Measuring Productivity using the Index Number Approach: An Introduction

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Abstract

This paper provides an introduction to productivity measurement using index number techniques. Attention is given to the construction of productivity series using common index number formulae, the economic and axiomatic approaches to selecting an index number formula, and the use of chaining. Special attention is also given to measuring physical capital inputs and quality adjusted labour inputs. Numerical examples are used throughout the paper to illustrate the analysis.

JEL CLASSIFICATION

C43 – Index numbers and aggregation

D24 – Production; capital and total factor productivity; capacity

E23 – Production

KEYWORDS

Productivity measurement; index numbers; capital, quality-adjusted labour inputs

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Measuring Productivity using the Index Number Approach: An Introduction

1 Introduction

Productivity measures are required in several areas of economic analysis. These range from exchange rate determination to examining the (proximate) sources of economic growth. A variety of methods are available to the productivity analyst in calculating productivity estimates, with the choice of method partly dependent on the objectives of the analysis. This paper provides an introduction to measuring productivity using the index number method. Consideration is given to this approach owing to the use of index numbers in constructing economic aggregates (such as GDP and the Consumers' Price Index) and because index number techniques are used by statistical agencies that publish official productivity measures. Readers interested in alternative approaches to productivity measurement should see Mawson, Carlaw and McLellan (2003).

There are two main approaches to choosing an index number formula: the economic and axiomatic approaches. The former approach bases the choice of index formula on a producer's underlying production technology, and therefore has theoretical microeconomic underpinnings. The axiomatic approach bases the choice of index formula on desirable properties that indexes should exhibit. Once the index formula is chosen, consideration then needs to be given as to whether the productivity index should be chained to reduce substitution bias associated with fixed weight indexes.

Good measures of outputs and inputs are needed in forming reliable productivity measures. This paper gives special attention to measuring physical capital and quality-adjusted labour inputs, as measuring these inputs present particular challenges for productivity analysts.[1] Measuring physical capital inputs requires the construction of productive capital stocks from data on past investments and the formation of rental prices for different asset types. This is achieved using an integrated framework in which the loss in the productive capacity of an asset is linked to economic depreciation and its rental price (or user cost of capital). The measurement of quality-adjusted labour inputs requires rich information on labour market earnings and worker characteristics.

[1] This is not to say the measurement of other inputs and outputs is not difficult. For a discussion of problems in measuring outputs and the implications for productivity measurement see Baily and Gordon (1988) and Diewert and Fox (1999).

The remainder of the paper is organised as follows. Section 2 discusses the index number approach to measuring productivity and choice of index number formula. Included in this section are numerical examples, based on hypothetical price and quantity data, illustrating the construction of productivity indices using various index formulae. Section 3 outlines the rationale and procedure for chaining fixed weight indices, illustrating this procedure with a simple numerical example. Measurement of capital inputs using an integrated framework that links the loss in productive capacity of an asset with economic depreciation and its rental price is discussed in Section 4. The measurement of quality-adjusted labour inputs is canvassed in Section 5. Section 6 provides a brief summary.

2 Index number approach

This section presents an introduction to measuring productivity using the index number method. Subsection 2.1 discusses various productivity measures and subsection 2.2 presents several index number formulae often used in constructing productivity indices. The economic and axiomatic approaches to choosing an index number formula are discussed in subsection 2.3. Finally, subsection 2.4 presents numerical examples using the index formulae from subsection 2.2 to illustrate the construction of productivity indices and the differences between different index formulations. Readers interested in a more detailed review of the index number approach to productivity measurement should consult Diewert and Nakumara (2004).

2.1 Productivity index

Productivity measures attempt to capture the ability of inputs to produce output (usually over time). In general a productivity index is defined as the ratio of an output quantity index to an input quantity index, that is:

$$A_t = \frac{Q_t}{X_t}; \qquad (1)$$

for $t = 0,...,T$ and where A_t is a productivity index, Q_t is an output quantity index and X_t is an input quantity index. Each index represents accumulated growth from period 0 to period t.

When X_t comprises a single input, for example labour or physical capital, A_t is a partial productivity index. The two well known partial productivity measures are labour and capital productivity. A limitation of partial productivity measures is that changes in productivity may reflect the impact of omitted inputs. For example, increases in labour productivity may be due to increases in the available amount of physical capital (one of the omitted inputs in the measurement of labour productivity) per worker, rather than increases in the underlying productivity of labour.

When X_t comprises two or more inputs, A_t is a multifactor productivity index.[2] Most often multifactor productivity is formed using labour and physical capital, although some

[2] Multifactor productivity and total factor productivity are often used synonymously. However, strictly total factor productivity is measured by dividing an output quantity index by an input quantity index that is constructed using *all* inputs in the production process. Rarely is this the case, hence the preference for the name multifactor productivity.

productivity studies have included additional variables such as land and inventories (see, for example, Diewert and Lawrence, 1999).

Productivity indices are usually constructed using disaggregate prices and quantities of outputs and inputs. Because outputs and inputs are heterogenous it is simply not possible to add all outputs to form an output quantity index or, likewise, to add all inputs to form an input quantity index. Disaggregate data on the volumes of outputs and inputs need to be weighted to form output and input quantity indices. Output and input prices, or nominal output and input shares, are typically used as representative weights when forming output and input quantity indices.

2.2 Index number formulae

When constructing productivity indices it is not immediately apparent which weighting procedure should be used to weight output and input quantities when forming output quantity and input quantity indices and on what basis the weighting structure should be chosen. There are numerous index formulae that can be used to construct output and input indices. The Laspeyres, Paasche, Fisher and Törnqvist indexes are some of the more widely used index formulae.

Suppose information on the price and quantity of M outputs is available for period $t = 0,...,T$. Denoting the output price and quantity vectors in period t as $p_t \equiv (p^1,...,p^M)$ and $q_t \equiv (q^1,...,q^M)$, the Laspeyres output quantity index (Q_t^L) is defined as follows:

$$Q_t^L = \frac{\sum_{m=1}^{M} p_0^m q_t^m}{\sum_{m=1}^{M} p_0^m q_0^m} = \sum_{m=1}^{M} w_0^m \frac{q_t^m}{q_0^m} \qquad (2)$$

where $w_t^m = \frac{p_t^m q_t^m}{\sum_{m=1}^{M} p_t^m q_t^m}$ is output m's nominal output share. Note that equation (2) shows the Laspeyres output quantity index is the period 0 share-weighted sum of quantity ratios.

The Paasche output quantity (Q_t^P) index is defined as follows:

$$Q_t^P = \frac{\sum_{m=1}^{M} p_t^m q_t^m}{\sum_{m=1}^{M} p_t^m q_0^m} = \left[\sum_{m=1}^{M} w_t^m \left(\frac{q_t^m}{q_0^m}\right)^{-1}\right]^{-1} \qquad (3)$$

The Paasche output quantity index uses period t prices as the weights, in contrast to the Laspeyres output quantity index that uses period 0 prices as weights.

The Fisher output quantity index (Q_t^F) is found by taking the geometric average of the Laspeyres and Paasche output quantity indexes, that is:

$$Q_t^F = (Q_t^L Q_t^P)^{\frac{1}{2}} \tag{4}$$

Finally, the Törnqvist output quantity (Q_t^T) index is defined as follows:

$$Q_t^T = \prod_{m=1}^{M} \left(\frac{q_t^m}{q_0^m} \right)^{\frac{1}{2}\left(w_0^m + w_t^m\right)} \tag{5}$$

Input quantity indexes are defined in a similar manner using input prices (c_t) and input quantities (x_t).

2.3 Selecting an index number formula: The economic and axiomatic (test) approaches

As discussed in the previous subsection, there are numerous index formulae that can be used to form aggregate productivity measures. This raises the question: are there any criteria that can be used to decide on the choice of index formula? The index number literature offers two main approaches: the economic approach and the axiomatic (test) approach.[3]

The economic approach bases the choice of index number formula on a producer's underlying production technology (that is, the production, cost, revenue or profit function). This approach assumes competitive optimising behaviour by producers. In other words producers are assumed to maximise profit (minimise costs) for a given production technology.[4]

Consider the following production technology:

$$Q_t = f(A_t, K_t, H_t) \tag{6}$$

where A_t is the level of multifactor productivity, K_t aggregate physical capital services, and H_t the aggregate labour input (in this case the total number of hours worked).

An index expressed in terms of the above technology is known as a theoretic index. In continuous time the theoretic index is a Divisia index.[5] As data are available in discrete time, rather than as continuous functions of time, it is necessary to use an index formula to approximate the Divisia index.[6]

A particular index is defined as an exact index when it corresponds directly to the theoretic index derived from the production technology (Diewert, 1976). For example, if production

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[3] A third approach, the stochastic approach, is less widely used. For a critical review of this approach see Diewert (1995).

[4] Although, recent research by Diewert and Fox (2004) shows that an index of multifactor productivity can be derived from the economic approach without the need to assume competitive optimising behaviour.

[5] Each of the index formulae in equations (2) to (5) use discrete data to measure quantity changes between period *0* and period *t*. The Divisia index is formed assuming data exists for all periods between period *0* and period *t*.

[6] The Törnqvist index is often described as a Divisia index. It is actually a discrete approximation to the Divisia index.

technology takes the translog functional form the Törnqvist quantity index is the corresponding index for the underlying production technology. Thus, the Törnqvist index is "exact" for translog production technology.

When an exact index corresponds to a production technology that has a flexible functional form, a functional form that is able to approximate a range of other functional forms, the index is defined as a superlative index (Diewert, 1976).[7] The Törnqvist index is a superlative index because the translog functional form can approximate a range of other functional forms. A superlative index must also be an exact index, however it is possible for an index to be an exact index but not a superlative index.

The axiomatic approach bases the choice of index number formula on properties that an index should exhibit, with these properties being embodied in axioms. One of the appealing features of this approach is that it does not make any assumptions about competitive optimising behaviour. The following four axiomatic tests are often used (see, for example, Diewert and Lawrence, 1999): the constant quantities test; the constant basket test; the proportionality test; and the time reversal test.[8]

The constant quantities test states that if quantities are identical in two periods, then the quantity index should be the same regardless of what prices are in both periods. The constant basket test states that if prices remain unchanged between two periods then the ratio of the quantity indexes between the two periods should be equal to the ratio of values between the two periods. The proportionality test requires that when all quantities increase or decrease by a fixed proportion between two periods, then the index should increase or decrease by the same fixed proportion. The time reversal test requires the index going from period 0 to period 1 to be the inverse of the index going from period 1 to period 0. In other words, if prices and quantities in period 0 and t are interchanged, the resulting index should be the inverse of the original index. The Fisher index passes all four of the above tests. The Törnqvist index does not pass the constant basket test, while the Laspeyres and Paasche indexes fail the time reversal test. These results are summarised in Table 1.

[7] More rigorously, Diewert (1976) defined a *flexible aggregator* as a linearly homogenous function that provides a second order approximation to an arbitrary twice continuously differentiable linearly homogenous function.

[8] This is not to suggest these four axioms are exhaustive. There are a variety of axiomatic tests. Diewert (1992) evaluates various index number formulae against twenty different axioms.

Table 1 – Index axioms

	Constant quantities	Constant basket	Proportionality	Time reversal
Laspeyres	✓	✓	✓	×
Paasche	✓	✓	✓	×
Törnqvist	✓	×	✓	✓
Fisher	✓	✓	✓	✓

In practice it is common to use both the economic and axiomatic approaches when choosing an index number formula, while also bearing in mind the end use of the index number series. Data availability will also influence the choice of index formula. Using both the economic and axiomatic approaches a strong case can be made in favour of using the Fisher index.[9]

2.4 Numerical example using different index number formulae[10]

This subsection illustrates the use of the index formulae in equations (2) to (5) to form productivity measures using hypothetical price and quantity data.

Consider a situation in which an economy produces two outputs, q_t^Y and q_t^Z, using two inputs, x_t^Y and x_t^Z, where both output prices (p_t^Y and p_t^Z) and input prices (c_t^Y and c_t^Z) are exogenously determined. Furthermore, suppose information on the prices and quantities of outputs and inputs is available for three periods $t = 0,1,2$. This information is presented in Table 2.

Table 2 – Prices and quantities of outputs and inputs

	Prices and quantities of outputs				Prices and quantities of inputs			
	p_t^Y	q_t^Y	p_t^Z	q_t^Z	c_t^Y	x_t^Y	c_t^Z	x_t^Z
$t=0$	3	6	3	5	2	6	3	7
$t=1$	3	7	4	6	3	5	3	10
$t=2$	4	8	6	8	6	4	4	14

To construct output quantity indexes using the formulae (2) to (5) we first construct nominal output shares. Nominal output shares are calculated by dividing nominal output (revenue) for each good by total nominal output for all goods. The nominal output shares for goods Y and Z are displayed in columns (1) and (2) in panel I of Table 3.

It is also necessary to calculate the ratio of period $t's$ quantity to period $0's$ quantity for both goods. This quantity ratio is known as a quantity relative. Quantity relatives for output Y and Z are presented in columns (3) and (4) in panel I of Table 3.

The Laspeyres output quantity index is constructed by first multiplying the nominal output shares in period 0 by the corresponding quantity ratios in each period (that is, the first entries in columns (1) and (2) are multiplied by the corresponding quantity ratios in columns (3) and (4), to yield the share-weighted quantity relatives contained in columns

[9] In fact the Fisher index passes all twenty axiomatic tests considered by Diewert (1992).

[10] A spreadsheet containing the numerical examples presented in this paper is available from the author.

(5) and (6)). The share-weighted quantity ratios are then added together for each period to yield the Laspeyres quantity index (that is, values in columns (5) and (6) are added for each period to yield the values in column (7)).

The Paasche output quantity index is constructed by multiplying the nominal output shares for each period by the corresponding reciprocal of the quantity ratios (that is, nominal output shares in columns (1) and (2) and multiplied by the reciprocal of the corresponding quantity ratios in columns (3) and (4)). Finally, the Paasche output quantity index is obtained by adding the share-weighted reciprocal of the quantity ratios and then taking the reciprocal (that is, adding the values in columns (8) and (9) and then taking the reciprocal to yield the values in column (10)).

The Fisher output quantity index is found by taking the geometric average of the Laspeyres and Paasche output quantity index. The Fisher output quantity index is shown in column (11) of Table 3.

Table 3 – Laspeyres, Paasche, Fisher and Törnqvist output quantity index

Panel I	(1) w_t^Y	(2) w_t^Z	(3) $\frac{q_t^Y}{q_0^Y}$	(4) $\frac{q_t^Z}{q_0^Z}$		
$t=0$	0.545	0.455	1.000	1.000		
$t=1$	0.467	0.533	1.167	1.200		
$t=2$	0.400	0.600	1.333	1.600		
Panel II	(5) $w_0^Y \frac{q_t^Y}{q_0^Y}$	(6) $w_0^Z \frac{q_t^Z}{q_0^Z}$	(7) Q_t^L	(8) $w_t^Y \left(\frac{q_t^Y}{q_0^Y}\right)^{-1}$	(9) $w_t^Z \left(\frac{q_t^Z}{q_0^Z}\right)^{-1}$	(10) Q_t^P
$t=0$	0.545	0.455	1.000	0.545	0.455	1.000
$t=1$	0.636	0.545	1.182	0.400	0.444	1.184
$t=2$	0.727	0.727	1.455	0.300	0.375	1.481
Panel III	(11) Q_t^F	(12) $\left(\frac{q_t^Y}{q_0^Y}\right)^{0.5(w_0^Y + w_t^Y)}$	(13) $\left(\frac{q_t^Z}{q_0^Z}\right)^{0.5(w_0^Z + w_t^Z)}$	(14) Q_t^T		
$t=0$	1.000	1.000	1.000	1.000		
$t=1$	1.183	1.081	1.094	1.183		
$t=2$	1.468	1.146	1.281	1.468		

The first step in calculating the Törnqvist output quantity index is to take the quantity ratios to the power of the arithmetic average of the nominal output shares in period 0 and period t. The resulting exponential weighted quantity ratios are reported in columns (12) and (13) of Table 3. The final step in calculating the Törnqvist output quantity index is found by taking the product of the exponential weighted quantity ratios (that is, multiplying column (12) by column (13) for each period to yield the values in column (14)).

Input quantity indices can be constructed in a similar manner using the input quantity and price data displayed in Table 1. Productivity indices are found by taking the ratio of the output quantity index to the respective input quantity index. Both the input quantity and productivity indices are reported in Table 4.

Table 4 – Input and productivity indices

	Input indices				Productivity indices			
	X_t^L	X_t^P	X_t^F	X_t^T	A_t^L	A_t^P	A_t^F	A_t^T
$t=0$	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
$t=1$	1.212	1.154	1.183	1.184	0.975	1.026	1.000	0.999
$t=2$	1.515	1.250	1.376	1.389	0.960	1.185	1.067	1.057

The multifactor productivity indices displayed in Table 4 show marked differences in productivity growth for the hypothetical economy depending on which index number formulae is used. For example, the Laspeyres productivity index (A_t^L) suggests multifactor productivity decreased by 4% between period 0 and period 2, while the Paasche multifactor productivity index (A_t^P) suggests productivity increased by 18.5%. The increase in the Fisher productivity index (A_t^F) lies between the increases in the Laspeyres (A_t^L) and Paasche (A_t^P) productivity indices (as it is their geometric mean). The Törnqvist productivity index (A_t^T) shows a similar increase to the Fisher productivity index, as the Törnqvist index usually approximates the Fisher index quite closely.

3 Chained indices

In addition to choosing an index number formula, a choice needs to be made whether to construct fixed-weight or chained indices. A fixed-weight quantity index compares quantities in period t relative to some fixed base period (which is why fixed-weight indices are also known as fixed-base indices). Information on price movements and therefore weighting changes in the intervening periods are ignored. In contrast, a chained index compares quantities between two periods taking into account information on weighting changes in the intervening period or periods. In other words, a chained index uses price information that is more representative of that faced by economic agents in each period than does a fixed-weight index.

When relative prices change, relative quantities also usually change. For example, if the price of a particular good rises relative to all other goods in an economy because of an increase in demand, then price taking firms will tend to produce more of this good relative to other goods. Alternatively, consumers will tend to substitute away from goods that have become relatively more expensive to less expensive goods. Using a fixed-weight index to measure quantity changes in the presence of relative price changes will introduce substitution bias into the quantity index because information on relative price changes is not taken into account when measuring quantity changes. Moreover, the substitution bias usually becomes larger over time, as the fixed weights become more unrepresentative of those faced by agents when measuring quantity changes in more recent periods. Chaining fixed-weight indices helps to alleviate the substitution bias.

The following numerical example illustrates how the use of a fixed-weight index may give a distorted measure of quantity movements, owing to substitution bias, and how chaining helps to alleviate this problem. Construction of the Laspeyres output quantity index, using nominal output shares in period 0 as weights for the quantity ratios, was shown in Table 3 (and is also reproduced in panel I of Table 5). This Laspeyres quantity index indicates that aggregate output in the two good economy increased by 18% between period 0 and period 1 and 23% between period 1 and period 2. Because the price of q^Z is increasing relatively more than the price of q^Y, firms will produce relatively more of q^Z to reap higher profits. However, because weights are fixed in period 0, too little weight is given to q^Z and too much weight to q^Y when constructing the aggregate output quantity index beyond period 1. In other words, in this economy with price taking firms, the Laspeyres output quantity index is biased downwards, tending to understate aggregate quantity movements.

Table 5 – Chaining using Laspeyres output indices

Panel I **Fixed-weight Laspeyres index (Base weights period 0)**	$w_0^Y \frac{q_t^Y}{q_0^Y}$	$w_0^Z \frac{q_t^Z}{q_0^Z}$	$Q_{t,0}^L$	Percentage change
$t=0$	0.545	0.455	1.000	
$t=1$	0.636	0.545	1.182	18.2%
$t=2$	0.727	0.727	1.455	23.1%
Panel II **Fixed-weight Laspeyres index (Base weights period 1)**	$w_1^Y \frac{q_t^Y}{q_1^Y}$	$w_1^Z \frac{q_t^Z}{q_1^Z}$	$Q_{t,1}^L$	Percentage change
$t=0$	0.400	0.379	0.779	
$t=1$	0.467	0.533	1.000	28.4%
$t=2$	0.533	0.800	1.333	33.3%
Panel III **Chained Laspeyres index**			$Q_{t,C}^L$	Percentage change
$t=0$			1.000	
$t=1$			1.182	18.2%
$t=2$			1.576	33.3%

The downward bias in aggregate output movements using the Laspeyres quantity index with nominal output shares fixed in period 0 is shown by constructing the Laspeyres output quantity index using nominal output shares from period 1 as the fixed weights. This is shown in panel II of Table 5. The increase in aggregate output between period 1 and period 2 is 33%, compared to an increase of 23% when the Laspeyres output quantity index is calculated using nominal output shares from period 0 as the fixed weights. The larger increase in aggregate output when weights are used from period 1 is owing to greater weight being given to increases in q^Z and lesser weight being given to increases in q^Y than when period 0 weights are used.

The chained output quantity index is formed by linking fixed weight quantity indices. In the above example, the percentage increase in aggregate output between period 0 and 1 is derived from the fixed-weight Laspeyres index constructed using fixed weights from period 0 (see panel III of Table 5). The percentage increase in the chained Laspeyres quantity index between period 1 and 2 is derived from the fixed-weight Laspeyres quantity index that was constructed using fixed weights from period 1.

More generally, a chained index is constructed as follows:

$$C_{0,t} = 1 \times D_{0,1} \times D_{1,2} \times ... \times D_{t-1,t} \quad (7)$$

where $C_{0,t}$ denotes the chained index between period 0 and t and $D_{t-1,t}$ the direct index between period $t-1$ and t. Chaining can be applied to any of the index number formulae outlined in equations (2) to (5). Using the example presented in Table 5, the chained Laspeyres index in period 2 is calculated as follows:

$$\begin{aligned} C_{0,2} &= 1 \times D_{0,1} \times D_{1,2} \\ &= 1 \times 1.182 \times 1.333 \\ &= 1.576 \end{aligned} \quad (8)$$

4 Physical capital stock and the user cost of capital

Measuring capital and multifactor productivity requires measures of physical capital inputs. As the flow of physical capital services is not directly observable, productivity analysts usually assume the flow of capital services is proportional to the capital stock. Ideally the capital stock measure should be formed taking into account the loss in productive capacity of capital assets that occurs over time. Subsection 4.1 discusses measurement of productive capital stocks, where stocks are constructed from past investments and where the loss in the productive capacity of capital assets is taken into account. Subsection 4.2 illustrates the construction of the productive capital stock using a simple numerical example. Finally, subsections 4.3 and 4.4 discuss and illustrate measurement of rental prices and how these are linked to economic depreciation and the loss in productive capacity of capital assets. Readers interested in further details on measuring physical capital stocks and rental prices should consult, for example, Hulten (1990), Hulten and Wykoff (1995), and Diewert and Lawrence (2000).

4.1 Productive capital stock

Productive capital stocks endeavour to measure the total productive capacity of different types of capital assets in existence at a point in time. Suppose information on investments in a particular asset type is available for period $t-s$ to period t for $s = 0,...,S$ and is denoted by the vector $I \equiv (I_{t-S}, I_{t-S+1},...,I_{t-1}, I_t)$. Furthermore assume that the productive capacity of an asset in period t that is now s periods old (that is, the s-vintage asset) is given by:

$$R_{t,s} = \phi_s I_{t-s} \quad (9)$$

where ϕ_s denotes the relative productive capacity of a s-vintage asset to the productive capacity of a new asset. The series ϕ_s is known as the age-efficiency schedule and is usually normalised so that $\phi_0 = 1$. The age-efficiency schedule shows the decline in the productive capacity of an asset over its economic life.[11]

Three commonly used age-efficiency patterns are the linear, 'one-hoss-shay' and geometric age-efficiency schedules. The linear age-efficiency schedule assumes that the productive capacity of an asset depreciates linearly over the asset's economic life. The 'one-hoss-shay' efficiency pattern assumes the productive capacity of an asset remains constant over its economic life but then falls to zero when the asset's economic life ends.[12] The geometric age-efficiency pattern assumes the productive capacity of an asset declines at a constant rate.

Given real investment data, the functional form of the age-efficiency schedule, and assuming past vintages of a particular asset can be aggregated, the productive capital stock for a particular asset type in period t is calculated as follows:

$$K_t = \sum_{s=0}^{S} \phi_s I_{t-s} \qquad (10)$$

Equation (10) is known as the perpetual inventory model of the productive capital stock.[13]

4.2 Numerical example of the productive capital stock

Table 6 presents hypothetical data on the (new) price of purchasing a capital asset (p_t) and the nominal investment in the asset (V_t). The volume of investment in the asset (I_t) is found by dividing the nominal investment series by the price series (that is, dividing column (2) by column (1)). The age-efficiency schedule, which is displayed in column (4), assumes the life of the asset is five years and the loss in productive capacity is linear (that is, the age-efficiency schedule (ϕ_t) is linear).

Table 6 – Productive capital stock

	(1) p_t	(2) V_t	(3) I_t	(4) ϕ_s	(5) I_0	(6) I_1	(7) I_2	(8) I_3	(9) I_4	(10) I_5	(11) K_t
$t,s=0$	1.0	100.0	100.0	1.00	100.0						100.0
$t,s=1$	1.1	120.0	109.1	0.80	80.0	109.1					189.1
$t,s=2$	1.3	150.0	115.4	0.60	60.0	87.3	115.4				262.7
$t,s=3$	1.5	160.0	106.7	0.40	40.0	65.5	92.3	106.7			304.4
$t,s=4$	1.6	180.0	112.5	0.20	20.0	43.6	69.2	85.3	112.5		330.7
$t,s=5$	1.7	195.0	114.7	0.00	0.0	21.8	46.2	64.0	90.0	114.7	336.7

To calculate the productive capital stock for the asset it is first necessary to account for the diminished productivity capacity of each investment over time. Columns (5) to (10)

[11] This exposition ignores survival probabilities for capital assets to simplify the analysis. If survival probabilities are introduced into the analysis, the age-efficiency schedule represents the loss in the productive capacity of a capital asset conditional on survival.
[12] The 'one-hoss-shay' efficiency schedule is also known as the 'light bulb' efficiency pattern because a light bulb delivers a constant flow of capital services before its life ends.
[13] Equation (10) is often augmented to include the initial capital stock in period *t-s*.

present the productive capacity of the asset for each investment. These series are calculated by multiplying the initial investment by the relevant value from the age-efficiency schedule. For example, to find the productive capacity of an investment in the asset in $t=3$ that was made in $t=0$, the initial investment (I_0) of 100 is multiplied by 0.4, the value of the age-efficiency schedule for a three period old asset (ϕ_3). Likewise, the productive capacity of an investment in the asset in $t=4$ that was made in $t=2$ (69.2) is found by multiplying 115.4 (I_2) by 0.6, the value of the age-efficiency schedule for a two period old asset (ϕ_2).

Finally, the productive capital stock of the asset is calculated using equation (10) by adding the efficiency adjusted investments for each period. The resulting productive capital stock for the asset is reported in column (11).

4.3 Age-efficiency and age-price schedules, economic depreciation, and the user cost of capital

When forming an aggregate quantity index, prices are used to weight the different quantities. Hall (1968) showed that the rental price of capital (user cost of capital) is the relevant price when aggregating different types of capital For some assets the rental price for different asset vintages is observable because there is an active rental market (for example, residential and non-residential buildings). However, for other assets, rental markets do not exist or are very thin (for example, rental markets may not exist for certain types of specialised machinery). In this situation firms purchase capital assets and pay an implicit rental for their use. Although the user cost of capital is not directly observable, it can be imputed using information on the price of a new asset, the rate of economic depreciation, and asset price inflation.

The assumption of perfect competition implies the price of a s-vintage asset in period t $(P_{t,s})$ is equal to the discounted stream of future rentals $(U_{t,s})$, that is:

$$P_{t,s}=\sum_{v=0}^{V}\frac{U_{t+v,s+v}}{(1+r)^{1+v}} \tag{11}$$

where r is the discount rate and V the time the asset's life ends.

Noting that $P_{t+1,s+1}=\frac{U_{t+1,s+1}}{1+r}+\frac{U_{t+2,s+2}}{(1+r)^2}+...+\frac{U_{t+V,s+V}}{(1+r)^{1+V}}$, equation (11) can be rewritten as:

$$P_{t,s}=\frac{U_{t,s}}{(1+r)}+\frac{P_{t+1,s+1}}{(1+r)} \tag{12}$$

If $s=0$ equation (12) states that the price of a new asset at the beginning of period t is equal to the discounted value of the rental for period t plus the discounted price of the one period old asset at the beginning of $t+1$.

When measuring the user cost of capital it is common to incorporate asset price inflation. When asset price inflation is incorporated the price of a s-vintage asset in period $t+1$ is equal to the price of the s-vintage asset in period t multiplied by one plus that rate of asset price inflation:

$$P_{t+1,s} = P_{t,s}(1+\pi_t) \quad (13)$$

where π_t is the asset price rate of inflation.

Substituting the similar expression for $P_{t+1,s+1}$ from equation (13) into equation (12) and solving for the user cost of capital yields:

$$\begin{aligned} U_{t,s} &= P_{t,s}(1+r) - P_{t,s+1}(1+\pi_t) \\ &= P_{t,s}r + (P_{t,s} - P_{t,s+1}) - (P_{t+1,s+1} - P_{t,s+1}) \end{aligned} \quad (14)$$

Equation (14) is the basic user cost formula. The first term of equation (14) is the finance cost associated with purchasing the asset (or the opportunity cost from investing the funds used to purchase the asset elsewhere). The second term is the loss in the value of the asset due to ageing and represents economic depreciation. The final term represents the capital gains or losses associated with owning the asset.

Equation (14) is often expressed in rate form as follows:

$$U_{t,s} = P_{t,s}(r - \pi_t + (1+\pi_t)d_{t,s}) \quad (15)$$

where $d_{t,s}$ is the depreciation rate from an s-vintage asset in period t.[14]

In equation (15) the price of a new asset ($P_{t,0}$) and *ex post* asset inflation (π_t) are observable. The discount rate (r) can be obtained from financial markets or an *ex post* internal rate of return can be computed using the approach suggested by Jorgenson and Griliches (1967). The latter approach involves equating capital income with the product of the user cost of capital and the productive capital stock and then solving for the discount rate.

To calculate the economic depreciation rate ($d_{t,s}$) information is used on the age-efficiency schedule that is used to calculate the productive capital stock. This recognises the fact that:

> One cannot select an efficiency pattern independently of the depreciation pattern and maintain the assumption of competitive equilibrium at the same time. And, one cannot arbitrarily select a depreciation pattern independently from the observed pattern of vintage asset prices P_s^t (suggesting a strategy for measuring depreciation and efficiency).
>
> Hulten 1990:129

To calculate economic depreciation using the age-efficiency schedule it is first necessary to calculate the age-price schedule from the age-efficiency schedule. The age-price schedule (θ_s) gives the relative value of a s-vintage asset to the value of a new asset. Economic depreciation for a particular asset is then calculated by tracing the loss in value of an investment which is derived by multiplying the initial investment by the relevant value from the age-price schedule. The age-price schedule is usually normalised so that $\theta_0 = 1$.

[14] The basic user cost of capital formula shown in equation (15) can be augmented to include additional information such as the impact of taxation on the user cost of capital.

The link between the age-efficiency schedule and age-price profile can be seen as follows. In a competitive market the ratio of the s-vintage rental price to the $s+1$-vintage rental price asset is equal to the relative efficiency of the s-vintage asset to the $s+1$-vintage asset:

$$\frac{U_{t,s+1}}{U_{t,s}} = \phi_{s+1} \tag{16}$$

Substituting equation (16) into equation (11) yields:

$$\begin{aligned} P_{t,s} &= \frac{\phi_s U_{t,s}}{(1+r)} + \frac{\phi_{s+1} U_{t+1,s}}{(1+r)^2} + \ldots + \frac{\phi_{s+V} U_{t+V,s}}{(1+r)^{1+V}} \\ &= \sum_{v=0}^{V} \frac{\phi_{s+v} U_{t+v,s}}{(1+r)^{1+v}} \end{aligned} \tag{17}$$

Hence the vintage asset price sequence is a function of the age-efficiency profile.

Assuming the user cost of capital grows at a constant nominal rate (that is, $U_{t+v,s} = U_{t,s}(1+g)^v$ for $v = 0,...,V$), then equation (17) can be rewritten as:

$$P_{t,s} = \sum_{v=0}^{V} \frac{\phi_{s+v} U_{t,s} (1+g)^v}{(1+r)^{1+v}} \tag{18}$$

Consider a new asset purchased at period t for $P_{t,0}$ and its subsequent sequence of vintage prices. The rental price in period 0 is found by solving equation (17) for $U_{t,0}$ (which is possible since $P_{t,0}$ is observed). The subsequent sequence of vintage prices is also calculated by using equation (17) and the assumption that the rental price grows at a constant rate.

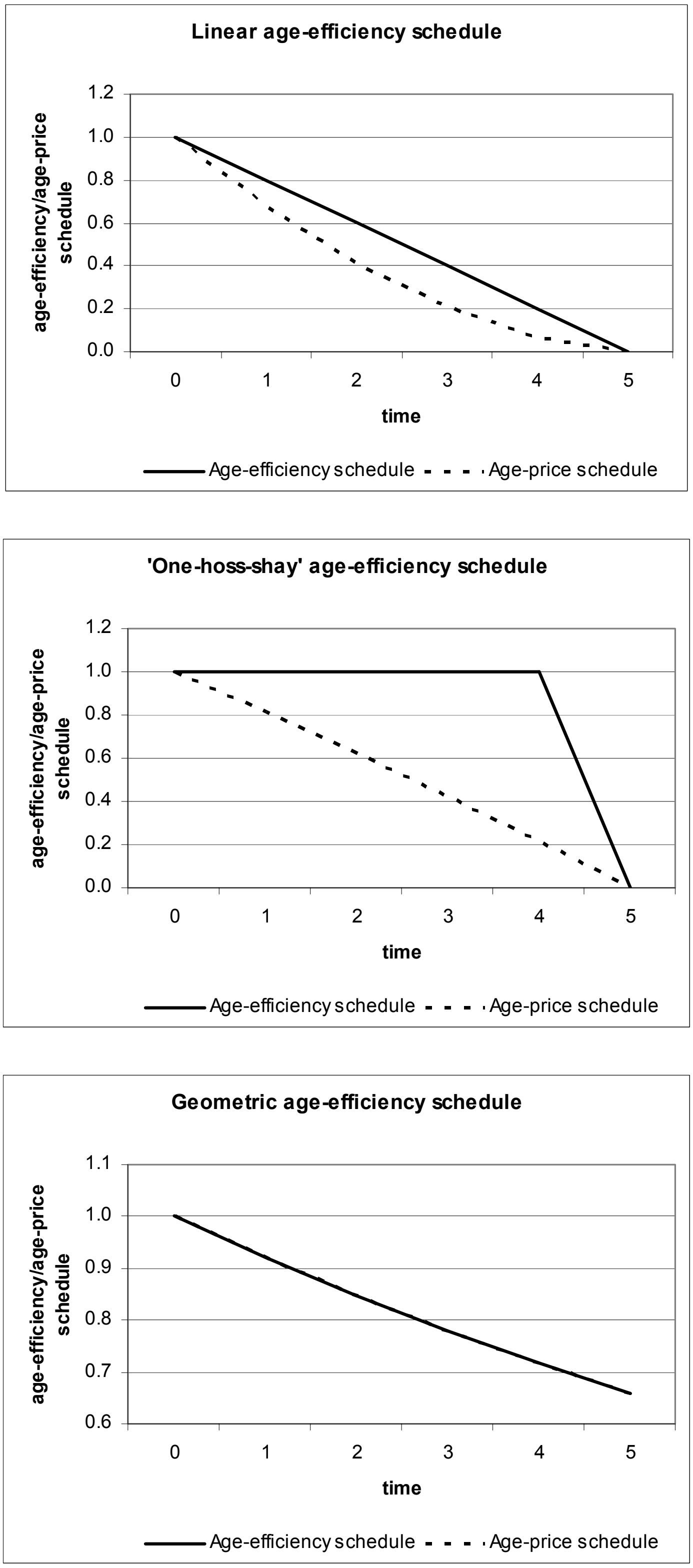
Values of the age-price schedule are found by taking the price of the new asset to the price of the s-vintage asset:

$$\theta_s = \frac{P_{t,s}}{P_{t,0}} \tag{19}$$

where $P_{t,0}$ is the price of the new asset in period t.

Figure 1 shows the age-efficiency and age-price schedules for the linear, 'one-hoss-shay', and geometric efficiency patterns. The age-price schedules are calculated from the age-efficiency schedules assuming rental prices grow at a constant 2% and that the discount rate is 6%. When using the linear and 'one-hoss-shay' age-efficiency schedules the economic life of the asset is assumed to be five periods. Figure 1 illustrates the linear age-efficiency schedule produces a non-linear age price schedule and the 'one-hoss-shay' age-efficiency schedule produces a linear age-price profile. The geometric age-efficiency schedule produces an identical age-price schedule.

Figure 1 – Age-efficiency and age-price schedules



Once the age-price schedule is obtained from the age-efficiency schedule, economic depreciation (D_t) can then be derived as follows:

$$D_t = \sum_{s=1}^{S} (\theta_s I_{t-s} - \theta_{s-1} I_{t-s-1}) \tag{20}$$

Equation (20) shows that economic depreciation is calculated by summing the loss in value of the s-vintage investments between two consecutive periods.

The real net capital stock, which is the logical base for the economic depreciation rate, is calculated by summing the value for each investment after adjusting for economic depreciation (using the age-price schedule):

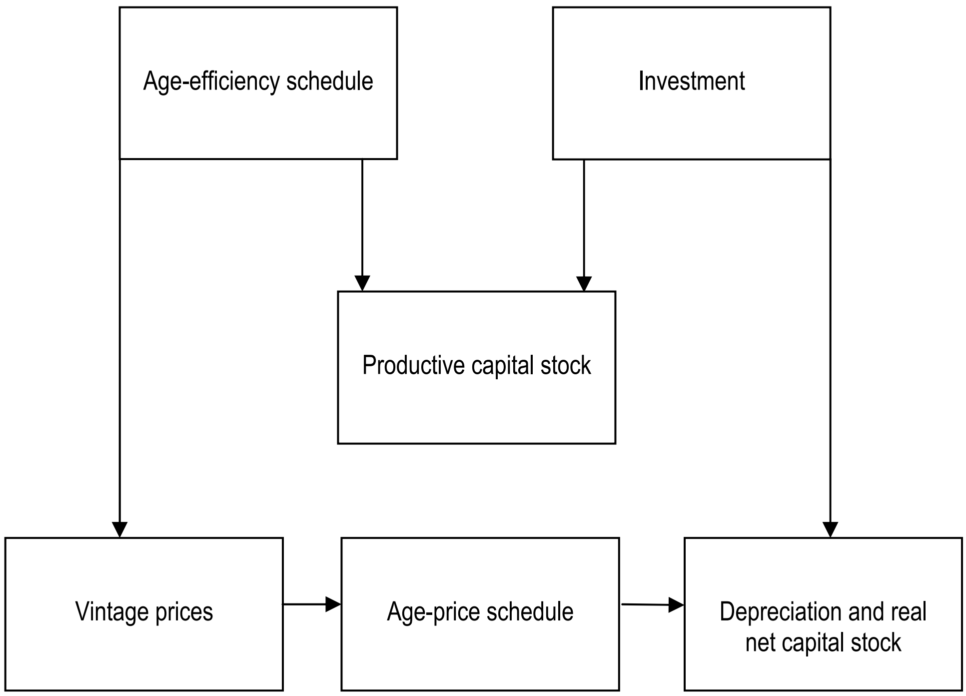
$$N_t = \sum_{s=0}^{S} \theta_s I_{t-s} \tag{21}$$

The economic depreciation rate in period t is then calculated by taking the ratio of economic depreciation to the real net capital stock, that is:

$$d_t = \frac{D_t}{N_t} \tag{22}$$

To summarise, Figure 2 shows the link between the age-efficiency schedule, the age price schedule and economic depreciation. It also serves to highlight the point that the loss in productive capacity of an asset cannot be determined independently of the depreciation pattern. Figure 2 also shows the relationship between productive capital stocks and real net capital stocks.

Figure 2 – Age-efficiency schedule, age price schedule and economic depreciation



4.4 Numerical example of the net capital stock and economic depreciation

Table 7 presents information to construct the economic depreciation rate for the asset (d_t), which is needed to construct the user cost of capital (U_t). Based on the linear age-efficiency schedule presented in column (4) of Table 6, column (1) of Table 7 displays the corresponding age-price schedule. The age-price schedule is calculated assuming a discount rate of 6% and that the rental price grows at a constant 2%. To calculate economic depreciation for the asset (D_t) it is first necessary to trace the loss in the value of each investment in the asset over time. This is presented in columns (2) to (7) of Table 7. Economic depreciation (D_t) is calculated by summing the loss in value of each investment between two consecutive periods. For example depreciation in $t = 2$ is equal to the loss in value of the initial investment (I_0) between $t = 1$ and $t = 2$ (67.5 – 41 ≈ 26.5) plus the loss in value of the investment made in $t = 1$ (109.1 – 73.6 ≈ 35.5), which equals 61.9.

Table 7 – Net capital stock and depreciation

	(1) θ_s	(2) I_0	(3) I_1	(4) I_2	(5) I_3	(6) I_4	(7) I_5	(8) D_t	(9) N_t	(10) d_t
$t,s=0$	1.00	100.0							100.0	
$t,s=1$	0.67	67.5	109.1					32.5	176.6	0.2
$t,s=2$	0.41	41.0	73.6	115.4				61.9	230.0	0.3
$t,s=3$	0.21	20.8	44.7	77.9	112.5			86.6	255.9	0.3
$t,s=4$	0.07	7.0	22.7	47.3	75.9	112.5		103.0	265.4	0.4
$t,s=5$	0.00	0.0	7.6	24.0	46.1	75.9	114.7	111.7	268.4	0.4

The base used to calculate the depreciation rate is the net capital stock. This is calculated by adding the age-price weighted investment for each period and is reported in column (9) of Table 7. The economic depreciation rate (d_t) is displayed in column (10) of Table 7.

5 Measuring quality-adjusted labour input

Multifactor productivity is most often measured using labour and physical capital inputs. The number of hours worked rather than the number of people employed or the number of hours paid is usually the preferred labour input when measuring productivity. This is because the number of people employed does not capture changes in the number of hours worked by each worker nor changes in the composition of part-time versus full-time workers, while the number of hours paid may not accurately capture the number of hours actually worked by salaried workers.

If the number of hours worked is used as the labour input when constructing productivity measures, differences in the human capital associated with each hour worked are not accounted for. Essentially the hours worked by different types of workers are treated as if they were all identical, with differences in the human capital, or the quality of workers subsumed within the productivity measure. For example, the difference in the human capital embodied in the hours worked by a heart surgeon and a school teacher will be ascribed to the productivity measure. Moreover, changes in the human capital of workers owing to further education or greater work experience will be captured by changes in productivity over time.

Productivity analysts are often interested in gauging the contribution to changes in aggregate output from changes in the human capital or the quality of labour inputs. This requires an adjustment for differences in the quality of hours worked by different types of workers. This is done by separately accounting for different types of labour inputs when forming productivity measures.[15]

In section 2, when discussing the economic approach to choosing an index number formula, the aggregate production function was denoted as follows:

$$Q_t = f(A_t, K_t, H_t) \tag{23}$$

where H_t denoted the aggregate number of hours worked, and was calculated by summing over hours worked at the sub-aggregate level (for example industries). It was also outlined that when the production function was given the translog functional form, the continuous time (Divisia) index could be approximated using the Törnqvist index formula.

An alternative specification to the production function presented in equation (23) is the following:

$$Q_t = g(B_t, k_t^1, ..., k_t^M, h_t^1, ..., h_t^N) \tag{24}$$

In this specification each of the capital inputs ($k_t^1, ..., k_t^M$) and each of the labour inputs ($h_t^1, ..., h_t^N$) are accounted for separately. B_t denotes the alternative measure of multifactor productivity (which is interpreted below). As discussed in section 3, when the productive capital stocks of various asset types are used, the aggregate capital stock measure is formed using the corresponding user cost of capital measures as weights in the index formula. Likewise, when different types of labour inputs are used, income shares for the different types of labour inputs are used as weights in the index formula.

The difference between the multifactor productivity measure (A_t) corresponding to the underlying production function in equation (23) and the multifactor productivity measure (B_t) corresponding to equation (24) is the latter measure accounts for changes in the composition or quality of labour inputs. This can be seen from the analysis that follows.

Consider the case where the Törnqvist index is used to measure multifactor productivity. Assuming that the aggregate capital stock has been formed using rental prices for different asset types, the multifactor productivity indices can be written as follows:

[15] Related to the measurement of changes in the quality or composition of labour inputs is measuring human capital stocks for inclusion in (comprehensive) national wealth accounts. Constructing human capital stocks using the lifetime-income method has recently been revived by Jorgenson and Fraumeni (1989 and 1992), although this approach has a long ancestry in economics dating back to work by Petty (1690). The lifetime-income approach to measuring human capital stocks has similarities with the method used to measure the productive capital stocks discussed in section 4, where the discounted stream of future rentals is used to value an asset at a point in time. The lifetime-income income approach values an individual's human capital using the discounted stream of future labour market income.

$$A_t = \frac{Q_t}{K_t^{\frac{1}{2}(w_0^K + w_t^K)} H_t^{\frac{1}{2}(w_0^L + w_t^L)}} \quad (25)$$

$$B_t = \frac{Q_t}{K_t^{\frac{1}{2}(w_0^K + w_t^K)} L_t^{\frac{1}{2}(w_0^L + w_t^L)}} \quad (26)$$

where w_t^K is capital's income share and w_t^L is labours income share, $H_t = \frac{\sum_{n=1}^{N} h_t^n}{\sum_{n=1}^{N} h_0^n}$ the total number of hours worked, and $L_t = \prod_{n=1}^{N} \left(\frac{h_t^n}{h_0^n} \right)^{\frac{1}{2}(w_0^n + w_t^n)}$ a Törnqvist index of aggregate labour input. Furthermore, the labour quality index (LC_t) can be written as:

$$LC_t = \frac{L_t}{H_t} \quad (27)$$

This index is the ratio of the aggregate labour input index to an index of total hours worked. This labour quality index is akin to that adopted in work by Jorgenson, Gallop and Fraumeni (1987) and Jorgenson and Fraumeni (1989, 1992).[16]

Finally, substituting equations (25), (26) and (27) yields the following index for multifactor productivity:

$$B_t = A_t LC_t^{\frac{1}{2}(w_0^L + w_t^L)} \quad (28)$$

Equation (28) shows the alternative multifactor productivity index (B_t) is simply the original multifactor productivity (A_t) adjusted for the quality composition of the labour input.

In forming the alternative multifactor productivity measure (B_t) it is necessary to have estimates of labour income shares for the various labour types. Labour shares for the various labour types can be estimated in one of two ways. One approach is to classify the labour inputs into different categories based on the characteristics of various workers and then use the average wage in forming labour shares for the various labour inputs. For example, workers could be classified into various categories based on their level of educational qualification. This approach has been adopted in work by Jorgenson, Gallop and Fraumeni (1987).

An alternative approach is to estimate wage equations econometrically using worker characteristics, such as the number of years worked, as explanatory variables and then use the predicted values from the wage equations to form weights for the various types of labour inputs. This approach has been used by the Bureau of Labour Statistics (1993)

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[16] In Jorgenson and Griliches (1967) the interpretation given to their version of equation (27) was that owing to errors in aggregation of labour services. More recently Jorgenson, Gallop and Fraumeni (1987) and Jorgenson and Fraumeni (1989, 1992) have interpreted this as a labour quality index.

when forming their multifactor productivity estimates that accounts for changes in the composition of labour over time (Bureau of Labour Statistics, 1993).

6 Conclusion

This paper provides an introduction to productivity measurement using the index number approach. Consideration was given to this approach, rather than alternative approaches to productivity measurement, owing to the widespread use of index techniques in constructing economic aggregates and because the index number approach is used by statistical agencies in constructing official productivity measures. Attention was given to common index number formulae, whose application were illustrated using simple numerical examples, and approaches to choosing an index number formula. Special attention was also given to measuring physical capital inputs using an integrated framework that links productive capital stocks, economic depreciation, and rental prices; and measuring quality adjusted labour inputs. In regard to the latter, the alternative measure of multifactor productivity that incorporated quality-adjusted labour inputs was shown to be the original multifactor productivity measure adjusted for the composition of the aggregate labour input.

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