Copula based simulation procedures for pricing basket Credit Derivatives

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Abstract

This paper deals with the impact of structure of dependency and the choice of procedures for rare-event simulation on the pricing of multi-name credit derivatives such as n^{th} to default swap and Collateralized Debt Obligations (CDO). The correlation between names defaulting has an effect on the value of the basket credit derivatives. We present a copula based simulation procedure for pricing basket default swaps and CDO under different structure of dependency and assessing the influence of different price drivers (correlation, hazard rates and recovery rates) on modelling portfolio losses. Gaussian copulas and Monte Carlo simulation is widely used to measure the default risk in basket credit derivatives. Default risk is often considered as a rare-event and then, many studies have shown that many distributions have fatter tails than those captured by the normal distribution. Subsequently, the choice of copula and the choice of procedures for rare-event simulation govern the pricing of basket credit derivatives. An alternative to the Gaussian copula is Clayton copula and t-student copula under importance sampling procedures for simulation which captures the dependence structure between the underlying variables at extreme values and certain values of the input random variables in a simulation have more impact on the parameter being estimated than others .

Keywords: Collateralized Debt Obligations, Basket Default Swaps, Monte Carlo method, One factor Gaussian copula, Clayton copula, t-student copula, importance sampling.

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1. Introduction

A credit derivative is an over-the-counter derivative designed to transfer credit risk from one party to another. By synthetically creating or eliminating credit exposures, they allow institutions to more effectively manage credit risks. Four of the most common credit derivatives are credit default swap, credit linked notes, total return swap and credit spread options. The dominant product in the credit derivatives market is the credit default swap. However, the last ten years or so has seen the growth of 'portfolio credit derivatives' such as basket default swap and Collateralised Debt Obligation (CDO)[1]. These financial instruments have been used successfully by large financial institutions to diversify and reduce credit risk. Many empirical works has been done on single name credit derivative products. Hull, Predescu & White (2004) analyze the impact of credit rating announcements in the pricing of credit default swap. Norden & Weber (2004) analyze the empirical relationship between credit default swap, bond and stock markets. Ericsson, Jacobs & Oviedo (2004) investigate the relationship between theoretical determinants of default risk (firm leverage, volatility and the riskless interest rate) and actual market spread of credit default swap using linear regression. Abid & Naifar (2006 (a)) explain empirically the determinants of credit default swap rates using a linear regression. They find that credit rating, maturity, riskless interest rate, slope of the yield curve and volatility of equities explain more than 60% of the total level of credit default swap.

Actually, more substantial empirical studies are devoted on structured credit derivatives instruments, in particular basket default swap and CDO. The main problem in the pricing of such instruments is modelling the structure of dependency of the default times. Defaults are rarely observed. Copulas can be introduced to model these correlations by using the correlations of corresponding default time. We know that Kendall's tau remains invariant under monotone transformations. This is the foundation of modelling the correlation of credit events by using the correlation of underlying default time via copulas. Li (2000) present a Gaussian copula method for the pricing of first to default swap. Other studies of elliptical copulas with higher tail dependence, such as the t-copula, can be found in Mashal and Naldi (2002). The Marshall-Olkin copula is yet another class of copula functions, which stems from the multivariate compound Poisson process. In this model, individual defaults are constructed from a series of independent common shock. Previous work on the use of the Marshall-Olkin copula in the context of credit risk modelling includes Duffie and Pan (2001), Wong (2000), Lindskog and McNeil (2003).Hull & White (2004) develop two procedures to pricing tranches of CDO and n^{th} to default swap. The first procedure involves calculating the probability distribution of the number of defaults by a time T and suited to the situation where companies have equal weight in the portfolio and recovery rates are assumed to be constant. The second involves calculating the probability distribution of the total loss from defaults by time T. Jobst (2002) propose a pricing model that draws expected loan loss of CDO based on parametric bootstrapping through extreme value theory under the impact of asymmetric information. Tavares et al. (2004) present a basket model to deal with the Gaussian copula smile. They combine the copula model (to model the default risk that is driven by the economy) with independent Poisson processes (to model the default risk that is driven by a particular sector and by the company in question).Hull and White (2005) introduce the technique of perfect copulas. Their copula model can be regarded as 'perfect' in that it hits the tranche quotes exactly. The hazard-rate-path probability distribution is the only input about the underlying copula in order to value a CDO. Burtschell et al. (2005) employ the technique of the double Student-t copula model for the calibration of CDO. They find that this copula model fit better the features to the CDO market in comparison to other models like Gaussian, t-Student, stochastic correlation, Clayton and Marshall-Olkin copulas. Madan (2004) provide details for the pricing of n^{th} to default contracts using the one factor Gaussian and the Clayton copulas. He model the marginal default time densities using Weibull and Frechet

[1] According to SIFMA[TM], Global CDO issuance through the third quarter of 2006, at \$322 billion, has exceeded full year 2005 issuance by 20%. Issuance in the third quarter of 2006, at \$117.8 billion, also exceeded issuance in the third quarter of last year by 30%. European Securitisation Forum (ESF) Forecasts Issuance to Grow to a New Record of €531 Billion in 2007, Led by residential mortgage-backed securities (RMBS), commercial mortgage-backed securities (CMBS) and CDO. Then, a 16.4 percent growth rate from the €456 billion issued in 2006.

families and the joint densities are obtained using the method of copula. Verschuere (2006) present a factor approach combined with copula functions to price tranches of synthetic Collateralized Debt Obligation (CDO) having totally inhomogeneous collateral (the obligors in the CDO pool have different spreads and different notional). Sircar and Zariphopoulou (2006) study the impact of risk aversion on the valuation of basket credit derivatives. They use the technology of utility-indifference pricing in intensity based models of default risk.

In our paper, we analyse the impact of structure of dependency and the nature of simulation procedures on the pricing of multi-name credit derivatives such as n^{th} to default swap and Collateralized Debt Obligations (CDO). To express dependencies between times of default, Gaussian, student and Clayton copulas have been considered. The copula function links the univariate margins with their full multivariate distribution. It presents a useful tool when modelling non Gaussian data since the Pearson's correlation coefficient is adapted for linear dependence and normal distribution. One appealing feature of a copula function is that the margins do not depend on the choice of the dependency structure and then, we can model and estimate the structure of dependency and the margins separately.

The remainder of this paper is organised as follows: section two describes some mathematical background about the concept of copula and its properties. In section three, we present some tools for modelling joint default times. Section four and five present a copula based simulation procedures for pricing basket default swaps and CDO, assessing the influence of different price drivers (correlation, hazard rates and recovery rates) on modelling portfolio losses. Section six summarizes the findings and concludes.

2. Stylised facts about Copula functions

The most important problem in the pricing of basket credit derivatives and CDO tranches is the modelling of the joint default times. In this section, we will introduce the concept of a copula function. Copula was first used in survival analysis and actuarial sciences. Copula functions are getting more and more popular credit correlation modelling due to its simplicity and fast computation. Embrechts, et al (1999) clarified many issues concerning dependence and its relationship to correlation, especially in financial data such as market crashes, credit crises. According to Gennheimer (2002) there are several reasons why copulas are such an attractive tool for modelling dependence:

1. They provide us with a powerful tool for building a large number of multivariate models and are extremely useful in the Monte Carlo simulation of dependent risk factors.
2. They allow us to overcome the fallacies and dangers of approaches to dependence that focus only on correlation.
3. They provide a way of studying scale-free measures of dependence.
4. They express dependence on a quantile scale, which we will find is useful for describing the dependence of extreme outcomes.

Frey & McNeil (2002) analyse default correlation and the pricing and hedging of credit sensitive instruments with copulas functions. Jouanin et al (2001) address the problem of incorporating default dependency in intensity-based credit risk models by using copulas functions to model the joint distribution of the default times. Abid & Naifar (2005, 2006(b)) study the impact of Stock returns volatility of reference entities on credit default swap rates using Archimedean copula.
For n uniform random variables $u_1, u_2, ..., u_n$, the joint distribution function C is defined as:

$$C(u_1, u_2, ..., u_n, \theta) = \Pr[U_1 \leq u_1, U_2 \leq u_2, ..., U_n \leq u_n] \qquad (2.1)$$

With θ is the dependence parameter.
We present the following definition for the bivariate case: A copula function is the restriction to $[0,1]^2$ of a continuous bivariate distribution function whose margins are uniform on $[0,1]$. A (bivariate) copula is a function $C:[0,1]^2 \rightarrow [0,1]$ which satisfies the boundary conditions:

$$C(t,0)=C(0,t)=0 \text{ and } C(t,1)=C(1,t)=t \text{ for } t\in[0,1] \quad (2.2)$$

Similarly, copula satisfies the 2-increasing property:

$$C(u_2,v_2)-C(u_2,v_1)-C(u_1,v_2)+C(u_1,v_1)\geq 0 \quad (2.3)$$

For all u_1, u_2, v_1, v_2 in $[0,1]$ and $u_1 \leq u_2$ and $v_1 \leq v_2$.

A copula is symmetric if:

$$C(u,v)=C(v,u) \text{ for all } (u,v) \text{ in } [0,1]^2 \text{ and is asymmetric otherwise.} \quad (2.4)$$

Sklar (1959) shows the importance of copulas as a universal tool for studying multivariate distributions[2]. By definition, applying the cumulative distribution function (CDF) to a random variable (r.v.) results in a r.v. that is uniform on the interval [0, 1]. Let X a random variable with continuous distribution function F_X, $F_X(X)$ is uniformly distributed on the interval [0,1]. This result is known as the probability integral transformation theorem and present many statistical procedures. With this result in hand, we may introduce the copula using basic statistical theory. In particular, the copula C for (X,Y) is just the joint distribution function for the random couple $F_X(X)$, $F_Y(Y)$ provided F_X and F_Y are continuous.

The previous representation is called canonical representation of the distribution. Thus, copulas link joint distribution functions to their margins. Then, in continuous distribution, the problem of obtaining the joint distribution has reduced to selecting the appropriate copula. We can build multidimensional distributions with different marginals.

Copula functions allow us to separate the structure of dependency between default times into two parts: the first part is the specification of the marginal distribution function (the distribution of default time of each obligor. The second part is the choice of the appropriate copula which describes the structure of dependency between default times.

Numerous copulas can be found in the literature (see Nelson (1999) and Joe (1997)). The most commonly applied copula function (especially in finance modelling) is the Gaussian copula[3]. This could be justified by the fact that the multivariate normal distribution has two appealing characteristics: first, their marginal distributions are normal and second, it can be fully described by their marginal distribution and a variance-covariance matrix. For univariate margins $F_1,\ldots,F_n$ which are Gaussians, the dependence structure among the margins is described by a unique normal copula function. Let $X_1,\ldots,X_n$ be random variables which are standard normal distributed with means $\mu_1,\ldots,\mu_n$, standard deviations $\sigma_1,\ldots,\sigma_n$ and correlation matrix Σ. Then, the distribution function $C_\Sigma(u_1,\ldots,u_n)$ of the random variables $U_i=\Phi\left(\frac{X_i-\mu_i}{\sigma_i}\right), i\in\{1,\ldots,n\}$ is a Gaussian copula with correlation matrix Σ. $\Phi(.)$ denotes the cumulative univariate standard normal distribution function.

The Gaussian copula can be written as:

$$C_\Sigma(u_1,\ldots,u_n)=\frac{1}{(2\pi)^{\frac{n}{2}}\sqrt{\det\Sigma}}\int_{-\infty}^{\Phi^{-1}(u_1)}\cdots\int_{-\infty}^{\Phi^{-1}(u_n)}\exp\left(-\frac{1}{2}(\vec{v}-\vec{\mu})^T\Sigma^{-1}(\vec{v}-\vec{\mu})\right)dv_1\ldots dv_n$$

(2.5)

With Φ^{-1} is the inverse of the standard univariate Gaussian distribution function.

By differentiating the precedent equation with respect to $u_1,\ldots,u_n$, we obtain the density of the Gaussian copula:

[2] The original definition of copula is given by Sklar (1959) and the Sklar's theorem is considered as the most important theorem about copula functions. The problem of obtaining a joint distribution is reduced to selecting the appropriate copula.

[3] Credit Metrics ™ and KMV model implicitly incorporate copula functions based on the multivariate Gaussian distribution of asset value process.

$$C_\Sigma(u_1,...,u_n)=\frac{1}{\sqrt{\det\Sigma}}\exp\left(-\frac{1}{2}(v_1,...,v_n)\left(\sum^{-1}-1\right)\begin{pmatrix}v_1\\ .\\ .\\ .\\ v_n\end{pmatrix}\right);\text{ With } v_i=\Phi^{-1}(u_i), i\in\{1,...,n\}. \qquad (2.6)$$

The following algorithm generate random variates $(u_1,...,u_n)$ which are determination of correlated uniform variates on [0,1] from the Gaussian copula with the correlation matrix Σ:

- Find the Cholesky decomposition[4] A of the correlation matrix Σ, such that $\Sigma = A \cdot A^T$;
- Simulate n independent standard normal random variates $Z=(z_1, z_2,..., z_n)^T$;
- Set $x = A \cdot Z$;
- Set x back to an n-dimensional vector u of uniform variates on $[0,1]$ by computing $u = \Phi(x)$. The vector u is a random variate from the n-dimensional Gaussian copula C_Σ.

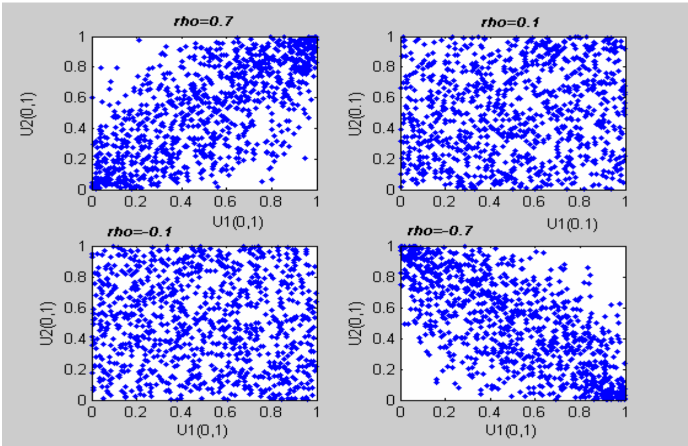

Figure 1:1000 simulated standard uniform random variables under Gaussian copula

If we use a Gaussian copula, we preserve the underlying distribution of the individual random variables but the joint distribution is like a multidimension Gaussian. This naturally assigns very little weight to the tails. In reality, we find that within the financial markets, tail events occur much more frequently. So we would like a joint distribution which has fatter tails but preserves the same (bell shaped, non-skewed) characteristics of the Gaussian, hence we use the t-Student copula.

$$t_v(x)=\int_{-\infty}^{x}\frac{\Gamma\left(\frac{v+1}{2}\right)}{\Gamma\left(\frac{v}{2}\right)\sqrt{v\Pi}}\left(1+\frac{s^2}{v}\right)^{-\frac{v+1}{2}}ds\,. \qquad (2.7)$$

The t-student copula with the correlation matrix Σ and v degrees of freedom is presented as follow:

$$C_{v,\Sigma}(u_1,...,u_n)=\int_{-\infty}^{t_v^{-1}(u_1)}\cdots\int_{-\infty}^{t_v^{-1}(u_n)}k_{v,\Sigma}\left(1+\frac{1}{v}\left(\vec{v}-\vec{u}\right)^T\sum^{-1}\left(\vec{v}-\vec{u}\right)\right)^{-\frac{v+n}{2}}dv_1...dv_2 \qquad (2.8)$$

With:

[4] A symmetric and positive definite matrix can be efficiently decomposed into a lower and upper triangular matrix. For a given matrix, this is achieved by the LU decomposition which factorizes A=LU. If A satisfies the above criteria, one can decompose more efficiently into $A = LL^T$, where L (which can be seen as the ``matrix square root" of A) is a lower triangular matrix with positive diagonal elements. L is called the Cholesky triangle.

$$k_{\nu,\Sigma} = \frac{\Gamma\left(\frac{\nu+n}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{(\Pi\nu)^n \det(\Sigma)}}. \tag{2.9}$$

To simulate random variates from the t-Student copula $C_{\nu,\Sigma}$ with the correlation matrix Σ and ν degrees of freedom, we can use the following algorithm:

- Find the Cholesky decomposition A of the correlation matrix Σ, such that $\Sigma = A \cdot A^T$;
- Simulate n independent standard normal random variates $Z = (z_1, z_2, ..., z_n)^T$;
- simulate a random variate, s, from χ^2_ν distribution, independent of Z;
- Set $y = A \cdot Z$;
- Set $x = y\sqrt{\frac{\nu}{s}}$;
- Set x back to an n-dimensional vector u of uniform variates on $[0,1]$ by computing $u = t_\nu(x)$. The vector u is a random variate from the n-dimensional t-Student copula $C_{\nu,\Sigma}$.

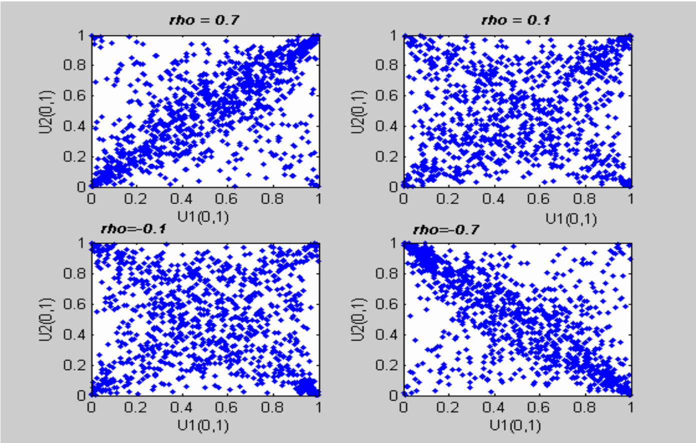

Figure 2: 1000 samples from t-Student copula with degree of freedom equal to 7.

In terms of the appropriate choice for the number of degrees of freedom, it is often necessary to carry out some statistical tests with historical data to ascertain how fat we require the tails to be, Galiani (2003) use an Exact Maximum Likelihood Method (EML). Other woks explain how to calibrate t-student copula to real market data (Mashal and Zeevi (2003), Romano (2002), Meneguzzo and Vecchiato (2002)…). The difference between Gaussian copulas and the t-Student copulas can be described with the concept of tail dependence. If a bivariate copula $C(u,v)$ such as:

$$\lim_{u\to 1} \frac{1 + C(u,u) - 2u}{1-u} = \lambda_U > 0, \tag{2.10}$$

Then C has upper tail dependence with parameter λ_U. If:

$$\lim_{u\to 0} \frac{C(u,u)}{u} = \lambda_L > 0 \tag{2.11}$$

Then C has lower tail dependence with parameter λ_L.

Numerous copulas can be found in the literature (see Nelson (1999) and Joe (1997)). The Archimedean copula has many families that are capable to present different structure of dependency and different methods are developed to estimates its parameters. We only need to find functions which will serve as generators and define the corresponding copula. Clayton copula is an example of Archimedean copula. This family proposed by Clayton (1978) is the following:

Let $\Phi(t) = \frac{\left(t^{-\theta} - 1\right)}{\theta}$ with $\theta \in [-1, \infty)/\{0\}$, then:

$$C_{\theta}^{\text{clayton}}(u, v) = \max\left[\left(u^{-\theta} + v^{-\theta} - 1\right)^{-\frac{1}{\theta}}, 0\right] \tag{2.12}$$

If $\theta > 0$, then $\phi(0) = \infty$ and we can simplify the above expression:

$$C_{\theta}^{\text{clayton}}(u, v) = \left(u^{-\theta} + v^{-\theta} - 1\right)^{-\frac{1}{\theta}} \tag{2.13}$$

With θ expresses the degree of dependence among the marginal components.
To illustrate the range of bivariate behaviour that can be represented by Clayton copula, consider the following figures:

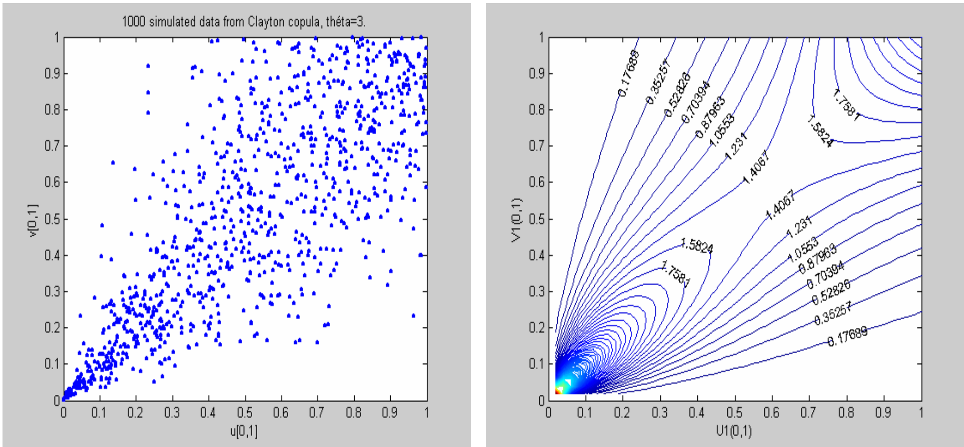

Figure 1: 1000 simulated standard uniform random variables under Clayton copula.

The Clayton copula has lower tail dependence but not upper tail dependence. The contour generated by the Clayton copula implies fat tailed distribution. The contour or the level curves of a copula C are given by $\{(u, v) \in I^2 / C(u, v) = t\}$.

3. Modelling joint default times

In this section, we present some mathematical background for modelling joint default times. We assume a probability space $(\Omega, \Im, \{\Im_t\}_{t \geq 0}, P)$, where Ω is the underlying probability space containing all possible events over a finite time horizon. $\Im$ is a σ-field representing the collection of all events. $\{\Im_t\}_{t \geq 0}$ is a filtration that carries information with times and P is a probability measure[5]. The pricing is assumed under no arbitrage and then, P is risk-neutral measure.
Default time τ_i for each obligor $i = \{1, \ldots, n\}$ should be a random variable and the event of default should be known for everybody at any times because we assume a perfect market with a free flow of information. Default is a stopping time τ with respect to the filtration: $\{\tau < t\} \subset \Im_t$, $\forall t \geq 0$. Let $F(t) = P(\tau \leq t)$ be the distribution function and $f(t)$ is the density function of stopping time, the hazard rate h of τ or intensity process is defined such that we have for the probability of default until time ($t + \Delta t$) given survival till t is given by:

[5] According to Bielecki et al (2005), the probability space is endowed with a filtration $\Im = H \vee F$, where the filtration H carries information about evolutions of credit events, such as changes in credit ratings of perspective credit names and F is some reference filtration.

$$h(t)=\frac{f(t)}{1-F(t)}=\frac{1}{1-F(t)}\times\frac{dF(t)}{dt}=\lim_{\Delta\to 0}\frac{P(t<\tau\le t+\Delta t/\tau>t)}{\Delta t} \tag{3.1}$$

With $dF=(t<\tau\le t+dt,\tau>t)$, $1-F(t)=P(\tau>t)=S(t)$ is called the survival function that gives the probability that a security will attain age t .the hazard rate function gives the instantaneous default probability for a security that has attained age t .The marginal survival distributions $S_i(t_i)$ is assumed to be smooth and strictly decreasing, this can be written as:

$$S_i(t_i)=1-F_i(t_i)=e^{-\int_0^t h_i(s)ds} \tag{3.2}$$

Then $$F_i(t_i)=1-e^{-\int_0^t h_i(s)ds} \tag{3.3}$$

$h_i(.)$ is the default intensity process for entity *i*. The default times τ_i are defined:

$$\tau_i:=\inf\left\{t\ge 0:\int_0^t h_i(s)ds\ge\theta_i\right\} \tag{3.4}$$

Where θ_i has an exponential distribution with unit intensity[6].

For a long time, finance and risk management treated dependence and correlation as basically equivalent. Based upon law of large number arguments, it was assumed that risk factors were normally distributed. This implies that the joint distribution function is determined by the vector of means and the covariance matrix. Effects regarding correlation are particularly strong for some of the most recent innovations in credit markets, namely single-tranche CDOs and basket default swap. Consequently, to value such a contract the joint distribution of default times τ_i :

$$F(t_1,...,t_n)=P(\tau_1\le t_1,...,\tau_n\le t_n) \tag{3.5}$$

The joint survival time distribution is given by:

$$S(t_1,...,t_n)=P(\tau_1>t_1,...,\tau_n>t_n) \tag{3.6}$$

$$=C\left(e^{\left(-\int_0^t h_1(s)ds\right)},e^{\left(-\int_0^t h_2(s)ds\right)},...,e^{\left(-\int_0^t h_n(s)ds\right)}\right) \tag{3.7}$$

For deterministic intensities, this framework converges to Li (200) model .The times of default τ_i are defined as the first time the default countdown processes:

$\lambda_i(t):=e^{\left(-\int_0^t h_i(s)ds\right)}$ reach the level of the trigger variables U_i :

$$\tau_i:=\inf\{t\ge 0:\lambda_i(t)\le U_i\} \tag{3.8}$$

The choice of a dependence structure between default times drives the prices of basket default swaps and CDO tranches. Copulas functions allow us to separate the problem of modelling the default times into two parts: first, the specification of the marginal distribution functions and second, the choice of a suitable copula which describes the dependence structure between the default times. Then, the marginal distributions together with the choice of a suitable copula are sufficient to specify the full joint distribution of the default times.

The benefits for using copulas to modelling joint default times:

- Maintains input correlation matrix reasonably well.
- Distribution-free approach.

[6] The exponential distribution is used to model Poisson processes, which are situations in which an object initially in state A can change to state B with constant probability per unit time λ. The time at which the state actually changes is described by an exponential random variable with parameter λ

- Can be employed in simulation procedures.
- Allows for various dependence structures (including tail dependence).
- Generates an exact joint distribution.

4. Basket default Swaps spread

The most common type of basket default swaps is the first-to-default swap (FTDS), where the seller compensates the buyer any loss of the principal and also, possibly, the accrued interest of the asset in the reference basket which defaults first. The main difference between (FTDS) and a credit default swap (CDS) is the event causing payout for the contract (in one case, it is the first default of any of a list of names and in the other is default of a single name). A n^{th} to default basket default swap gives protection against the n^{th} default in the underlyings pool of credits. We are going to present different pricing methodologies for this product.

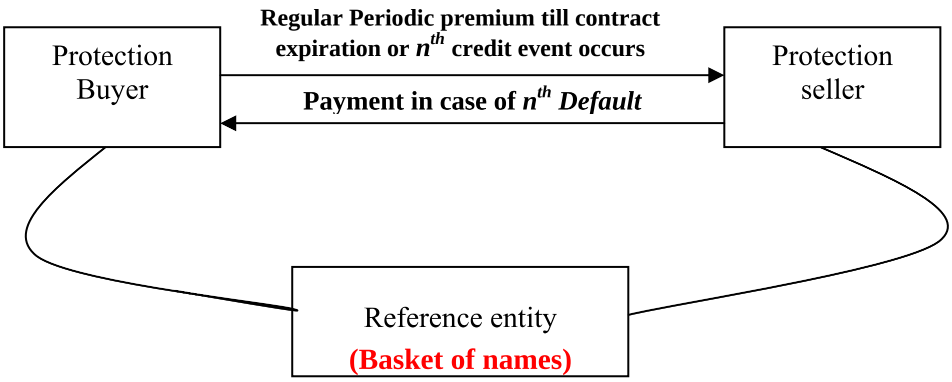

Figure 1: *n^{th} to Default Basket*

4.1. Pricing under Gaussian copula and t-student copulas using Monte Carlo simulations

The pricing of n^{th} to default basket default swap depends on the time the n^{th} credit defaults. The default times of different obligors are connected to each other by a Gaussian or t-student copula. The marginal default times of all credits in the basket must be known.
Suppose a basket of credit default swap with the following characteristics:

Basket default swap	
Initial Par Value	A is the notional amount of the contract. The total value of the basket is $V = \sum_{i=1}^{N} A_i$
Number of obligors	N
Payment frequency	δ =1 for annual payment frequency, δ_i i δ =0.25 for quarterly payment frequency…
Maturity date	T
Basket seniority	n=1 for first to default basket, n=2 for second to default basket
Fair price	s

Table 1: *Basket default swap notations*

According to Galiani (2003), the risk neutral price of the n^{th} to default basket swap is computed by equating the expected value of the discounted premium payment leg (fixed cash flow to be paid till contract expiration T or n^{th} credit event occurs) with the expected value of the discounted default leg (contingent payment in case of default), under the equivalent martingale measure P^*. Under this measure, the price processes of any tradeable security, discounted by the money market account, are P^*-martingales with respect to some filtration.

The premium legs are paid as long as the underlying credit has not defaulted until the maturity of the contract. The present value of the premium leg of the nth to default basket default swap can be computed as follows:

$$E(\mathrm{PL}_n) = E\left[p\delta\beta(t_i)\sum_{i=1}^{K} \mathrm{I}_{\{\tau^n > t\}} \right] \tag{4.1.1}$$

Where ($p = f.A$) the premium leg as function as the fair spread of the contract (f) as a fraction of is notional amount (A) in basis point, $t_i, i \in \{1,...,k\}$ are the payment dates either until T or until $\tau < T$ in case of default, δ_i is the frequency of payment and $\beta(t)$ is the discount factor.

Let Let $F_{(n)}(t) = P^*(\tau^n \leq t)$ be the distribution function of τ^n, then we can rewrite the premium leg as:

$$E(\mathrm{PL}_n) = p\delta\beta(t)\sum_{i=1}^{K}\left[1 - F^n(t_i)\right] \tag{4.1.2}$$

The second part for pricing n^{th} to default swap is the default leg $E[DL_n]$. The default leg ban be expressed as the difference between the expected discounted default payment $E[DP_n]$ and the expected discounted accrued premium $E[AP_n]$. Then, $E[DL_n] = E[DP_n] - E[AP_n]$.

With:

$$E[DP_n] = E\left[A.(1 - R^n)\beta(\tau^n)\sum_{j=1}^{N} \mathrm{I}_{\{\tau^n \leq T\}} \right]$$

$$= A\sum_{j=1}^{N}(1 - R^n)\int_0^T \beta(\tau^n) F_{(n)}^{n^{th}=j}(dt) \tag{4.1.3}$$

We notice that $F_{(n)}^{n^{th}=j}(t)$ is the distribution function of n^{th} basket default relative to the j^{th} defaulter for allowing different recovery rates for the obligors.

$$E[AP_n] = E\left[p\sum_{i=1}^{K}\left(\frac{\tau^n - t_{i-1}}{t_i - t_{i-1}}\delta\right)\beta(\tau^n)\mathrm{I}_{\{t_{i-1} < \tau^n \leq t_i\}} \right]$$

$$= p\sum_{i=1}^{K}\int_{t_{i-1}}^{t_i} \frac{v - t_{i-1}}{t_i - t_{i-1}}\delta\beta(v)F_{(n)}(dv) \tag{4.1.4}$$

The fair spread of the basket default swap is given as:

$$s = \frac{E[DL_n]}{E[PL_n] + E[AP_n]} = \frac{\sum_{j=1}^{N}(1 - R^n)\int_0^T \beta(\tau^n)F_{(n)}^{n^{th}=j}(dt)}{\sum_{i=1}^{K}\delta\beta(t)\left[1 - F^n(t_i)\right] + \int_{t_{i-1}}^{t_i}\frac{v - t_{i-1}}{t_i - t_{i-1}}\delta\beta(v)F_{(n)}(dv)} \tag{4.1.5}$$

Pricing a n^{th} to default basket default swap under Gaussian and t-student copula using Monte Carlo simulations can be presented as the following steps:

Step 1: simulate N-dimensional vector of correlated uniform random variates from a copula (C_Σ or $C_{v,\Sigma}$) as described in § 2.

Step 2: Translate the corresponding uniform variates into default time for each obligors.

Step 4: Sort the credits with respect to their default time τ_i and determine the n^{th} default time τ^n. For first to default swap, we find the first default time $\tau^* = \min_i \tau_i$.

Step 5: Based on specific realization of τ^n Determine the present value of the premium leg $E[PL_n]$.
Step 6: Determine the present value of the default leg $E[DL_n]$.
Step 7: Repeat all steps above until the required number of scenarios has been simulated and the sample average fair spread of the n^{th} to default basket swap as described in the equation (4.1.5).

To determine the impact of the structure of dependency via Gaussian and t-student copulas on the n^{th} to default basket swap spread, a simulation study was performed for different baskets of different A 50000 Monte Carlo simulations is performed.

Basket default swap (**BDS 1**): Homogeneous baskets of five names, maturity date T=5, constant default intensities h=0.01; constant interest rate 5%, a deterministic recovery rate of 40%, δ =0.25 for quarterly payment frequency and correlation between each pair of entities rho=0.3

Basket default swap (**BDS 2**): Homogeneous baskets of five names, maturity date T=5, constant default intensities h=0.01; constant interest rate 5%, a deterministic recovery rate of 40%, δ =0.25 for quarterly payment frequency and correlation between each pair of entities *rho=0.6*

Basket default swap (**BDS 3**): Homogeneous baskets of 10 names, maturity date T=5, constant default intensities *h= 0.05*; constant interest rate 5%, a deterministic recovery rate of 40%, δ =0.25 for quarterly payment frequency and correlation between each pair of entities *rho=0.5*

	Fair price (Gaussian copula)			Fair price (t-student copula		
	First to default	Second to default	Third to default	First to default	Second to default	Third to default
(**BDS 1**)	446.7645	140.4528	52.8958	385.7386	153.3905	40.3990
(**BDS 2**)	294.6252	138.3890	78.8875	302.1418	122.9784	93.5541
(**BDS 3**)	1.1597e+003	690.5959	480.8674	1.7200e+003	810.5678	447.4202

Table 2: *Basket default swap premiums(on basis points)*

We notice that the premium computed under Gaussian and t-student copulas are different. Burtschell et al (2005) find that Gaussian and t-student copulas lead to quite similar premium for first to default swap when they change the number of names in the basket. Similarly, the differences are minor when they change the rank of default

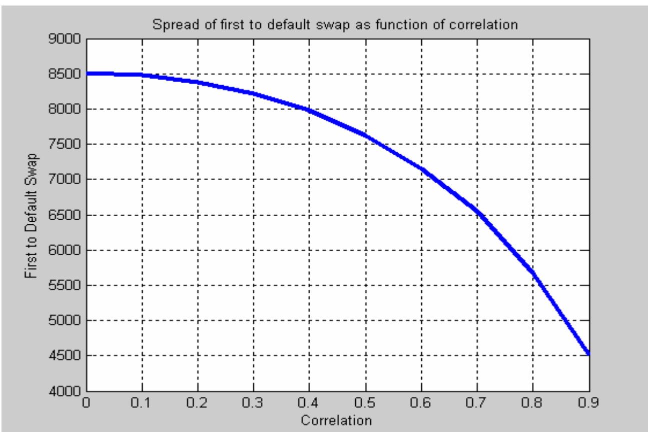

Figure 2: *Gaussian copula spread of first default swap as function of correlation using Monte Carlo simulation.*

4.2. Pricing under one factor Gaussian copula model

In this section, we implement a one factor model approach first presented by Laurent and Gregory (2003), and Andersen, Sidenius and Basu (2003). The one factor Gaussian copula is the copula associated with multivariate normal random variables that display the correlation structure induced by linear dependence on a single common normally distributed factor. This model results in semi-analytical expressions for the price of basket credit derivatives, avoiding the time consuming simulation step needed with the Monte Carlo method. In the semi-explicit approach, we must determine the probability $P(N(t)=m)$ that a certain number of credits has defaulted at time t and the probability that at time t, credit i has defaulted as n^{th} credit in the basket. Given these probabilities, we can compute analytically the present value of the premium and default leg and then, the price of the n^{th} basket default swap.

Suppose that $V_1,...,V_m$ are marginally standard normal variates. In a one factor model we write:

$$V_i(t)=a_iC+\sqrt{1-a_i^2}\varepsilon_i \tag{4.2.1}$$

With C is a common factor for all firms and ε_i is the error terms. C and ε_i are independent standard normally distributed random variables and then, $\mathrm{cov}(V_i,V_j)=a_ia_j$ and $\mathrm{Var}(V_i)=1$. The term a_i determines how strong V_i is linked to the evolution of the common factor C.

The default of an obligor is triggered when $V_i \le K_i$ with K_i is the default barrier.

The link between the V_i and the default time τ_i is given via the marginal distribution of the default time:

$$F_i(t)=P(\tau_i \le t)=P(V_i \le K_i) \tag{4.2.2}$$

$K_i = \Phi^{-1}[F_i(t)]$, with Φ^{-1} is the inverse cumulative distribution function.

The conditional default probability that credit i defaults at time t conditional on C is :

$$p_i(t\setminus C)=P(V_i \le K_i \setminus C)=\Phi\left[\frac{\Phi^{-1}[F_i(t)-a_iC]}{\sqrt{1-a_i^2}}\right] \tag{4.2.3}$$

The times of default conditionally on C are independent. The probability of default joint is written as:

$$F(t_1,....,t_n)=\int_{-\infty}^{+\infty}\prod_{i=1}^{n}\Phi\left[\frac{\Phi^{-1}[F_i(t)]-a_ic}{\sqrt{1-a_i^2}}\right]g(c)dc, \tag{4.2.4}$$

$$S(t_1,....,t_n)=\int_{-\infty}^{+\infty}\prod_{i=1}^{n}\Phi\left[\frac{a_ic-\Phi^{-1}[F_i(t)]}{\sqrt{1-a_i^2}}\right]g(c)dc. \tag{4.2.5}$$

With $g(c)=\frac{e^{-c^2/2}}{\sqrt{2\pi}}$ represent the density function of C.

The pricing of n^{th} to default basket default swap involve the knowledge of risk neutral probability of n^{th} default according to time:

$$S(t_1,t_2,...,t_n)=\int_{-\infty}^{+\infty}\prod_{i=1}^{n}q_i(t\setminus c)g(c)dc, \tag{4.2.6}$$

With $q_i(t\setminus C)$ is the survival function conditional to credit i. The number of default until time t is presented via the following process:

$$N(t)=\sum_{i=1}^{n}I_{\{\tau_i\le t\}}\ . \tag{4.2.7}$$

The probability of n^{th} to default in the basket following the one common factor can be written as:

$$P[N(t)=n]=\int P[N(t)=n\setminus c]g(c)dc \tag{4.2.8}$$

We must determine $P[N(t)=n\setminus C], n=\{0,1,...,N\}$ with N is the total number of credits in the basket $n\le N$. For $n=0$, then no default in the basket. The conditional probability of not default can be written as :

$$P[N(t)=0\setminus C]=\prod_{i=1}^{n}q_i(t\setminus C). \tag{4.2.9}$$

The conditional probability of one default $n=1$, can be written as:

$$P[N(t)=1\setminus C]=\sum_{i=1}^{n}\left[(1-q_i)\prod_{\substack{j=1\\ j\neq i}}^{n}q_j\right]$$

$$P[N(t)=1\setminus C]=\sum_{i=1}^{n}\left[(1-q_i)\prod_{j=1}^{n}\frac{q_j}{q_i}\right] \tag{4.2.10}$$

We use (4.2.9) and (4.2.1), then:

$$P[N(t)=1\setminus C]=P[N(t)=0\setminus C]\sum_{i=1}^{n}w_i(t\setminus C), \tag{4.2.11}$$

$w_i(t\setminus C)=\dfrac{1-q_i(t\setminus C)}{q_i(t\setminus C)}$.

We repeat the same procedure to n^{th} default in the basket, $n=\{2,3,...,N\}$ we can easily verify:

$P[N(t)=1\setminus C]=P[N(t)=0\setminus C]\sum w_p(1),...,w_p(k)$

The n^{th} survival function can be presented as follows: $S^n(t)=P(\tau^n>t)$,

$$S^n(t)=P[N(t)<n],$$
$$S^n(t)=\sum_{j=0}^{n-1}P[N(t)=j]. \tag{4.2.12}$$

With τ^n is the n^{th} default time.

The premium legs are paid as long as the underlying credit has not defaulted until the maturity of the contract. The accrued premium payments are taken into account. We suppose that free risk interest rate and recovery rate are constant. The present value of the premium leg of the n^{th} to default basket default swap can be computed as:

$$E(PL_n)=E\left[p_n\delta_i\beta(t_i)\sum_{i=1}^{I}I_{\{\tau^n>t\}}\right] \tag{4.2.13}$$

Where p_n is the n^{th} primium leg, $t_i, i\in\{1,...,I\}$ are the payment dates of the premium leg, δ_i is the frequency of payment and $\beta(t)$ is the discount factor. We combine (4.2.12) and (4.2.13), we can rewrite the present value of the premium leg of the n^{th} to default basket default swap as:

$$E[PL_n]=p_n\sum_{i=1}^{I}\delta_i\beta(t_i)\sum_{n=0}^{n-1}P[N(t_i)=n] \tag{4.2.14}$$

The second part for pricing n^{th} to default swap is the default leg $E[DL_n]$. We suppose that the basket is composed by homogeneous credit with the same notional and then, the same recovery rate. The loss in case of default can be written as:

$$E[DL_n]=E\left[(1-R)\beta(\tau^n)I_{\{\tau^n\leq T\}}\right]=-(1-R)\int_0^T\beta(t)dS^n(t), \tag{4.2.15}$$

With T is the maturity date. The integration of the precedent equation can be written as:

$$E[DL_n]=(1-R)\left[1-\beta(T)S^n(T)+\int_0^T S^n(t)d\beta(t)\right] \tag{4.2.16}$$

Following the hypothesis of risk neutral probability and constant free risk interest (r) The loss in case of default can be written as:

$$E[DL_n]=(1-R)\left[1-e^{-rT}S^n(T)-r\int_0^T S^n(t)e^{-rt}dt\right], \tag{4.2.17}$$

The term of pricing a basket default swap (spread: S) can be computed by dividing the present value of the default leg $E[DL_n]$ through the present value of the premium leg $E[PL_n]$:

$$s = \frac{E[DL_n]}{E[PL_n]}$$

Suppose a basket default swap with 20 names with constant nominal amount and assuming correlation matrix with one factor $a = \sqrt{0.6}$ ($a = \sqrt{\rho}$) and a recovery rate of R= 40%. We can find the ongoing premiums and see the effect of hazard rate.

n^{th} *to default*	h=0.01	h=0.04	h=0.06	h=0.08
1	419.11	1313.2	1860.6	2410.9
2	229.91	852.5	1219.5	1607.3
3	159.88	648.5	943.3	1238.1
4	118.38	511.1	779.5	1019
5	87.49	406.6	659.8	872

Table3: *Spread of n^{th} to default swap with n=20, correlation factor* $a = \sqrt{0.6}$ *and recovery rate R=40%.*

Note that the effect of hazard rate is important for the pricing of the n^{th} to default swap. For h=0.01, the spread of first to default swap is 419.11 basis point. However, for h=0.08, the spread of first to default swap is2410.9. Indeed, the highest spread always defaults first since the value of hazard rate reflect the intensity of default probability.

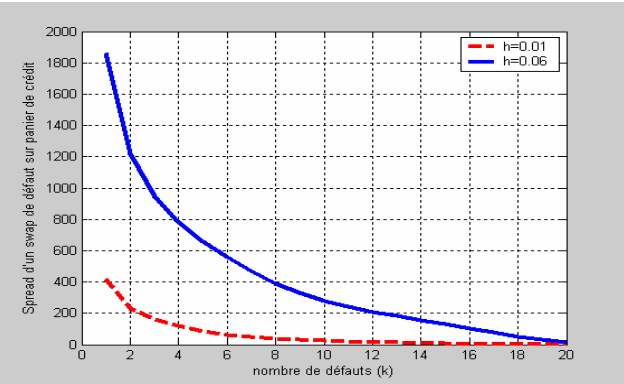

Figure 3: *Spread of n^{th} to default swap for different hazard rates.*

Figure 4 shows that spread of basket default swap decrease according to the number of defaults with different hazard rate. From the point of view of the protection seller, the higher the n^{th} to default, the less likely the spread is to pay. Similarly, we notice that the level of spread increase when the hazard rate increase.

Now, assuming hazard rate *h=0.06* and a recovery rate of R= 40%. We can find the ongoing premiums and see the effect of correlation factor.

n^{th} *to default*	$a = 0$	$a = \sqrt{0.1}$	$a = \sqrt{0.3}$	$a = \sqrt{0.6}$
1	8449.1	6130.5	3635.6	1860.6
2	3695.2	2925.1	2004.2	1219.5
3	2205.8	1817.3	1360.5	943.3
4	1421.9	1219.9	993.8	779.5
5	903.9	834.8	751.6	659.8

Table 4: *Spread of n^{th} to default swap with n=20, h=0.06 and recovery rate R=40%.*

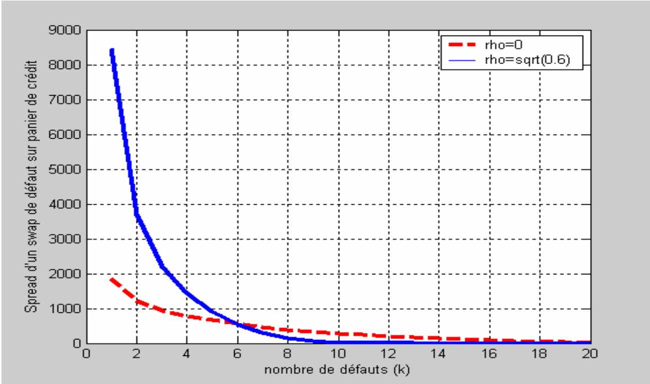

Figure 4: *Spread of n^{th} to default swap for different correlation factors.*

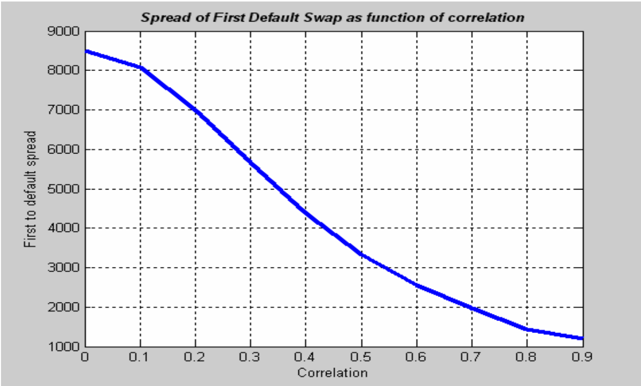

Figure 5: *one factor Gaussian copula spread of first default swap as function of correlation*

Similarly, Figure 5 shows that spread of basket default swap decrease according to the number of defaults with different correlation coefficient. However, we notice that the level of spread increase when the correlation coefficient increase.

4.3. Pricing under one factor Clayton copula

The structure of dependency via Clayton copula in credit risk analysis has been considered by many authors like Laurent and Gregory (2003), Madan et al (2004), Burtschell et al (2005). Let C any positive random variable with $f(c)$ be the marginal density. C is called the common factor. The conditional distribution function as conditionally independent given C is some distribution function $G_i^c(x)$:

$$p_i(X_i < x \backslash \mathrm{C} = \mathrm{c}) = G_i^c(x). \tag{4.3.1}$$

The joint density of variables $\mathrm{X_i}$ is given by:

$$F(x_1,..., x_M) = \int_0^\infty \prod_{i=1}^M G_i^c(x_i) f(c) d(c) == \int_0^\infty e^{c\sum_{i=1}^M Ln(G_i(x_i))} f(c) d(c) \tag{4.3.2}$$

$$= \psi\left(-\sum_{i=1}^M \mathrm{Ln}(\mathrm{G_i}(\mathrm{x_i}))\right) \tag{4.3.3}$$

where $\psi(s)$ is the Laplace transform of C and $\psi(s) = \int_0^\infty e^{-sc} f(c) d(c)$. From (4.3.2), we can present the implied marginal densities:

$$\mathrm{F_i}(\mathrm{x_i}) = \int_0^\infty \mathrm{G_i^c}(\mathrm{x_i}) f(c) d(c) == \int_0^\infty e^{c\mathrm{Ln}(\mathrm{G_i}(\mathrm{x_i}))} f(c) d(c)$$

$$= \psi\left(-\text{Ln}\left(G_i\left(x_i\right)\right)\right) \tag{4.3.4}$$

Then: $G_i\left(x_i\right) = e^{-\psi^{-1}\left(F_i\left(x_i\right)\right)}$

Substituting (4.3.4) into (4.3.3),, we obtain:

$$F\left(x_1,\ldots,x_M\right) = \psi\left(\sum_{i=1}^{M} \psi^{-1}\left(F_i\left(x_i\right)\right)\right)$$

A typical example is the Clayton copula where the variable C has a Gamma distribution with parameter $\frac{1}{\theta}$, where $\theta > 0$. Then, we obtain:

$$f\left(v\right) = \frac{1}{\Gamma\left(\frac{1}{\theta}\right)} e^{-v} C^{(1-\theta)/\theta}, \text{ with } \psi\left(s\right) = \left(1+s\right)^{-\frac{1}{\theta}}, \tag{4.3.5}$$

$$\psi^{-1}\left(u\right) = u^{-\theta} - 1.$$

Then,

$$C\left(u_1,\ldots,u_M\right) = \left(\sum_{i=1}^{M} u_i^{-\theta} - M + 1\right)^{-\frac{1}{\theta}} \tag{4.3.6}$$

The conditional default probability that credit i defaults at time t conditional on C is given by equation(4.2.3) in case of one factor Gaussian copula. In the one factor Clayton copula:

$$p_i\left(t \setminus C\right) = \exp\left(C\left(1 - F_i\left(t\right)^{-\theta}\right)\right) \tag{4.3.7}$$

The joint survival function on one factor Gaussian copula is given by (4.2.5). In the one factor Clayton copula:

$$S\left(t_1,\ldots,t_n\right) = \int_{-\infty}^{+\infty} \prod_{i=1}^{n} \left(1 - p_i\left(t \setminus C\right)\right) \frac{1}{\Gamma\left(\frac{1}{\theta}\right)} e^{-C} C^{\left(\frac{1-\theta}{\theta}\right)} .dc \tag{4.3.8}$$

	Clayton $\theta = 0.18$	Gaussian Rho=0.3	Clayton $\theta = 0.36$	Gaussian Rho=0.5	Clayton $\theta = 0.66$	Gaussian Rho=0.7
First to default	462.6	454.9	293.8241	287.8186	110.6611	92.6339

Table 5: *Spread of n^{th} to default swap under one factor Clayton copula*

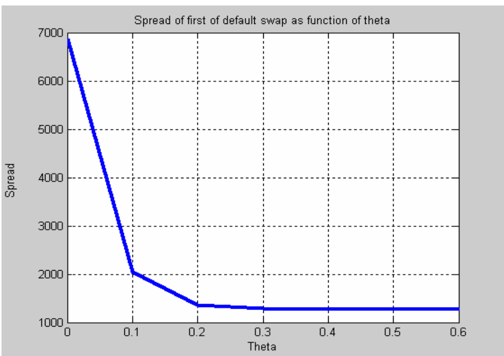

Figure 6: *Spread of first default swap as function of theta (one factor Clayton copula)*

4.4. Pricing under Optimised Monte Carlo simulations via Importance sampling

Variance reduction has always been a central issue in Monte Carlo experiments. Importance sampling is a variance reduction technique that can be used in the Monte Carlo simulation. The idea behind Importance sampling is that certain values of the input random variables in a simulation have more impact on the parameter being estimated than others. If these "important" values are emphasized by sampling more frequently, then the estimator variance can be reduced.

Joshi & Kainth (2004) apply importance sampling to the pricing of nth to default credit swaps within the Li model and obtain stable and sizeable speed ups. They show that Monte Carlo simulations in the Li model can be slow to converge and present procedures for accelerating the computation of prices and sensitivities to hazard rates. Glasserman &Li (2005) develop importance sampling (IS) procedures for rare-event simulation for credit risk measurement. They focus on the normal copula model originally associated with J.P. Morgan’s CreditMetrics system. Dependence between obligors is captured through a multivariate normal vector of latent variables; a particular obligor defaults if its associated latent variable crosses some threshold. Glassserman &Juneja (2006) have considered the problem of simultaneous estimation of the probabilities of multiple rare events. Successful applications of importance sampling for rare event simulation typically focus on the probability of a single rare event. As a way of demonstrating the effectiveness of an importance sampling technique, the probability of interest is often embedded in a sequence of probabilities decreasing to zero. They shows that Importance sampling based on exponential twisting produces aymptotically efficient estimates of rare event probabilities in a wide range of problems.

According to Milicia (2006), Importance Sampling is a change of probability measure which takes as from the original probability measure to a new, biased, one which better suits our purposes
Assume we want to estimate by simulation the probability p of an event $(X \geq t)$ where X is a random variable with probability density function $f(x)$. Importance sampling is concerned with the determination of an alternate density function $f^{\Theta}(x)$, usually referred to as a biasing density, for the simulation experiment. This density allows to the above event to occur more frequently, so the sequence lengths δ gets smaller for a given estimator variance:

$$p = E(X \geq t) = \int (x \geq t)\frac{f(x)}{f^{\Theta}(x)} f^{\Theta}(x)dx = E^{\Theta}((X \geq t)L(X)) \tag{4.4.1}$$

$$L(x) = \frac{f(x)}{f^{\Theta}(x)} \tag{4.4.2}$$

$L(x)$ is the *the Radon-Nikodym derivative* or likelihood ratio of the change of measure. The Importance sampling estimator is then:

$$p_{IS} = \frac{1}{K}\sum_{i=1}^{K}(X_i \geq t)L(X_i) \tag{4.4.3}$$

Where $X_i \sim f^{\Theta}$

In order to identify successful IS techniques, the variance ratio $\frac{\sigma_{MC}^2}{\sigma_{IS}^2}$, and this can be interpreted as the speed-up factor by which the Importance sampling estimator achieves the same precision as the Monte Carlo estimator.
Joshi and Kainth (2004) explain the implementation difficulties of Li (2000) model. For a short team deal, many Monte Carlo paths result in a zero or constant pay-off. Then, they sample more intensively the areas where the pay-off is rapidly changing. For basket credit derivative, this means ensuring that every path has enough defaults to result in a non-trivial pay-off and achieving a reasonable distribution

within that area. Subsequently, they achieve this by redefining the probability measure, and weighting the pay-off's value to compensate for the adjustment.
Assume a basket default swap of size N, The Joshi and Kainth method first determines whether a particular asset defaults within the life of the swap, and, if a default is to occur, it then determines the time of the default in the interval [0*;* *T*]. Only the default probabilities are changed[7].

Let default indicator variables $Y_i = I(\tau_i \le T)$, (i=1,…,N) be the indicator variables for defaults which occur within the lifetime of the contract. Conditional default probabilities are then given by $P_1 = F_1(T)$ and $P_i = P(Y_i = 1/\tau_1,...,\tau_{i-1}) = F_i(T/\tau_1,...,\tau_{i-1})$, i=2,…N.
The importance sampling probabilities are defined as:

$$p_1 = \frac{n}{N},\ \tilde{p}_i \begin{bmatrix} \dfrac{n - \sum_{j=1}^{i-1} Y_j}{N-i+1} & \sum_{j=1}^{i-1} Y_j < n \\ \\ pi & \sum_{j=1}^{i-1} Y_j \ge n \end{bmatrix} \tag{4.4.4}$$

With $\sum_{j=1}^{i-1} Y_j$ the number of assets which have defaulted by time T.
Joshi and Kainth (2004) algorithm can be presented as:

Step1: draw a N-dimentional vector U uniform variates

$$\text{Step2: Let } V_i \begin{bmatrix} \dfrac{p_i U_i}{\tilde{p}_i} & \text{if } U_i \le \tilde{p}_i \\ \\ pi + \dfrac{1-p_i}{1-\tilde{p}_i}(U_i - \tilde{p}_i) & \text{otherwise} \end{bmatrix} \tag{4.4.5}$$

$$\text{And } \tau_i = F_i^{-1}(V_i / \tau_1,...,\tau_{i-1}) \tag{4.4.6}$$

Step 3: Calculate the weight (likelihood ratio):

$$L_i = \begin{bmatrix} \dfrac{p_i}{\tilde{p}_i} & \text{if } Y_i = 1 \\ \dfrac{1-p_i}{1-\tilde{p}_i} & \text{if } Y_i = 0 \end{bmatrix} \tag{4.4.7}$$

Once $\tau_1,...,\tau_N$ have been generated, we evaluate $V(\tau_1,...,\tau_N)$ and return the weighted estimate $V(\tau_1,...,\tau_N)L$ with $L = L_1 L_2 ... L_N$ the weight for the path.
To implement this algorithm, we must sample from the conditional default time distribution in (4.4.6). Let $\Sigma = A \cdot A'$ be the positive definite matrix, let $W = A \cdot Z$, then $W_i = \Phi^{-1}(F_i(\tau_i))$.Then conditioning on $\tau_1,...,\tau_i$ is equivalent to conditioning on $Z_1,...,Z_i$

[7] Chen and Glasserman (2006) show that the importance sampling probabilities used by Joshi and Kainth (2004) may be interpreted as follows: Consider an urn initially containing *n* black balls and *N-n* white balls. Balls are drawn from the urn at random, without replacement. If j < n of the first i-1 balls are black, then the probability that the i[th] draw produces a black ball is (n-j)/(N-i+1). All n black balls will eventually be drawn.

$$P(\tau_i \leq t / Z_1,\ldots,Z_{i-1}) = \Phi\left(\frac{\Phi^{-1}\left(F_i(t) - \sum_{j=1}^{i=1} A_{ij} Z_j\right)}{A_{ii}}\right) \tag{4.4.8}$$

Milicia (2006) adopt the Importance Sampling method[8] to other elliptical copula like t-student copula. Let $W = \sqrt{\frac{\nu}{s}} \cdot A \cdot Z$ with $s \sim \chi^2_\nu$ and $\tau_i = F_i^{-1}(T_\nu(W_i))$, then :

$$P(\tau_i \leq t / Z_1,\ldots,Z_{i-1}) = \Phi\left(\frac{\frac{T_\nu^{-1}(F_i(t))}{\sqrt{\frac{\nu}{s}}} - \sum_{j=1}^{i-1} A_{ij} Z_j}{A_{ii}}\right) \tag{4.4.9}$$

	Fair price (Gaussian copula) **MC**	Fair price (t-student copula) **MC**	Fair price Gaussian copula) **JK IS**	Fair price Gaussian copula) **JKM IS**
(BDS 1)	446.7645 (0.0084788)	385.7386 (0.008478)	438.11 (0.08478)	439.36 (0.000612)

Table 6: *Spread of First to default swap under Monte Carlo and Importance sampling*

We notice that the spread of first to default swap change with the structure of dependency and the simulation techniques. Then, the choice of copula and the choice of procedures for rare-event simulation govern, also, the pricing of basket credit derivatives.

5. Collateralized debt obligation spread

Collateralized debt obligation (CDO) refers to securitization[9] of pools of assets. A CDO cashflow structure allocates interest income and principal repayments from a collateral pool of different debt instruments to a prioritized collection (tranches) of CDO securities. Following the classification of Tvakoli (2003), a CDO is backed by portfolios of assets that may include a combination of bonds, loans, securitised receivables, asset-backed securities, tranches of other CDO's, or credit derivatives referencing any of the former. Some market practitioners define a CDO as being backed by a portfolio including only bonds. A Collateralized loan obligation (CLO) is a type of CDO that is backed by a portfolio of loans. A Collateralized bond obligation (CBO) is a type of CDO that is backed by a portfolio of bonds issued by a variety of corporate or sovereign obligors. The development of structured credit derivatives leads to the emergence of synthetic Collaterized Debt Obligations which transfer the risk of a pool of single-name Credit Default Swaps. This realizes an exposure to a variety of names.

[8] We use the Matlab code for Importance Sampling as implemented in the paper of Milicia (2006). An electronic version of the article and the corresponding Matlab code are available online at http://www.brics.dk/~milicia/libraryGM.htm

[9] According to Tvakoli (2003), securitization has been a means for banks to reduce the size of their balance sheets and to reduce the risk on their balance sheets. This allowed banks to do more business and allowed investors accsess to diversified pools of assets to which they otherwise not have had access.

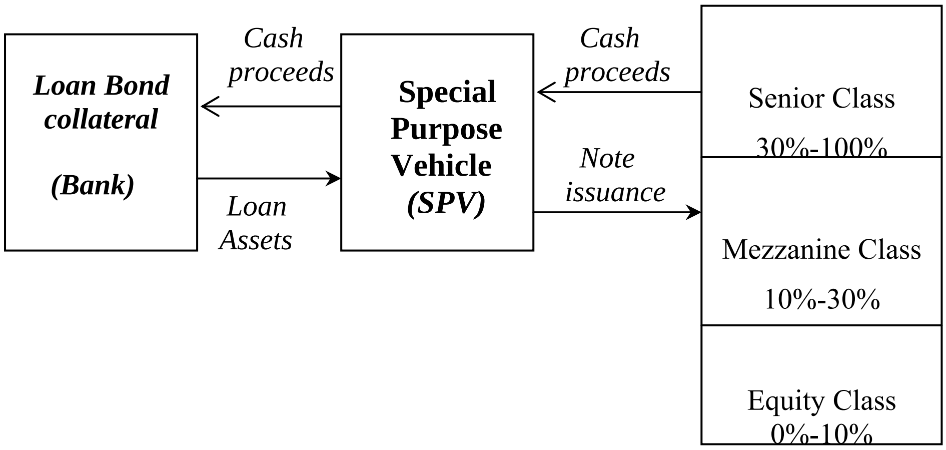

Figure 7: *CDO*

Figure 6 illustrate an example of a simple CDO structure with three tranches: equity, mezzanine and senior. Suppose that the total CDO notional is 100 millions and during the lifetime of CDO some debts in the collateral portfolio might default. At maturity, if the total default loss is less than 10 millions, only the equity tranche is affected. If the total losse is between 10 and 30 millions, the equity tranche does not get the principal back and the mezzanine gets only part of it. If the loss is more than 30 millions than the equity and mezzanine do not get anything back and senior tranche gets is left.
From an economic perspective, CDO structures thus create custom exposures that investor's desire and cannot achieve in any other way. However, CDO address some important market imperfections. First, banks and certain other financial institutions have regulatory capital requirements that make it valuable for them to securitize and sell some portion of their assets, reducing the amount of (expensive) regulatory capital that they must hold. Second, individual bonds or loans my be illiquid, leading to a reduction in their market values. Securitization may improve liquidity, and thereby raise the total valuation to the issuer of the CDO structure. Third, adverse selection[10] can be mitigated by securitization of assets in a CDO. The seller achieves a higher total valuation (for what is sold and what is retained) by designing the CDO structure so as to concentrate the "lemon's premium"[11] into small subordinate tranches, leaving the large senior tranch relatively immune to the effects of adverse selection.According to Duffie & Gârleanu (2001), the effect of adverse selection can be discovered in the transfer of bank loans or junk bonds. There is an informational asymmetry between the potentially better-informed seller of such assets and the potentially uninformed buyer, so there is a price reduction sometimes called a lemon's premium.

There are a number of ingredients involved in pricing individual tranches: default probability, recovery rate and default correlation. The first two are self-explanatory. Correlation, however, deals with the distribution of defaults throughout a portfolio and the likelihood of a single default causing a succession of defaults. Mahadevan and Schwartz (2001) identify three broad types of CDO pricing methodologies: *rating methodologies* that infer a credit rating for the CDO from the ratings of its constituent parts and the relationships between them[12]which is then used to price the CDO off similarly rated bonds and CDOs. *Market value methodologies* that essentially equate the CDO price to the sum of the market values of the constituent parts (Duffie & Gârleanu (2001), Mashal (2002)…). Finally, *Cash flow methodologies* that involve discounting back simulated future cash flows.

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[10] There may be a significant amount of private information regarding the bank loan. An investor may be concerned about being "picked off" when trading such instruments.
[11] The reduction in price due to adverse selection is sometimes called a "lemon's premium".
[12] For example the process used by Moody's, they calculate "diversity scores" by which the analysis of a portfolio of correlated assets is effectively simplified into an analysis of a portfolio of uncorrelated assets.

5.1. Pricing under Gaussian and t-student copula using Monte Carlo simulations

Consider an homogeneous CDO with n obligors with nominal amount A_i and recovery rate R_i with i= 1,2,…,n, (assumed deterministic), maturity T years and we assume constant risk free interest rate. The total value of the portfolio is $V_T = \sum_{i=1}^{n} A_i$ and $L_i = (1 - R_i)A_i$ will denote the loss given default for the i^{th} credit. Let τ_i be the default time of the i^{th} name and $N_i(t) = \sum_{i=1}^{n} I_{\{\tau_i \le t\}}$ be the counting process which jumps from 0 to 1 at default time of name i. let $L(t)$ will denote the cumulative loss on the collateral portfolio at time t:

$$L(t) = \sum_{i=1}^{n} L_i N_i(t). \tag{5.1.1}$$

The tranche [a,b] suffers a loss at time t if $a\%V_T < L(t) \le b\%V_T$, where $a\%$ and $b\%$ are respectively lower and upper bound. Suppose that $a\%V_t = a'$ and $b\%V_t = b'$, then, the tranche loss :

$$L_{a',b'}(t) = \left[L(t) - a'\right] I_{\{L(t)\in[a',b']\}} + \left(b' - a'\right) I_{\{L(t)\in[b',V_t]\}} \tag{5.1.2}$$

Using Monte Carlo simulation, the estimation of tranche loss becomes a straightforward task. According to Peixoto (2004), Pricing a CDO using Monte Carlo simulation involves creating sample paths of correlated default times. These default times are used to calculate the payments on two legs and value each leg. The first is the present value of tranche losses triggered by credit events during the CDO lifetime and is called default leg [DL] and the second is the present value of the premium payments weighted by the outstanding capital (original tranche amount minus accumulated losses) and is called premium leg [PL]. The fair spread of CDO can be computed by dividing the present value of the default leg $E[DL]$ through the present value of the premium leg $E[PL]$:

$$S = \frac{E[DL]}{E[PL]} \tag{5.1.3}$$

The K^{th} default leg can be computed as:

$$DL^k = \sum_{i=1}^{n} e^{-r\tau_i^k} L_{a,b}\left(\tau_i^k\right) \tag{5.1.4}$$

Where r is the free risk interest rate and $\{\tau_1^k, \tau_2^k, ..., \tau_N^k\}$ the sequence of default times with K^{th} iteration of a Monte Carlo simulation. The accumulated loss is given by:

$$\Im^k(t) = (1 - R)\sum_{i=1}^{n} I_{\{\tau_i^k < t\}} \tag{5.1.5}$$

The premium leg is paid over the outstanding capital in the tranche. If during the lifetime of the CDO the tranche is wiped out, there are no more premium payments:

$$PL^k = N\sum_{j=1}^{w} \delta_j e^{-rt_j} \min\left\{\max\left[b - \Im^k(t_j), 0\right], b - a\right\} \tag{5.1.6}$$

Where $\{t_1, ..., t_w\}$ are the premium payment dates with frequency δ_j.

Table 3 presents fair spread of an homogeneous CDO with Monte Carlo simulation. Standard errors of estimates are less than 1 basis point.

Tranche	Spread (basis point)
0% à 10% (Equity)	2952,4
10% à 30% (Mezzanine)	779.3024
30% à 100 % (Senior)	43.4713

Table 8: Fair *spread of an homogeneous CDO with* h = 0.06, *recovery rate R=0.4, correlation coefficient rho=0.4, 50,000 iterations and quarterly spread payment.*

Similarly to the basket default swap, we use t-student copulas to take into account the occurrence of joint extreme events among obligors.

Tranche	Spread (basis point)
0% à 10% (Equity)	3172.895
10% à 30% (Mezzanine)	762.065
30% à 100 % (Senior)	30.210

Table 9: Fair *spread of an homogeneous CDO with* $h = 0.06$, *recovery rate R=0.4, correlation coefficient rho=0.4, 50,000 iterations and* Degree of freedom(DoF= 10)

Equity tranche prices appear to be higher when computed through the t-copulas, with the opposite verified for the Senior tranche. Mezzanine tranches seem not to be particularly affected by the copula function.

The principal payment and interest income are allocated to the notes according to the following rule: Senior CDO notes are paid before mezzanine and lower subordinated notes are paid, with any residual cash flow, to an equity tranche. Therefore, equity tranche offers a larger spread than the more senior notes because is the first to be affected by a default in the portfolio. The price sensitivity with respect to price driving factors such as correlation, recovery rate, and the credit-worthiness of the underlying portfolio is examined. The following table present spread of CDO with different borns (a% and b%)

Tranche	Spread (basis point)
0% à 5% (Equity)	4438.5
5% à 15% (Mezzanine)	1705.5
15% à 100 % (Senior)	130.4230

Table10: Fair *spread of an homogeneous CDO with* $h = 0.06$, *recovery rate R=0.4, correlation coefficient rho=0.4, 50,000 iterations and quarterly spread payment.*

When we change the upper and the lower boundary of each trench, the CDO's spread change also. In fact, when we change the boundary of Equity tranch of (0%-10%) to (0%-5%), we note that the CDO's spread increase (2952,4 to 4438.5) because the expected loss in case of default incre for all the equity tranche.. The same argument for Mezzanine and Senior tranche.
Now, we study the sensitivity of CDO's spread to recovery rate. We assume that all other variables are constant.

Tranche	Spread (basis point), *R=0.1*	Spread (basis point), *R=0.7*
0% à 10% (Equity)	3860.6	1830.4
10% à 30% (Mezzanine)	1237	191.5912
30% à 100 % (Senior)	161.5789	9.5762e-004

Table 11: Fair *spread of an homogeneous CDO with* $h = 0.06$, *recovery rate, correlation coefficient rho=0.4, 50,000 iterations and quarterly spread payment.*

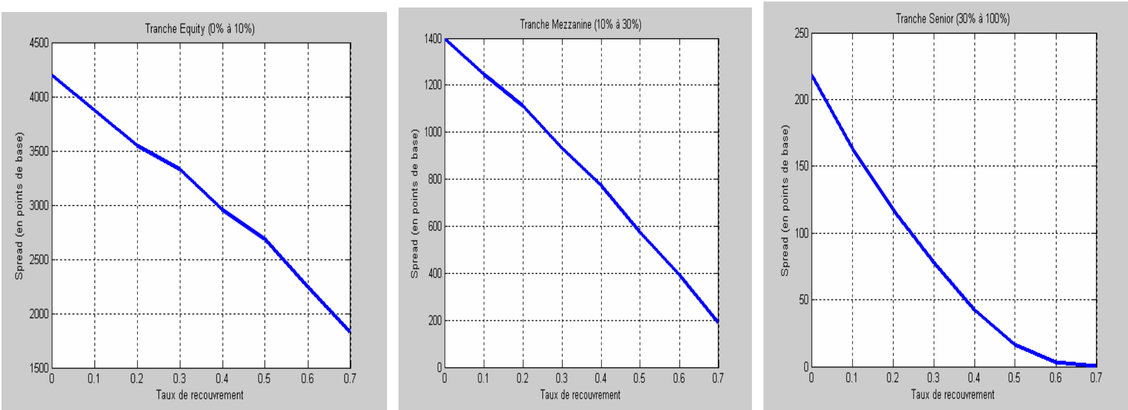

Figure 8: Fair *spread of an homogeneous CDO with correlation coefficient rho=0.4, 50,000 iterations and quarterly spread payment.*

Currently, we study the sensitivity of CDO's spread to correlation coefficient[13]. We assume that all other variables are constant.

Tranche	Spread (basis point), *rho=0.3*	Spread (basis point), *rho=0.7*
0% à 10% (Equity)	3.4904e+003	1.7873e+003
10% à 30% (Mezzanine)	760.2878	776.2190
30% à 100 % (Senior)	27.1457	104.2831

Table 12: Fair *spread of an homogeneous CDO with* $h = 0.06$, *recovery rate R=0.4 and 50,000 iterations and quarterly spread payment.*

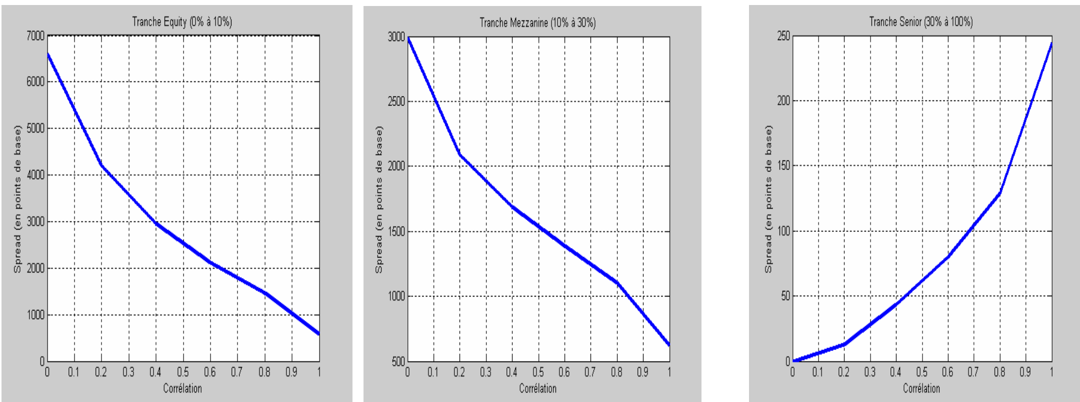

Figure 9: Fair *spread of an homogeneous CDO with recovery rate R=0.4, 50,000 iterations and quarterly spread payment.*

Unfortunately when correlations are implied from all of the tranche prices observed in the market, each tranche has its own individual correlation which is different from the others. The value of the lowest tranches of a CDO, say an equity tranche, increases as the correlation between defaults falls, and decreases as default correlation rises. According to Gibson (2004), the effect of correlation on CDO tranches is intuitive. The more the defaults within a portfolio become correlated, the more the portfolio behaves like a single credit. A higher correlation of defaults implies a greater likelihood that losses will wipe out the equity and mezzanine tranches and inflict losses on the senior tranche. According to Rachev (2006), the value of the equity tranche rises as correlation rises. Equity tranche

[13] CDO tranches are quoted in the market, they incorporate a correlation calculation. For example, a junior mezzanine tranche (6%–9% of loss) might be quoted at a bid offer of 75/95bp, with a 16.5% correlation. An equity tranche, however, might be seen as 1,400/1,600bp with a 25% correlation on the bid and a 20% correlation on the offer.

investors gain more in a scenario with very few defaults than they lose from a scenario with many defaults (they are only exposed to the first few defaults). Mezzanine tranches are subject to both effects, which can broadly cancel each other out and make mezzanine tranches less sensitive to correlation. The relationship is the other way round for the most senior tranches: as correlation decreases, value decreases because the probability of a large number of defaults decreases.

As option premiums translate into a volatility smile through the Black & Scholes formula, CDO spreads generate a correlation smile. Turc & Very (2004) use the beta model to imply a correlation from the market spreads of CDO tranches. They present a bootstraping technique as a means for incorporating the smile into CDO prices and hedges.

Tavares et al (2004) explain the difference on the behaviour of the correlation for each tranches firstly, by the supply and demand argument that investors are vervous of the risk inherent in equity tranches while Mezzanine tranches are extremely popular with investors. Sellers of protection on senior tranches seem to require a minimum coupon irresepective of subordination. Secondly, the Gaussian copula loss distribution underestimates the perceived chances of a very low or very high number of defaults whilst significantly over-estimating the chances of observing a few defaults. They proposed a new method named "Composed basket model" while it attemts to rationalise how market participants allocate the risk.

Finally, we present the relationship between expected loss and recovery rate. In the Monte Carlo simulations, the losses as well as the tranche notional are computed for every scenario individually and the value of CDO tranches is the average of the present values. We use a Gaussian copula to model the dependence structure in the portfolio. Following Galiani (2003), the pricing of CDO tranches with Monte Carlo method can be presented as the following steps:

Step 1: simulate v_i correlated N(0,1) random variables, i =1,…n.

Step 2: Find U_i's such that $U_i = \Phi(v_i)$ and Φ is the cumulative normal distribution function.

Step 3: Find the default times $\tau' s$ by $\tau = \frac{-\ln U_i}{\lambda}$, i =1,…,n.

Step 4: Given the simulated default times**,** we can compute the value of losses at maturity for [a,b] tranche $L_T[a,b]$.

Step 5: Repeat all steps above until the required number of scenarios has been simulated (m=50000 iterations in our example). The estimator of expected loss can be computed as the average for every scenario individually: $E[L_{a,b}(T)] = \frac{1}{m}\sum_{K=1}^{m} L_{a,b}^{k}(T)$.

Tranche	Expected loss given default (%) with R=0.4	Expected loss given default (%) with R=0.1	Expected loss given default (%) with R=0.7
0% à 10% (Equity)	81.3169	87.0294	70.3784
10% à 30% (Mezzanine)	32.6471	52.4393	17.8161
30% à 100 % (Senior)	1.3022	4.7677	0.1222

Table 13 : *Expected loss tranche with h=0.06, correlation coefficient rho=0.3 and 50.000 iterations.*

According to Tables 9, 10 and 11 we notice that the expected losses of equity tranches is higher than the mezzanine and senior tranche because the risk is higher. The equity note is also called the first loss position because it is the first to be affected by a default in the portfolio. Also, we notice the dependence of the expected loss given default and recovery rates for a fixed correlation. Table 14 shows that for recovery rate R=0.1, we obtain the expected losses of equity tranches is 87.0294%.The expected losses of equity tranches of the same CDO but with recovery rate R=0.7

Figure 9 shows the relationship between expected loss tranche and recovery rate for different correlation coefficient. Notice that the expected loss for equity tranche is larger meaning that

Mezzanine and senior tranches get less affected by the losses. This can be observed where the senior expected loss gets close to zero at high recovery rates.

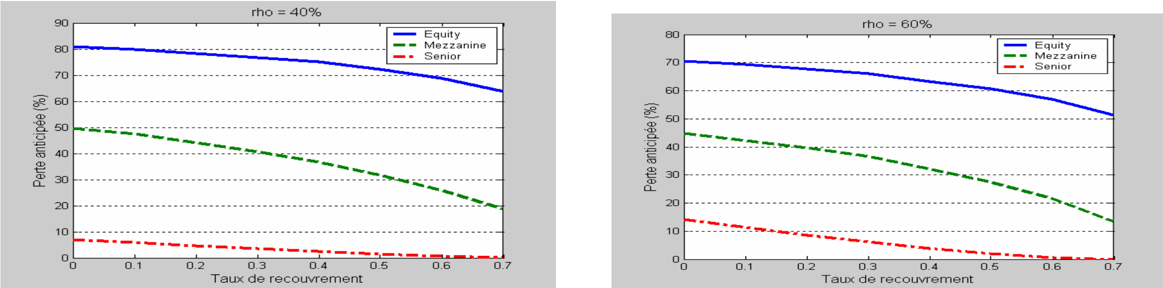

Figure 10: *The relationship between Expected loss tranche and recovery rate.*

Others extensions can be done with stochastic recovery rates. Garcia et al (2004) analyzes the effect of stochastic recoveries on individual ratings. They develop a Monte Carlo simulation model that generates credit events with both constant and stochastic recovery rates. They shown that non-constant recovery rates can have a significant impact on binomial expansion technique (BET) ratings, particularly for mezzanine tranches of CDOs with noninvestment- grade underlying collateral pools.

5.3. Pricing under One factor Gaussian copula method

The current standard for pricing CDO is the one-factor Gaussian copula. The market-standard version of this copula is characterized by a single parameter to summarize all correlations among default times. These individual default variables of the homogeneous portfolio are dependent on one systematic risk factor which results both in a complexity reduction due to the factor-model and also in conditionally independent defaults, whose properties simplify computations. For modelling purpose, the conditional default probability that credit i defaults at time t conditional on C (equation (4.2.3)), is the only inputs which include all model specification. The one-factor structure of this model implies that, conditional on the realization of the common factor, the *m* individual credits are independent. When pricing a standard CDO, this conditional independence greatly facilitates the calculation of the conditional loss distribution of the tranche.

Tranche	Spread on basis point
0% à 10% (Equity)	3628.1
10% à 30% (Mezzanine)	757.6
30% à 100 % (Senior)	46.91

Table 14: *Fair spread of homogenous CDO with h=0.06, correlation factor of Gaussian copula* $a = \sqrt{0.6}$ *recovery rate R=0.4.*

For the pricing of CDO, a one-factor Gaussian copula model with constant and equal pairwise correlations, default intensities and recovery rates for all assets in the reference portfolio has become the standard market model. Some drawbaks in the applicability of the one factor Gaussian model are enumerated by Gregory and Laurent (2004). They present an extension to the Gaussian one-factor copula model that allows a clustered correlation structure by specifying inter and intra sector correlation. Additionally they introduce dependence between recovery rates and default events which leads to an improved modeling of the smile. Hull and White (2005) argue that the market‘s focus on implied correlations is misplaced. Andersen and Sidenius (2005) extend the one factor Gaussian copula in order to match the “correlation smiles” in the CDO market. Instead of equation (4.2.1), the latent variables are given by: $V_i(t) = B_i\left(a_i C + \sqrt{1-a_i^2}\varepsilon_i\right) + (1-B_i)\left(\beta_i C + \sqrt{1-\beta_i^2}\varepsilon_i\right)$; Where B_i are

Bernoulli random variables, a_i and β_i determine how strong V_i is linked to the evolution of the common factor C ($(0 \le \beta_i \le a_i \le 1)$. Subsequently, there two states: one corresponding to a high correlation and the other to a low correlation: $V_i(t) = (B_i a_i + (1 - B_i)\beta)C + \sqrt{1 - (B_i a_i + (1 - B_i)\beta_i)^2}\,\varepsilon_i$.
They focus on a stochastic correlation Gaussian model which a_i is linked to the evolution of the common factor C with probability p and β_i with probability (1- p).
Others extension of the one factor Gaussian copula is the one factor t-student copula and double t-copula. In the t-Student approach, the random vector V follows a t-student distribution with ν degrees of freedom. In the Double t-copula, C and ε_i are independent random variables following t-student distribution with respectively ν and ν' degrees of freedom (Hull and White (2004), Burtschell et al (2005)).

6. Conclusion

The aims of this paper is is to assess the importance of the dependence structure and the choice of simulations procedures regarding the valuation on the of multi-name credit derivatives such as basket default swaps and CDO tranches. The key idea of modelling correlated default is the usage of copulas functions. The valuation models are set up with Gaussian, Clayton and Student t- copulas. Two different methods for valuation are described: The first is the standard Monte Carlo method for simulating the default times, with which multi-name credit derivatives can be priced. The advantage of Monte-Carlo is its simplicity and generality. Its main drawbacks, however, are the quality of the convergence, especially when one computes sensitivities. A good convergence is particularly hard to achieve for credit products since default events are usually rare, and probabilities in the tail of the distribution are difficult to estimate. On the other hand, the direct implementation in closed form is very accurate but is less trivial to implement; it is also very expensive computationally. Indeed, this method is based on enumerating the 2n default configurations of the basket and computing the probability of each configuration. This algorithm explodes exponentially as the number of credits increases.

In the second approach the correlation structure is simplified by a factor copula approach. This approach enables us to provide semi-explicit expressions that reduce the computational times compared with Monte Carlo simulations techniques. We investigate the influence of different price drivers (correlation, hazard rates and recovery rates) on basket credit derivatives modelling portfolio losses and basket credit derivatives spreads.

Furthermore, the Gaussian distribution has thin tails compared to other distributions. As we are concerned of default events that are by nature tail events, we use distributions with fat tails such as the t-student and Clayton distribution. Similarly, the choice of procedures for rare-event simulation governs the pricing of basket credit derivatives. Joshi and Kainth (2004) shows that For baskets of high-quality credits or short-maturity swaps, Monte Carlo simulation produces few paths with n or more defaults; nearly all paths simply return a value of 0. Then, they introduce Importance Sampling methods that forces all paths to produce at least n defaults. An alternative to the Monte Carlo simulation is Clayton copula and t-student copula under importance sampling procedures for simulation which captures the dependence structure between the underlying variables at extreme values and certain values of the input random variables in a simulation have more impact on the parameter being estimated than others.

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Appendix

Some Matlab codes used in the paper

```
function y1 = Gaussrnd(rho,N);
% This function generates random numbers from Gaussian copula as presented in Figure 1.
% rho = the Gaussian copula correlation matrix
% N = number of random numbers to be generated
% y1 = random numbers from Gaussian copula, N by 2 matrix
if N == 0
y1=[];
else
   A = chol(rho);
   z = randn(N,2)*A;
   u = cdf('Normal', z(:,1),0,1);
   v = cdf('Normal', z(:,2),0,1);
y1 = [u v];
end
```

```
function y2 = Claytonrnd(Theta, N);
% This function generates random numbers from Clayton copula as presented in Figure 1.
% input:
% Theta: the Clayton copula parameter with  Theta ∈ [−1,∞)/{0}
% N = number of random numbers to be generated
% output:
% y = random numbers from Clayton copula, n by 2 matrix
if N== 0
   y2=[];
else
   y2 = NaN*ones(N,2);
   q = rand(N,1);
   u = rand(N,1);
   v = ((q.^(-Theta/(1+Theta))-1).*u.^(-Theta)+1).^(-1/Theta);
   y2 = [u v];
end
```

```
function y3 = studrnd(rho, Dof, N);
% This function generates random numbers from Student's t copula as presented in Figure 2.
% rho = the correlation matrix
% Dof: degrees of freedom
% N = number of random numbers to be generated
% y3 = random numbers from Student's t copula, N by 2 matrix
if N == 0
y3=[];
else
   A = chol(rho);
   z=randn(N,1);
   s=chi2rnd(Dof);
    y = z' · A
    x = (sqrt(Dof)/ sqrt(s))· y;
y3 = tcdf(x,Dof);
end
```

```
function[Fairspread]=Gaussn2dMc[N,n2d,rho,T,r,R,h]
%  inputs:
% h: Default intensity for all obligors (assumed deterministic)
% N= number of obligors on the basket default swap (basket size)
% k: number to default in the basket default swap eg k=1 for first to default swap
% runs: number of Monte Carlo simulations
% rho: correlation matrix between obligors
% T: maturity of credit default swap, typical maturity is three, five and ten years.
% r: risk free interest rate (assumed deterministic)
% R: recovery rate
```

```
% delta: day count fraction on year eg delta=0.25  quarterly period
% ntimesubsteps=10; %no. of sub timesubsteps within each delta for integration
% outpout:
% Fairspread: return fair spread of nth to default swap as described in § 4.1

n=T/delta %time steps for indexing preminum payments eg. T=5, delta=0.25, n=5/0.25=20
dt=delta;
dt2=dt/ntimesubsteps;

%%%%%%%%%%%%%%%%Simulation of default times%%%%%%%%%%%%%%%%%%%%%%
mu=zeros(N,1);
sig=ones(N,N)*rho + (1-rho)*eye(N);
%%%%%%%%%%%%%%%% Simulation of default times via Gaussian copula  %%%%%%%%%%%%%%
matrix=mvnrnd(mu,sig,runs);
defTime=-log(cdf('Normal',matrix,0,1))/h;
%%%%%%%%%%%%%%%% Simulation of default times via t-student copula  %%%%%%%%%%%%%%
%CorMat=(eye(N)-eye(N)*rho) + ones(N,N)*rho;
%A=chol(CorMat); %CorMat=A'*A here
%z=randn(scenarios,N);
%s = chi2rnd(DoF,1,N);
%y=z*A;
%x=repmat((sqrt(DoF)./sqrt(s)),scenarios,1).*y;
%u=tcdf(x,DoF);
%defTime = -log(u) ./ repmat(h,scenarios,N);
%%%%%%%% %%%%%%%%%% The value of premium leg %%%%%%%%%%%%%%%%%%%%%
t=dt;
PL=0; %Expected value of average Premium leg payments
for i=1:n,
          %find probability of survival of kth default until each indexing period
          B=exp(-r*t); %B is the discount factor at time t
          SurvivalProb=1-sum(sum(defTime<t,2)>=k,1)/size(defTime,1);
          PL=PL+delta*B*SurvivalProb;
          t=t+(T/n);
end
PL
%%%%%%%%%%%%%% The value of default leg %%%%%%%%%%%%%%%%%%%%%%%%%%
DP=0; %Expected value of average default leg payments
kdefaultTime=sort(defTime,2);
kdefaultTime=kdefaultTime(:,k); %default time for each scenario
DPvec=(1-Recovery_rate)*exp(-r*kdefaultTime).*(kdefaultTime<T);
DP=sum(DPvec)/scenarios

%%%%%%%%%%%%%% The value of accrued premium leg %%%%%%%%%%%%%%%%
AP=0; %Expected value of average Accrued premium
t=0;
for i=1:n,
          curt=t;
          for ti=1:ntimesubsteps,
                    %find probability that k th default happened
                    %between ti and ti+1
                    DefaultsVec=(kdefaultTime>(curt)) & (kdefaultTime<=(curt+dt2));
                    ProbOfDefault=sum(DefaultsVec)/scenarios;
                    curt=curt+dt2/2;
                    B=exp(-r*curt);
                    AP=AP+(curt-t)*B*ProbOfDefault;
                    curt=curt+dt2/2;
          end
          t=t+dt;
end
AP
spread=10000*DP/(PL+AP)
```