2.1. *G_1 has no Nash Equilibria in pure strategies.*

Proof. See Appendix. □

Proposition 2.2. *G_1 is payoff secure.*

Proof. See Appendix. □

By [16][Proposition 4.2], G_1 is diagonally better reply secure since it is quasi symmetric (in particular, symmetric), diagonally payoff secure (in particular, payoff secure) and each $u_i(x,x,x) = \dfrac{1}{3} = cost$ is upper semicontinuous. Besides, such a game would admit a Nash Equilibrium in pure strategy if the game was diagonally quasi concave by [16][Proposition 4.1]; but, it is not true by Proposition 2.1. This implies that G_1 is not diagonally quasi concave (see [16][pp. 1010]). At the end, we state the following Proposition.

Proposition 2.3. *G_1 has a mixed symmetric Nash Equilibrium.*

Proof. See Appendix. □

Remark 2.1. *Economic games like G_1 were studied in [8][Proposition 1] and [6].*

2.2 A two-person non symmetric game.

Let $G_2 = ([0,1]^2, u_1, u_2)$ be a game defined as follows:

$$u_1(x_1,x_2) := \begin{cases} (1-x_2)x_1 & 0 \le x_1 \le \dfrac{x_2}{2} \\ (x_2-1)x_1 + x_2(1-x_2) & \dfrac{x_2}{2} < x_1 < x_2 \\ -\dfrac{b(x_2)}{1+x_2}{x_1}^2 + b(x_2)x_1 - \dfrac{b(x_2)\,x_2}{1+x_2} & 1 \ne x_2 \le x_1 \le 1 \\ \epsilon & x_1 = x_2 = 1 \end{cases} \tag{1}$$

where $\epsilon > 0$ sufficiently small, $b \in C^1([0,1[, \mathbb{R}_+^*)$ which satisfy the following conditions:

$$3 - b\left(\frac{1}{2}\right) > 0 \tag{2}$$

$$\dot{b}\left[\frac{(1-x_2)^2}{1+x_2}\right] + b\left[\frac{(x_2-1)(x_2+3)}{(1+x_2)^2}\right] < (>)0 \quad \forall x_2 < (>)\frac{1}{2},\ x_2 \ne 1. \tag{3}$$

Moreover, u_2 is a strict concave function in the variable x_2 and continuous on the subset $[0,1]^2$. Besides, we claim that BR_2 is a decreasing, surjective function.

Let $M(x_2) = \max_{x_1 \in [0,x_2]} u_1(x_1,x_2)$ and $N(x_2) = \max_{x_1 \in [x_2,1]} u_1(x_1,x_2)$. It's easy to prove that u_1 is continuous respect to x_1 at the point $x_1 = x_2$ for every $x_2 \in X_2 \setminus \{1\}$. By continuity and Berge's maximum Theorem, the condition (2) assures that there exists a $\bar{x}_2 \in]0, \frac{1}{2}[$ such that $M(\bar{x}_2) = N(\bar{x}_2)$; and the (3) one assures that the function $N(x_2)$ is strict decreasing for all $x_2 \in [0, \frac{1}{2}]$ and strict increasing for all $x_2 \in [\frac{1}{2}, 1[$. Moreover, since $\lim_{x_2 \to 1^-} M(x_2) = 0$, there exists $\bar{\bar{x}}_2 \in]\frac{1}{2}, 1[$ such that $M(\bar{\bar{x}}_2) = N(\bar{\bar{x}}_2)$ and $M(x_2) < N(x_2)$ for all $x_2 \in]\bar{\bar{x}}_2, 1[$. Finally, we can claim these two following conditions on G_2:

$$\frac{1+\bar{x}_2}{2} = \max \lim_{y \to \bar{x}_2} BR_1(y) < BR_2^{-1}(\bar{x}_2) \tag{4}$$

$$\frac{\bar{\bar{x}}_2}{2} = \min \lim_{y \to \bar{\bar{x}}_2} BR_1(y) < BR_2^{-1}(\bar{\bar{x}}_2) < \max \lim_{y \to \bar{\bar{x}}_2} BR_1(y) = \frac{1+\bar{\bar{x}}_2}{2}. \tag{5}$$

We state the following Proposition.

Proposition 2.4. *G_2 has no Nash Equilibria in pure strategies.*

Proof. We prove that the multifunction $Z := BR_1 - BR_2^{-1} \,:\, [0,1] \to [0,1]$ has no zeros. This implies that G_2 has no Nash equilibrium point. It's easy to note that Z is continuous and is reduced to a singleton for every $x_2 \neq \bar{x}_2, \bar{\bar{x}}_2$. By remarking that $\dfrac{1}{2} = BR_1(0) < BR_2^{-1}(0) = 1$ and that the condition (4) holds; since BR_1 is strict increasing on the subset $[0, \bar{x}_2[$ and, moreover, BR_2^{-1} is strict decreasing on $[0,1]$, we can state that $Z \neq 0$ on $[0, \bar{x}_2]$. Since BR_1 is increasing on the subset $]\bar{x}_2, 1[$ and BR_2^{-1} decreasing on $[0,1]$; and by condition (5), we can state that $Z \neq 0$ on $[\bar{x}_2, 1]$. Therefore, under the hypotheses (2),(3),(4) and (5), G_2 has no Nash equilibrium in pure strategies. □

However, the payoffs are continuous on the unit square except for the point $(1,1) \in [0,1]^2$. From (3), we note that

$$\frac{\dot{b}}{b} > \frac{(x_2+3)}{(1+x_2)\,(1-x_2)}$$

and, by integrating from $\frac{1}{2}$ to x_2,

$$\ln\left[\frac{b(x_2)}{b(\frac{1}{2})}\right] > -2\,\ln(1-x_2) + \ln(1+x_2) - \ln(6)$$

and, finally,

$$b(x_2) > \frac{1+x_2}{6\,(x_2-1)^2}\, b\left(\frac{1}{2}\right) \quad \forall x_2 > \frac{1}{2},\ x_2 \neq 1. \tag{6}$$

In fact, by (6), we have

$$\lim_{x_2 \to 1^-} u_1(1, x_2) = 0 < \epsilon = u_1(1,1) < \frac{b(\frac{1}{2})}{24} \leq \lim_{x_2 \to 1^-} \frac{(1-x_2)^2\, b(x_2)}{4\,(1+x_2)} = \lim_{x_2 \to 1^-} u_1\left(\frac{1+x_2}{2}, x_2\right). \tag{7}$$

However, in any case, the function $u_1(\cdot, 1)$ is not lower semicontinuous at the point 1 by the left-hand side of (7). Since u_2 is continuous at the point $(1, 1)$ and the right-hand side of (7) holds, then $\sum_{i=1}^{2} u_i$ is not upper semicontinuous at the point $(1, 1)$. Therefore, the hypothesis [9][Theorem 3.1, b)] fails.

We introduce the following Proposition.

Proposition 2.5. *G_1 is better reply secure and reciprocally upper semicontinuous game but not payoff secure one.*

Proof. We can only focus our attention on the point $(1, 1) \in [0, 1]$. We prove that G_1 is better reply secure. In fact, by strict concavity, we have

$$u_2(1, 0) = \max_{x_2 \in [0,1]} u_2(1, x_2) > u_2(1, 1)$$

Fix $\epsilon > 0$ such that $u_2(1, 0) - \epsilon > u_2(1, 1)$; and, by continuity respect to the opponent's variable, we have

$$|u_2(x_1, 0) - u_2(1, 0)| < \epsilon$$

and, trivially,

$$u_2(x_1, 0) = [u_2(x_1, 0) - u_2(1, 0)] + u_2(1, 0) \geq u_2(1, 0) - \epsilon > u_2(1, 1)$$

for all $x_1 \in U^-$ a suitable left neighborhood of 1. We prove that G_1 is reciprocally upper semicontinuous game. Whenever u_1 is lower semicontinuous at $(1, 1)$ along suitable directions toward $(1, 1)$, u_2 is continuous along all the sequences converging to $(1, 1)$. We prove that G_1 is not payoff secure. It's sufficient to observe that

$$\liminf_{x_2 \to 1^-} \{x_1 \in [0, 1] \mid u_1(x_1, x_2) \geq \epsilon = u_1(1, 1)\} = \emptyset$$

holds. □

Remark 2.2. *G_1 shows that payoff security and reciprocally upper semicontinuity assumptions jointed together are not necessary conditions for better reply security but, only, sufficient ones according to [16][Proposition 3.2].*

Therefore, Nash Equilibria inexistence is due to the quasiconcavity assumption's failure, as it's shown in (1).

3 Main Results on Nash Equilibria

In this section, we want to introduce results giving sufficient conditions for existence of Nash Equilibria in pure strategy without constraints, by weakening the classical quasi concavity hypothesis.

3.1 Nash Equilibria without Constraints

At first, we introduce this useful Lemma.

Lemma 3.1. *Let* $G := (X_i, u_i)_{i=1,\dots,n}$ *a game and* $\phi(x, y) : X \times X \to \mathbb{R}$ *the equilibrium bifunction for* G. *Suppose that* ϕ *is diagonally transfer continuous in* $y \in X$. *Assigned* $H(x) := \{y \in X : \phi(x, y) > \phi(y, y)\}$ *for all* $x \in X$, *then* $\bigcup_{x \in X} \overset{\circ}{H}(x) = \bigcup_{x \in X} H(x)$ *holds.*

Now, we present the fundamental definition of a multiconnected topological space due to Llinares[10][Definition 1].

Definition 3.1. *A topological space* X *is a multiconnected space if for any nonempty finite subset* $A = \{a_0, a_1, \dots, a_n\}$ *of* X, *there exists a family of elements* $\{b_0, b_1, \dots, b_n\}$ *in* X *and a family of functions*

$$P_i^A : X \times [0, 1] \to X \qquad i = 1, 2, \dots, n$$

such that

$$P_i^A(x, 0) = x, \;\; P_i^A(x, 1) = b_i \quad \forall x \in X \tag{8}$$

and the following function

$$G_A : [0, 1]^n \to X$$

given by

$$G_A(t_0, t_1, \dots, t_{n-1}) = P_0^A(\dots P_{n-1}^A(P_n^A(a_n, 1), t_{n-1}), t_0) \tag{9}$$

is a continuous function. Henceforth, if X *is endowed by such functions* P_i^A *satisfying the condition (8) and (9), we say, simply, that* X *has an mc-structure.*

Now, we introduce the Main Definition and Main Theorem.

Definition 3.2 (**Main Definition I**). *Let X a multiconnected-topological space and Y a set. We shall say $\phi : X \times Y \to \mathbb{R}$ mc-concave on $A \subseteq X$ in $y \in Y$ if, and only if, $\forall\, x_1, x_2 \in A$ $\exists\, \xi_{x_1,x_2} : [0,1] \to 2^X$ with open inverse image and with nonempty multiconnected values, such that $\forall \lambda \in [0,1]$, $\forall x^{'} \in \xi_{x_1 x_2}(\lambda)$*

$$\phi(x^{'}, y) \geq \lambda \phi(x_1, y) \,+\, (1-\lambda)\phi(x_2, y)$$

We shall say that ϕ is mc-concave in $y \in Y$ when $A = X$.

Theorem 3.1 (**Main Result I**). *Let $G = (X_i, u_i)_{i=1,\dots,n}$ be an n-person game satisfying the following:*

i) *X a compact mc-topological space;*

ii) *ϕ is diagonally transfer continuous in $y \in X$;*

iii) *Let*

$$B := \{y \in X \mid \phi(x,y) \text{ is mc concave in } y\} \neq \emptyset;$$

and, denoted by

$$C := \{y \in X \mid \phi(y,y) \geq \phi(x,y)\ \forall x \in co_{mc}(A),\, \forall A \in F(X)\}\,^{1},$$

suppose that $B \cap C \neq \emptyset$. Otherwise, if the previous intersection is empty, suppose that $B \cap \overline{C} \neq \emptyset$; and ϕ is upper semicontinuous on $\overline{C} \times \overline{C}$; and $\phi(x, \cdot)$ is lower semicontinuous on $\overline{C}$ $\forall x \in co_{mc}(A)$, $\forall A \in F(X)$.

Then, G has a Nash equilibrium.

Proof. By absurd and by Theorem [2][Proposition 4;pp. 269], we have that for every $x \in X$ there exists $y \in X$ such that $\phi(y,x) > \phi(x,x)$. Then, we should have $X = \bigcup_{x \in X} H(x)$ and $H(x) \neq \emptyset$ for all $x \in X$, where $H(x)$ is defined at Lemma 3.1. But, by compactness of X, by hypothesis **ii**) and by Lemma 3.1, we have that $X = \bigcup_{i=1}^{n} \overset{\circ}{H}(x_i)$ with $x_1, x_2, \dots, x_n \in X$.

[1] $co_{mc}(A)$ is the convex hull generated by $A \subseteq X$ respect to mc structure on X.

By using a unit partition argument on $\overset{\circ}{H}(x_i)$, we built these continuous functions

$$0 \leq \alpha_i \leq 1 \tag{10}$$

$$\sum_{i=1}^{n} \alpha_i = 1 \tag{11}$$

$$x \notin \overset{\circ}{H}(x_i)\colon \alpha_i(x) = 0. \tag{12}$$

By the hypothesis **iii)**, $\exists\, \xi_{x_1,x_2,\ldots,x_n} := \xi_n : [0,1]^n \longrightarrow 2^X$ [2] and $\exists\, y^{'} \in B \cap C \neq \emptyset$; or, if $B \cap C = \emptyset$, $\exists\, y^{'} \in B \cap \overline{C} \neq \emptyset$ such that

$$\phi(\xi_n(\alpha_1(x), \alpha_2(x), \ldots, \alpha_n(x)), y^{'}) \geq$$

$$\alpha_1(x)\phi(x_1, y^{'}) + \alpha_2(x)\phi(x_2, y^{'}) + \ldots \cdots + \alpha_n(x)\phi(x_n, y^{'}). \tag{13}$$

We define $p : X \longrightarrow 2^X$

$$p(x) = \xi_n(\alpha_1(x), \alpha_2(x), \ldots, \alpha_n(x)).$$

By α_i's continuity and by the regular property on ξ_n, the correspondence p is an open inverse image one; and it assumes mc convex values. Therefore, by [10] [Theorem 1], there exists a nonempty subset $A \in F(X)$ and $\overline{x} \in X$ such that

$$\overline{x} \in p(\overline{x}) \tag{14}$$

$$\overline{x} \in co_{mc}(A \cap p(\overline{x})) \subset co_{mc}(A). \tag{15}$$

If $y^{'} \in B \cap C \neq \emptyset$, by the properties (10), (11) and (12) on α_i, by the inequalities (13), (14) and (15), we have

$$\phi(y^{'}, y^{'}) \geq \phi(\overline{x}, y^{'}) \geq \sum_{i=1}^{n} \alpha_i(\overline{x})\phi(x_i, y^{'}) > \sum_{i=1}^{n} \alpha_i(\overline{x})\phi(y^{'}, y^{'}) = \phi(y^{'}, y^{'}). \tag{16}$$

But, this is an absurd. If $y^{'} \in B \cap \overline{C} \neq \emptyset$, there exists a $y_n \in C$ converging to $y^{'}$ such that

$$\phi(y^{'}, y^{'}) \geq \phi(y_n, y_n) \geq \phi(\bar{x}, y_n) \geq \phi(\bar{x}, y^{'})$$

by the hypothesis **iii)**. However, we can proceed as before. Since $y^{'} \in B$, we can state the same inequalities in (16) after the term $\phi(\bar{x}, y^{'})$. □

[2] It can be shown that the mc-concavity property holds for a finite number of points $x_1, x_2, \ldots, x_n \in A$.

3.2 Nash Equilibria with Constraints

We introduce some generalizations of Main Definition in topological vector space involving duality structures.

Definition 3.3 (**Main Definition II**). *Let X be a locally convex topological vector space with a multiconnection structure and X^* its dual. A bifunction $\phi : X \times X \to \mathbb{R}$ is named mc-concave linear invariant in $y \in X$ on $A \subseteq X$ if and only if for each $p \in X^*$ the bifunction $\phi(x, y) + p(x)$ is mc-concave in $y \in X$ on $A \subseteq X$. We shall say, simply, that ϕ is mc-concave linear invariant in y when $A = X$.*

Definition 3.4. *Let X be a locally convex topological vector space with a multiconnection structure and X^* be its dual; and $G = (X_i, u_i, C_i)$ a game with constraints. We define the following subsets:*

$$B^* := \{y \in X \mid \phi(x, y) \text{ is mc-concave linear invariant in } \; y\};$$
$$C_1^* = \left\{ y \in X \;\middle|\; p(y) \leq \sup_{z \in C(y)} p(z) \;\; \forall p \in X^* \right\};$$
$$C_2^* = \left\{ y \in X \;\middle|\; p(y) \leq \sup_{x \in co_{mc}(A)} p(x) \;\; \forall p \in X^*, \forall A \in F(X) \right\}.$$

Now, we state a result in the context of generalized quasi variational inequalities.

Theorem 3.2 (**Main Result II**). *Let a game with constraints $G = (X_i, u_i, C_i)$ where X_i is a compact convex subset of a real topological vector space E which has sufficiently many linear continuous functionals. Suppose $C_i : X_{-i} \to X_i$ an upper hemicontinuous with nonempty closed and convex values. Suppose that the function $\phi : X \times X \to \mathbb{R}$ is diagonally transfer continuous in $y \in X$; and*

$$A := \{x \in X | \sup_{y \in C(x)} \phi(y, x) \leq \phi(x, x)\} \tag{17}$$

is closed. Let C as in the hypothesis iii) in Theorem 3.1, suppose that

$$B^* \cap [(C \cap C_1^*) \cup (A \cap C_2^*)] \neq \emptyset.$$

Otherwise, if the previous intersection is empty, suppose that

$$B^* \cap \left[\overline{(C \cap C_1^*) \cup (A \cap C_2^*)}\right] \neq \emptyset$$

and ϕ is upper semicontinuous on $\overline{C} \times \overline{C}$; and $\phi(x, \cdot)$ is lower semicontinuous on $\overline{C}$, for all $x \in co_{mc}(A)$, for all $A \in F(X)$. Then G has a Nash Equilibrium.

Proof. We define $\psi(x, y) : X \times X \to \mathbb{R}$ as $\psi(x, y) = \phi(y, x) - \phi(x, x)$. We'll prove that

$$\exists\, x^* \in C(x^*) \, : \, \sup_{y \in C(x)} \psi(x^*, y) \leq 0 \tag{18}$$

By [2][Proposition 4;pp. 269], it's sufficient for proving Nash Equilibrium existence. By absurd, for each $x \in X$, either $x \notin C(x)$ or there exists $u \in C(x)$ such that $\psi(x, u) > 0$. In the case $x \notin C(x)$, note that E has sufficiently many continuous linear functionals and, by Hahn Banach Theorem, there exists $p \in E^*$ such that $p(x) - \sup_{z \in C(x)} p(z) > 0$. Let

$$V_p := \{x \in X : p(x) - \sup_{z \in C(x)} p(z) > 0\} \neq \emptyset.$$

As C is upper hemicontinuous, V_p is a neighborhood of $x \in X$. In the case that there exists $u \in C(x)$ such that $\psi(x, u) > 0$, then $\sup_{y \in C(x)} \psi(x, y) > 0$. Let $V_0 := \{x \in X \mid \sup_{y \in C(x)} \psi(x, y) > 0\} \neq \emptyset$. Then V_0 is open by (17). It's clear that $X = V_0 \bigcup \cup_{p \in E^*} V_p$. It's possible to extract an open finite refinement $(V_0, V_{p_i} := V_i)$ with $i \in I$. Now, we assume $\{\alpha_0, \alpha_i\}$ with $i \in I$ a family of continuous non negative real valued function on X such that α_i vanishes on $X \setminus V_i (X \setminus V_0)$ with $i \in I$. Now, we define

$$\Pi(x, y) = \alpha_0(y)\psi(y, x) + \sum_{i \in I} \alpha_i(y) p_i(y - x)$$

for each $(x, y) \in X \times X$. Therefore, the funtion Π is diagonally transfer continuous in $y \in X$ since ϕ satisfies the same condition in $y \in X$ and p_i are continuous. Let be

$$B_\Pi := \{y \in X \mid x \in X \to \Pi(x, y) \text{ is mc concave in } y\}$$

and

$$C_\Pi := \{y \in X \mid \Pi(y, y) \geq \Pi(x, y) \; \forall x \in co_{mc}(A), \forall A \in F(X)\}.$$

The function $\Pi(\cdot, y)$ is mc-concave in $y \in B_\Pi$ since ϕ is mc-concave linear invariant in $y \in B^*$ and p_i are continuous; and,

$$B_\Pi \cap C_\Pi(\cap\overline{C_\Pi}) \supseteq B^* \cap [(C \cap C_1^*) \cup (A \cap C_2^*)] \left(\cap \left[\overline{(C \cap C_1^*) \cup (A \cap C_2^*)}\right]\right) \neq \emptyset$$

holds; then, the condition $iii)$ in the Theorem 3.1 holds for B_Π, C_Π and Π. Therefore, there exists $\overline{x} \in X$ such that

$$0 \geq \sup_{y\in X} \Pi(y, \overline{x}) - \Pi(\overline{x}, \overline{x}) = \sup_{y\in X} \alpha_0(\overline{x})\psi(\overline{x}, y) + \sum_{i\in I} \alpha_i(\overline{x})p_i(\overline{x} - y) \tag{19}$$

On the other hand, since $(\alpha_i)_{i\in I}$ is a partition of unit, there exists at least one index $i \in I$ such that $\alpha_i(\overline{x}) > 0$. We prove that the right hand side of inequality (19) is strictly positive. If $\overline{x} \in V_0 \cap V_i$ for some $i \in J \subseteq I$, there exists $y^* \in C(\overline{x})$ such that $\psi(\overline{x}, y^*) > 0$. Since $\overline{x} \in V_i$, we have

$$p_i(\overline{x}) > \sup_{z\in C(\overline{x})} p_i(z) \geq p_i(y^*).$$

It follows that $\alpha_i(\overline{x})p_i(\overline{x} - y^*) > 0$. Hence, we have that

$$\alpha_0(\overline{x})\psi(\overline{x}, y^*) + \sum_{i\in J} \alpha_i(\overline{x})p_i(\overline{x} - y^*) > 0.$$

If $\overline{x} \in V_0 \bigcup_{i=1,\ldots,n} V_i^C$, we have $\alpha_0(\overline{x})\psi(\overline{x}, y^*) > 0$. If $\overline{x} \in V_0^C \bigcap V_i$ for some $i \in J \subseteq I$, there exists a $y^* \in C(\overline{x}) \neq \emptyset$ such that $\sum_{i\in J} \alpha_i(\overline{x})p_i(\overline{x} - y^*) > 0$. However, the previous conditions contradict (19). Therefore, the condition (18) holds. $\square$

4 Some examples in Game Theory

In this section, we want to give two examples of static games in which the classical quasi concavity hypothesis fail but the assumptions stated in Theorem 3.1 hold. In the first example, simultaneously, we give an example of multiconnection structure on a compact subset.

4.1 A not quasi-concave game

Let $G_3 := ([-1,+1],[0,1],u_1,u_2)$ defined as follows

$$u_1(x_1,x_2) = \begin{cases} -x_1^2 + g(x_2) & \text{if } \mid x_1 \mid < 1 \\ \frac{1}{4} & \text{otherwise} \end{cases}$$

$$u_2(x_1,x_2) = \ln(1+x_2)\,\sin(\pi\, x_2\,(1+|x_1|)\,x_2$$

where $g : [0,1] \longrightarrow \mathbb{R}$ is a lower semicontinuous function such that $g \geq \frac{1}{4}$ on $\left[\frac{1}{4},1\right]$; $g \leq 1$; $g(n_{min}) = \frac{1}{4}$ with $n_{min} = BR_2(1)$. Besides, we denote by

$$m_{min} = \arg\max_{y_1 \geq 0} \left\{ \arg\min_{x_2 \in [0,1]} \phi(0,x_2,y_1,y_2) = \{1\} \right\}$$

and ${x_2}^*(y_1)$, ${x_2}^{**}(y_1)$ the absolute maximum and minimum points for the function $\phi(0,\cdot,y_1,y_2)$ unchangingly respect to y_2. We define the following multifunction $D:\ [-1,+1]\times[0,1] \to [-1,+1]\times[0,1]$ as follows:

$$D(x_1,x_2) := \begin{cases} (0,x_2) & |x_1| = 1,\ x_2 < \frac{4}{5} \\ (x_1,x_2) & |x_1| \neq 1, x_2 < \frac{4}{5} \\ (0,{x_2}^*(m_x)) & x_2 \geq \frac{4}{5} \end{cases}$$

where $m_x \in [0,1]$ is the convex coordinate in the equality $x_2 = (1-m_x)\frac{4}{5} + m_x$. Now, for every $z \in X$ we define the path that joins any arbitrary point $x \in X$ with $D(z) \in X$ by this parametric function $P^z : X \times [0,1] \to X$ in the following three cases, as prescribed below:

1. $\boxed{|z_1| = 1,\ z_2 < \frac{4}{5}}$

$$P^z(x,t) = \begin{cases} (x_1,x_2) & |x_1| = 1,\ x_2 < \frac{4}{5}, 1, t = 0 \\ co_{t\neq 0}[(0,x_2),(0,z_2)] & |x_1| = 1,\ x_2 < \frac{4}{5},\ t \neq 0 \\ co_t[(x_1,x_2),(0,z_2)] & |x_1| \neq 1,\ x_2 < \frac{4}{5} \\ (x_1,x_2) & x_2 \geq \frac{4}{5},\ t = 0 \\ co_{t\neq 0}[(0,{x_2}^*(m_x)),(0,z_2)] & x_2 \geq \frac{4}{5},\ t \neq 0 \end{cases}$$

2. $\boxed{|z_1| \neq 1,\ z_2 < \frac{4}{5}}$

$$P^z(x,t) = \begin{cases} (x_1, x_2) & |x_1| = 1,\ x_2 < \frac{4}{5},\ t = 0 \\ co_{t\neq 0}[(0,x_2),(0,z_2),(z_1,z_2)] & |x_1| = 1,\ x_2 < \frac{4}{5},\ t \neq 0 \\ co_t[(x_1,x_2),(0,z_2),(z_1,z_2)] & |x_1| \neq 1,\ x_2 < \frac{4}{5} \\ (x_1, x_2) & x_2 \geq \frac{4}{5},\ t = 0 \\ co_{t\neq 0}[(0,{x_2}^*(m_x)),(0,z_2),(z_1,z_2)] & x_2 \geq \frac{4}{5},\ t \neq 0 \end{cases}$$

3. $\boxed{z_2 \geq \frac{4}{5}}$

$$P^z(x,t) = \begin{cases} (x_1, x_2) & |x_1| = 1,\ x_2 < \frac{4}{5},\ t = 0 \\ co_{t\neq 0}[(0,x_2),(0,{x_2}^*(m_z))] & |x_1| = 1,\ x_2 < \frac{4}{5},\ t \neq 0 \\ co_t[(x_1,x_2),(0,x_2),(0,{x_2}^*(m_z))] & |x_1| \neq 1,\ x_2 < \frac{4}{5} \\ (x_1, x_2) & x_2 \geq \frac{4}{5},\ t = 0 \\ co_{t\neq 0}[(0,{x_2}^*(m_x)),(0,{x_2}^*(m_z))] & x_2 \geq \frac{4}{5},\ t \neq 0 \end{cases}$$

We present the following Propositions.

Proposition 4.1. *The topological space X endowed with this family of functions $(P^z)_{z\in X}$ has a multiconnected structure. Moreover, the convex hull $co_{mc}(A) = \{0\} \times \left[0, \frac{4}{5}\right]$ for every $A \in F(X)$.*

Proposition 4.2. *The function $\phi(\cdot, y)$ has multiconnected uplevels*

$$A_{k,y} = \{x \in X \ |\phi(x,y) \geq k\}$$

for every

$$y \in \left\{ z \in [-1,1] \times [0,1] \ \middle| \ |z_1| \geq m_{min},\ z_2 \geq \frac{1}{4} \right\}.$$

Proof. See Appendix. □

Proposition 4.3. *G_3 has a Nash equilibrium in pure strategy.*

Proof. See Appendix. □

Remark 4.1. *Note that $BR_1(n_{min}) = \{-1, 0, 1\}$ is not reduced to a singleton and is not connected; therefore, [1][Theorem 2.1] and [18][Theorem 10] fail. Besides, note that u_1 is not continuous although the payoffs' uplevels are acyclic subsets; therefore, [15][Theorem 7] fails.*

4.2 An oscillating problem

Let $G_4 := \left([0, \pi], \left[1, \frac{\pi}{2}\right], u_1, u_2\right)$ whose payoffs are defined as follows:

$$u_1(x_1, x_2) = \sin(x_1\, x_2)$$
$$u_2(x_1, x_2) = \cos(x_2 - x_1).$$

Now, we construct the function $\phi(x, y) = \sin(x_1\, {y_2}^2) + \cos(x_1 - y_1)$. The latter function is diagonally transfer continuous in y since u_1, u_2 are continuous on the strategy space $X = [0, \pi] \times [1, \pi/2]$. For every $y, z \in X$, we put $H := \left[0, \frac{\pi}{{y_2}^2}\right] \times \left[\max\left\{1, y_1 - \frac{\pi}{2}\right\}, \frac{\pi}{2}\right]$; and we define $D : X \to H \subset X$ and $P^z : X \times [0, 1] \to X$ as follows:

$$D(x) =:= \begin{cases} x & x \in H \\ \left(\dfrac{\pi}{2y_2^2}, y_1\right) & \text{otherwise} \end{cases}$$

$$P^z(x, t) = \begin{cases} co_t[x, z] & x, z \in H \\ x & z \in H,\, x \notin H,\, t = 0 \\ co_{t \neq 0}\left[\left(\dfrac{\pi}{2y_2^2}, y_1\right), z\right] & z \in H,\, x \notin H,\, t \neq 0 \\ x & z \notin H,\, x \in H,\, t = 0 \\ co_{t \neq 0}\left[z, \left(\dfrac{\pi}{2y_2^2}, y_1\right)\right] & z \notin H,\, x \in H,\, t \neq 0 \end{cases}$$

The topological space X endowed with this family of functions $(P^z)_{z \in X}$ has a multiconnected structure for every $y \in X$. According to the notations in Theorem 3.1, it can be shown that ϕ is, at most, mc concave in the singleton $B := \{y\}$ since $\phi(\cdot, y)$ is strict concave on H; and, by arbitrariety of y and by Theorem 3.1, $B \cap C \neq \emptyset$ if, and only if, the following nonlinear optimization problem with constraints

$$\begin{cases} \max \{\sin(y_1\, y_2^2) + \cos(y_2 - y_1)\} = 2 \\ 0 \leq y_1 \leq \pi \\ 1 \leq y_2 \leq \dfrac{\pi}{2} \end{cases} \tag{20}$$

has solutions. We impose that the gradient is equal to zero. The critical points are $P_0(0,0)$, $P_1(\frac{1}{3}z, -\frac{2}{3}z)$ and $P_2(z^{'}, z^{'})$ where z and $z^{'}$ are, respectively, solutions of $H_0(z) = -9\sin(z) + 4\cos\left(\frac{4}{27}z^3\right) z^2 = 0$ and $H_1(z) = 2\,z^3 - \pi = 0$. We note that $H_1(z) = 2z^3 - \pi$ is increasing and continuous on $[1, \pi/2]$ and $H\,(1)\ H\left(\frac{\pi}{2}\right) < 0$. Then, after simple calculations, $P_2 \in X$ is a solution of the system (20). In fact, the game achieves its Nash equilibrium at the point $\left(\sqrt[3]{\frac{\pi}{2}}, \sqrt[3]{\frac{\pi}{2}}\right)$.

Remark 4.2. *The two players' strategy spaces are not equal. Therefore, [21][Theorem 1] fails. Moreover, if $X_2 = [\frac{5}{4}, \pi/2] \subset [1, \pi/2]$, the problem (20) has no solutions.*

5 Appendix

Proof of Proposition 2.1.

Proof. First of all, every opponents' allocation $(x_i, x_j) \in [0,1]^2$ is represented in the Figure 1. In the first case (1), we have other four subcases described as follows:

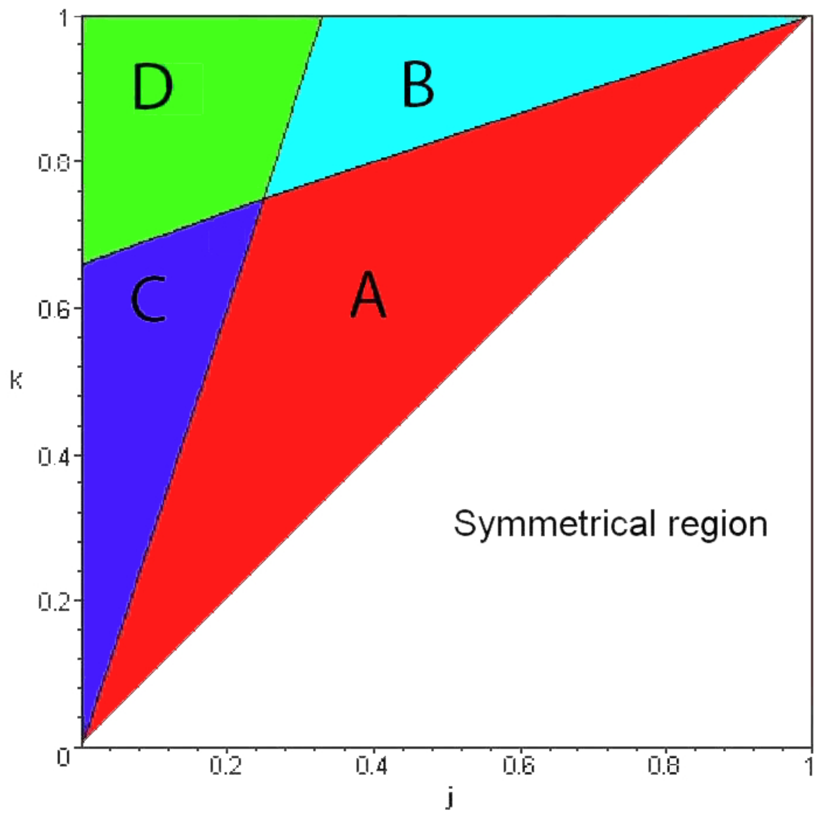

Figure 1: The upper triangle in the square includes all the possible subcases $A)$, $B)$, $C)$ and $D)$ according to which u_i is described by (1). The diagonal line represents the allocations according to which u_i is described by (2). The lower triangle in the square represents the symmetrical region obtained if $x_j < x_k$.

A) If $(x_j, x_k) \in A := \left\{ (x_j, x_k) \in [0,1]^2 \;\middle|\; x_j > \frac{x_k}{3},\, x_k < \frac{x_j}{3} + \frac{2}{3} \right\}$, we have

$$\begin{gathered}
\lim_{x_i \to x_j^-} u_i(x_i, x_j, x_k) > u_i(x_j, x_j, x_k) > \lim_{x_i \to x_j^+} u_i(x_i, x_j, x_k) = \\
\lim_{x_i \to x_k^-} u_i(x_i, x_j, x_k) < u_i(x_k, x_j, x_k) < \lim_{x_i \to x_k^+} u_i(x_i, x_j, x_k)
\end{gathered} \tag{21}$$

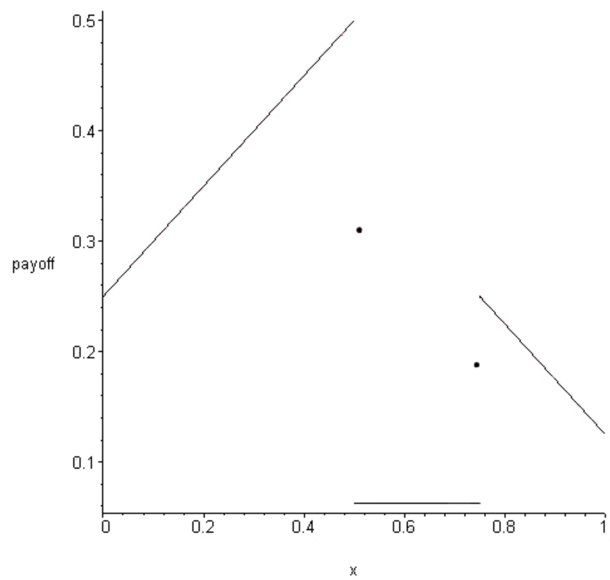

Figure 2: Payoff in the subcase $A)$ with $x_j = 0.5$ and $x_k = 0.75$.

B) If $(x_j, x_k) \in B := \left\{ (x_j, x_k) \in [0,1]^2 \;\middle|\; x_j > \frac{x_k}{3},\; x_k \geq \frac{x_j}{3} + \frac{2}{3} \right\}$, we have

$$\begin{aligned} &\lim_{x_i \to x_j^-} u_i(x_i, x_j, x_k) > u_i(x_j, x_j, x_k) > \lim_{x_i \to x_j^+} u_i(x_i, x_j, x_k) = \\ &\lim_{x_i \to x_k^-} u_i(x_i, x_j, x_k) \geq u_i(x_k, x_j, x_k) \geq \lim_{x_i \to (x_k \neq 1)^+} u_i(x_i, x_j, x_k) \end{aligned} \tag{22}$$

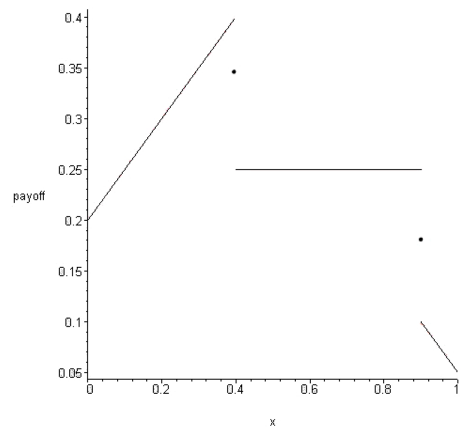

Figure 3: Payoff in the subcase B) with $x_j = 0.4$ and $x_k = 0.9$.

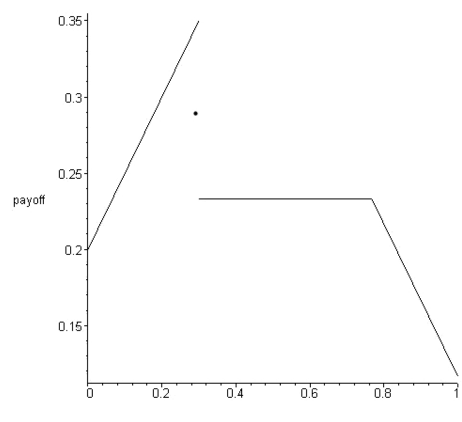

Figure 4: Payoff in the limit subcase B); if $(x_j\, x_k) \in [0,1]^2$ are on the borderline between the regions A and B in the Figure 1.

C) If $(x_j, x_k) \in C := \left\{ (x_j, x_k) \in [0,1]^2 \;\middle|\; x_j \le \frac{x_k}{3},\, x_k < \frac{x_j}{3} + \frac{2}{3} \right\}$, we have

$$\begin{aligned} &\lim_{x_i \to (x_j \neq 0)^-} u_i(x_i, x_j, x_k) \le u_i(x_j, x_j, x_k) \le \lim_{x_i \to x_j^+} u_i(x_i, x_j, x_k) = \\ &\lim_{x_i \to x_k^-} u_i(x_i, x_j, x_k) < u_i(x_k, x_j, x_k) < \lim_{x_i \to x_k^+} u_i(x_i, x_j, x_k) \end{aligned} \tag{23}$$

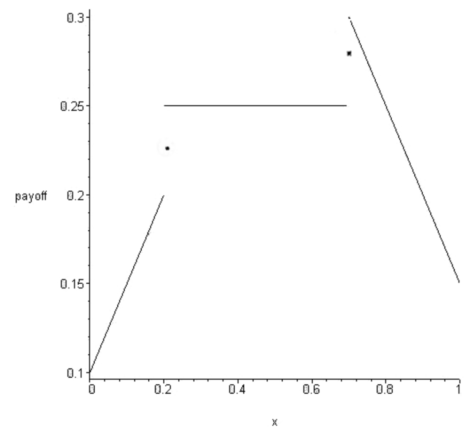

Figure 5: Payoff in the subcase C) with $x_j = 0.2$ and $x_k = 0.7$.

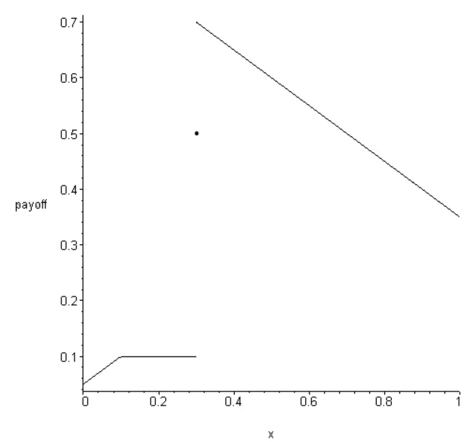

Figure 6: Payoff in the limit subcase C); if $(x_j\, x_k) \in [0,1]^2$ are on the borderline between the regions A and C in the Figure 1.

D) If $(x_j, x_k) \in D := \left\{ (x_j, x_k) \in [0,1]^2 \,\middle|\, x_j \le \frac{x_k}{3},\ x_k \ge \frac{x_j}{3} + \frac{2}{3} \right\}$, we have

$$\lim_{x_i \to (x_j \neq 0)^-} u_i(x_i, x_j, x_k) \le u_i(x_j, x_j, x_k) \le \lim_{x_i \to x_j^+} u_i(x_i, x_j, x_k) =$$
$$\lim_{x_i \to x_k^-} u_i(x_i, x_j, x_k) \ge u_i(x_k, x_j, x_k) \ge \lim_{x_i \to (x_k \neq 1)^+} u_i(x_i, x_j, x_k) \qquad (24)$$

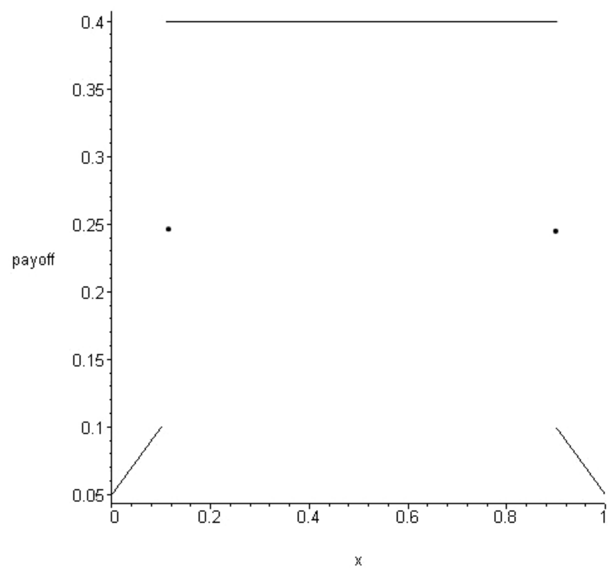

Figure 7: Payoff in the subcase D) with $x_j = 0.1$ and $x_k = 0.9$.

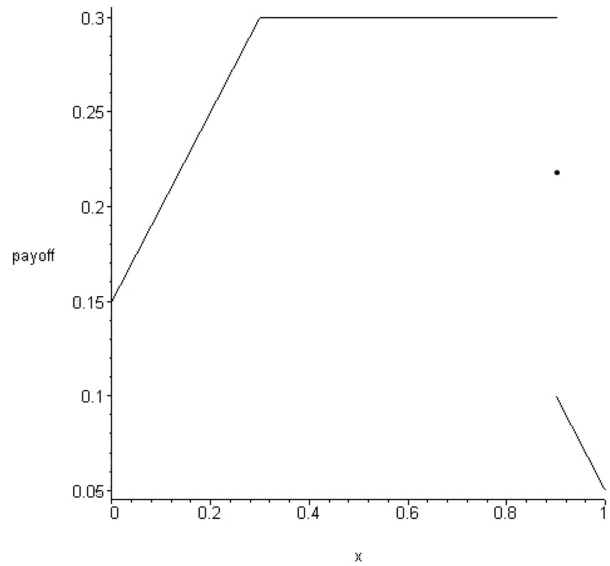

Figure 8: Payoff in the limit subcase D); if $(x_j\, x_k) \in [0,1]^2$ are on the borderline between the regions B and D in the Figure 1.

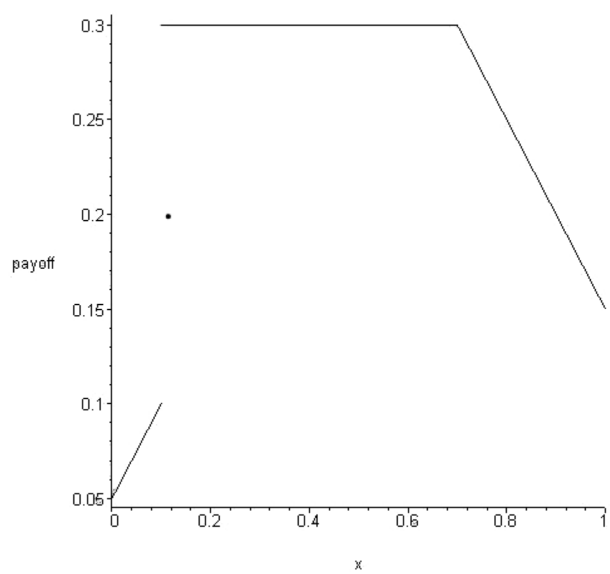

Figure 9: Payoff in the limit subcase D); if $(x_j\, x_k) \in [0,1]^2$ are on the borderline between the regions C and D in the Figure 1.

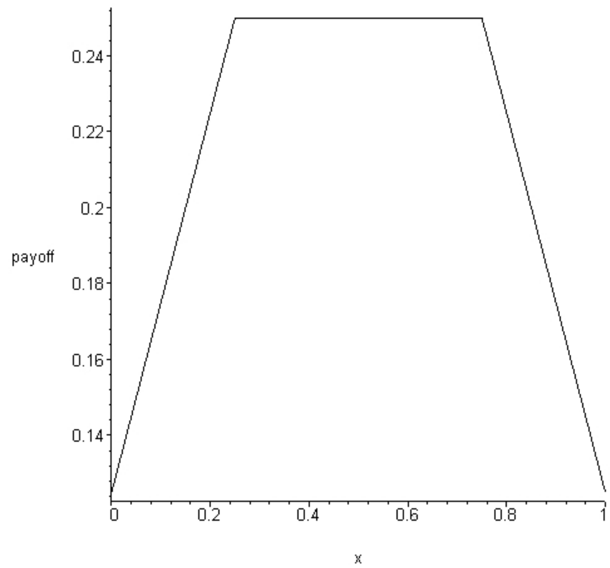

Figure 10: Payoff in the limit subcase D); if $(x_j, x_k) = (0.25, 0.75)$ is on the closure of the four regions A), B), C) and D) in the Figure 1.

We study the subcases **A**), **B**) and **C**) by assuming strictly the previous inequalities (21), (22) and (23). In this case, $u_i(\cdot, x_j, x_k)$ presents a real discontinuity at $x_i = x_j$ and $x_i = x_k$. In the case **A**), whenever $u_i(\cdot, x_j, x_k)$ is lower semicontinuous for $x_i \to x_j^-$ and $x_i \to x_k^+$ then it's upper semicontinuous for $x_i \to x_j^+$ and $x_i \to x_k^-$; in the case **B**), whenever $u_i(\cdot, x_j, x_k)$ is lower semicontinuous for $x_i \to x_j^-$ and $x_i \to x_k^-$, then it's upper semicontinuous for $x_i \to x_j^+$ and $x_i \to (x_k \neq 1)^+$; in the case **C**), whenever $u_i(\cdot, x_j, x_k)$ is upper semicontinuous for $x_i \to (x_j \neq 0)^-$ and $x_i \to x_k^-$ then it's lower semicontinuous for $x_i \to x_j^+$ and $x_i \to x_k^+$; We analyze the subcases in **B**) and **C**) by taking the inequalities in (22) and (23) as equalities. In the limit subcase **B**), $u_i(\cdot, x_j, x_k)$ is continuous at $x_i = x_k$, but it's upper semicontinuous for $x_i \to x_j^+$ and lower semicontinuous for $x_i \to (x_j \neq 0)^-$. In the limit subcase **C**), $u_i(\cdot, x_j, x_k)$ is continuous at $x_i = x_j$, but it's upper semicontinuous for $x_i \to x_j^-$ and lower semicontinuous for $x_i \to (x_j \neq 0)^-$. Since $u_i(\cdot, x_j, x_k)$ is strict increasing and strict decreasing, respectively, on $[0, x_j[$ and $]x_k, 1]$ [3]; and constant on $]x_j, x_k[$, we have that

$$BR_i(A \cup B \cup C) = \emptyset \tag{25}$$

holds. We study the last subcase **D**). It's trivial, as shown in (24), that

$$]x_j, x_k[\subset BR_i(D) \subset [x_j, x_k] \tag{26}$$

Let $(\bar{x}_i, \bar{x}_j, \bar{x}_k) \in [0,1]^3$ be a strategy with $(\bar{x}_j, \bar{x}_k) \in D$. If $\bar{x}_i \in \overset{\circ}{BR_i}(D)$ [4], there exists a strictly increasing sequence $x_{j_n} < \bar{x}_i$ and a strictly decreasing $\bar{x}_i < x_{k_n}$ one, such that $x_{j_1} = \bar{x}_j$, $x_{k_1} = \bar{x}_k$; and the sequences $u_j(\bar{x}_i, x_{j_n}, x_{k_n})$, $u_k(\bar{x}_i, x_{j_n}, x_{k_n})$ [5] are strict increasing. By monotony, the sequences x_{j_n} and x_{k_n} converge to $\bar{x}_i$. Then, there exists $\nu \in \mathbb{N}$ such that $(x_{j_n}, x_{k_n}) \in A$ for all $n \geq \nu$. Therefore, by (25), we have that

$$BR_i\left(\{(x_{j_n}, x_{k_n})\}_{n \geq \nu}\right) \subset BR_i(A) = \emptyset. \tag{27}$$

[3] In the case **C**), the first subset is empty if $x_j = 0$; in the case **B**), the second subset is empty if $x_k = 1$. But, at least, one of them is never empty.

[4] This implies that $\bar{x}_i \neq 0, 1$.

[5] By simmetry, we observe that $u_j(\bar{x}_i, \cdot, x_{k_n})$ does not depend on x_{k_n} in the subset $]0, \bar{x}_i[$; and $u_k(\bar{x}_i, x_{j_n}, \cdot)$ does not depend on x_{j_n} in the subset $]\bar{x}_k, 1[$.

If $\bar{x}_i = \bar{x}_j$ [6], we have that $x_{j_n} = \bar{x}_j$ for all n but $\bar{x}_i < x_{k_n}$ is strict decreasing; and $u_k(\bar{x}_i, x_{j_n}, x_{k_n})$ is strict increasing. As before, the sequences $(x_{j_n}, x_{k_n}) \in A$ for n sufficiently large. We can conclude as before. If $\bar{x}_i = \bar{x}_k$ [7] , the proof is the same by exchanging the role of the sequence x_{j_n} by x_{k_n}. Therefore, by (25),(26) and (27), G_1 has no pure Nash equilibria in the case (1).

In the second case (2), the player i tends to move towards the same allocations chosen by the other two players; since its payoff is, at least, increasing in its own variable if $x_i < x_j = x_k$ or decreasing in its own variable if $x_i > x_j = x_k$. However, its payoff is never upper semicontinuous at $x_i = x_j = x_k$. □

Proof of Proposition 2.2.

Proof. We prove that the game is payoff secure. Fix $(x_i, x_j, x_k) \in [0,1]^3$ and ϵ a strict positive real number. We'll prove that there exists a payoff secure strategy $\bar{x}_i \in [0,1]$ such that $u_i(\bar{x}_i, x_j^{'}, x_k^{'}) \geq u_i(x_i, x_j, x_k) - \epsilon$ for small deviations $(x_j^{'}, x_k^{'})$ from the point (x_j, x_k). Let $F = \{x_j = x_i\} \times \{x_k = x_i\}$; and $TC^{-}_{(x_i|x_j,x_k)}$ be the tangent cone at (x_j, x_k) along which the function $u_i(x_i, \cdot, \cdot)$ is lower semicontinuous at (x_j, x_k). In our case, for all $x_i \in [0,1]$, the function $u_i(x_i, \cdot, \cdot)$ is continuous except for this closed subset

$$E_{x_i} := \{x_j = x_i\} \times \{x_k > x_i\} \cup \{x_j = x_i\} \times \{x_k < x_i\} \cup \{x_j < x_i\} \times \{x_k = x_i\} \cup$$

$$\cup \{x_j > x_i\} \times \{x_k = x_i\} \cup F \subset [0,1]^2. \tag{28}$$

Then, for all $(x_j, x_k) \in [0,1]^2 \setminus E_{x_i}$ there exists a neighborhood of the point (x_j, x_k) in $[0,1]^2$ such that

$$u_i(x_i, x_j^{'}, x_k^{'}) \geq u_i(x_i, x_j, x_k) > u_i(x_i, x_j, x_k) - \epsilon$$

$$\forall (x_j^{'}, x_k^{'}) \in TC^{-}_{(x_i|x_j,x_k)} \cap U = U.$$

It's enough to choose $\bar{x}_i := x_i$ as payoff secure strategy for the i^{th}-player [8]. Let be $(x_i, x_k) \in$

[6] If $\bar{x}_j = 0$, then $\bar{x}_i = 0 \neq 1$.

[7] If $\bar{x}_k = 1$, then $\bar{x}_i = 1 \neq 0$.

[8] In general, payoff security sssumption at $(\bar{x}_i, \bar{x}_{-i}) \in X$ is much hard to be checked if the function $x_{-i} \in X_{-i} \to u_i(\bar{x}_i, x_{-i})$ is upper semicontinuous at $\bar{x}_{-i}$ (see [16][Cor.3.4]).

E_{x_i} such that $x_k < 3\,x_i \neq 0$ and a sequence $x_{j_n} \to x_i^-$, we have

$$u_i(x_i, x_{j_n}, x_k) = \frac{x_k - x_{j_n}}{2} \overset{n\to\infty}{\longrightarrow} \frac{x_k - x_i}{2} < \frac{x_i + x_k}{4} = u_i(x_i, x_i, x_k).$$

But, the previous lack of lower semicontinuity can be overpassed. We focus our attention on the case $\{x_j = x_i\} \times \{x_k > x_i\} \subset E_{x_i}$. Suppose that $(x_j, x_k) \in A \cup B$. By lower semicontinuity of $u_i(\cdot, x_j, x_k)$ at x_j from the left side, there exists $\delta > 0$ and $\bar{x}_i > 0$ with $0 < \bar{x}_i < x_i$ and $x_i - \bar{x}_i < \delta$ such that

$$u_i(\bar{x}_i, x_j, x_k) > u_i(x_i, x_j, x_k). \tag{29}$$

By continuity for the function $u_i(\bar{x}_i, \cdot, \cdot)$ onto $\{x_j = x_i\} \times \{x_k > x_i\} \setminus E_{\bar{x}_i} = \{x_j = x_i\} \times \{x_k > x_i\}$, there exists $\delta_1 > 0$ such that $B_{\delta_1}(x_j, x_k) \cap E_{\bar{x}_i} = \emptyset$ and

$$u_i(\bar{x}_i, x_j^{'}, x_k^{'}) > u_i(\bar{x}_i, x_j, x_k) - \epsilon \quad \forall\, (x_j^{'}, x_k^{'}) \in B_{\delta_1}(x_j, x_k) \cap [0,1]^2. \tag{30}$$

By (29), (30) and by choosing $\bar{x}_i$ as payoff secure strategy for the player i, we have the thesis. Suppose that $(x_j, x_k) \in C$. By transfer lower semicontinuity (see [20][Definition 1]) of $u_i(\cdot, x_j, x_k)$ at x_j, there exists $\delta > 0$ and $\bar{x}_i$ with $x_k < \bar{x}_i < 1$ and $\bar{x}_i - x_k < \delta$ such that

$$u_i(\bar{x}_i, x_j, x_k) > u_i(x_i, x_j, x_k). \tag{31}$$

By continuity for the function $u_i(\bar{x}_i, \cdot, \cdot)$ at $(x_j, x_k) \notin E_{\bar{x}_i}$ [9], there exists $\delta_1 > 0$ such that $B_{\delta_1}(x_j, x_k) \cap E_{\bar{x}_i} = \emptyset$ and

$$u_i(\bar{x}_i, x_j^{'}, x_k^{'}) > u_i(\bar{x}_i, x_j, x_k) - \epsilon \quad \forall (x_j^{'}, x_k^{'}) \in B_{\delta_1}(x_j, x_k) \cap [0,1]^2. \tag{32}$$

By (31), (32) and by choosing $\bar{x}_i$ as payoff secure strategy for the player i, we have the thesis. Suppose that $(x_j, x_k) \in D$. By right upper semicontinuity for the function $u_i(\cdot, x_j, x_k)$ at x_j, there exists $\bar{x}_i > x_j$ and $\bar{x}_i < x_k$ such that

$$u_i(\bar{x}_i, x_j, x_k) \geq u_i(x_i, x_i, x_k) \tag{33}$$

and, by continuity for $u_i(\bar{x}_i, \cdot, \cdot)$ at $(x_j, x_k) \notin E_{\bar{x}_i}$, there exists a neighborhood U of the point (x_j, x_k) in $[0,1]^2$ such that $U \cap E_{\bar{x}_i} = \emptyset$ and

$$u_i(\bar{x}_i, x_j^{'}, x_k^{'}) > u_i(\bar{x}_i, x_j, x_k) - \epsilon \quad \forall (x_j^{'}, x_k^{'}) \in U. \tag{34}$$

[9] The choise of $\bar{x}_i$ depends on (x_j, x_k).

Then, by (33) and (34), we have that

$$u_i(\bar{x}_i, x_j^{'}, x_k^{'}) \geq u_i(x_j, x_j, x_k) - \epsilon \quad \forall (x_j^{'}, x_k^{'}) \in U \cap E_{\bar{x}_i}.$$

Now, we refer to the following subset $\{x_j < x_i\} \times \{x_k = x_i\} \subset E_{x_i}$. The proof is similar to the previous case. Suppose that $(x_j, x_k) \in A \cup C$. In this case, there exists $\delta > 0$ such that

$$A_\delta = \bigcap_{(x_j^{'}, x_k^{'}) \in B_\delta(x_j, x_k)} \{x_k + \delta \leq x_i^* < 1 \mid u_i(x_i^*, x_j^{'}, x_k^{'}) \geq u_i(x_i, x_j, x_i)\} \neq \emptyset$$

is closed in a subset included in $[x_k + \delta, 1[$. Then, we choose $\bar{x}_i := \max A_\delta$ as payoff secure strategy for the player i. Suppose that $(x_j, x_k) \in B$. In this case, there exists $\delta > 0$ such that

$$A_\delta = \bigcap_{(x_j^{'}, x_k^{'}) \in B_\delta(x_j, x_k)} \{0 < x_i^* \leq x_i - \delta \mid u_i(x_i^*, x_j^{'}, x_k^{'}) \geq u_i(x_i, x_j, x_i)\} \neq \emptyset$$

is closed in a subset included in $]0, x_i - \delta]$. Then, we choose $\bar{x}_i := \min A_\delta$ as payoff secure strategy for the player i. Suppose that $(x_j, x_k) \in D$. By left semicontinuity for the function $u_i(\cdot, x_j, x_k)$ at x_k, there exists $\bar{x}_i < x_k$ and $\bar{x}_i > x_j$ such that

$$u_i(\bar{x}_i, x_j, x_k) \geq u_i(x_i, x_j, x_i) \tag{35}$$

and, by continuity for $u_i(\bar{x}_i, \cdot, \cdot)$ at the point (x_j, x_k), there exists a neighborhood U of the point (x_j, x_k) such that $U \cap E_{\bar{x}_i} = \emptyset$ and

$$u_i(\bar{x}_i, x_j^{'}, x_k^{'}) > u_i(\bar{x}_i, x_j, x_k) - \epsilon \quad \forall (x_j^{'}, x_k^{'}) \in U. \tag{36}$$

Then, by (35) and (36), we have that

$$u_i(\bar{x}_i, x_j^{'}, x_k^{'}) > u_i(x_i, x_j, x_i) - \epsilon \quad \forall (x_j^{'}, x_k^{'}) \in U.$$

As regards the cases $\{x_j = x_i\} \times \{x_k < x_i\}$ and $\{x_j > x_i\} \times \{x_i = x_k\}$, we fall in the previous two cases since $u_i(x_i, \cdot, \cdot)$ is symmetric [10]. As regards to the last subset F in (28), let U a neighborhood of (x_j, x_j) in $[0, 1]$ and $\bar{x}_j = \inf Pr_{x_j}(U \cap \{x_j^* \leq x_k^*\})$, $\bar{x}_k =$

[10] i.e $u_i(x_i, x_j, x_k) = u_i(x_i, x_k, x_j)$ holds for all x_i, x_j, x_k.

$\sup Pr_{x_k}(U \cap \{x_j^* \leq x_k^*\})$ such that $\frac{1}{3} < \bar{x}_j$ or $\frac{2}{3} > \bar{x}_k$ holds. In the first case, by choosing $\bar{x}_i$ such that $\frac{2}{3} - \bar{x}_j < \bar{x}_i < \bar{x}_j$, we have

$$u_i(\bar{x}_i, x_j^{'}, x_k^{'}) > u_i(\bar{x}_i, \bar{x}_j, \bar{x}_k) - \epsilon = \frac{\bar{x}_i + \bar{x}_j}{2} - \epsilon >$$

$$\frac{1}{3} - \epsilon = u_i(x_i, x_i, x_i) - \epsilon \quad \forall (x_j^{'}, x_k^{'}) \in U \cap \{x_j^* \leq x_k^*\}.$$

In the second case, by choosing $\bar{x}_i$ such that $\frac{4}{3} - \bar{x}_k > \bar{x}_i > \bar{x}_k$, we have

$$u_i(\bar{x}_i, x_j^{'}, x_k^{'}) > u_i(\bar{x}_i, \bar{x}_j, \bar{x}_k) - \epsilon = 1 - \frac{\bar{x}_i + \bar{x}_j}{2} - \epsilon >$$

$$\frac{1}{3} - \epsilon = u_i(x_i, x_i, x_i) - \epsilon \quad \forall (x_j^{'}, x_k^{'}) \in U \cap \{x_j^* \leq x_k^*\}$$

By simmetry, we can obtain the same properties on $U \cap \{x_j^* \geq x_k^*\}$. □

Proof of Proposition 2.3.

Proof. It's easy to check that u_i is bounded and continuous except for the following regular subset

$$A_i^{**} \subset \left\{(x_i, x_j, x_k) \in [0,1]^3 \;\middle|\; x_j = I_{[0,1]}(x_i) \vee x_k = I_{[0,1]}(x_i)\right\}$$

Moreover, u_i is strictly weakly lower semicontinuous (see [16][Definition 6]) thanks to the inequalities (21),(22), (23) and (24) and $\sum_{i=1}^n u_i = 1$ is upper semicontinuous. In fact, all the discontinuities values for u_i at x_i are included in the convex hull generated by the $\limsup$ and $\liminf$ around the points x_j and x_k. By symmetrical evidence for the other indexes $s = j, k$ and by applying results in [4] [Theorem 5 pp.14; Lemma 7 pp.19], G_1 has a mixed symmetric Nash equilibrium. □

Proof of Proposition 4.2.

Proof. First of all, let $\phi(x, y) = u_1(x_1, y_2) + u_2(y_1, x_2)$ be the equilibrium bifunction

$$\phi(x,y) = \begin{cases} -x_1^2 + g(y_2) + \ln(1 + x_2)\ \sin(\pi\, x_2\, (1 + |y_1|)\ x_2 & |x_1| \neq 1 \\ \\ 1/4 + \ln(1 + x_2)\ \sin(\pi\, x_2\, (1 + |y_1|)\ x_2 & \text{otherwise.} \end{cases}$$

For sake of simplicity, we assume $y_1 > 0$. Now, we consider the following properties:

$$x_1\, \partial_{x_1}\phi(x_1, x_2, y_1, y_2) < 0 \quad \forall x_1 \neq -1, 0, +1,\, \forall x_2, y_1, y_2 \tag{37}$$

$$\partial_{x_1}\phi(0, x_2, y_1, y_2) = 0 \quad \forall x_2, y_1, y_2 \tag{38}$$

$$\phi(0, x_2, y_1, y_2) \geq \phi(\pm 1, x_2, y_1, y_2) \quad \forall y_2 \geq \frac{1}{4},\, \forall x_2, y_1 \tag{39}$$

$$\partial_{x_2}\phi(0, 0, y_1, y_2) = 0 \quad \forall y_1, y_2 \tag{40}$$

$$g(y_2) = \phi(0, 0, y_1, y_2) > (=)\phi(0, 1, y_1, y_2) \quad \forall y_1 \neq 0, 1, (\ y_1 = 0, 1),\ \forall y_2 \tag{41}$$

$$\phi(0, \cdot, y_1, y_2)\ \text{ is locally strictly increasing at the point } x_2 = 0 \quad \forall y_1, y_2 \tag{42}$$

$$\phi(0, \cdot, y_1, y_2)\, \text{is increasing on } \ ]x_2^{**}(y_1), 1[\quad \forall y_1 \in]m_{min}, 1], \forall y_2 \tag{43}$$

$${x_2}^{*}(y_1), {x_2}^{**}(y_1)\, \text{are strict decreasing, respectively, on } [0, 1]\text{and}[m_{min}, 1] \tag{44}$$

$$y_1 \in]m_{min}, 1] \rightarrow \phi(0, {x_2}^{*}(y_1), y_1, y_2)\ \text{ is strict decreasing} \qquad \forall y_2 \in [0, 1] \tag{45}$$

$$y_1 \in]m_{min}, 1] \rightarrow \phi(0, {x_2}^{**}(y_1), y_1, y_2)\ \text{ is strict increasing} \quad \forall y_2 \in [0, 1] \tag{46}$$

$$0 < \max_{y_1 \in [0,1]} x_2^{*}(y_1) < \frac{4}{5} \leq \min_{y_1 \in]0,1]} x_2^{**}(y_1). \tag{47}$$

From now onwards, let $(y_1, y_2) \in [m_{min}, 1] \times [\frac{1}{4}, 1]$ be a fixed strategy. It's easy to prove that the equation

$$\phi(0, x_2, y_1, y_2) = c \in]g(y_2), \phi(0, x_2^{*}(y_1), y_1, y_2)[$$

has two solutions. The first one belonging to the subset $]0, x_2^{*}(y_1)[$ and the second one belonging to the subset $]x_2^{*}(y_1), \bar{x}(y_1, y_2)[$; with $\bar{x}(y_1, y_2) \in [0, \frac{4}{5}[$ the zeros of $\phi(0, x_2, y_1, y_2) = g(y_2)$ for every $y_1 \in [m_{min}, 1]$. At the same way, the following equation

$$\phi(0, x_2, y_1, y_2) = c \in]\phi(0, x_2^{**}(y_1), y_1, y_2), \phi(0, 1, y_1, y_2)]$$

has two solutions; the first belonging to the subset $]\bar{x}(y_1, y_2), x_2^{**}(y_1)[$ and the second one belonging to the subset $]x_2^{**}(y_1), 1]$. Besides, we denote by $n_{sup} = \lim_n x_{2,n}$ such that $\lim_n g(x_{2,n}) = \sup_{x_2 \geq \frac{1}{4}} g(x_2)$. We observe that

$$A_{k,y} - \{|x_1| = 1\} = \left\{ x \in\ X \ \middle|\, |x_1| \leq \sqrt{g(y_2) + \ln(1 + x_2)\ \sin\left(\pi\, x_2\, (1 + y_1)\right)\ x_2 - k} \right\}.$$

Clearly, if the quantity under the square root's sign is strict negative, the previous subset is empty. First of all, we can prove that $Pr_{x_2}(A_{k,y})$ is an mc subset.

i) If, by applying Ky Fan's Min-Max [7][Theorem 1] and by (46),

$$\inf_{x_1\in[-1,+1]} \min_{x_2\in[0,+1]} \min_{y_1\geq m_{min}} \min_{y_2\geq\frac{1}{4}} \phi(x_1,x_2,y_1,y_2) < k \leq$$

$$\sup_{x_1\in[-1,+1]} \min_{x_2\in[0,+1]} \min_{y_1\geq m_{min}} \min_{y_2\geq\frac{1}{4}} \phi(x_1,x_2,y_1,y_2) =$$

$$\min_{x_2\in[0,+1]} \min_{y_1\geq m_{min}} \phi(0,x_2,y_1,n_{min}) = \phi(0,1,m_{min},n_{min})$$

and, by (46), trivially

$$k \leq \phi(0,1,m_{min},n_{min}) \leq \phi(0,x_2^{**}(y_1),y_1,n_{min}) \leq \phi(0,x_2,y_1,n_{min}) \leq$$

$$g(n_{min}) + \ln(1+x_2)\ \sin\left(\pi\, x_2\,(1+y_1)\right)\ x_2 \quad \forall x_2\in[0,1], y_1\in[m_{min},1] \tag{48}$$

Therefore, by the inequality (48), $Pr_{x_2}\left(A_{k,y}\right) = [0,1]$.

ii) By (45), if

$$\phi(0,1,m_{min},n_{min}) < k < \sup_{x_1\in[-1,+1]} \max_{x_2\in[0,+1]} \max_{y_1\geq m_{min}} \sup_{y_2\geq\frac{1}{4}} \phi(x_1,x_2,y_1,y_2) =$$

$$\max_{x_2\in[0,+1]} \max_{y_1\geq m_{min}} \phi(0,x_2,y_1,n_{sup}) = \phi(0,{x_2}^{*}(m_{min}),m_{min},n_{sup})$$

we have other three subcases:

iia) Suppose $k\in\mathbb{R}$ such that

$$\phi(0,1,m_{min},n_{min}) < k \leq \phi(0,x_2^{**}(y_1),y_1,y_2).$$

In this case, $Pr_{x_2}(A_{k,y}) = [0,1]$ is an mc subset.

iib) Suppose $k\in\mathbb{R}$ such that

$$\phi(0,{x_2}^{**}(y_1),y_1,y_2) < k \leq \phi(0,1,y_1,y_2) \tag{49}$$

We denote by

$$co_{\frac{4}{5},1,k} = \left\{x_2 \,\middle|\, x_2(m_x) = (1-m_x)\frac{4}{5} + m_x,\ \ \phi(0,x_2(m_x),y_1,y_2) \geq k,\ \ 0\leq m_x\leq 1\right\}.$$

By inequality (49), we state that $\emptyset \neq co_{\frac{4}{5},1,k} \neq co_{\frac{4}{5},1}$. Let $m^{**} := 5\,x_2^{**}(y_1) - 4$ be a positive number. By choosing $x_2 \in co_{\frac{4}{5},1,k}$ and $x_2 < {x_2}^{**}(y_1)$, we can write

$x_2 = x_2(m_x)$ with $m_x \in [0, m^{**}[$. Suppose that $y_1 \geq m^{**}$. Since ${x_2}^*$ is decreasing and continuous, we can imply

$$ {x_2}^*(y_1) \leq {x_2}^*(m^{**}) \leq {x_2}^*(0) < {x_2}^{**}(y_1) $$

by (47); and, then

$$ {x_2}^*(m_x) \subset [{x_2}^*(y_1), {x_2}^{**}(y_1)[\quad \forall m_x \in [0, m^{**}[. $$

Since $\phi(0, \cdot, y_1, y_2)$ is strict decreasing on the subset $[{x_2}^*(y_1), {x_2}^{**}(y_1)]$, we have

$$ \phi(0, x_2^*(m_x), y_1, y_2) \geq \phi(0, {x_2}^*(0), y_1, y_2) = \phi(0, 0.71, y_1, y_2) > $$

$$ \phi(0, \frac{4}{5}, y_1, y_2) \geq \phi(0, x_2(m_x), y_1, y_2) \geq k \quad \forall m_x \in [0, m^{**}[. \tag{50} $$

On the contrary, suppose that $y_1 < m^{**}$. Let $m_x \in]y_1, m^{**}[$. It's clear that ${x_2}^*(y_1) > {x_2}^*(m_x)$; but there exists $x_{2s} \in]{x_2}^*(y_1), \bar{x}(y_1, y_2)[$ such that

$$ g(y_2) < \phi(0, {x_2}^*(m_x), y_1, y_2) = \phi(0, x_{2s}, y_1, y_2) < \phi(0, x_2^*(y_1), y_1, y_2). $$

Then, by remarking that $x_2^{**}(y_1) > x_{2s} > \bar{x}(y_1, y_2)$, we have

$$ \phi(0, {x_2}^*(m_x), y_1, y_2) = \phi(0, x_{2s}, y_1, y_2) > \phi(0, \bar{x}(y_1, y_2), y_1, y_2) > $$

$$ \phi(0, \frac{4}{5}, y_1, y_2) \geq \phi(0, x_2(m_x), y_1, y_2) \geq k \quad \forall m_x \in]y_1, m^{**}[. \tag{51} $$

Besides, as in the case $y_1 \geq m^{**}$, we have that

$$ \phi(0, {x_2}^*(m_x), y_1, y_2) \geq \phi(0, x_2(m_x), y_1, y_2) \geq k \quad \forall m_x \in [0, y_1]. \tag{52} $$

holds. Now, we choose $x_2 \in co_{\frac{4}{5},1,k}$ and $x_2 > {x_2}^{**}(y_1)$. We can write $x_2 = x_2(m_x)$ with $m_x \in]m^{**}, 1]$. We assume $m^{**} \geq y_1$. We obtain

$$ 0 \neq {x_2}^*(1) \leq {x_2}^*(m_x) \leq {x_2}^*(y_1). $$

Since the function $\phi(0, \cdot, y_1, y_2)$ is increasing on the subset $[0, {x_2}^*(y_1)]$, we obtain, by (41) and (43),

$$ \phi(0, {x_2}^*(m_x), y_1, y_2) > \phi(0, 0, y_1, y_2) = g(y_2) \geq \phi(0, 1, y_1, y_2) = $$

$$\max_{m_s \in [m^{**},1]} \phi(0, x_2(m_s), y_1, y_2) \geq \phi(0, x_2(m_x), y_1, y_2) \geq k \tag{53}$$

On the contrary, we assume $m^{**} < y_1$. In this case, we prove that

$$\phi(0, {x_2}^*(m), y_1, y_2) \geq \phi(0, x_2(m), y_1, y_2) \quad \forall m \in]m^{**}, y_1]. \tag{54}$$

By absurd, we suppose that there exists $m \in]m^{**}, y_1]$ such that

$$\phi(0, {x_2}^{**}(y_1), y_1, y_2) \leq \phi(0, {x_2}^*(m), y_1, y_2) < \phi(0, x_2(m), y_1, y_2) \leq \phi(0, 1, y_1, y_2).$$

Then, there exists $m_1 \in]m^{**}, m[$ such that

$$\phi(0, {x_2}^*(m), y_1, y_2) = \phi(0, x_2(m_1), y_1, y_2)$$

and, then

$$\phi(0, {x_2}^*(m_1), y_1, y_2) < \phi(0, {x_2}^*(m), y_1, y_2) = \phi(0, x_2(m_1), y_1, y_2)$$

and, again, by choosing m_1 instead of m, there exists $m_2 \in]m^{**}, m_1[$ such that

$$\phi(0, {x_2}^*(m_2), y_1, y_2) < \phi(0, {x_2}^*(m_1), y_1, y_2) = \phi(0, x_2(m_2), y_1, y_2).$$

By keeping this way on, we can construct a sequence of points $m_{k+1} \in]m^{**}, m_k[$ such that

$$\begin{aligned} & y_1 \geq m = m_0 > m_1 > \dots m_{k-1} > m_k > m_{k+1} \overset{k \to +\infty}{\longrightarrow} m^{**} \\ & {x_2}^*(m^{**}) > {x_2}^*(m_{k+1}) > {x_2}^*(m_k) > {x_2}^*(m_{k-1}) \cdots > {x_2}^*(m_1) > {x_2}^*(m_0) \\ & \phi(0, {x_2}^*(m_{k+1}), y_1, y_2) < \phi(0, {x_2}^*(m_k), y_1, y_2) = \phi(0, x_2(m_{k+1}), y_1, y_2) \end{aligned} \tag{55}$$

and, passing the above inequalities to the limit,

$$\phi(0, x_2(m^{**}), y_1, y_2) = \lim_{k \to \infty} \phi(0, x_2(m_k), y_1, y_2) = \lim_{k \to \infty} \phi(0, {x_2}^*(m_k), y_1, y_2) =$$

$$\phi(0, {x_2}^*(m^{**}), y_1, y_2) = \lim_{k \to \infty} \phi(0, {x_2}^*(m_{k+1}), y_1, y_2) \leq \phi(0, x_2(m^{**}), y_1, y_2)$$

we have

$$\phi(0, {x_2}^*(m^{**}), y_1, y_2) = \phi(0, x_2(m^{**}), y_1, y_2) = \phi(0, {x_2}^{**}(y_1), y_1, y_2).$$

But, this implies that ${x_2}^*(m^{**}) = {x_2}^{**}(y_1)$ which contradicts (47). Then, the equality (54) holds.

If $m > y_1$, then $x_2^*(m) < x_2^*(y_1)$. Therefore, there exists $x_{2m} \in]{x_2}^*(y_1), \bar{x}(y_1, y_2)[$ such that

$$g(y_2) < \phi(0, {x_2}^*(m), y_1, y_2) = \phi(0, x_{2m}, y_1, y_2) < \phi(0, x_2^*(y_1), y_1, y_2).$$

By absurd, we can construct such sequences as before by choosing as the initial point $m_0 = m$. At first step, if $m > m_1 > y_1$ then $x_2^*(m) < x_2^*(m_1) < x_2^*(y_1)$. Therefore

$$\phi(0, {x_2}^*(m_1), y_1, y_2) > \phi(0, {x_2}^*(m), y_1, y_2)$$

holds since $\phi(0, \cdot, y_1, y_2)$ is strictly increasing on $[0, x_2^*(y_1)]$. By (55), this is an absurd. If $m_1 < y_1$, we can proceed as in the (54)'s proof. Finally,

$$\phi(0, {x_2}^*(m), y_1, y_2) \geq \phi(0, x_2(m), y_1, y_2) \quad \forall m \in]y_1, 1]. \tag{56}$$

holds.

Now, we can conclude $co_{\frac{4}{5},1,k}$ is an mc subset, since the inequalities (50), (51), (52), (53),(54) and (56) hold. We denote by

$$co_{0,\frac{4}{5},k} = \left\{ x_2 \in \left[0, \frac{4}{5}\right] \mid \phi(0, x_2, y_1, y_2) \geq k \right\}$$

The latter one is a convex subset included in $[0, \frac{4}{5}[$ since the function $\phi(0, \cdot, y_1, y_2)$ is quasi concave on $[0, \frac{4}{5}]$. Then, it is an mc subset. By gathering these two mc subsets, on the real line, we obtain that $Pr_{x_2}(A_{k,y}) = co_{0,\frac{4}{5},k} \cup co_{\frac{4}{5},1,k}$ is an mc subset thanks to the definition of mc-structure given at page 15.

iic) Suppose $k \in \mathbb{R}$ such that $\phi(0, 1, y_1, y_2) < k < \phi(0, {x_2}^*(m_{min}), m_{min}, n_{sup})$. In this case, $co_{\frac{4}{5},1,k} \subset \left[\frac{4}{5}, {x_2}^{**}(y_1)\right[$. Then, we can proceed as in the case $x_2(m_x) < {x_2}^{**}(y_1)$ in **iib)**.

By (37), (39) and by the definition of multiconnection structure, $A_{k,y}$ is an mc subset. $\square$

Proof of Proposition 4.3.

Proof. First of all, we prove that the bifunction ϕ is diagonally transfer continuous in y. In fact, the diagonalized bifunction $\phi(y,y)$ is upper semicontinuous on the strip regions $(\{\pm1\}\times[0,1])^2$ since

$$\lim_{y_1\to\pm1}\phi(y_1,y_2,y_1,y_2)-\phi(1,y_2,1,y_2)=g(y_2)-\frac{5}{4}<0$$

holds. Therefore, if $y\in\{z\in[-1,1]\times[0,1]\mid |z_1|=1\}$, we have

$$\limsup_{y'\to y}\phi(y',y')\le\phi(y,y)<\phi((1,x_2^*(1)),y)=\lim_{y'\to y}\phi((0,x_2^*(1)),y').$$

If $y\in\{z\in[-1,1]\times[0,1]\mid |z_1|\neq 1\}\setminus\{(0,x_2^*(0))\}$, we have

$$0<\phi(y,y)-\phi((0,x_2^*(0)),y)=\lim_{y'\to y}\phi(y',y')-\phi((0,x_2^*(0)),y').$$

Moreoveor, there exists no points $x\in X$ such that

$$\phi((x,(0,x_2^*(0)))>\phi((0,x_2^*(0)),(0,x_2^*(0)))$$

$$\phi((x,(1,x_2^*(1)))>\phi((1,x_2^*(1)),(1,x_2^*(1))).$$

Let $x,x',\,y\in X$. We introduce the following multifunction $\xi\,:\lambda\in[0,1]\to 2^X$

$$\xi(\lambda)=\begin{cases}\overbrace{A_{k,y}\setminus\{x\in X\ ||x_1|=1\}}^{\circ} & A_{k,y}\neq\{(0,x_2^*(y_1))\}\\ \{(0,x_2^*(y_1))\} & \text{otherwise}\end{cases}$$

where $k=\lambda\phi(x,y)+(1-\lambda)\phi(x',y)$ and the subsets $A_{k,y}$ are the k-uplevels for $\phi(\cdot,y)$. By Proposition 4.2, ξ is a nonempty open inverse image multifunction with multiconnected values for suitable y's values. Besides, there exists a point $\bar{P}\equiv(1,n_{min})\in[m_{min},1]\times\left[\frac{1}{4},1\right]$ such that

$$\phi(\bar{P},\bar{P})=\max_{A\in F(X)}\max_{x\in co_{mc}(A)}\phi(x,\bar{P})=\max_{x\in\{0\}\times[0,\frac{4}{5}]}\phi(x,\bar{P}).$$

by using the property (37). Therefore, by Main Theorem 3.1, the thesis is given. □

We introduce this simple lemma.

Lemma 5.1. *Let* $f:[0,a]\to\mathbb{R}^+$ *be twice continuous differentiable on the interval* $[0,a[$ *and continuous on* $[0,a]$*. Suppose that*

i) $f^{'}(0) = 0$;

ii) *f is strict increasing on* $]0, \min \arg \max_{x \in [0,a]} f(x)] \neq \emptyset$*;*

iii) $\min \arg \max_{x \in [0,a]} f(x) \neq a$.

Then

$$\{(x_2, x_1) \in [0,a] \times R^+ \mid x_2 \in [0,a], \sqrt{f(x_2)} \geq x_1\}$$

is not a convex subset.

Remark 5.1. *In the case* ***i)*** *of the proof of Theorem 4.2, let a k such that*

$$\lim_{x_1 \to 1} \phi(x_1, x_2^*(y_1), y_1, y_2) < k. \tag{57}$$

for some (y_1, y_2)'s values [11]*. Then, we can define the family of functions*

$$x_2 \in [0,1] \to \phi(0, x_2, y_1, y_2) - k \in [0,1[.$$

Every function of this family satisfies ii) and iii) of Lemma 5.1 by (47) and (42); the condition i) of the same Lemma by (40). Therefore, by a simple symmetric argument, $A_{k,y} \setminus \{x \in X \mid |x_1| = 1\}$ is not a convex subset. Moreover, in the case ***ii)*** *of the proof of Theorem 4.2, some $A_{k,y} \setminus \{x \in X \mid |x_1| = 1\}$ are not connected subsets for suitable k's values.*

[11]This subset is not empty. For example, (m_{min}, n_{min}) satisfies the inequality (57) since $\phi(0, 1, m_{min}, n_{min}) > \phi(1, x_2^*(m_{min}), m_{min}, n_{min})$ holds.

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