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Some Examples and Counterexamples about Continuity on Equilibrium Problems *

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Abstract

This paper attempts to underline how the Diagonal Transfer Continuity hypothesis (Baye, Tian and Zhou, 1993) and Better-Reply Security (Reny, 1999) are unconnected between themselves as sufficient conditions for stating the existence of Nash equilibria. Besides, various examples and counterexamples regarding Nash equilibria existence Theorem (Baye, Tian and Zhou, 1993) and extensions of maximum existence results for bifunctions established for a function of one variable (Baye and and Zhou,1995). We present, also, a sufficient conditions for Diagonal Transfer Continuity. Moreover, an example of quasi-concave game having multiple Nash equilibria, in which the aforesaid hypotheses and other improvements (Lignola, 1997) fall, is presented.

Keywords: Nash Equilibria Existence; Generalized Convexity; 2 Person Game; Generalized Continuity; Diagonal Transfer Quasi Concavity; Diagonal Transfer Continuity.

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1 Introduction

In the Sections 2 and 3, we study the relationships between the *Diagonal Transfer (Upper Semi) Continuity* introduced in [1], [2], and *Better-Reply Security* introduced in [5]. Both of them represent the main attempts to relax the continuity hypothesis on Nash Equilibria Theorems.

We introduce some notations and definitions. Let $G = (X_i, u_i)_{i=1,\ldots n}$ a maximum game with $X_i \subset \mathbb{R}^{h_i}$ the individual strategy space and $X = \prod_{i=1}^{n} X_i \subset \mathbb{R}^{h\left(=\sum_{i=1}^{n} h_i\right)}$ the whole strategy space. Let U a neighborhood of a point $x_0 \in \mathbb{R}$, we denote by $U^- := \{x \in U \mid x < x_0\}$ and $U^+ := \{x \in U \mid x > x_0\}$. Let $A \subset \mathbb{R}$, we denote by χ_A the characteristic function of the subset A which assumes value 1 if $x \in A$ or 0 if $x \notin A$. For sake of simplicity, we denote by $\chi_{\{l\}} = \chi_l$ where $l \in \mathbb{R}$; and let $a, b \in \mathbb{R}$ we denote $J(a,b) =]\min\{a,b\}, \max\{a,b\}[$. Let $x \in X$ be a multistrategy, we denote $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{i+1}) \in X_{-i} = \prod_{i \in I \setminus \{i\}} X_i$. Moreover, let be $BR_i : X_{-i} \longrightarrow X_i$

$$BR_i(x_{-i}) = \arg \max_{x_{-i} \in X_i} u_i(x_i, x_{-i})$$

the *Best Reply multifunction* for the the player i.

Moreover, the function $\phi : (x,y) \in X \times X \to \sum_{i=1}^{n} u_i(x_i, y_{-i})$ is the equilibrium bifunction for the game G. Such a function ϕ is *diagonal transfer continuous on $A \subseteq X$ in $y \in Z \subseteq X$* if, by assuming that for every point $(x,y) \in A \times Z$ such that $\phi(x,y) > \phi(y,y)$, there exists $\bar{x} \in A$ and $U \subset Z$ a neighborhood of y in Z such that $\phi(\bar{x}, y') > \phi(y', y')$ for all $y' \in U$. We shall simply say that ϕ is diagonal transfer continuous in y when $A = X$ and $Z = X$.

For the following definition, we claim that $X \subset \mathbb{R}^h$ and $C \subseteq X$ are convex subsets. Therefore, $\phi(x,y)$ is *diagonal transfer quasi concave in x on $A \subseteq X$* for any finite subset $X^m = \{x^1, \ldots, x^m\} \subset A$ there exists a corresponding finite subset $Y^m = \{y^1, \ldots, y^m\} \subset C$ such that for any finite subset $\left\{y^{k^1}, y^{k^2}, \ldots, y^{k^s}\right\}$, $1 \le s \le m$ and any $y^{k^0} \in \mathrm{co}\left\{y^{k^1}, y^{k^2}, \ldots, y^{k^s}\right\}$ we have

$$\min_{1 \le l \le s} \phi(y^{k^l}, y^{k^0}) \le \phi(y^{k^0}, y^{k^0}).$$

We will simply say ϕ diagonally transfer quasi concave in x when $A = X$ and $C = X$.

In terms of individual payoffs, we remark these definitions. For the following definition, we claim that $X_i \subset \mathbb{R}^{h_i}$ is a convex subset. A payoff u_i is said to be *uniformly transfer quasi concave on* X if, for any finite subset $X^m = \{x^1, \ldots, x^m\} \subset X$ there exists a corresponding finite subset $Y_i^m = \{y_i^1, \ldots, y_i^m\} \subset C$ such that for any finite subset $\left\{y_i^{k^1}, y_i^{k^2}, \ldots, y_i^{k^s}\right\}$, $1 \leq s \leq m$ and any $y_i^{k^0} \in \text{co}\left\{y_i^{k^1}, y_i^{k^2}, \ldots, y_i^{k^s}\right\}$ we have

$$\min_{1\leq l\leq s}\left[u_i(x_i^{k^l}, x_{-i}^{k^l}) - u_i(y_i^{k^0}, x_{-i}^{k^l})\right] \leq 0.$$

A payoff u_i is said to be *uniformly quasi concave on* X if $y_i^j = x_i^j$ for all $j = 1, \ldots m$ and $i = 1, \ldots n$. A payoff u_i is said to be *transfer upper semicontinuous in* x_i if, for every $y_i \in X_i$ and $x \in X$, $u_i(x_i, x_{-i}) > u_i(y_i, x_{-i})$ implies that there exists a a point $\bar{x} \in X$ and a neighborhood U of y_i such that $u_i(\bar{x}) > u_i(y_i^{'}, \bar{x}_{-i})$ for all $y_i^{'} \in U$. A function $f : X \to \mathbb{R}$ is said to be *transfer (weakly) upper continuous on* X if for points x, $y \in X$, $f(y) < f(x)$ implies that there exists a point $x^{'} \in X$ and a neighborhood U of y such that $f(z) \leq (<) f(x^{'})$ for all U. A game G is Better-Reply Secure if whenever (x^*, u^*) is in the closure of the graph of its vector payoff function and x^* is not an equilibrium and other players deviate slightly from x^*_{-i}, some player i can secure a payoff strictly above u_i^* at x^* [5][pp.1033]. His hypothesis generalized the *Complementary Discontinuities* (*Reciprocally Upper Semicontinuity*) assumption introduced by Simon in [7]; and the *Payoff Security* introduced by himself in [8]. In particular, payoff security requires that for every strategy $x \in X$, each player has a strategy $\bar{x}_i \in X_i$ that, virtually, guarantees the payoff he receives at x even if the others deviate slightly from x [5][pp. 1032]. In mathematical words, for every strategy $x \in X$ and $\epsilon > 0$, there exists $\bar{x}_i \in X_i$ such that $u_i(\bar{x}_i, y_{-i}) > u_i(x) - \epsilon$ for all y_{-i} in a neighborhood of x_{-i} and for all $i = 1, \ldots, n$. Reciprocal upper semicontinuity requires that some players payoff jumps up whenever some other players payoff jumps down [5][pp. 1034]. In mathematical words, if whenever (x, u) is in the closure of the graph of its vector payoff function and $u_i(x) \leq u_i$ for every player i, then $u_i(x) = u_i$ for every player i.

In the Section 4, an example of quasi-concave game in which the Diagonal Transfer Continuity's and Better-Reply Security's hypotheses and Lignola's ones [6] [Th.3.1] fail, notwithstanding such a game has a countable Nash equilibria set.

2 Does Diagonal Transfer Continuity Hypothesis imply Better-Reply Security one?

In this section, a diagonally transfer quasi concave game, in which the diagonal transfer continuity hypothesis holds while better-reply security one fail, is introduced. At that aim, we present the following Proposition.

Proposition 2.1. *Let $A \subseteq X$ be an open subset in X and $\phi : X \times X \longrightarrow \mathbb{R}$. Let $\phi\,|_{A\times A}$ be an upper semicontinuous function and $\phi\,|_A\,(x,\cdot)$ be lower semicontinuous for all $x \in A$. Then, ϕ is diagonally transfer continuous on X in $y \in A$.*

Proof. Suppose that there exists a point $(x, y) \in X \times A$ such that

$$\phi(x, y) > \phi(y, y).$$

By lower semicontinuity, there exists a neighborhood $U_{1,y} \subseteq A$ of y in A such that

$$\phi(y, y) \geq \phi(z, z) \quad \forall\, z \in U_{1,y}$$

and, by upper semicontinuity, there exists a neighborhood $U_{2,y} \subseteq A$ of y in A such that

$$\phi(x, z) \geq \phi(x, y) \quad \forall\, z \in U_{2,y}$$

and, by gathering all the previous equations, we obtain

$$\phi(x, z) \geq \phi(x, y) > \phi(y, y) \geq \phi(z, z) \quad \forall\, z \in U_{1,y} \cap U_{2,y} \subseteq A.$$

□

Now, let $G_1 = ([-1, 1], [-1, 1], u_1, u_2)$ whose payoffs are defined as follows:

$$u_1(x_1, x_2) = \begin{cases} -x_1^2 + 1 & x_1 \neq 0 \\ \\ 1 + \epsilon + \chi_{]-\epsilon,0[\cup]0,+\epsilon[}(x_2)\, f(x_2) & x_1 = 0 \end{cases}$$

$$u_2(x_1, x_2) = \begin{cases} -x_2^2 + 1 + \chi_{[-1,0]\cup[\epsilon,1]}(x_1)\, g_\epsilon(x_2) & x_2 \neq 0 \\ \\ 1 - \epsilon & x_2 = 0 \end{cases}$$

where $g : [-1, +1] \setminus \{0\} \to \mathbb{R}^+$ is the following function

$$g_\epsilon(x_2) = \begin{cases} \epsilon \sqrt{{x_2}^3} & 1 \geq x_2 > 0 \\ \\ 0 & -1 \leq x_2 < 0 \end{cases}$$

and $f :]-\epsilon, \epsilon[\setminus \{0\} \to \mathbb{R}$ an even continuous function satisfying the following properties:

i) $f \geq -\epsilon$;

ii) $\exists!\, x_2^* = \arg\max_{x_2>0}\{f(x_2) - x_2^2 + g_\epsilon(x_2)\} = \arg\max_{x_2>0}\{-x_2^2 + g_\epsilon(x_2)\} = \frac{9}{16}\epsilon^2$;

iii) f is positive on $]0, x_2^*]$ and, locally, at the points $x_2 = +\epsilon$.

Now, we construct the aggregate function for G_1 and its diagonalized version:

$$\phi(x_1,x_2,y_1,y_2)=\begin{cases} 2-x_1^2-x_2^2 & x_1\neq 0,\ x_2\neq 0,\ y_1\in]0,\epsilon[\\ 2-x_1^2-x_2^2+g_\epsilon(x_2) & x_1\neq 0,\ x_2\neq 0,\ y_1\notin]0,\epsilon[\\ 2+f(y_2) & x_1=0,\ x_2=0,\ y_2\in]-\epsilon,0[\cup]0,\epsilon[\\ 2 & x_1=0,\ x_2=0,\ y_2\notin]-\epsilon,0[\cup]0,\epsilon[\\ 2+\epsilon-x_2^2+f(y_2) & x_1=0,\ x_2\neq 0,\ y_1\in]0,\epsilon[,\ y_2\in]-\epsilon,0[\cup]0,\epsilon[\\ 2+\epsilon-x_2^2+g_\epsilon(x_2)+f(y_2) & x_1=0,\ x_2\neq 0,\ y_1\notin]0,\epsilon[,\ y_2\in]-\epsilon,0[\cup]0,\epsilon[\\ 2+\epsilon-x_2^2 & x_1=0,\ x_2\neq 0,\ y_1\in]0,\epsilon[,\ y_2\notin]-\epsilon,0[\cup]0,\epsilon[\\ 2+\epsilon-x_2^2+g_\epsilon(x_2) & x_1=0,\ x_2\neq 0,\ y_1\notin]0,\epsilon[,\ y_2\notin]-\epsilon,0[\cup]0,\epsilon[\\ 2-\epsilon-x_1^2 & x_1\neq 0,\ x_2=0 \end{cases} \tag{1}$$

$$\phi(y_1,y_2,y_1,y_2)=\begin{cases} 2-y_1^2-y_2^2+g_\epsilon(y_2) & -1\leq y_1<0,\ \epsilon\leq y_1\leq 1,\ y_2\neq 0 \\ 2-y_1^2-y_2^2 & 0<y_1<\epsilon,\ y_2\neq 0 \\ 2 & y_1=0,\ y_2=0 \\ 2+\epsilon-y_2^2+g_\epsilon(y_2) & y_1=0,\ |y_2|\geq\epsilon \\ 2-y_2^2+\epsilon+g_\epsilon(y_2)+f(y_2) & y_1=0,\ y_2\neq 0,\ |y_2|<\epsilon \\ 2-\epsilon-y_1^2 & y_1\neq 0,\ y_2=0 \end{cases} \tag{2}$$

Now, we prove the following proposition.

Proposition 2.2. *The function $\phi(x,y)$ is diagonally transfer continuous in y.*

Proof. By properties i) and iii), the following inequalities

$$\limsup_{y_1 \to 0} \phi(y_1, y_2, y_1, y_2) \leq \phi(0, y_2, 0, y_2) \quad \forall y_2 \in [-1, 1] \tag{3}$$

$$\limsup_{y_1 \to \epsilon} \phi(y_1, y_2, y_1, y_2) = \phi(\epsilon, y_2, \epsilon, y_2) \quad \forall y_2 \in [-1, 1] \tag{4}$$

$$\lim_{y_2 \to \pm\epsilon} \phi(y_1, y_2, y_1, y_2) = \phi(y_1, \pm\epsilon, y_1, \pm\epsilon) \quad \forall y_1 \in [-1, 1] \setminus \{0\} \tag{5}$$

$$\liminf_{y_2 \to \pm\epsilon} \phi(x_1, x_2, y_1, y_2) \geq \phi(x_1, x_2, y_1, \pm\epsilon) \quad \forall y_1 \in [-1, 1] \setminus \{0, \epsilon\}\,,\ \forall (x_1, x_2) \in X \tag{6}$$

hold trivially. By Proposition 2.1 and (5), (6), ϕ is diagonally transfer continuous on X in $y \in [-1, 1] \setminus \{0, \epsilon\} \times [-1, 1] \setminus \{0\}$ [1]. By assuming that for every point

$$(x_1, x_2, y_1, y_2) \in X \times (\{0\} \times [-1, +1] \cup \{\epsilon\} \times [-1, +1] \cup [-1, +1] \times \{0\})$$

such that $\phi(x_1, x_2, y_1, y_2) > \phi(y_1, y_2, y_1, y_2)$, [2] we can show that there exists a point $(\bar{x}_1, \bar{x}_2) \in X$ and $U \subset X$ a neighborhood of (y_1, y_2) such that $\phi(\bar{x}_1, \bar{x}_2, y_1^{'}, y_2^{'}) > \phi(y_1^{'}, y_2^{'}, y_1^{'}, y_2^{'})$ for all $(y_1^{'}, y_2^{'}) \in U$. For sake of simplicity, we divide our analysis into four subcases:

Area 1. $A_1 = \{y_1 \in [-1, +1] \mid y_1 = 0\} \times \{y_2 \in [-1, +1] \mid y_2 \neq -\epsilon, 0, x_2^*, \epsilon\}$;

Area 2. $A_2 = \{y_1 \in [-1, +1] \mid y_1 \neq 0, \epsilon\} \times \{y_2 \in [-1, +1] \mid y_2 = 0\}$;

Area 3. $A_3 = \{y_1 \in [-1, +1] \mid y_1 = \epsilon\} \times [0, 1]$;

Area 4. $A_4 = \{(0, \epsilon), (0, -\epsilon), (0, 0)\}$.

We put in **Area 1**.

We suppose that $y_2 \in]-\epsilon, 0[\cup]0, +\epsilon[\setminus \{x_2^*\}$. By i) and ii), there exists an open neighborhood $U_{1,0} \subset]-\epsilon, \epsilon[$ of 0, $V_{2,y_2} \subset]-\epsilon, 0[\cup]0, +\epsilon[\setminus\{x_2^*\}$ of y_2, such that

$$\phi(0, x_2^*, y_1{'}, y_2{'}) = 2 + \epsilon - x_2^{*2} + g_\epsilon(x_2^*) + f(y_2{'}) >$$

$$> 2 + \epsilon\, \chi_0(y_1{'}) - y_2{'}^2 - y_1{'}^2 + g_\epsilon(y_2{'}) + \chi_0(y_1{'}) f(y_2{'}) = \phi(y_1{'}, y_2{'}, y_1{'}, y_2{'})$$

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[1] This subset is open in X.

[2] Note that $\arg\max_{(y_1,y_2)\in X} \phi(y_1, y_2, y_1, y_2) = (0, x_2^*)$; and $\arg\max_{(x_1,x_2)\in X} \phi(x_1, x_2, 0, x_2^*) = (0, x_2^*)$. Therefore, the diagonal transfer continuity in $(0, x_2^*) \in X$ is satisfied. It needs to check it for the other points belonging to the square $[-1, 1]^2$.

$$\forall\, (y_1^{'}, y_2^{'}) \in\, U_{1,0}^{-} \cup \{0\} \times V_{2,y_2} \tag{7}$$

and, by i) and iii), such that

$$\phi(0, x_2^{*}, y_1^{'}, y_2^{'}) = 2 + \epsilon - x_2^{*2} + f(y_2^{'}) > 2 - y_2^{'\,2} = \lim_{y_1^{'} \to 0^{+}} \phi(y_1^{'}, y_2^{'}, y_1^{'}, y_2^{'})$$

$$\forall\, (y_1^{'}, y_2^{'}) \in U_{1,0}^{+} \times V_{2,y_2}. \tag{8}$$

We suppose that $|y_2| > \epsilon$. By (3) and ϵ sufficiently small, there exists a suitable neighborhood $V_{2,y_2} \subset \{y_2 \mid\mid y_2 \mid > \epsilon\}$ of y_2 such that

$$\inf_{y_1^{'} \in U_{1,0}} \phi(0, {x_2}^{*}, y_1^{'}, y_2^{'}) = 2 + \epsilon - \frac{81}{256}\,\epsilon^4$$

$$2 + \epsilon - \epsilon^2 + \sqrt[2]{\epsilon^5} > \phi(0, {y_2}^{'}, 0, y_2^{'}) \geq \limsup_{y_1^{'} \to 0} \phi(y_1^{'}, y_2^{'}, y_1^{'}, y_2^{'}) \quad \forall\, y_2^{'} \in {V_{2,y_2}}^{3} \tag{9}$$

By (7), (8) and (9), ϕ is diagonally transfer continuous on X in $(y_1, y_2) \in A_1 \subset X$.

We put in **Area 2**.

Since the property iii) holds and by choosing $(\bar{x}_1, \bar{x}_2) = (0, 0)$, there exists a neighborhood $V_{2,0} \subset]-\epsilon, \epsilon[$ of 0 such that

$$\phi(0, 0, y_1^{'}, y_2^{'}) = 2 + f(y_2^{'}) \geq 2 = \sup_{y_1 \in [0,1] \setminus \{0, \epsilon\}} {}^{4} \lim_{y_2 \to 0} \phi(y_1, y_2, y_1, y_2)$$

$$\forall\, (y_1^{'}, y_2^{'}) \in [0, 1] \setminus \{0, \epsilon\} \times V_{2,0} \setminus \{0\} \tag{10}$$

and

$$\inf_{y_1 \neq 0, \epsilon} \phi(0, 0, y_1, 0) = 2 > 2 - \epsilon = \sup_{y_1 \neq 0, \epsilon} \phi(y_1, 0, y_1, 0). \tag{11}$$

By (10) and (11), ϕ is diagonally transfer continuous on X in $(y_1, y_2) \in A_2 \subset X$.

We put in **Area 3**.

Since $\phi(\cdot, \cdot, \cdot, \cdot)$ is lower semicontinuous at $(\epsilon, 0, \epsilon, 0)$ and the properties (4) and (5) hold, there exists $U_{1,\epsilon}$ of ϵ in X and a suitable neighborhood V_{2,y_2} of y_2 such that

$$\sup_{x_1 \neq 0,\, x_2 \neq 0} \phi(x_1, x_2, y_1^{'}, y_2^{'}) = 2 > 2 - \epsilon^2 + \frac{27}{256}\,\epsilon^4 = \max_{y_2 \in [0,1]} \limsup_{y_1 \to \epsilon} \phi(y_1, y_2, y_1, y_2) \geq$$

$$\geq \max_{y_2 \in [0,1]} \phi(\epsilon, y_2, \epsilon, y_2) \quad \forall\, (y_1^{'}, y_2^{'}) \in U_{1,\epsilon} \times \in V_{2,y_2}. \tag{12}$$

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[3]Note that $\epsilon^2 - \frac{81}{64}\epsilon^4 - \epsilon^{\frac{5}{2}} > 0$ for ϵ sufficiently small.

[4]This superior value is not a maximum one.

By (12), ϕ is diagonally transfer continuous on X in $(y_1, y_2) \in A_3 \subset X$.

We put in **Area 4**.

We deal with the case regarding $(0, \epsilon)$ and $(0, -\epsilon)$. Since the property iii) holds, there exists $U_{1,0}$ a neighborhood of 0, $V_{2,\epsilon\,(-\epsilon)}$ a neighborhood of $\epsilon\,(-\epsilon)$ in X; and $\bar{x}_2 > 0$ in a suitable neighborhood of 0 such that

$$\inf_{(y_1,y_2)\in U_{1,0}\setminus\{0\}\times V_{2,\epsilon\,(-\epsilon)}} \phi(0, \bar{x}_2, y_1, y_2) = 2+\epsilon-\bar{x}_2^2 > 2-\epsilon^2+\epsilon^{\frac{5}{2}} = \sup_{y_2\in V_{2,\epsilon\,(-\epsilon)}} \limsup_{y_1\to 0} \phi(y_1, y_2, y_1, y_2) \tag{13}$$

and, trivially,

$$\phi(0, \bar{x}_2, 0, y_2^{'}) = 2 + \epsilon \overbrace{-\bar{x}_2^2 + g_\epsilon(\bar{x}_2)}^{>0} + \chi_{\{|y_2^{'}|<\epsilon\}} f(y_2^{'}) > 2 + \epsilon \overbrace{-y_2^{'\,2} + g_\epsilon(y_2^{'})}^{<0} + \chi_{\{|y_2^{'}|<\epsilon\}} f(y_2^{'}) =$$

$$= \phi(0, y_2^{'}, 0, y_2^{'}) \quad \forall\, y_2^{'} \in V_{2,\epsilon\,(-\epsilon)}. \tag{14}$$

We deal with the case regarding $(y_1, y_2) = (0, 0)$. Since the property (iii) holds, there exists $V_{2,0} \subset]-\epsilon, x_2^*[\subset]-\epsilon, \epsilon[$ an open neighborhood of 0 and $\bar{x}_2 = \sup V_{2,0}$, such that

$$\phi(0, \bar{x}_2, y_1^{'}, y_2^{'}) = 2 + \epsilon - \bar{x}_2^2 + g_\epsilon(\bar{x}_2)\, \chi_{\{y_1^{'}<0\}}(y_1^{'}) + f(y_2^{'}) > 2 - y_1^{'\,2} - y_2^{'\,2} + g_\epsilon(y_2^{'})\, \chi_{\{y_1^{'}<0\}}(y_1^{'}) =$$

$$= \phi(y_1^{'}, y_2^{'}, y_1^{'}, y_2^{'}) \quad \forall (y_1^{'}, y_2^{'}) \in [-1, 1] \setminus \{0\} \times V_{2,0} \setminus \{0\} \tag{15}$$

and

$$\phi(0, \bar{x}_2, 0, y_2^{'}) = 2 + \epsilon - \bar{x}_2^2 + g_\epsilon(\bar{x}_2) + f(y_2^{'}) >$$

$$> 2 + \epsilon - y_2^{'\,2} + g_\epsilon(y_2^{'}) + f(y_2^{'}) = \phi(0, y_2^{'}, 0, y_2^{'}) \quad \forall y_2^{'} \in V_{2,0} \setminus \{0\} \tag{16}$$

and, finally,

$$\inf_{y_1^{'}\in[-1,1]} \phi(0, \bar{x}_2, y_1^{'}, 0) = 2 + \epsilon - \bar{x}_2^2 > 2 = \sup_{y_1^{'}\in[-1,1]} \phi(y_1^{'}, 0, y_1^{'}, 0) \tag{17}$$

By (13), (14), (15), (16) and (17), ϕ is diagonally transfer continuous on X in $(y_1, y_2) \in A_4 \subset X$. □

Remark 2.1. It can be noted that $G_1(\epsilon) \overset{\epsilon\to 0^+}{\longrightarrow} G_0$ in the punctual convergence of the payoffs. This limit quasi concave game G_0 satisfies the diagonal transfer continuity and better-reply secure game properties. Therefore, a new question arises: What are the nonlinear perturbation properties of these two fundamental hypotheses? What are their closure properties respect to the punctual convergence or other kinds?

Proposition 2.3. *G_1 is a diagonally transfer quasi concave game.*

Proof. By [1][Prop.1,1(e)], it's sufficient to prove that, at least, one payoff is transfer upper continuous in its own strategy [1][Def.4] and is uniformly transfer quasi concave [1][Def.3].

Now, let $({x_1}^i, {x_2}^i)_{i\in\{1,\dots m\}} \subset [-1,1]^2$ be a family of distinct elements and

$$ {x_1}^{\bar{i}} = \min \arg_{x_2} \max_{i\in\{1,\dots m\}} u_1({x_1}^i, {x_2}^i) $$

there exists $({y_1}^i)_{i\in\{1,\dots m\}} \subset [-1,1]$ a family of elements choosen in the following subsets as follows:

$$ {y_2}^i \in \begin{cases} J\left(0, {x_1}^{\bar{i}}\right) & {x_2}^{\bar{i}} \neq 0 \\ \{0\} & {x_2}^{\bar{i}} = 0 \end{cases} $$

such that, for any finite subset $\{{y_1}^i\}_{i\in A}$ and for any ${y_1}^0 \in co(\{{y_1}^i\}_{i\in A})$ and for $A \subseteq \{1, 2 \dots, m\}$,

$$ \min_{i\in A} \left\{u_1({x_1}^i, {x_2}^i) - u_2({y_2}^0, {x_2}^i)\right\} \leq 0. \tag{18} $$

Suppose that ${x_1}^{\bar{i}} \neq 0$. Since $u_1(\cdot, {x_2}^i)$ is strict increasing on $[-1, 0[$ and strict decreasing on $[0, 1[$; and ${y_1}^0 \in J\left(0, {x_1}^{\bar{i}}\right)$ we have

$$ \min_{i\in A} \left\{u_1({x_1}^i, {x_2}^i) - u_1({y_1}^0, {x_2}^i)\right\} = \min_{i\in A} u_1({x_1}^i, {x_2}^i) - \max_{i\in A} u_1({y_1}^0, {x_2}^i) \leq $$

$$ \leq u_1({x_1}^{\bar{i}}, {x_2}^i) - 1 < 0 \tag{19} $$

Suppose that ${x_1}^{\bar{i}} = 0$, then ${y_1}^0 = 0$. Therefore, the first term in the equation (19) is lesser or equal than

$$ u_1(0, {x_2}^i) - u_1(0, {x_2}^i) = 0 \tag{20} $$

□

For sake of sufficient conditions, we introduce the following Proposition.

Proposition 2.4. *The payoff u_2 is transfer upper continuous in its own strategy but not uniformly transfer quasi concave on X.*

Proof. The function $u_2(x_1, \cdot)$ is upper semicontinuous on $[-1,1] \setminus \{0\}$ for all $x_1 \in [-1,1]$; and, then, it is transfer upper semicontinuous on $[-1,1] \setminus \{0\}$. By choosing $(x_1, x_2) \in [-1,1] \times]-\sqrt{\epsilon}, \sqrt{\epsilon}[$ we have $u_2(x_1, x_2) > 1-\epsilon = u_2(x_1, 0)$ for all x_1. But, there exists a point $(\bar{x}_1, \bar{x}_2) = (0, x_2^*) \in [-1,0] \times]0, \epsilon[$ and $V_2 \subseteq]-\epsilon, x_2^*[$ a neighborhood of 0 such that

$$u_2(\bar{x}_1, \bar{x}_2) = -x_2^{*2} + g_\epsilon(x_2^*) + 1 > \sup_{x_2^{'} \in V_2} u_2(0, x_2^{'}) \geq u_2(\bar{x}_1, x_2^{'}) \quad \forall x_2^{'} \in V_2$$

Therefore, u_2 is transfer upper continuous respect to its own variable. Now, let $({x_1}^1, {x_2}^1) \in]0, \epsilon[^2$ and $({x_1}^2, {x_2}^2) = (0, {x_2}^*)$ such that ${x_2}^* > {x_2}^1$ a finite family of points in X. It's easy to remark that, necessarily, the transferred points ${y_2}^i$ suitable for satisfying the uniformly transfer quasi concave on X have to satisfy

$${y_2}^1 \leq {x_2}^1, \quad {y_2}^2 = {x_2}^*$$

but, if we choose ${y_2}^0 \in]{x_2}^1, {x_2}^*[\subset [{y_2}^1, {y_2}^2]$, we have

$$0 < u_2({x_1}^1, {x_2}^1) - u_2({x_1}^1, {y_2}^0)$$
$$0 < u_2(0, {x_2}^*) - u_2(0, {y_2}^0)$$

since $u_2({x_1}^1, \cdot)$ is strict decreasing on $]0, \epsilon[$ and $u_2(0,)$ is strict increasing on $]0, {x_2}^*[$. Therefore $u_2(\cdot, \cdot)$ is not uniformly transfer quasi concave on X. □

Now, we introduce the following Proposition.

Proposition 2.5. *G_1 is not a better-reply secure game.*

Proof. We choose $(0, x_2) \in [-1,1] \times]0, x_2^*[$. For all neighborhood $V_2 \subset]0, +\epsilon[$ of x_2, the following

$$\max_{x_1 \in [0,1]} u_1(x_1, {x_2}^{'}) = 1 + \epsilon + f(x_2^{'}) \geq 1 + \epsilon + f(x_2) = u_1(0, x_2) \quad \forall {x_2}^{'} \in V_2. \tag{21}$$

holds. But, if ${x_2}^{'} = x_2 \in V_2$ the previous inequality is not strict. For all neighborhood $U_1 \subset]-\epsilon, +\epsilon[$ of 0, the following

$$\inf_{x_1 \in U_1} \sup_{{x_2}^{'} \in [-1,1]} u_2(x_1, {x_2}^{'}) = \min\left\{1, 1 + \frac{27}{256}\epsilon^4\right\} = 1 < 1 - {x_2}^2 + \epsilon \sqrt[2]{{x_2}^3} = u_2(0, x_2) \tag{22}$$

holds. Therefore, by (21) and (22), G_1 is not better-reply secure at $(0, x_2)$. □

However, G_1 has a Nash equilibrium at the point $(0, x_2^*)$ according to Theorem [1][Th.1]. Now, let us modify the f's values continuously in a neighborhood of the point $x_2 = \epsilon$; and, at a second time, globally on the whole subset $]-\epsilon, +\epsilon[\setminus\{0\}$ without preserving the conditions (i), (ii) and (iii). For sake of simplicity, we denote, again, this new function by f.

Therefore, we assume that $f\left(x_2^*\right) = -\epsilon$. In that case, it can be shown, easily, that ϕ is *not diagonally transfer continuous in y* but diagonally transfer quasi concave in x; and G_1 has a Nash equilibrium. Therefore, the following schema

$$\text{Diagonal transfer quasi concavity} \Longleftrightarrow \text{Nash equilibrium existence.}$$

$$\boxed{\text{implication established by [1][Th.1]}} \Uparrow \not\Downarrow \boxed{\text{implication not valid for } G_1}$$

$$\text{Diagonal transfer continuity}$$

holds. Moreover, we assume that $f\left(x_2^*\right) < -\epsilon$. In the last case, it can be shown, easily, that ϕ is *not diagonally transfer continuous in y* but diagonally transfer quasi concave in x; and G_1 *has no pure Nash equilibria.* If diagonal transfer continuity hypothesis doesn't hold, then the following

$$\text{Diagonal transfer quasi concavity} \not\Rightarrow \text{Nash equilibria existence.}$$

holds. It represents a counterexample on the Theorem [1][Th.1].

In the Theorem [2][Th.1], Tian & Oth. prove that if a function achieves its maximum value then it is weakly transfer upper (semi)continuous [2][Def.2] [5] on a compact subset. The same *necessary condition* for existence of maximum points doesn't hold for diagonalized bifunctions on compact subsets and the diagonalized version of transfer continuity condition. In fact, by choosing $f_\epsilon(x_2) = \left|\arctan\left(\frac{x_2}{\epsilon^3}\right)\right|$, the diagonalized bifunction in (1) has a maximum point without preserving the diagonal transfer continuity. For understanding that, it's easy to observe that the function $H(x_2, \epsilon) = -x_2^2 + \epsilon\, x_2^{3/2} + f_\epsilon(x_2)$ has a maximum point belonging to the subset $]0, \epsilon[\setminus\{x_2^*\}$ for ϵ sufficiently small. In fact, let the well defined differentiable function $C(\epsilon) := \left\{x_2 \in]0,1] \;\middle|\; \frac{\partial H}{\partial x_2}(x_2, \epsilon) = 0\right\}$ for ϵ sufficiently small [6] ; we

[5]If the function has a unique point of maximum, we can substitute weakly transfer upper (semi)continuous by transfer upper (semi)continuous.

[6]Note that $C(\epsilon) = \left\{x_2 \in]0,1] \;\middle|\; \left(4\,{x_2}^2\epsilon^6 + 4\,{x_2}^6 - 3\,\epsilon^7 x_2 - 3\,\epsilon {x_2}^5 - 2\,\epsilon^3\right)^2 = 0\right\}$.

note that

$$\lim_{\epsilon \to 0^+} C(\epsilon) = 0, \qquad \lim_{\epsilon \to 0^+} \frac{d\,x_2^*}{d\epsilon}(\epsilon) = 0 < \frac{4}{5} \cong \lim_{\epsilon \to 0^+} \frac{d\,C}{d\epsilon}(\epsilon) < 1$$

However, f_ϵ does not satisfy the property ii). It can be shown, easily, that the last property is necessary one for diagonal transfer continuity.

3 Does Better-Reply Security imply Diagonal Transfer Continuity?

In this section, a quasi concave game, in which the better-reply security hypothesis holds while diagonal transfer continuity one fails, is introduced.

Let $G_2 = ([-1,1], [-1,1], u_1, u_2)$ defined as follows:

$$u_1(x_1, x_2) = \begin{cases} -x_1^2 + 1 & x_1 \neq 0 \\ 1 + \epsilon & x_1 = 0 \end{cases}$$

$$u_2(x_1, x_2) = \begin{cases} -x_2^2 + 1 + x_1 & x_1 \neq 0 \\ 0 & x_1 = 0,\ x_2 < 0 \\ -x_2 - \frac{1}{2} & x_1 = 0,\ x_2 \geq 0 \end{cases}$$

with $\epsilon > 0$; and the aggregate bifunction and its diagonalized verion are the following:

$$\phi(x_1, x_2, y_1, y_2) = \begin{cases} -x_1^2 - x_2^2 + y_1 + 2 & x_1 \neq 0,\ y_1 \neq 0 \\ -x_1^2 + 1 & x_1 \neq 0,\ y_1 = 0,\ x_2 < 0 \\ -x_1^2 + \dfrac{1}{2} - x_2 & x_1 \neq 0, y_1 = 0,\ x_2 \geq 0 \\ 2 + \epsilon - x_2^2 + y_1 & x_1 = 0,\ y_1 \neq 0 \\ 1 + \epsilon & x_1 = 0,\ y_1 = 0,\ x_2 < 0 \\ \dfrac{1}{2} + \epsilon - x_2 & x_1 = 0,\ y_1 = 0,\ x_2 \geq 0 \end{cases}$$

$$\phi(y_1, y_2, y_1, y_2) = \begin{cases} -y_1^2 - y_2^2 + y_1 + 2 & y_1 \neq 0 \\ 1 + \epsilon & y_1 = 0,\ y_2 < 0 \\ \dfrac{1}{2} + \epsilon - y_2 & y_1 = 0,\ y_2 \geq 0 \end{cases}$$

Now, we present the following Proposition.

Proposition 3.1. *G_2 is better-reply secure but not diagonally transfer continuous.*

Proof. The function $u_2(\cdot, x_2)$ is lower semicontinuous at the point 0 for all $x_2 \in X_2$, while $u_1(x_1, \cdot)$ is constant at 0 for all $x_1 \in X_1$. By [5][Cor. 3.4], G_2 is a payoff secure game. The vector payoffs field (u_1, u_2) has the subset $\{(0, x_2) \in [-1, 1]^2 \mid x_2 \in [-1, 1]\}$ as discontinuities set. Our attention can be focused on the previous subset. We choose a point $(0, x_2)$ and a sequence $(x_{1n}, x_{2n}) \in [-1, 1]^2$ converging to $(0, x_2)$.

Suppose that $x_{1n} \neq 0$ for n sufficiently large, we have that $u_2(x_{1n}, x_{2n}) \overset{n}{\to} 1 > u_2(0, x_2)$ while $u_1(x_{1n}, x_{2n}) \overset{n}{\to} 1 < 1 + \epsilon = u_1(0, x_2)$.

Suppose that $x_{1n} = 0$ for n sufficiently large and $x_2 \neq 0$, we have that $u_i(x_{1n}, x_{2n})$ converging to $u_i(0, x_2)$ for all $i = 1, 2$.

Suppose that $x_{1n} = 0$ for n sufficiently large and $x_2 = 0$, we have that $u_1(x_{1n}, x_{2n})$ converging to $u_1(0, 0)$ but $\limsup_n u_2(x_{1n}, x_{2n}) = 0 > -\frac{1}{2} = u_2(0, 0) = \liminf_n u_2(x_{1n}, x_{2n})$

[7]. Therefore, in all of three cases, at least one payoff u_i is converging to a value greater or equal than $u_i(0, x_2)$ along the sequence (x_{1n}, x_{2n}). We can conclude that G_2 is reciprocally upper semicontinuous. By [5][Prop. 3.2], G_2 is better-reply secure. Now, we prove that G_2 is not diagonally transfer continuous at the point $(0, 0)$. It's easy to verify that

$$\phi(0, -\frac{1}{2}, 0, 0) = 1 + \epsilon > \frac{1}{2} + \epsilon = \phi(0, 0, 0, 0).$$

holds. Let U_0 a neighborhood of $(0, 0)$ and for all $(y_1, y_2) \in U$ with $y_1 \neq 0$. Necessarily, by continuity, there exists V_2 a suitable neighborhood of 0 such that

$$\phi(0, x_2, y_1, y_2) = 2 + \epsilon - x_2^2 + y_1 > -y_1^2 - y_2^2 + y_1 + 2 = \phi(y_1, y_2, y_1, y_2) \ \ \forall\, x_2 \in V_2$$

but, by considering all the points $(0, y_2) \in U_0$, we obtain

$$\max_{x_2 \in V_2} \phi(0, x_2, 0, y_2) = 1 + \epsilon \not> 1 + \epsilon = \phi(0, y_2, 0, y_2) \ \ \forall\, y_2 \in Pr_2(U_0)^-.$$

□

Moreover, G_2 is a quasi concave game and has multiple Nash equilibria of this kind $(0, x_2)$ with $x_2 < 0$.

4 How much Does Generalized Continuity Assumption Need ?

In this section, we want to introduce a game G_3 about which the most recent continuity assumptions, stated in [1], [5] and [6], fail and the quasi concavity assumption is preserved; and, simultaneously, a countable Nash equilibria set exists. A new question arises: Are there new kind of generalized continuity concepts which represent sufficient conditions for Nash Equilibrium existence in the setting of quasi concave games? By a slight variation on G_3, we show a G_3^* which furnishes a counter example of the Theorem [5][Th.3.1]. Let

[7] Any sequence (x_{1n}, x_{2n}) converging to a point $(0, x_2)$ in a different way as prescribed before is such as the sequence $(u_1, u_2)(x_{1n}, x_{2n})$ is not converging.

$c = \tan(1)$; $\epsilon > 0$ sufficiently small; x_1^* the unique solution of the eq. $e^{-\frac{x}{\epsilon}} = x^2$ in $]0,1[$; and $x_2^* := \sqrt{\frac{\tan\, x_1^*}{c}}$. We define payoffs on the set $[0,1]^2$ as follows:

$$u_1(x_1,x_2) = \begin{cases} -\epsilon & x_2 = 0,\ x \geq x_1^* \\ 0 & x_2 = 0,\ x_1 < x_1^* \\ [1 - x_2 \ln x_2 - 2\,x_2 + x_1]^+ & x_2 \neq 1,\ x_1 \leq \arctan(c\,x_2^2) \\ \dfrac{1 - x_2 \ln x_2 - 2\,x_2 + \arctan(c\,x_2^2)}{\arctan(c\,x_2^2) - 1}\ (x_1 - 1) & x_2 \neq 0,1,\ \ x_1 > \arctan(c\,x_2^2) \\ 0 & x_2 = 1,\ x_1 \neq 1 \\ \epsilon & x_2 = 1,\ x_1 = 1 \end{cases}$$

$$u_2(x_1,x_2) = \begin{cases} \left(e^{-\frac{x_1}{\epsilon}} - x_1^2\right)\, x_2 + x_1^2 & x_2 \leq 1 - x_1,\ x_1 \leq x_1^* \\ -\left(e^{-\frac{x_1}{\epsilon}} - x_1^2\right)(x_2 + 2\,x_1 - 2) + x_1^2 & 1 - x_1 < x_2,\ x_1 \leq x_1^* \\ -\left|\sin\left(\dfrac{x_1 - x_1^*}{1 - x_1}\right)\right| \left(x_2 - \dfrac{1}{2}\right) + {x_1^*}^2 & x_1^* < x_1 \neq 1 \\ \liminf_{x_1 \to 1^-} u_2\,\big|_{x_1^* < x_1 \neq 1}\,(x_1,x_2) & x_1 = 1,\ x_2 \leq \dfrac{1}{2} \\ \limsup_{x_1 \to 1^-} u_2\,\big|_{x_1^* < x_1 \neq 1}\,(x_1,x_2) & x_1 = 1,\ x_2 > \dfrac{1}{2} \end{cases}$$

where x_1^* satisfies the following property

$$\frac{1}{2\,c} \int_0^{x_1^*} \frac{(\tan y)^{-\frac{1}{2}}}{\sqrt{c}} + \frac{\tan y}{c} + 2\ d\,y < 1. \tag{23}$$

Now, we prove the following Proposition.

Proposition 4.1. G_3 *is neither better-reply secure nor diagonally transfer continuous game.*

Proof. It's easy to prove that G_3 is not better-reply secure at the not equilibrium point

$(1,1,\epsilon,0) \in \overline{\mathrm{Graph}(u_1,u_2)}$. We note that

$$\lim_{x_2\to 1^-} \max_{x_1\in[0,1]} u_1(x_1,x_2) = 0. \tag{24}$$

By permanence on sign's Theorem, we can choose U_2 a right suitable neighborhood of 1 such that $\max_{x_1\in[0,1]} u_1(x_1,x_2) < \epsilon = u_1(1,1)$ for all $x_2 \in U_2 \setminus \{1\}$; besides, for all right neighborhood U_1 of 1 there exists $\bar{x}_1 \in U_1$ such that

$$\max_{x_2\in[0,1]} \limsup_{x_1\to 1^-} u_2(x_1,x_2) = \max_{x_2\in[0,1]} u_2(\bar{x}_1,x_2) = \max_{x_2\in[0,\frac{1}{2}]} u_2(\bar{x}_1,x_2) = 0 \not> 0 = u_2(1,1) \tag{25}$$

We prove that the condition (c) in [6][Th.3.1] fails. In fact, we have

$$\sup_{x_1\in[0,1]} \liminf_{x_2\to 1^-} u_1(x_1,x_2) = \sup_{x_1\in[0,1]} \left\{ \lim_{x_2\to 1^-} [1 - x_2\ln x_2 - 2\,x_2 + x_1]^+, \right.$$

$$\left. \lim_{x_2\to 1^-} \frac{1 - x_2\ln x_2 - 2\,x_2 + \arctan(c\,x_2^2)}{\arctan(c\,x_2^2) - 1} \lim_{x_1\to 1^-} x_1 - 1 \right\} =$$

$$= \max\left\{ \sup_{x_1\in[0,1]} [x_1 - 1]^+, \frac{1}{2}\,\frac{2\,c - 3 - 3\,c^2}{c} \lim_{x_1\to 1^-} (x_1 - 1) \right\} = 0 \not\geq \epsilon = u_1(1,1)$$

Now, it will be proved that the game is not diagonally transfer continuous at the point $(1,1) \in [0,1]^2$. It's remarkable that

$$\phi(1,1,1,1) = u_1(1,1) + u_2(1,1) = \epsilon + x_1^{*2} <$$

$$< \frac{1}{2} + x_1^{*2} = u_1(\frac{1}{2},1) + u_2(1,0) = \phi(\frac{1}{2},0,1,1). \tag{26}$$

holds. Moreover, let be a function $g \in C^1([0,1],[0,1])$ such that

$$g(1) = 1, \qquad 0 < \frac{c}{1+c^2} \leq \dot{g}_-(1) < \frac{2c}{1+c^2}. \tag{27}$$

By computing this limit, we have

$$\lim_{x_2\to 1^-} u_1(g(x_2),x_2) = \lim_{x_2\to 1^-} \frac{1 - x_2\ln x_2 \; - 2\,x_2 + \arctan(c\,x_2^2)}{\arctan(c\,x_2^2) - 1} \; (g(x_2) - 1) =$$

$$= \lim_{x_2\to 1^-} \left[\frac{g(x_2) - 1}{\arctan(c\,x_2^2) - 1} + \frac{(-x_2\,\ln x_2 - 2\,x_2 + 2)\,(g(x_2) - 1)}{\arctan(c\,x_2^2) - 1} \right] = \lim_{x_2\to 1^-} \frac{\dot{g}(x_2)\,(1 + c^2\,x_2^4)}{c\,x_2} +$$

$$+ \lim_{x_2\to 1^-} \frac{(-\ln x_2 - 3)\,(g\,(x_2) - 1) + (-x_2\ln x_2 - 2\,x_2 + 2)\,\dot{g}(x_2)}{2\,(1 + c^2 x_2^4)^{-1}\;c\,x_2} =$$

$$= \dot{g}_{-}(1)\,\frac{(1+c^2)}{c}. \tag{28}$$

By Implicit Function's Theorem and by properties (24) and (27), there exists a neighborhood U of 1 in $[0,1]$ and $g^{-1} : U \to U$ a local inverse function such that

$$\max_{(\bar{x}_1,\bar{x}_2)\in[0,1]^2} \phi(\bar{x}_1,\bar{x}_2,g(x_2),x_2) - \phi(g(x_2),x_2,g(x_2),x_2) =$$

$$= \max_{\bar{x}_1\in[0,1]} u_1(\bar{x}_1,g^{-1}(x_1)) - u_1(x_1,g^{-1}(x_1)) + \max_{\bar{x}_2\in[0,1]} u_2(g(x_2),\bar{x}_2) - u_2(g(x_2),x_2) \le$$

$$\le \left[\max_{\bar{x}_1\in[0,1]} u_1(\bar{x}_1,g^{-1}(x_1)) - u_1(x_1,g^{-1}(x_1))\right] + \left[u_2(g(x_2),0) - \inf_{x_2\in\in U\setminus\{1\}} u_2(g(x_2),x_2)\right] <$$

$$< -\dot{g}(1^-)\frac{1+c^2}{c} + 2\,\epsilon + 1 < 0 \qquad \forall\, x_1,\, x_2 \in U\setminus\{1\}. \tag{29}$$

By the properties (26) and (29), we conclude the proof. □

Now, we introduce the following two Proposition.

Proposition 4.2. *G_3 has infinite Nash Equilibria.*

Proof. In fact, there exists a sequence

$$x^*_{1,n} = \left(\frac{x^*_1 + 2\,\pi\, n}{1+2\,\pi\, n}\right)_{n\in\mathbb{N}_+} \subset\ [x^*_1, 1[\subset]0,1[$$

converging to 1, such that $\arg\max_{x_2\in[0,1]} u_2(x^*_{1,n}, x_2) = [0,1]$;but, by surjectivity of Best Reply function associated to u_1, there exists a sequence

$$x^*_{2,n} = \left(\sqrt{\frac{\tan\, x^*_{1,n}}{c}}\right)_{n\in\mathbb{N}_+} \subset]0,1[$$

converging to 1, such that $x^*_{1,n} = \arg\max_{x_1\in[0,1]} u_1(x_1, x^*_{2,n})$. We prove that that the previous sequence includes all the Nash equilibria for G_3. In fact, if $x_2 = 0$, then

$$0 \notin BR_2\,(BR_1(0)) = BR_2\,([0,x^*_1[) =]1-x^*_1, 1];$$

if $x_2 \neq x^*_{2,n}$ and $\arctan(c\,x_2^2) \ge x^*_1$, then

$$x_2 \notin BR_2\,(BR_1(x_2)) = BR_2\left(\{\arctan(c\,x_2^2)\}\right) = \{0\}\,;$$

if $x_2 \neq 0$ and $\arctan(c\,x_2^2) < x_1^*$, by property (23), we can imply

$$x_2^* - 1 + \arctan\left(c\,x_2^*\right) = \int_0^{\sqrt{\frac{\tan x_1^*}{c}}} \frac{1 + c^2\,s^4 + 2\,c\,s}{1 + c^2 s^4}\,d\,s\ - 1 =$$

$$= \frac{1}{2\,c} \int_0^{x_1^*} \frac{(\tan y)^{-\frac{1}{2}}}{\sqrt{c}} + \frac{\tan y}{c} + 2\ d\,y - 1 < 0$$

and, then

$$x_2 \notin BR_2\left(BR_1(x_2)\right) = BR_2\left(\{\arctan(c\,x_2^2)\}\right) = \left\{1 - \arctan(c\,x_2^2)\right\}.$$

□

Besides, it would seem that better-reply security assumption is not a necessary condition for Nash equilibrium existence. For example, by changing, only, the u_2's value at the points $(x_{1,n}^*, x_2)$, as follows

$$u_2(x_{1,n}^*, x_2) = \liminf_{x_1 \to 1^-} u_2\left|_{x_1^* < x_1 \neq 1}\right. (x_1, x_2) \quad \forall n \in \mathbb{N}_+,\ \forall x_2 \in [0,1] \tag{30}$$

This new game is named G_3^*.

Proposition 4.3. *G_3^* is not better-reply secure, only, at the point $(x_1^*, x_2^*)_{n\in\mathbb{N}_+}$ but quasi concave game without Nash Equilibria.*

Proof. For testing the better-reply security assumption, it needs to check it on all the discontinuity points for u_1 or u_2 on $[0,1]^2$.

First of all, G_3^* becomes better-reply secure at the point $(1,1)$. In the above case, the condition (25) does not hold. Besides, G_3^* becomes better-reply secure at all discontinuous points $\left\{x_{1,n}^*\right\} \times \left]\frac{1}{2}, 1\right]$. [8] We observe that $u_2(\cdot, x_2)$ is lower semicontinuous but not continuous at $x_{1,n}^*$ for all $x_2 \in \left]\frac{1}{2}, 1\right]$; and $u_2(x_{1,n}^*, \cdot)$ is strict decreasing function. Let $x_2 \in \left]\frac{1}{2}, 1\right]$ and $\frac{1}{2} < \bar{x}_2 < x_2$, we obtain

$$u_2(x_{1,n}^*, x_2) < u_2(x_{1,n}^*, \bar{x}_2) < \liminf_{x_1 \to x_{1,n}^*} u_2(x_1, \bar{x}_2).$$

[8]Note that $\inf_{n\in\mathbb{N}_+^*} \lim_{\epsilon\to 0} x_{2,n}^* > \frac{1}{2}$. Therefore, $(x_{1,n}^*, x_{2,n}^*)_{n\in\mathbb{N}_+^*} \subset \left]\frac{1}{2}, 1\right]$.

G_3^* is better-reply secure at all discontinuous points $\left(\left\{x_{1,n}^*\right\} \times \left]0, \frac{1}{2}\right] \setminus \left\{x_{2,n}^*\right\}\right)_{n \in N_+}$. Let be $x_{1,n}^*$, $x_2 \in \left]0, \frac{1}{2}\right] \setminus \{x_{2,m}^*\}$ with $m \neq n$; and we choose $\bar{x}_1$ such that

$$\max u_1(\cdot, x_2) > \bar{x}_1 > x_{1,n}^*$$

if $\max u_1(\cdot, x_2) > x_{1,n}^*$; or

$$\max u_1(\cdot, x_2) < \bar{x}_1 < x_{1,n}^*$$

if $\max u_1(\cdot, x_2) < x_{1,n}^*$. Therefore, by observing that $u_1(x_1, \cdot)$ is continuous on $\left]0, \frac{1}{2}\right]$ for all $x_1 \in [0, 1]$, we obtain

$$u_1(x_{1,n}^*, x_2) < u_1(\bar{x}_1, x_2) = \lim_{x_2' \to x_2} u_2(\bar{x}_1, x_2') \quad \forall n \in \mathbb{N}_+ \tag{31}$$

G_3^* is better-reply secure at all discontinuous points $[x_1^*, 1] \times \{0\}$. Let $x_1 \geq x_1^*$. Since $u_1(0, \cdot)$ is continuous at 0, we obtain

$$-\epsilon = u_1(x_1, 0) < 0 = u_1(0, 0) = \liminf_{x_2 \to 0} u_1(0, x_2). \tag{32}$$

G_3^* is better-reply secure at all discontinuous points $\{1\} \times]0, 1[\setminus \left\{\frac{1}{2}\right\}$.

Since $\max x_1 \in [0, 1] u_1(x_1, x_2) > 0$ is continuous for all $x_2 \in]0, 1]$, we obtain

$$\inf_{x_2' \in U_2} \max_{x_1 \in [0,1]} u_1(\bar{x}_1, x_2') > 0 = u_1(1, x_2)$$

with $\bar{x}_1 = BR_1(x_2)$. G_3^* is *not better-reply secure* at all discontinuous points (x_1^*, x_2^*) [9]. In fact, we can note that

$$u_1(x_1^*, x_2^*) \geq u_1(x_1, x_2^*) \geq \limsup_{x_2 \to x_2^*} u_1(x_1, x_2) \quad \forall x_1 \in [0, 1]. \tag{33}$$

and

$$u_2(x_1^*, x_2^*) \geq 0 = \max_{x_2} \limsup_{x_1 \to x_1^*} u_2(x_1, x_2) \tag{34}$$

holds. In spite of G_3, G_3^* has no Nash equilibria in pure strategy. It's sufficient to check on the points of the sequence $(x_{1,n}^*, x_{2,n}^*)$. In fact, we have

$$x_{1,n}^* \notin BR_1\left(BR_2(x_{1,n}^*)\right) = BR_1(0) = [0, x_1^*[.$$

since $x_{1,n}^* \geq x_1^*$, $\forall n \in \mathbb{N}_+$. □

[9] If $\lim_{\epsilon \to 0} x_2^* = 0$, then $x_2^* \subset [0, \frac{1}{2}]$.

Remark 4.1. The reader can note that G_3 is not better-reply secure, only, at the point $(1,1)$, while G_3^* is better-reply secure at the point $(1,1)$ but not at the points (x_1^*, x_2^*).

Remark 4.2. By following the same path for constructing G_3^*, we can change the u_1's value at the points $(x_{1,n}^*, x_{2,n})$. In fact, it's trivial to make $u_1(\cdot, x_{2,n}^*)$ lower semicontinuous at the points $x_{1,n}^*$ as follows

$$u_1(x_1^*, x_2^*) = [1 - x_2^* \ln(x_2^*) - 2\, x_2^* + x_1^* - \epsilon_n]^+$$

with $\epsilon_n > 0$. But, this slight and more simple modification ruins the quasi concavity assumption. In fact, the game has no Nash equilibrium in pure strategy.

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