Fixed-income instrument pricing.

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Abstract:

In this article we discuss the fundamentals of pricing of the popular financial instruments. The basic point of our approach is to extend the present value benchmark concept. The present value valuation approach plays the similar role as The Newton Laws in the Classic Mechanics. Thus our primary goal is to present a new outlook on valuation of the debt securities and its derivatives. We also, demonstrate why the present value is not a complete method of pricing either securities or derivatives. Then, as illustration we present a valuation of the floating rate, callable and convertible bonds. Next we discuss major drawbacks of the risk neutral interpretation of the derivatives pricing. At the end of the article we discuss interest rate swap and derivative valuation of some classes of the fixed income securities.

Basic of fixed-income pricing.

U.S. Treasury debt instruments are issued to raise money needed to operate the Government and to pay off maturing obligations. Treasury bills or T-bills are short-term securities that mature in one year or less from their issue date. An investor buys a T-bill for a price less than its' face (par) value, and when it matures the investor receives T-bill par (face) value. US Treasury notes and bonds are securities that pay a fixed rate of interest every six months until the maturity that is when an investor gets their par value. The only difference between the instruments is their length to maturity. Treasury notes mature within 10 years from their issue date. Bonds mature in more than 10 years from the issue date. Treasury bills, notes, bonds are transferable, so they can be sold or bought in the security market.

Strips are zero-coupon securities that don't have periodic interest payments. Market participants create strips by separating the interest payments and principal of Treasury note or bond. When the security is "striped" the interest payments and the principal becomes separate market instruments and can be held and transferred independently.

A 0-coupon bond price B (t , T) by definition is a function of two variables t and T. In many practical situations it seems more convenient to use as independent variables t and T – t instead of t , T. It represents the value at date t of the $1 at date T. Thus B (t , T) is the present value at date t of receiving one dollar at T with no risk of default. The standard form used for pricing the Treasury security is

$$B(t,T) = 1 - i_d \frac{T-t}{360} = \frac{1}{1 + i_s \frac{T-t}{365}}$$

Here B (t , T) is the 0-coupon T-bill, note, or bond price at date t , and B (T , T) = $1; parameters i_d and i_s are constant here but they also can depend on time and in a complex environment may be assumed to be stochastic. They are the discount rate and the simple interest rate respectively. Given B (t , T) the values of interest and discount rates can be easily calculated and vice versa. The finical tables usually provide investors information about the bond prices. In continuous compounding we assume that

$$\frac{d\,B(t,T)}{d\,t} = r\;B(t,T)$$

t < T , with a boundary condition B (T , T) = 1. A 0-coupon debt-security price B (t , T), t ≤ T and B (T , T) = 1 with no risk of default is sometimes referred to as present value or discount factor. That is the value at t of receiving one dollar at maturity. The benchmark formula

$$PV(t;t_1,c,...,t_{N-1},c,\,t_N,c+F) = \sum_{j=1}^{N} c\,B(t,\,t_j) \;+\; F\,B(t,\,T) \qquad (1)$$

represents the present value of the cash flow associated with the coupon bearing bond. Here:

*) c is the coupon paid by issuer of the security to bond holders at the moments predetermined dates $t = t_0 < t_1 < t_2 < \ldots < t_N = T$

**) B (t , T) is the value of strips (0-coupon bond) with no risk of default.

***) F is the face (par) value of the bond.

We denote the left hand side of (1) as PVB (t , c , T) and note that (1) presents one of the possible presentation of the price of the coupon bearing bond.

Thus, the bond price that is by definition the present value of the all payments attached to the security over its lifetime. If the value of a 0-coupon bond with various maturities is given, then the above formula (1) represents the so-call term structure of the coupon-bearing bond. That is by definition the function given by (1) of the variable T when time t is fixed.

We represent other interpretation of the coupon-bearing bond price. The formula we introduce below differs from (1) and it coincides with (1) in a very particular case. For example two types of the bond price would be equal when the risk free interest rate assumed to be a constant. To highlight a motivation of the difference between two approaches one should remark that bond's issuer (bond seller) estimates the spot bond price based on present value cash flow presented in (1). That means that if a bond issuer would pay $c coupon at a date s ∈ (t , T) then this amount can be risk-free generated by investing $cB(t , s) , B(t , s) ≤ 1 in the bond at the date t. Note that the bond issuer is a owner of the investment over period [t , s). On the other hand a bond buyer (investor) receiving the coupon payment at the moment s owns this sum over the adjacent period (s , T] until the bond expired. Real world market data show that the rates of return over non-coinciding intervals do not equal. That proves necessity to distinguish seller's and buyer's cash flows generated by the bond contract. It is clear that the bondholder return is estimated based on the discounted face value of the same cash flow. The value at time T of the amount of $c paid at t_j is by definition equals to $c \times B(t_j, T)$. Hence the bondholder's capital at the date T is

$$FV(t; t_1, c, \ldots, t_{N-1}, c, t_N, c+F) = \sum_{j=1}^{N} c\, B^{-1}(t_j, \mathrm{T}) + F$$

Recall that by definition B (T , T) = 1. The pricing problem is to establish the price of the coupon-bearing bond. To derive this price we assume that all needed market information is available. Thus the bond buyer considers alternatives invest in 0-coupon or c-coupon government bonds. To avoid arbitrage opportunity market should provide equal rate of return on government bonds with the same expiration dates regardless of the coupon value. This remark leads us to the equation

$$\frac{FV(t; t_1, c, \ldots, t_{N-1}, c, t_N, c+F)}{B_c(t, T)} = \frac{B(T, T)}{B(t, T)} \tag{2}$$

The solution of the equation (2) is the price at time t of the coupon-bearing bond. From (2) it follows that

$$B_c(t, T) = \left[\sum_{j=1}^{N} c\, B^{-1}(t_j, T) + F \right] B(t, T) \tag{3}$$

The equality (3) states that the price of the coupon bond is the present value of the total cash over the lifetime of the bond at maturity date. Here it was assumed that functions on the right-hand side (3) are given. Also note that the value $B_c(t, T)$ does not equal to the commonly used present value (1).

Unfortunately, the values $B(t_j, T)$, $t < t_j$, $j = 1, 2, \ldots N$ are unknown at the date t and therefore the assumption used in derivation (3) is not realistic. One way to proceed is a randomization of the problem setting. Admitting stochastic setting one can apply the statistical estimates drown from historical data prior the date t. Other way is to use either forward contract market data on government bonds or an analytic assumption regarding stochastic bond price dynamics. Let s be a difference between (3) and (1). Then $s = s(\omega)$ is a random variable. This setting leads to interpretation of the real price as a settlement price between two counterparties. If a settlement price is the present value PVB (t , c , T) of the coupon bond then the market risk of the counterparties can be expressed with the help of the commulative distribution function *F (x)* of the random variable *s* (ω)

$$F(x) - F(0) = P\{\, 0 \le B_c(t, T) - PVB(t, c, T) < x \,\}$$

$$F(-x) = P\{\, B_c(t, T) - PVB(t, c, T) < -x \,\}$$

In order to present an analytic form of the function *F (x)* an assumption on the bond future values distribution is needed. There is a common way to avoid such difficulties is to apply the 'implied' approach that is widely used in the modern finance. Recall that 'implied' approach admits a hypothetical distribution model without its statistical testing. The proof of the formula (3) is straightforward and based on the mathematical induction method. We begin with the last interval $(t_{N-1} , T]$. Over this interval the values of the coupon bond can be received from the 0-coupon bond curve by multiplying it by the factor (F + c). Indeed, the bond supplied by either 0 or c > 0 coupon issued by the same institution should promise equal rate of return for any t from $(t_{N-1} , T]$. Otherwise, there exist an arbitrage opportunity. Note; the arbitrage is used as a necessary condition, but not as the price definition argument as it frequently does in derivative applications. From the equation

$$\frac{c + F}{B_c(t, T)} = \frac{1}{B(t, T)}$$

it follows that for any t from $(t_{N-1} , T]$

$$B_c(t, T) = (c + F)\, B(t, T)$$

At the date t_{N-1} the bondholders receive a coupon of $c and from the formula above we see that

$$B_c(t_{N-1}, T) = B_c(t_{N-1} + 0, T) + c$$

where $B_c(t_{N-1} + 0 , T) = \lim B_c(t_{N-1} + h , T)$ when number h > 0 tends to 0. Next let us repeat the pricing procedure over the semi open interval $(t_{N-2} , t_{N-1}]$. Then from the equation

$$\frac{B_c(t_{N-1}, T)}{B_c(t, T)} = \frac{B(t_{N-1}, T)}{B(t, T)}$$

follows that

$$B_c(t, T) = \frac{[\, B_c(t_{N-1} + 0, T) + c \,]\, B(t, T)}{B(t_{N-1}, T)} = [(c + F) + c\, B^{-1}(t_{N-1}, T)]\, B(t, T)$$

for arbitrary t from the interval (t_{N-2} , t_{N-1}] and where B (t_{N-1} + 0 , T) = B (t_{N-1} , T) as the 0 coupon bond price is assumed to be continuous function. Since there is a finite number subintervals (t_{j-1} , t_j] , j = 1, 2, …N this construction can be completed by induction for any finite number of steps. Indeed this formula holds on (t_j , T]. Then

$$B_c(t, T) = [\sum_{k=j+1}^{N} c B^{-1}(t_k, T) + F] B(t, T)$$

for any t from (t_j , t_{j+1}]. In particular

$$B_c(t_j + 0, T) = [\sum_{k=j+1}^{N} c B^{-1}(t_k, T) + F] B(t_j, T)$$

Hence

$$B_c(t_j, T) = B_c(t_j + 0, T) + c$$

and then

$$\frac{B_c(t_j, T)}{B_c(t, T)} = \frac{B(t_j, T)}{B(t, T)}$$

for any t from (t_{j-1} , t_j];

$$B_c(t, T) = \frac{[B_c(t_j + 0, T) + c] \; B(t, T)}{B(t_j, T)} =$$

$$= \frac{[B(t_j, T) (\sum_{k=j+1}^{N} c B^{-1}(t_k, T) + F) + c] \; B(t, T)}{B(t_j, T)} =$$

$$= B(t, T) [\sum_{k=j}^{N} c B^{-1}(t_k, T) + F]$$

That justifies the formula (3).
We have highlighted the difference between coupon bond price given by (3) and the present value of the cash floor generated by coupon bond that commonly used as the price of the coupon-bond. Now, we present risk management that covers stochastic relationship between coupon bond price and its present value. The probability that bond price exceed present value is F (x) – F (0) specifies the chance that purchasing bond offers return higher than selling it. Indeed, from seller's point of view the cost of the coupon bond at time t could be presented by (1). The bond value given by (3) is what the bond buyers assumes to be the bond price at the date t . The present value of the bond in

neutral market having symmetric distribution to go up or down could be a good unbiased estimate for the bond price. A condition below illustrates such situation.
In order that (1) and (3) to be equal for an arbitrary chosen dates of coupon payments and maturity, it is necessary and sufficient that the 0-coupon bond satisfy the follow equality

$$B(t, s)\, B(s, T) = B(t, T)$$

for arbitrary $0 \le t \le s \le T$. Unfortunately, historical data shows that this equality does not take place in the real world.

Putting in (3) c = 0 we arrive at the 0 coupon bond with face value \$F and its present value therefore is \$FB (t , T). This price represents an instrument referred to as a 'strip'. In this case we see that the bond price is equal to its present value. If F = 0, then the price of this component of the stripped treasury can be also obtained from (3). This financial instrument is a claim on pure coupon payments. The Wall Street Journal uses abbreviations: "np" and "bp" for the Treasury note and bond respectively, and "ci " for the claim on pure coupon payments. One can easy figure out that using historical data that the present value given by formula (1) and the bond price (3) lead to the different values. A source of the discrepancy is a variability of the interest rates. Assume now that $B(t_j, T)$, j = 1, 2, … N are random variables and therefore the spread

$$s(\omega) = B_c(t, T) - PVB(t, c, T)$$

is a random variable. The buyer's risk value then is associated with the probability of the event $\{\omega : s < 0\}$. On the other hand the bond seller's risk is associated with the event $\{\omega : s > 0\}$. The distribution of the random function $s = s(t, T; \omega)$ depends on the unobservable at date t random variables $B(t_j, T)$, $t_j > t$. To derive statistical characteristics of the spread the unobserved random variables should be replaced by their statistical. The data related to forward contacts on bonds with expiration at t_j can be used to construct an estimate of the conditional expectation $E\{B(t_j, T) \mid \mathcal{F}_t\}$. Here $\mathcal{F}_t$ is the σ-algebra generated by the bond prices prior the date t , $t \le t_j$. We will not discuss here details of the modeling of the conditional expectations.

Let us now take a look at another popular debt contract known as *floating rate bond*. Our construction will be based on the scheme that makes a difference between coupon bond price (3) and the correspondent present value of the bond represented by (1). The cash flow is specified as follows. Consider an example: a two years floating rate note with 26-weeks interest payments and \$1 is the face value. The contract issue date is $t = t_0$ = 01/06/00. The table below specifies information that will be used

Security Term	Issue Date	Maturity Date	Discount Rate%	Investment Yield%	Price Per \$100
26-week	01-06-2000	07-06-2000	5.585	5.844	97.176
26-week	07-06-2000	01-04-2001	5.975	6.247	96.979
26-week	01-04-2001	07-05-2001	5.360	5.586	97.290
26-week	07-05-2001	01-03-2002	3.500	3.612	98.231
2-year	12-31-1999	12-31-2001	6.125	6.233	99.800

The 26-weeks interest at date t_0 is 5.844 percent per annum therefore the coupon payment $c(t_1) = 0.05844 / 2 = 0.02922$. In 26 weeks at t_1 = 07/06/00 the new 26-week interest is 6.247%. Therefore, the coupon payment $c(t_2) = 0.031235$ would be paid at date t_2 and the new interest rate 5.586 percent is covered next period. Therefore $c(t_3) = 0.02793$. At date t_3 = 01/04/01 the interest rate is 3.612%, and $c(t_4) = 0.01806$. Thus, the cash flow valuation can be presented as

Date	t	t_1	t_2	t_3	T
Coupon payments	0	2.922%	3.1235%	2.793%	1.806%

This point of view reflects bond issuer commitments. Indeed, at each reset date the floating rate bond always being valued at par that is \$1. Indeed, at date t_3 the present value of the payment \$(1+ 0.01806) received at T is \$1. Then at date t_2 the value of the payment \$(1+ 0.02793) at t_3 is \$1. That is

$$(1+0.02793)\$(t_3) = \$1(t_2)$$

The present value at t_1 of the payment \$(1+ 0.031235) at t_2 is again \$1 and the present value at t_0 of the payment \$(1+ 0.02922) at t_1 is \$1. Hence the interest of \$1 paid by the bond issuer to the floating bond holder at T is equivalent of receiving cash flow of \$0.02922, \$0.031235, \$0.02793, \$1.01806 at moments t_1, t_2, t_3, T correspondingly.

From a bond buyer perspective the pricing of the floating contract looks different. At date t the buyer pays the seller \$1 and receives a coupon for \$0.02922 in 26 weeks. Then at t_2 the investor receives a new coupon of \$0.031235 and a coupon of \$0.02793 at t_3. The last coupon of \$0.01806 arrives at maturity T along with the principal payment of \$1. Then the buyer can estimate profit of the investment at date T when the last payment is received. We assume it is a 26 weeks compounding interest, otherwise we would need additional information. Then the coupon payments are equivalent to the value

$$0.02922\,[1 + i(t_1, T)] + 0.031235\,[1 + i(t_2, T)] + 0.02793\,[1 + i(t_3, T)] + 1 \approx$$

$$\approx 1.092655$$

at T, where $i(s, t)$ denotes the interest received over the time interval $[s, t]$ and the values of interest are $i(t_1, T) = 0.079182$, $i(t_2, T) = 0.046494$, $i(t_3, T) = 0.01806$. Therefore, analysts should arrive at %9.2655 interest received by investor at the end of the second year. On the other hand if \$1 is invested for a two-year period with the interest of %6.233 per annum then the results are

$$(1+0.06233)^2 = 1.128545$$

This observation confirms that the rate of return from the bond buyer and seller perspectives is not equal. Therefore their estimates of the bond value are also different. In this example the economy position for the bond seller looks more favorable than for the bond's buyer.

In general setting consider an interval [t , T] and $t = t_o < t_1 < \ldots < t_N = T$ be an interest rate reset dates and the step $\Delta = t_{j+1} - t_j$ does not depend on j. Let $i (t_j , t_j + \Delta)$ be the a floating rate at t_j used for the next period. For writing simplicity assume that notional principal is $1 otherwise all cash transactions should be proportionally changed. The floating cash flow from the bond buyer is

Dates	t_0	t_1	t_2	…	$t_N = T$
Floating flow	-1	$i (t_o , t_o + \Delta)$	$i (t_1 , t_1 + \Delta)$	…	$1 + i (t_{N-1}, t_{N-1} + \Delta)$

One can see that $\$(T) [1 + i (t_{N-1}, T)] = \$(t_{N-1})$. Hence in particular

$$\$(t_{N-1})\, i (t_{N-2}, t_{N-1}) + \$(T) [1 + i (t_{N-1}, T)] = \$(t_{N-1}) [1 + i (t_{N-2}, t_{N-1})]$$

Therefore cumulative cash flow to the bond buyer over [t , T] is

$$\$(t_1)\, i (t_o , t_1) + \$(t_2)\, i (t_1 , t_2) + \ldots + \$(T) [1 + i (t_{N-1}, T)] =$$

$$= \$(t_1)\, i (t_o , t_1) + \$(t_2)\, i (t_1 , t_2) + \ldots + \$(t_{N-1}) [1 + i (t_{N-2}, t_{N-1})] = \ldots$$

$$\ldots = \$(t_1) [1 + i (t_o , t_1)] = \$1(t)$$

These calculations prove that $1 at date t is the price of the floating contract. The bond buyer who pays $1 at t will receive equivalent cash payments over the period (t , T]. The variability of the interest rate does not change the valuation. The issuer of the floating bond will receive an equivalent cash flow with an opposite sign. That is

Dates	t_0	t_1	t_2	…	$t_N = T$
Floating flow	1	$- i (t_o , t_o + \Delta)$	$- i (t_1 , t_1 + \Delta)$	…	$-[1 + i (t_{N-1}, t_{N-1} + \Delta)]$

This floating rate bond valuation used a present value exposure to justify pricing model. On the other hand bond buyer may use the other approach outlined above. Indeed formula

$$Fl(T) = \sum_{j=1}^{N} i(t_{j-1}, t_j)\, B^{-1}(t_j, T) + [\, 1 + i(t_{N-1}, t_N)]$$

presents the date-T value of the floating bond payments. One might expect that with the equal maturity the floating bond rate of return is the same as the rate of return of the 0-coupon bond. That implies that

$$Fl (T) / Fl (t) = 1 / B (t , T)$$

The solution of this equation is

$$Fl(t) = B(t, T)\,[\, \sum_{j=1}^{N} i\,(t_{j-1}, t_j)\, B^{-1}(t_j, T) + 1 + i\,(t_{N-1}, t_N)\,]$$

As far as the rates $i(t_{j-1}, t_j)$, $j \geq 2$ are unknown at the date t therefore it may be reasonable to interpret them as a sequence of random variables. For numeric calculations one can use their statistical estimates.

There is a popular class of debt securities referred to as to *convertible, callable bonds*. A convertible bond can be interpreted as a hybrid security. A convertible, callable bond is type of security that combines the company's bond and stocks. Recall some definitions that will be used bellow.

A convertible bond issued by a company gives the holder the right to convert or exchange the face bond value for the company's common shares in a particular period, or at any time beginning from predetermine moment in the future. The exchange ratio, i. e. amount of stocks obtained in exchange for a bond, may depend on the time.

The callable feature means that the bond issuer has the right to buy the bond back at a call price. The holder of the bonds has the right to convert the convertible, callable bonds once they have been called. Thus, the call forces the conversion earlier than the holder might choose.

In buying a convertible bond investors have an opportunity to exchange the bond for a certain amount of stocks at any time in the future. This represents the lowest boundary of the convertible bond price. The callable provision is not holder's benefit. This is an issuer's privilege. It makes the valuation more complicated. Theoretically one can admit that the bond price is higher than callable bond value because an issuer of the bond can immediately establish the call claim. First we simplify the setting assuming that the bond can be either exchanged or called at any time in the future. Convertibility of the bond combines two fundamental properties. First, it is a debt instrument and like any bond its value depends on the face value of the bond. On the other hand it could be exchanged for stock. Hence bond values depend on the stock price too. We will show how this intuitive idea can be expressed in mathematical formulas.

Denote ccb(l) the price of the convertible callable bond at a time $l \in [t, T]$. At any time l there are always two possibilities. One possibility is to exchange bond at any time until maturity. The holder of the convertible bond has the right to convert the bond in company stocks applying the prespecified convertible rate κ. For instance if the bond has denomination of $1000 and at a time of issue a conversion price was set at $125 then the conversion price implies that the conversion ratio $\kappa = 8$. The second possibility that the issuer may expose at the date l is the call to buy back bond for c. Let the company stock at date t be $S(t) = x$ and N > 0 be a small number. Following are the details of the bond pricing. Consider a rational investor with a common sense in a decision making. Now let us highlight the mutually disjoined events that determine an investor's strategy.

*) Suppose that the company stock will not promise visible profit at the date $l \in [t, T]$. This can be expressed for example in a form

$$\sup_{l \leq s \leq T} \kappa S(s) \leq \kappa x + \delta$$

for some prespecified by investor number N > 0. In this case it looks reasonable to exercise bond immediately at *l*. Note that the parameter N can represent some additional fees, charges or a risk cost. Next let A be a holder's lowest level, at which a bond holder thinks it makes sense to convert bond into stock. Thus if

$$\sup_{l \le s \le T} \kappa S(s) \ge A$$

then there exist a moment of time during the bond's lifetime when it would be reasonable to convert bond into stock. The conversion price A may not be convenient for the bond issuer and therefore at some level prior the barrier A the bond issuer can take advantage of a calling bond. Let C be the issuer's call-back level. The realization of this strategy can be expressed in the form

$$C \le \sup_{l \le s \le T} \kappa S(s) < A$$

Scenarios when the convertible callable bond is exercised as a bond itself are

$$\kappa x + \delta < \sup_{l \le s \le T} \kappa S(s) < C$$

Thus the exercised value of the convertible callable bond can be written in the form

$$\kappa x \, \chi \{ \sup_{t \le l \le T} \kappa S(l) \le \kappa x + \delta \} + A \, \chi \{ \sup_{t \le l \le T} \kappa S(l) \ge A \}$$
$$+ C \chi \{ C \le \sup_{t \le l \le T} \kappa S(l) \le A \} + N \chi \{ \kappa x + \delta < \sup_{t \le l \le T} \kappa S(l) < C \} \qquad (4)$$

where N is a bond denomination. One can probably note that parameter values A and C are probably unknown for the both parties. Therefore the case when issuer's value C is larger than bondholder's value A is also needs to be included. In this case formula (4) should be adjusted by jointly exchanging places of A and C. The time at which the investor should exercise a bond also may vary, and dependable on trajectory ω = S (∗). To represent an analytical valuation formula in a general case we put A ∨ C = max (A , C) and A ∧ C = min (A , C). Then (4) can be rewritten in the form

$$\kappa x \, \chi \{ \sup_{t \le l \le T} \kappa S(l) \le \kappa x + \delta \} + A \vee C \, \chi \{ \sup_{t \le l \le T} \kappa S(l) \ge A \vee C \} +$$

$$+ A \wedge C \, \chi \{ A \wedge C \le \sup_{t \le l \le T} \kappa S(l) \le A \vee C \} + N \chi \{ \kappa x + \delta < \sup_{t \le l \le T} \kappa S(l) < A \wedge C \}$$

If a scenario can describe the stock behavior when the first term is not equal to 0 then the bond should be exercised at initiation. Note; in this case that all other terms are equal to 0. If the second term is positive then the bond should be converted into stock at some moment τ_A when stock price S ($*$) hits the level A initially . If the stock price is assumed to be random then this moment is a random variable. If the company stock can be characterized by the third term in (4) then the bond should be called immediately after the moment τ_C. It is also possible by taking additional risk that the issuer will extend waiting period after τ_C but no later than τ_A. The last term represents the bond itself literally. In this case the bond would be exchanged for the notional principal N at the bond maturity. The American option valuation formula introduced in [2, 3] can be written as follows

$$f_\nu(t, x) = [\nu^{-1}\chi(\tau \le T) + S^{-1}(T)\chi(\tau > T)] \times$$
$$\times \max\{[(\nu - K)(\tau \le T) + (S(T) - K)\chi(\tau > T)], 0\}\chi\{S(T \wedge \tau) > K\} \tag{5}$$

Here $\tau = \tau_1(\nu) = \tau_x(\nu x)$ is the first passage time of the barrier νx by the random process $S(l; t, x)$ when ν equal A or C. The log-normal distributed random process $S(l; t, x)$ is assumed to be a solution to the linear stochastic differential equation. Formula (5) can be applied for calculation convertibility and call contributions in bond pricing. Let $f_A(t, x)$ and $f_C(t, x)$ be the values of American options exercised at the moments τ_A and τ_C correspondingly and let L (t , T) be a company's cost of borrowing \$N over [t , T] period. Note that in such setting L (,) represents the price of the company's 0-coupon corporate bond. Then the value

$$ccb(t,T) = \kappa x \chi\{\sup_{t \le l \le T} \kappa S(l) \le \kappa x + \delta\} + f_{A \vee C}(t, x)\chi\{\sup_{t \le l \le T} \kappa S(l) \ge A \vee C\} +$$
$$+ f_{A \wedge C}(t, x)\chi\{A \wedge C \le \sup_{t \le l \le T} \kappa S(l) \le A \vee C\} + \tag{6}$$
$$+ L(t, T)\chi\{\kappa x + \delta < \sup_{t \le l \le T} \kappa S(l) < A \wedge C\}$$

is the price at date t , t < T of the callable convertible bond. This formula is correct either for deterministic or stochastic settings. A contingent claim which price is (6) can be interpreted as a hybrid of American type option and corporate bond. It also makes sense to consider the European type of the bond. In this case the callable bond can be exercised only at maturity. One can see that in this case the payoff (6) can be easily adjusted to cover the changes. Namely, in (6) the expression " sup $\kappa S(l)$ " should be replaced by "κS (T)".
We observed that each term of the payoff (5) corresponds to the specific events (scenarios) and the union of these mutually exclusive events constitutes the entire probability space. Therefore, only one term in (6) does not equal to zero for any scenario.

In particular, the second and the third term in (6) are equivalent to the call option payoff. These terms indicate a portion of the company's stocks that can be bought for $A or $C correspondingly. There is a difference. Option's strike price is predetermined and known for the both counterparties of the option contract in contrast to the value A that is known only for the bond seller. Historical data can provided to estimate own risk for a particular value A to the seller of the bond. Then the bond buyer can make an estimate <A> of the A and use it for a construction an appropriate value C. The <A> of course may differ from A. Admitting a hypothetical distribution of the company's stock one can calculate the risk and statistical characteristics of the instrument. In more general setting parameters κ, A, and C may also depend on time or even on S.

Now, we can present an adjustment of the valuation formulas that admits the coupon interest paid by convertible callable bond. We will apply the valuation method that was used above for a plain bond. Note; that at maturity T the bond can only be converted or exercised as a corporate bond. Therefore,

$$ccb\,(T, T) = \kappa S(T)\,\chi\{\kappa S(T) > \mathrm{N}\} + \mathrm{N}\,\chi\{\kappa S(T) \le \mathrm{N}\}$$

Following derivation of the formula (3) we have

$$\frac{ccb(T,T)}{ccb(t,T)} = \frac{ccb_c(T,T)}{ccb_c(t,T)}$$

where $ccb_c(T, T)$ is the value of the convertible callable bond offering periodically coupon c% of the notional principal. Then

$$ccb_c(T, T) = ccb(T, T) + \mathrm{c}$$

Hence,

$$ccb_c(t, T) = [\,1 + \frac{c}{ccb(T, T)}\,] \times ccb(t, T) \qquad (7)$$

where $ccb(t, T)$ is given by (6). This formula holds for the time interval $(t_{N-1}, T]$. In order to receive the complete formula we apply the induction method used earlier for the proof of the equality (3). Note that

$$ccb_c(t_{N-1}, T) = ccb_c(t_{N-1} + 0, T) + c$$

where $ccb_c(t_{N-1} + 0, T)$ is the right hand side limit of the (7), when time t tends to the t_{N-1} and $t > t_{N-1}$. The formula

$$\frac{ccb_c(t_{N-1}, T)}{ccb_c(t, T)} = \frac{ccb(t_{N-1}, T)}{ccb(t, T)}$$

uniquely defines the value of the coupon convertible bond over the next interval. Hence

$$ccb_c(t, T) = ccb(t, T)\,[\,1 + \sum_{j=1}^{N} \frac{c}{ccb(t_j, T)}\,] \qquad (8)$$

Now we will apply the construction that has been developed to the swap valuation. Consider an *interest rate swap* referred to as "plain vanilla", with no chance of default. There are two counterparties involved in a swap contract. The counterparty A makes fixed periodic semiannual or quarterly payments to counterparty B. The magnitude of each fixed payment is usually a prespecified percent of the notional principal. In return, counterparty B pays floating rate payments to A. All payments are made in the same currency. The only netted amount is paid at reset dates. Consider for example information that helps us to present the swap transactions numerically

Security term	Issue Date	T-Yields	T-bond Price	LIBOR rate	LIBOR-Price
6-months	01-01-2000	5.76	0.972081	6.136	0.970312
6-months	07-01-2000	6.27	0.969684	7.014	0.966208
6-months	01-01-2001	5.15	0.974963	6.208	0.969974
6-months	07-01-2001	3.56	0.982558	3.827	0.981275
6-months	01-01-2002	1.77	0.991251	1.983	0.990209
2-years	12-31-1999	6.125	0.970364		

The swap is initiated at 01-01-2000. Assume that fixed rate is set 6.125% and notional principal is a million dollars. Then in a half-a-year later at 07-01-2000 party A should pay to party B payment

$$1,000,000 \times 0.06125 \times 0.5 = \$30,625$$

In return counterparty B should pay to the counterparty A

$$1,000,000 \times 0.06136 \times 0.5 = \$30,680$$

We used a year format in above calculations, though day format is common for numeric presentation. Note that the first payment is known in 6 months prior to this payment. The payment of the swap by agreement is netted and the only difference of $55 goes from B to A. The first common question is what rate needs to be applied to the present value of the sum. There are two rates that can be applied. Present value based on T-bond rate or variable LIBOR rate. It is a standard rule to use T-bond rate or money account as the basis presenting value for the known future value given in US dollars in the USA. We have presented the first fixed side of the swap. To get the value of the other component we should calculate differential of the second payment. The real world data from the above table states

$$1,000,000 \times (0.06125 - 0.07014) \times 0.5 = -\ \$4,445$$

Hence the amount of $4,445 is paid by B to A at 01-01-2001. Note that this component of the swap value is unknown on the date of 01-01-2000 and forward rates estimate can be applied. Following these calculations more closely we receive that

$$1{,}000{,}000 \times (\ 0.06125 - 0.06208\ ) \times 182/365 = -\ \$413.86$$

$$1{,}000{,}000 \times (\ 0.06125 - 0.03827) \times 183/365\ = \$11{,}521.48$$

paid by A to B on next reset dates. The minus sign signifies the reverse payment from B to A. Thus next table summarizes the real payments

Payments Date	Paid by A in $
01-01-2000	0
07-01-2000	- 55
01-01-2001	- 4,445
07-01-2001	- 413.86
01-01-2002	11,521.48
Total on the date 01-01-2002	6607.62

In general let $t = t_o < t_1 < \ldots < t_N = T$ be a partition of the interval [t , T] where t_j , $j = 0, 1, \ldots N$ are the reset dates; q and l (*, *) denote a fixed and a floating rates correspondingly. The fixed flow line in the table below represents the payments made by A to B and the floating line is the payments made by counterparty B to A.

Dates	t_0	t_1	t_2	…	$t_N = T$
Fixed flow	1	q	q	…	1 + q
Floating flow	1	$l\,(t_o, t_o + \Delta)$	$l\,(t_1, t_1 + \Delta)$	…	$1 + l\,(t_{N-1}, t_{N-1} + \Delta)$

If only netted payments are paid then the profit-loss table between parties is

Dates	t_0	t_1	t_2	…	$t_N = T$
A paid	0	$q - l\,(t_o, t_o + \Delta)$	$q - l\,(t_1, t_1 + \Delta)$	…	$q - l\,(t_{N-1}, t_{N-1} + \Delta)$
B paid	0	$l\,(t_o, t_o + \Delta) - q$	$l\,(t_1, t_1 + \Delta) - q$	…	$l\,(t_{N-1}, t_{N-1} + \Delta) - q$

Note, that the negative value in a cell means that the corresponding party pays the amount to the opposite side of the swap. If the notional principal is $N then all entries should be multiplied by the N in order to present real cash stream. Let us highlight some important points in the swap valuation. US institutions may only use the US risk free rate for the calculations. Thus the cash present value received by the counterparty A's is equal to

$$\sum_{k=1}^{N} [\ l(t_k, t_{k+1}) - q\ ]\ \$(t_{k+1}) = \$(t) \sum_{k=1}^{N} [\ l(t_k, t_{k+1}) - q\ ]\ B(t, t_{k+1}) \qquad (9)$$

The present value of the counterparty B's is the same number with an opposite sign.

Recall that the rate q is known at the date t_0. Therefore, at the beginning of the swap, party A is able to invest \$q in T-bonds with maturities t_k, k = 1,2, …,N. Thus the present value of the fixed side payments is

$$\$(t) \sum_{k=0}^{N-1} q\, B(t, t_{k+1})$$

On the other hand the future values of the floating rate side payments is

$$\$(T) \sum_{k=0}^{N-1} l(t_k, t_{k+1})\, B^{-1}(t_{k+1}, T)$$

Hence the fixed rate q is the solution to the equation

$$B(t,T) \sum_{k=0}^{N-1} l(t_k, t_{k+1})\, B^{-1}(t_{k+1}, T) - \sum_{k=0}^{N-1} q\, B(t, t_{k+1}) = 0 \qquad (10)$$

Note that the equation (10) derives value of q based on protection buyer side in the swap. It represents the cash flow to the party A. The cash flow to the counterparty B is the same amount with an opposite sign. The solution to the equation is

$$q = \frac{B(t,T)}{\sum_{k=0}^{N-1} B(t, t_{k+1})} \sum_{k=0}^{N-1} \frac{l(t_k, t_{k+1})}{B(t_{k+1}, T)} \qquad (11)$$

The exact values of the T-bond and floating rates in the second factor on the right are unknown at t and can be estimated based on historical data using statistical testing. Recall that fixed rate is often set at t as

$$q = i(t, T) + \delta \cong l(t, T)$$

where i (t , T) and l (t , T) are the annual Treasury and LIBOR interest rates over the period [t, T] and δ is an extra basic points. Note that the fixed rate q given by the formula (11) does not coincide with the value q presented by common handbook formulas. For the floating rate, formula (3) is used which differs from the usually used formula (1) in common derivation.

Definition: The swap value is the difference between fixed and floating present values of the lifetime payments.

In stochastic setting the solution q of the equation (10) is a random function that implies market risk. Thus the counterparty A market risk is associated with the event $\{\omega : q < 0\}$, whereas the counterparty B market risk value is associated with the probability of the event $\{\omega : q > 0\}$.

The presented in (11) swap value is not accurate solution to the swap valuation problem because the present value of the cash floor is not a variable rate coupon bond price. Recall that according to the swap rules only the net payoff is paid. We assigned the fixed rate to the initial moment of time and floating rates to the final date of the swap contract. It is easy to see that this assignment does not separate netted payoff and the reduction to the present value does not perfectly follow real swap transactions.

We will provide a more accurate outlook of the swap pricing. This can be accomplished by applying the option valuation method. As it was highlighted above, the present value valuation concept is not complete for a variable market. Here, we briefly highlight incompleteness of the present value concept and sketch an alternative approach. A benefit of the new method would be more accurate swap pricing. Indeed following the swap definition the only differential of the periodic payments must be paid. Details of the option pricing that we would be applied here follows [2-5].

Let us now return to the interest rate swap valuation. Note that the cash flow to the counterparty B in the interest rate swap can be written as a sum

$$C_B(*) = \Phi[q - l(t_k, t_{k+1})]\chi\{l(t_k, t_{k+1}) < q\}$$

and to A is

$$C_A(*) = \Phi[l(t_k, t_{k+1}) - q]\chi\{l(t_k, t_{k+1}) > q\}$$

Each term in the sums is of the same type as the options payoff represented above. In particular terms of the sum on the right hand side of the $C_A(*)$ represent a series of the call option payoff on forward variable interest rate $l(t_k) = l(t_k, t_{k+1})$ with the strike price q and maturity t_k .The terms on the right hand side of the $C_B(*)$ are payoffs of the put options with the same variable rate, strike price and the same maturity.

Follow [2,5] the pricing equation can be presented in the form

$$\frac{S(T)}{S(t)}\chi\{S(T) > K\} = \frac{C(T, S(T))}{C(t, S(t))} \quad (12)$$

where call option payoff $C(T, S(T)) = \max\{S(T) - K, 0\}$. Note that this equation makes sense for example for continuous functions regardless a probability measure. The solution of the equation (12) is a measurable function of its arguments and can be expressed in the form

$$C(t, x, \omega) = \frac{x}{S(T, \omega)}[S(T, \omega) - K]\chi\{S(T, \omega) > K\} \quad (13)$$

To get the comprehensive characteristics of this function a probability measure should be involved. In this case several common parameters such as mean standard deviation will comprise most important standard information regarding the option price.

The correspondent equation and its solution to the put option can be expressed analogously in the form

$$\frac{S(T)}{S(t)}\ \chi\ \{S(T) < K\}\ =\ \frac{P(T,\ S(T))}{P(t,\ S(t))} \tag{14}$$

and therefore

$$P\ (t,\ x)\ =\frac{x}{S(T)}\ [\ K\ -\ S(T)\ ]\ \chi\{\ S(T) < K\ \} \tag{15}$$

Note that these option price formulas (14, 15) are valid not only for log-normal price process S (*) but for an arbitrary distribution of the random variable at maturity S (T , ω).

The pricing formulas (13, 15) can be used to evaluate market risk of the options. Let us for example assume that an investor pays a $q premium for the option. Then the investor's market risk at the date t is associated with the event { C (t , x , ω) < q }. This event combines all possible outcomes in which investor pays more then the call option is worth. This risk is associated with the possible outcomes in which price given by (16) is less then the amount q. On the other hand the event { C (t , x) ≥ q } describes favorable for the investor scenarios. In these cases investor pays lowest price that ensues from (15). Having estimates regarding the mean and standard derivation of the theoretical option price the investor could establish a premium $q, appropriately for the investor risk. Details of the continuous time option pricing were represented first in [2].

Taking into account call and put option-pricing formulas (13, 15) let us return to the swap valuation. Thus at the date t = t_0 we see that

$$C_B^{[t,\,T]}\ (t)\ =\ \sum_{j=1}^{N}\ \frac{l\ (\ t_0\ )}{l\ (\ t_j\ )}\ [\ q\ -\ l\ (t_j\ )\ ]\ \chi\ \{\ l\ (\ t_j\ )\ <\ q\ \}$$

$$C_A^{[t,\,T]}\ (t)\ =\ \sum_{j=1}^{N}\ \frac{l\ (\ t_0\ )}{l\ (\ t_j\ )}\ [\ l\ (\ t_j\ )\ -\ q\ ]\ \chi\ \{\ l\ (\ t_j\ )>\ q\ \}$$

Thus by definition the swap value is

$$C_A^{[t,\,T]}(t) - C_B^{[t,\,T]}(t) = \sum_{j=1}^{N}\ \frac{l\ (\ t_0\ )}{l\ (\ t_j\ )}[\ l\ (t_j\ ) - q\ \ ][\ 2\chi\{\ l\ (t_j\ ) > q\ \} - 1\ ] \tag{16}$$

Note that either the expression on the left-hand side (10) or formula (16) represent the interest rate swap value. It is clear that these formulas present different values. The formula (16) represents the swap value more accurately. In particular (16) does not involve risk free discount factors. Another difference between the two formulas is (16) uses the netted payoff. Unfortunately, to derive the value q for which the swap value (16) is 0 is not easy. That of course implies significant difficulties in the risk management of the swap market analysis.

Remarks.

*) The swap valuation formula (16) differs from those that have been used in common handbooks and research papers.

*) The formula (16) implies that all random or deterministic functions involved are known. Actually only the rates established at the date $t = t_0$ are known. All others are unknown, and in the stochastic setting statistical estimates should be used to approximate unknown functions. Therefore the rates $B(t_{k+1}, T)$ and $l(t_k) = l(t_k, t_{k+1})$ in (6,7) should be interpreted as random variables and the fixed rate q defined by (11) and the swap value (16) is also a random variable. When both counterparties A and B admit a non-random fixed rate <q> to the swap, both counterparties are subjected to the market risk. The counterparty A risk value is $P\{\omega : q < 0\}$, whereas the counterparty B risk value is the probability of the event $\{\omega : q > 0\}$.

Now we will study the fixed income options pricing in continuous time highlighting details of the risk neutrality and other related notions used in modern finance. The critical viewpoints on Black-Scholes approach and risk-neutral interpretation of the pricing equation were introduced earlier in [2-5]. There are two major errors in the option pricing benchmark. By following original derivation of the Black-Scholes equation, it was shown [2-5] the term that does not equal to zero was lost in the derivation. Note that is not enough to state that the Black Scholes option price formula can not be used. The second problem is the option price definition itself. Simple examples discussed above show that the option price is a random function, and that any particular amount paid for the option price always implies the market risk. The Black-Scholes pricing is often interpreted by using other probability space in which real trend of an underlying security is replaced by risk-free interest rate. This transformation can be correctly realized by using measure change along with Girsanov theorem. This is a well-known fundamental result of the stochastic calculus, and its adjustment for the continuous time finance should lead to the risk-neutral world and probabilities. The risk-neutrality is used as a self-sufficient vehicle in pricing interest rates, exotics, and credit derivatives.

Risk neutralization and its affect on risky derivatives pricing.

In this section we present a critical viewpoint on benchmark understanding of the derivative pricing. We begin with the general comments regarding option pricing. In continuous time the Black Scholes equation solution represents the option price and provides the benchmark valuation commonly applied for further generalizations. The solution of the Black Scholes equation represents the spot price of the European option given maturity, strike price, and specific structure of the underlying security. The solution of the Black Scholes equation is the present value of the expected neutralized option payoff and might be used even when derivation technically is not quite perfect. Our comments will be related to the option price itself rather than to methods it has developed. Bearing in mind the interest rate price construction introduced above we remark that the method presented by Black and Scholes is one more illustration that the present value is not an universal pricing method.

We begin with a simple example and show that primary drawback of the option pricing models is their non-sensitivity with respect to the expected rate of return either on an underlying security or themselves. Indeed, the option price performed by Black Scholes equation solution does not depend on expected real return on underlying security. Therefore, given volatility the option the price suggested by Black and Scholes is the same regardless of expected rates of return. Thus if the underlying security of the option offers expected return say 5%, 0%, –5% or even –100% the Black and Scholes method recommends the same price. Later this pricing non-sensitive idea became a dominant pricing law and this approach is now the unique method applied for the valuation of any types of options. One can note that option's price non-sensitivity on real rate of return contradicts both theoretical and practical experience of the market participants.

Next example illustrates an error of the benchmark option definition just using simple algebra. Let S (t), t = 1, 2 be a security price at date t and K a strike price. Assume for simplicity that S(1) = K = $2 and there are two hypothetical securities value at the date t = 2 which are also an option maturity date

$$S_1 (2 , \omega) = \begin{cases} \$4 & \text{with probabilities } 0.99 \\ \$1 & \text{with probabilities } 0.01 \end{cases}$$

$$S_2 (2 , \omega) = \begin{cases} \$4 & \text{with probabilities } 0.01 \\ \$1 & \text{with probabilities } 0.99 \end{cases}$$

The average return on the first security is equal to 98.5% and - 48.5% on the second security. The volatility in both cases is the same 0.0891. One can easily recall that the binomial scheme presents the same call option price C = 2/3 for either security. Thus, for investors the binomial scheme suggests the option price that does not take into account return on security. Indeed one can note that in the call option on the first security in 99 cases out of 100 promises a positive payoff and only in 1 case a loss. With the second security the situation is opposite. Nevertheless in both scenarios binomial scheme suggests the same price. In addition we note that there is a significant difference in expected option returns. The expected rate of return of the call option for the first security is about 197% = [(4 – 2)*0.99 – 2/3] / (2/3) and for the second security is –97% = = [[(4 – 2)*0.01 – 2/3] / (2/3) that explicitly demonstrates failing of the option pricing method.

Though this example does not realistically represent instrument prices, though it explicitly shows misleading of the binomial scheme in understanding option price. For example lets consider a more realistic security for which

$$S_3 (2, \omega) = \begin{cases} \$4 & \text{with probabilities } 0.4 \\ \$1 & \text{with probabilities } 0.6 \end{cases}$$

The expected value of the stock at date 2 is $2.2. In this case the option price not surprisingly for Black Scholes followers is the same as before $2/3 and it coincides with 'unrealistically' securities S_i , i = 1,2. On the other hand the stocks with possible expected values at the date t = 2, say $2.13 or $1.95 does not change anything in the option pricing method. The call option suggests expected rate of return equal 20%.

We also wish briefly comment the arbitrage argument. It plays the center role in finance and has been applied to justify correctness of pricing methods. Mathematicians could easily recognize that arbitrage argument is a necessary condition in pricing. That is if a definition of an instrument price is wrong then non-existence of the arbitrage opportunity can not justify a pricing model.

Let us return to construction using for the above example. Let $\Omega = \{ \omega_u , \omega_d \}$, where the scenario is $S (1 , \omega_u) = 4$. For the scenario $\omega_u = \{ 2, 4 \}$ there is a unique possibility to determine the option price with the strike price K = 2 is assume that $C (t = 0 , S (0) = 2 , \omega_u)$ is a solution of the equation

$$\max \{(4 - 2), 0 \} / C = 4 / 2$$

This equation suggests that for the particular scenario ωu that the rate of return on option and on underlying security is the same. The solution of the equation is $C (0 , 2 , \omega u) = 1$. Then for the scenario $\omega d = \{ 2, 1 \}$there is no sense to buy option. Indeed as far as the option payoff is $\max \{ (1 - 2), 0 \} = 0$ the option value at maturity T = 1 is 0. Thus the call option price can be determine as the solution of the equation (12). Thus the option price is a random function and therefore when an investor pays nonrandom $c for the option he is subject to the market risk. The value of the risk is the probability that security price at maturity T will be bellow than it was assumed initially at t = 0. Note that 'real' probabilities of the states at maturity is used to establish appropriate value $c and calculate corresponding market risk of this choice.

Let us now consider the implications of the risk neutralization of the real world. There is another significant mistake that stems from the attempt to interpret the Black Scholes equation using the measure change method for the stochastic differential equations. In the derivatives pricing it is known as the risk-neutral option valuation. For details see [7,9]. We have observed how binomial method and its risk-neutral interpretation lead to the incorrect option valuation. The critical viewpoint on "binomial" pricing approach just introduced highlights two major drawbacks. The first one is a technical problem in the original derivation. By taking the explicit form of the solution of the Black-Scholes equation and substitute it into the Black and Scholes derivation it was shown [2,5] that the term that does not equal to zero was lost in original derivation. The second problem is the definition of the option premium itself. A simple example introduced above shows that the option price is a random function, and any choice of the price should be supplied by the risk characteristics implied by the chosen price.

In continuous time risk-neutral interpretation of the Black Scholes equation uses more sophisticated mathematical techniques. The interpretation of this technique is mathematically incorrect. Authors who used risk neutral interpretation have missed the fact that changing drift in stochastic differential equations does not change the distribution of the transformed solution with the respect to a new measure that provides drift changing. Changing the drift automatically changes original probability measure. This result from stochastic calculus probably has been overlooked.

The drift changing transformation can be correctly realized by using Girsanov's approach. This is a well known probabilistic result plays one of the most significant roles in the modern stochastic calculus. Applying this theory it is possible to present a correct critical viewpoint on the risk-neutral world and risk-neutral probabilities used in derivatives modeling. As the neutrality approach and corresponding thinking is used now independently from its origins it makes sense to focus on some details related to stochastic calculus rather than on its financial interpretation.

First we present relationship between (BSE) solution

$$\partial C(t,x)/\partial t + r x \partial C(t,x)/\partial x + \tfrac{1}{2}\sigma^2 x^2 \partial^2 C(t,x)/\partial x^2 = r C(t,x) \qquad \text{(BSE)}$$

and so-called "martingale representation". Let $S(t)$ be a random process on a probability space $\{\Omega, F, P\}$ and suppose that

$$d S(t) = \mu S(t)\, dt + \sigma S(t)\, dw(t) \qquad (17)$$

with constant coefficients μ, σ. Black and Scholes European call option $C(t, x)$ is the solution of the (BSE) problem in the domain (t, x) 5 $[0, T)$ 5 $(0, +\infty)$ with terminal boundary condition $C(T, x) = \max(x - K, 0)$. The next statement is often used to represent the Black Scholes option price in other form.

Statement 1: In the risk neutral world, exists a unique probability where normalized asset price follows a martingale.

We now establish the relationship between the (BSE) solution and the Statement. The solution of the (BSE) admits probabilistic representation

$$c(t,x) = \exp - r(T-t)\, E \max\{ Sr(T) - K, 0 \}$$

where E denotes expectation and random process $Sr(t)$ follows the equation

$$d\, Sr(t) = r\, Sr(t)\, dt + \sigma\, Sr(t)\, dw(t)$$

In this equation the real rate of return is replaced by the risk-free counterpart. For simplicity we use the same Wiener process $w(t)$ as in (17). The solution to this equation with initial value $S(t) = x$ can be written in the form

$$Sr(t) = x \exp\{ (r - \tfrac{1}{2}\sigma \cong)(T - t) + \sigma [w(T) - w(t)] \}$$

Then the (BSE) solution can be written in the form

$$c(t,x) = E \max\{ x \exp\{ \sigma [w(T) - w(t)] - \tfrac{1}{2}\sigma \cong (T - t) \} - K \exp - r(T-t), 0 \} \qquad (18)$$

Applying Ito formula one can see that the random process

$$\lambda\ (T) = x \exp\{\sigma\ [\ w(T) - w(t)\ ] - ½\,\sigma\cong(T - t)\}$$

is the solution of the equation

$$d\lambda\ (s) = \sigma\ \lambda\ (s)\, dw(s) \qquad ,\ s > t$$

and $\lambda\ (t) = x$. Note that the process $\lambda\ (s)$ is the martingale representation of the 'underlying' asset in Black-Scholes pricing formula. Note that this asset is not a real world asset it is actually the underlying of the option. Also, the strike price K on the right hand side (18) is replaced by its present value. This proofs the statement.

In the above statement one simply replaced the real asset (17) on other asset $S_r(t)$ on the original probability space. Thus without any loss the superfluous term "In the risk neutral world" can be omitted.

Now lets briefly comment how this statement has been applied for the pricing exotics or interest rate option. Recall that the function $S_r(t)$ can be interpreted as the value of the original asset price S (t) on probability space { Ω , F , ρP } where ρ is Girsanov density

$$\rho(x) = \exp\{(\frac{r-\mu}{\sigma}x)[\ w(T) - w(t)] - \frac{1}{2}(\frac{r-\mu}{\sigma}x)^2\ (T-t)\}$$

Recall that the distributions of the process S ( ) on { Ω , F , P } coincides with the distributions of the process $S_r(t)$

$$d S_r(t) = r S_r(t)\, dt + \sigma S_r(t)\, dw_r(t)$$

on probability space { Ω , F , ρP }where the Wiener process $w_r(t)$ is defined

$$w_r(t) = w(t) + \int_0^t \frac{r-\mu}{\sigma} S(l)\, dl$$

The 'risk-neutral world' commonly used to represent Black Scholes equation solution ignores the existence of the density ρ. In this case the 'risk-neutral world' has been interpreted as a new probability space { Ω' , F' , P' } having no relationship to original { Ω , F , P }. This interpretation of the 'risk-neutral world' is evidently incorrect.

Let us consider some inaccurate statements including the risk-neutrality related to the structural approach to the credit derivative valuation. This approach was presented by R. Merton [10] and then developed primarily by O.Vasicek [11-13]. These papers form the basis for the structural model of the credit derivative theory. In the credit derivatives field there is a positive probability of default on an underlying security. The scheme that leads to company's default can be briefly outlined as follows. Assume that a corporation is financed through a single debt D(t). Denote a single company's equity

E(t). Assume that the debt is a 0-coupon corporate bond maturing at T, with the equity owners paying the bond holders the amount of D(T). Let V(t) denote the market value of the corporation assets at time t. At the maturity T the value V(T) can be above or bellow the claim value D(T). If V(T) > D(T) then the firm payoffs to bondholders reduce the corporation value to V(T) - D(T). Otherwise in the case when V(T) < D(T) the corporation would default on its debt and bondholders can take ownership of the company.

In the paper [10] provides a framework for deriving the default probability on a firm. It was assumed that the asset value V (t) is a random process on probability space { Ω, F, P } and follows an equation

$$d V (t) = \mu V (t) d t + \sigma V (t) d W (t) \qquad (19)$$

The probability of default at date T is

$$P \{ V (T) < D (T) \} = \Phi (- d_1) ,$$

where Φ ( ) normal cdf and

$$d_1 = \frac{\ln \frac{V(0)}{D(T)} + (\mu - \frac{\sigma^2}{2}) T}{\sigma \sqrt{T}}$$

In [11] it was assumed that companies assets follow

$$d A_i (t) = r A_i (t) d t + \sigma i A i (t) d Z_i (t) \qquad (20)$$

i = 1,2,…n. Here r is a risk-free interest rate and Z_i (t) are correlated Wiener processes. This is so-called a risk-neutral setting. Then the probability of default of a firm was written in the form

$$P \{ A(T) < D (T) \} = \Phi (- c)$$

where the constant c is similar to the constant d_1. The index i in the above formula is omitted due to the assumption that the default probability on any loan does not depend on i and is the equal to p. The only difference between c and d1 is the constants μ and r in (19) and (20) correspondingly. In [13] the asset price initially follows

$$d A_i (t) = \mu_i A_i (t) d t + \sigma_i A_i (t) d Z_i (t) \qquad (21)$$

Then it was remarked that "for the purposes of pricing the tranches it is necessary to use the risk-neutral probability distribution. The risk-neutral distribution is calculated in the same way as above, except that default probabilities are evaluated under the risk-neutral measure P*,

$$p^* = P^*[\,A(t) < B\,] = N(-c)$$

It is a common misinterpretation of the risk-neutral world. As it was pointed out the solution to the equation (4) on original probability space $\{\Omega, F, P\}$ has the same distribution as the solution of the equation

$$dV(t) = rV(t)dt + \sigma V(t)\,dZ(t)$$

on $\{\Omega, F, P^*\}$ where $P^* = \rho P$ and W, Z are two Wiener processes. It is clear that in the formula for the value p^* the density ρ is missed. Therefore the probability of default of the company is unexpectedly different and depends on the probability space.

Now let us take a close look at the primary result [11]. In this paper a new canonical distribution was introduced. This distribution plays key role in the structural models as well as for CreditMark product of the Moody's KMV. In the New Basel Accord [14] the regulatory capital of a Bank should be calculated based on Internal Rating Based method. Under this method the canonical distribution must be applied for regulatory capital calculations.

It was assumed [11] that Wiener Z_i processes in the formula (20, 21) are correlated and

$$E[\,\Delta Z_i(t)\,]^2 = \Delta t$$

$$E\;\Delta Z_i(t)\Delta Z_j(t) = \rho\,\Delta t\,, \quad i \neq j$$

where $\Delta Z_i(t) = Z_i(t + \Delta t) - Z_i(t)$, i = 1,2,…n.

Statement 2,[11]. The processes $Z_i(t)$ admit representation

$$Zi(t) = \sqrt{\rho}\,x(t) + \sqrt{1-\rho}\;\varepsilon_i(t) \qquad (22)$$

where $x(t)$ and $\varepsilon_i(t)$, i = 1,2,…n are jointly independent Wiener processes. The Vasicek's idea of the proof is presented bellow. Let $U(t)$ be a Wiener process independent on given Wiener processes $Z\,i(t)$, i = 1,2,…n. Putting

$$x(t) = a\sum_{i=1}^{n} Z_i(t) + bU(t), \qquad \varepsilon_i(t) = \frac{1}{\sqrt{1-\rho}}\,(\,Z_i(t) - x(t)\sqrt{\rho}\,)$$

where

$$a = \frac{\sqrt{\rho}}{1+(n-1)\rho} \quad \text{and} \quad b = \frac{\sqrt{1-\rho}}{\sqrt{1+(n-1)\rho}}$$

one can check that $x(t)$ and $\varepsilon\,i(t)$), i = 1,2,…n are independent Wiener processes. There are several pitfalls that we remark bellow.

Remark 1. It is clear that the proof of statement 2 should be refined. Indeed the decomposition used in the statement 2 is correct only when the Wiener process U (t) exists. If the Wiener process U (t) does not exist and only assumed to exist then the stated decomposition for the given system { Z i (t) } may be true and may fail. In [13] it is stated that (6) is not an assumption but a property of the equicorrelated normal distribution. This statement does not correspond to the real situation.
Besides that the hypothetical Wiener process U (t) depends on the number n. Indeed if it is a well known fact for an infinite system of Wiener processes then we need a reference. On the other hand if for any finite ρ-correlated system of Wiener processes { Z_i (t) }, Wiener process U (t) is independent on the system { Z_i (t) }then note that this U (t) can be added to the original system. Therefore we should arrive at the statement that another Wiener process U ′ (t) exists again, that is independent on the extended system of the (n + 1) Wiener processes and so on. Also note that whether or not U (t) depends on n, the Wiener process x (t) explicitly depend on the number n by the construction.
Remark 2.
There is an additional difficulty in applications of the limit distribution

$$\Lambda(x) = N\{ \frac{1}{\rho} [\sqrt{1-\rho}\, N^{-1}(x) - N^{-1}(p)] \}$$

developed in [11]. The parameter *p* here denotes the probability of default of any asset A_i (t) i = 1,2,…n. Note that the assumption regarding an equal probability p is nonrealistic. Indeed in [11] it was admitted that the all risky securities have the same probability of default. That is

$$p = P \{ A_i (T) < D_i (T) \} = \Phi (- d_1) \qquad (23)$$

where the constant d_1 is defined above. Note that the value d_1 is the same for all i and it explicitly depends on common for all securities maturity expiration date T. It is difficult to imagine that (23) is true for any date until maturity.
Remark 3.
There is also a very challenging statement on page 2, [13].
“ Let L_i be a gross loss (before recoveries) on the i-th loan, so that L_i = 1 if the i-th borrower defaults and L_i = 0 otherwise. Let L be the portfolio percentage gross loss

$$L = \frac{1}{n} \sum_{i=1}^{n} L_i$$

If defaults on the loans are jointly independent then the portfolio loss distribution would converge by central limit theorem, to a normal distribution as the portfolio size n increases. “

The reference on the central limit theorem is incorrect. Asymptotic of the random sum L with independent terms normalizes by n governed by the Law of Large Numbers but not the Central Limit Theorem (CLT). Therefore the random losses L would converge to the number p.

Recall that the random variables Li are ρ-correlated. Thus the variance of the total losses will not subsidize to 0 and therefore the limit of the sum would be a random variable. Nevertheless some accuracy should be applied to prove this convergence.

The new approach that somewhat close to the structural model was introduced in [8]. In this paper the presentation (22) is used indirectly in order to describe joint defaults of a finite number different obligators. Thus the problem related to the construction of the process U (t) does not exist in this setting. It was shown that the decomposition does not cover all equally correlated Gaussian random variables to which needed decomposition could be applied to taking the assumption regarding the random variable U.

We comment that construction. Let M and Z_i , i = 1, 2, …, n be independent random variables with mean 0 and variance 1. Define random variables X_i , i = 1,2,…,n

$$X_i = a_i M + \sqrt{1 - a_i^2}\, Z_i$$

where the constants a_i are such that $| a_i | < 1$. Let t_i be the time of default of a i-th obligator and Q_i is the cumulative distribution function (cdf) of the random time t_i and $H_i (x)$ is the cdf of Z_i . Then

$$P\{X_i < x \mid M\} = P\{a_i M + \sqrt{1 - a_i^2}\, Z_i < x \mid M\} = \tag{24}$$

$$= P\{ Z_i < \frac{x - a_i m}{\sqrt{1 - a_i^2}} \}|_{m=M} = H_i (\frac{x - a_i M}{\sqrt{1 - a_i^2}})$$

Let $F_i (x)$ denotes the cdf of the random variable X_i . Define mapping

$$x = F_i^{-1}(Q_i(t)), \qquad t = Q_i^{-1}(F(x)) \tag{25}$$

Hence x = x (t) and t = t (x). Then from (25) it follows
Conclusion [8]. Conditionals on M defaults are independent. Indeed

$$Q_i(t \mid M) = P\{t_i < t \mid M\} = H_i \{ \frac{F_i^{-1}(Q_i(t)) - a_i M}{\sqrt{1 - a_i^2}} \} \tag{26}$$

This conclusion would follow from the fact that H_i are cumulative distribution functions that are independent of variables Z_i .
One probably noted that the link from equality (25) to (26) is not clear. The equalities (25) are related to unconditional probabilities and it is not enough to conclude that it is correct for conditional probabilities. For example the unconditional expectation of the Wiener process at any time t is 0 and $E \{ w (t) \mid \Gamma_t \} = w (t) \neq 0$ where $\Gamma_t = \sigma \{ w (s) , s \leq t \}$. On the other hand the random variables t_i and M are not connected to each other and therefore the random variables X_i and t_i may be ether dependent or independent.

Nevertheless the transition from (25) to (26) remains correct [8]. Thus the construction introduced in [8] where t_i is specified random times, and random variables M and Z_i are neither specified nor appropriate for applications.

Appendix.

In this appendix we present a constructive orthogonalization method of an equally correlated system of the Wiener processes. This method does not use the superfluous assumption regarding the existence of the Wiener processes $U_n(t)$, n = 1, 2, ... and therefore can be used for the portfolio loss studies.

Let us first consider the case n = 1. The stated result in contrast to the presented above decomposition is either strange or useless. It is useless because a Wiener process is what we need and there is no sense to decompose it. On the other hand the Wiener process in the left-hand side of (22) (i = n = 1) does not depend upon ρ. Therefore setting ρ equal first to 0 then 1 in Vasicek decomposition we arrive at obvious contradiction. The case n = 2 is quite known. Follow [1] we recall the orthogonalization procedure. Define Wiener processes $W_i(t)$, i = 1,2 putting

$$Z_1(t) = W_1(t)$$

$$Z_2(t) = \rho W_1(t) + \sqrt{1-\rho^2}\, W_2(t)$$

$$W_2(t) = \frac{Z_2(t) - \rho Z_1(t)}{\sqrt{1-\rho^2}}$$

From the second equality follows that

$$E\, W_2(t) = \frac{E\, Z_2(t) - \rho\, E\, Z_1(t)}{\sqrt{1-\rho^2}} = 0$$

Note that Wiener processes $W_1(t)$ and $W_2(t)$ are independent. Indeed,

$$E\,[\, W_2(t)\,]^2 = \frac{1}{1-\rho^2}\{\, E\, Z_2^2(t) - 2\rho\, E\, Z_1(t)Z_2(t) + \rho^2\, E\, Z_1^2(t)\} =$$

$$= \frac{t - 2\rho^2 t + \rho^2 t}{1-\rho^2} = t$$

Now introduce the general case. Let us $Z(t) = \{ Z_1(t), Z_2(t), \ldots Z_n(t) \}$. We can start with any Wiener process from the set $Z(t)$ but for simplicity let it will be $Z_1(t)$. We put

$$Z_1(t) = W_1(t)$$

$$Z_j(t) = \rho W_1(t) + \sqrt{1-\rho^2}\, W_j^{(1)}(t), \quad j = 2,3,\ldots,n$$

It follows that

$$W_j^{(1)}(t) = \frac{Z_j(t) - \rho W_1(t)}{\sqrt{1-\rho^2}}, \quad j = 2,3,\ldots,n$$

The Wiener processes $W_j^{(1)}(t)$ are independent upon $W_1(t)$ but still remain correlated among themselves. Indeed,

$$E[W_j^{(1)}(t)]^2 = \frac{1}{1-\rho^2}\{E[Z_j(t) - \rho Z_1(t)]^2\} = t$$

$$\rho_1 = E\, W_i^{(1)}(t) W_j^{(1)}(t) = \frac{1}{1-\rho^2}\{E[Z_j(t) - \rho Z_1(t)]^2\} = \frac{\rho}{1+\rho}$$

for $i \neq j$, and $i, j = 2, 3, \ldots, n$. Note that correlation $\rho_1 < \rho$. Thus we arrive to the new Wiener system

$$W^{(1)}(t) = \{W_2^{(1)}(t), \ldots, W_n^{(1)}(t)\}$$

of the size $n-1$ with equal joint correlation ρ_1 and this Wiener system that is independent on $W_1(t) = Z_1(t)$. Now we can apply transformations that were used for original system $Z(t)$ taking into account that new size is $n-1$. We put

$$W_2^{(1)}(t) = W_2(t)$$

$$W_j^{(1)}(t) = \rho_1 W_2(t) + \sqrt{1-\rho_1^2}\, W_j^{(2)}(t) \qquad j = 3, 4, \ldots, n$$

Then the Wiener system

$$W^{(2)}(t) = \{W_3^{(2)}(t), \ldots, W_n^{(2)}(t)\}$$

of the size $n-2$ is independent on Wiener processes $W_2(t)$ and $W_1(t)$ and has a joint correlation ρ_2

$$\rho_2 = \frac{\rho_1}{1+\rho_1} = \frac{\rho}{1+2\rho}$$

One can easy remark that the Wiener system received on (k-1)-th step

$$W^{(k-1)}(t) = \{ W_k^{(k-1)}(t),..., W_n^{(k-1)}(t) \}$$

is independent on system

$$W_{(k-1)}(t) = \{ W_1(t),..., W_{k-1}(t) \}$$

and has joint correlation

$$\rho_k = \frac{\rho_{k-1}}{1+\rho_{k-1}}$$

Using mathematical induction we can prove that

$$\rho_k = \frac{\rho}{1+k\rho}$$

Indeed, assuming that equality holds for k – 1 we get

$$\rho_k = \frac{\rho_{k-1}}{1+\rho_{k-1}} = \frac{\dfrac{\rho}{1+(k-1)\rho}}{1+\dfrac{\rho}{1+(k-1)\rho}} = \frac{\rho}{1+k\rho}$$

Thus the joint correlation formula has proved for any finite number k. The last step we have

$$W_{n-1}^{(n-2)}(t) = W_{n-1}(t)$$

$$W_n^{(n-2)}(t) = \rho_{n-2} W_{n-1}(t) + \sqrt{1-\rho_{n-2}^2}\, W_n^{(n-1)}(t)$$

$$W_n^{(n-1)}(t) = W_n(t)$$

where the Wiener process $W_n(t)$ is independent on $W_{n-1}(t)$. Thus starting from correlated system $Z(t) = \{ Z_1(t), \dots Z_n(t) \}$ and using a special form of the linear transformation we have arrived at the system $W(t) = \{ W_1(t), \dots W_n(t) \}$ of independent Wiener processes. Now it is not difficult to present the closed form of these transformations. Indeed

$$Z_1(t) = W_1(t)$$

$$Z_2(t) = \rho_0 W_1(t) + \sqrt{1-\rho_0{}^2}\ W_2(t)$$

$$Z_3(t) = \rho_0\ W_1(t) + \rho_1\sqrt{1-\rho_0{}^2}\ W_2(t) + \sqrt{1-\rho_1^2}\ \sqrt{1-\rho_0{}^2}\ W_3(t)$$

$$Z_k(t) = \rho_0\ W_1(t) + \rho_1\sqrt{1-\rho_0{}^2}\ W_2(t) + \rho_2\ \sqrt{1-\rho_1^2}\ \sqrt{1-\rho_0{}^2}\ W_3(t) + ...$$

$$... + \rho_{k-2}\prod_{i=0}^{k-3}\sqrt{1-\rho_i^2}\,W_{k-1}(t) + \prod_{i=0}^{k-2}\sqrt{1-\rho_i^2}\ W_k\ (t)\,, \qquad k = 2,3,...,n$$

This representation can be used for calculation of probability that k companies out of n will default at a particular date. On the other hand if assumption regarding existence of the Wiener process U insignificant then two different decomposition methods should lead to the same limit-loss distribution.

Interest Rate Derivatives.

Underlying securities for the options can be government debt instruments with 0 probability of default such as Treasury bills, notes, and bonds. The main difference between equity options and interest rate options is that for equity there is no information available about its values in the future times whereas the future value of the fixed-income instruments at maturity is given. This gives a possibility to calculate a promised yield of a debt security. We introduce first the term structure of interest rates notion that plays a fundamental role in the valuation of the debt instruments. The term structure of interest rates is defined as the relationship between the yield-to-maturity and the bond maturity. Recall that the yield-to-maturity is defined as simple interest rate compound yearly until maturity. Let B (t , T) be a security price at the date t with maturity T, given that B (T , T) = \$1. Thus the term structure is by definition the function that represent functional dependence of the price B (t , T) on variable T when the current time t is fixed. Note that the notation B (t , T) is consistent with any debt instrument such as money account or a long term bond. It was shown above that Black-Scholes approach along with the notorious risk neutrality vehicle is inappropriate tools for the option valuation.

Consider the European option that matures at date T, with strike price K written on a Treasury security that matures at a date T_b , T < T_b. By definition the option payoff at T is defined as

$$C(T, B(T, T_b)) = \begin{cases} B(T, T_b) - K, & \text{if} \quad B(T, T_b) > K \\ 0, & \text{if} \quad B(T, T_b) \le K \end{cases}$$

Next table illustrates the term structure for the fixed date July 7, 2005. Assume that no other security instruments were issued at the date t = 07/07/05. Then the term structure can be represented in the form

Table 1

Security Term	Issue Date	Maturity Date	Discount Rate %	Investment Yield %	Price Per $100
4-WEEK	07/07/05	08/04/05	3	3.049	99.766667
13-WEEK	07/07/05	10/06/05	3.145	3.214	99.205014
26-WEEK	07/07/05	01/05/06	3.325	3.429	98.319028

Note that the bond price and investment yields are inverse related parameters with respect to the value of the price. That means that lower bond price corresponds to higher interest rate. Let us present interpretation of the data in the Table 1 starting for example from discount rate. For U.S. Treasuries the investment yield is defined above as a simple interest rate. Thus one can easy to check that for example data related to the 4-weeks (28 days) T-instrument in the first raw of the Table 1 satisfies equality

$$[\, 1 - 0.03 \times 28 / 360 \,] = 0.997666\ldots = 1 / [1 + 0.0304879 \times 28 / 365 \,]$$

that consistent with the data in the table.

Let us assume that the interest rate is a variable function in time. Consider a lattice that will be used for a spot interest rate approximation. Recall that the lattice we used for the equity option valuation to represent values of the random equity price [2,3]. The debt security exposure at the date t is a term structure or in another words it is a set of Treasury instruments with all maturities available at this date t. Denote $t_{i+1} = t_i + \delta$, $i = 0, 1, 2$ and assume first that the option maturity T_{op} belongs to the period $t = t_0 < T_{op} \le t_1 = T$. The solution of the equation (12) can be used in which the equity price should be replaced by the debt-security price. Then

$$\frac{B(T_{op}, T)}{B(t, T)} \chi \{B(T_{op}, T) > K\} = \frac{C(T_{op}, B(T_{op}, T))}{C(t, B(t, T))} \qquad (27)$$

Detailed analysis takes into account uncertainty of the value of the government security at the option maturity. The uncertainty can be refined statistically by using observations on discount rate data over a preceding period. The length of the chosen period can vary from several days to the years.

Let us consider a numeric example. Let $t = t_0 < t_1 < T_{op} \le T$. The option pricing problem is to find the value of the call option at the date t. The first step is to construct the lattice. This lattice is used to present discrete in space and time approximation of the underlying security price. This construction is bearing in mind statistics of the discount rates over a preceding period. It represents pricing in backward direction of time. One can adjust the equity option scheme valuation introduced above for option on bond. Note that the lattice we will use in this example has legs of the different lengths. Let us begin with the information given in the Table1. Put for example $t = t_0 = 07/07/05$, $t_1 = t + 13$ weeks, $T_{op} = t + 18$ weeks, and $T = t + 26$ weeks. The value $B(T_{op}, T)$ used in (27) is unknown at t and is needed to be estimated. Next calculation represents an example of implied estimate of the unknown rates. The meaning of the notion "implied" here is that the value of the estimates should be consistent with the observed historical data. Let $B(s, T; t)$, $t \leq s \leq T$ denote a value of the estimate of the bond at the date s with delivery date at T. Using discount rate and investment yield we arrive at the following estimates related to the date t

$$B(t+18, t+26; t) = 1 - 0.03325 \times (26 - 18)*7/360 = 0.994828$$

$$B(t+13, t+26; t) = 0.991595$$

Then from equation

$$B(t, t+18; t)(1 + 0.03214 \times 13 \times 7/365)(1 + 0.03214 \times (18 - 13) \times 7/365) = 1$$

it follows that 'implied' value of the bond at date t is

$$B(t, t+18; t) = 0.995108$$

This number is a present value at t of the bond with18 months until delivery. Note that we do not state the uniqueness of the 'implied' present value. There may be another way or other data for an approximation of the present value. Given this estimate one can easy apply equation (27) and find the value of the call option written on the bond. Indeed, if strike price higher than 0.994828 then option value is 0 and if the strike price is bellow of this number then one can consider the opportunity of investing in an option. For example let the bond value at $T_{op} = t + 18$ is $B(t + 18, T) > 0.994828$. Then solution of the (27) brings the option price equal to

$$C(t, B(t, T)) = B^{-!}(t + 18, T)[B(t + 18, T) - 0.994828]B(t, T)$$

The previous algebraic calculations are non-stochastic and do not use probability distribution of states. Now we present a draft that applies stochastic setting. Our problem is to find date t an estimate $B(t+13, t+18; t)$ of the bond price $B(t+13, t+18)$. As far as the value $(t+13, t+18)$ is unknown at t we interpret it as a random variable. Historical data available we interpret as an observed sample set of the random variable. We supply an estimate with the corespondent sample distribution. Note that correctness of such setting admits independence of the observations in unchanged media. This assumption follows from the mathematical statistics that we should apply.
Collecting of the discount rates data we can derive statistical estimates of the bond price issued at $h + 13$ with expiration at $h + 26$ for available values of h such that $h + 26 \leq t$. Then we approximate the value of the bond at the dates t, $t + 13$, $t + 18$ with the fixed expiration at $t + 26$ by a discrete random variable with the help of the function

$$B(k, t+26) = \Sigma b_j \chi \{ b_j \le B(k, t+26) < b_{j+1} \}$$

where b_j are given numbers that for simplicity assumed are independent on h and k = t, t +13, t +18. The probabilities $P\{ b_j \le B(k, t+26) < b_{j+1} \}$, j = 1, 2,…L can be estimated with the help of the correspondent frequency $\nu_j(k)$ of the observation of the events $b_j \le B(k, t+26) < b_{j+1}$. Applying the equation (27) we arrive at the option prices at t or t + 13 with maturity t + 18. Note first that the option payoff at the T_{op} = t +18 is

$$\max \{ B(t+18, t+26) - K, 0 \}$$

where K is a strike price of the option on the bond that matured at T = t + 26. Then the call option price at the date t is equal to

$$C(t, B(t, T); T_{op}) = C(t, b_j; T_{op})$$

for the scenarios implied by the event $\{ b_j \le B(t, t+26) < b_{j+1} \}$. Denote $\nu_j(t)$, j = 1,2,..,L the frequency of these events. Thus the option value $C(t, B(t, T), T_{op})$ can be approximated by the expression

$$B(t, T)\, \Phi\, b_j^{-}!\, \chi \{ b_j \le B(t, t+26) < b_{j+1} \} \max \{ b_j - K, 0 \}$$

This construction can be extended for a multiple step discrete scheme. Recall that the commonly used binomial model fails to present pricing for the one period with arbitrary number of the states.
Assume now that the option can be exercise at any moment. Let us perform a randomization of the pricing problem by considering a numeric example. Denote

$$\alpha(g, h) = \min \{ s \in \Delta : i_d(s+g, s+h, t) \}$$
$$\beta(g, h) = \max \{ s \in \Delta : i_d(s+g, s+h, t) \}$$

and let for $\alpha(18, 26)$ = %3.1, $\beta(18, 26)$ = %3.4. We begin with the interval [t + 18 , t + 26]. Assume for simplicity that approximation of the security value admits only upper and lower values. Actually the scheme admits arbitrary finite different sates of the bond. Let the discount rates corresponding to the upper and lower note price be

$$B_u(t_0 + 18, t_0 + 26) = 0.99518, \qquad B_l(t_0 + 18, t_0 + 26) = 0.99483,$$

Stochastic approximation of the bond price at the date t +18 is a random variable $b_\lambda(*, \omega)$ such that

$$b_\lambda(t+18, t+26, \omega) = \begin{cases} B_u(t+18, t+26), & \text{with probability } p_u \\ B_l(t+18, t+26), & \text{with probability } p_l = 1 - p_u \end{cases}$$

The value of probability p_u is approximately equal to the frequency entering observed data into the correspondent neighborhood of the value B_u (t + 18 , t + 26). Then the European call option price C (t , b) given that the option does not exercised prior to maturity is a solution of the equation

$$\frac{b_\lambda(t+18, t+26)}{B(t, t+26)} \chi\{b_\lambda(t+18, t+26) > K\} = \frac{\max\{b_\lambda(t+18, t+26) - K, 0\}}{C(t, B(t, t+26))}$$

Letting the contract size be a million dollars, K = 0.995, B (t , t + 26) = 0.98319 we arrive at the solution of the equation

$$C(t, B(t, t+26)) = \begin{cases} \$177.83, & \text{with probability } p_u \\ 0, & \text{with probability } 1 - p_u \end{cases}$$

Denote $p_u = p_u$ (1) and let for example α (13, 18) = %3.18, β (13, 18) = %3.38. Then it follows that

$$B_u(t_0 + 13, t_0 + 18) = 0.99691, \qquad B_l(t_0 + 13, t_0 + 18) = 0.99671.$$

with probabilities p_u , 1 – p_u correspondingly. Denote here $p_u = p_u(2)$. Then we can refine pricing follow [2-5]. The pricing adjustment includes possibility to sell or exercise option at the intermediate date t_1 = t + 13. This setting correspond to American type of the option that gives the owner of the option the right to exercise option at the date t_1 earlier then the maturity date $T_{op} = t_2$. Thus the American option can be interpreted as a valuable adjustment of the European counterpart. This additional opportunity might increase the price of the option. Note that the opportunity to sell option prior maturity also as it prior exercising can increase the price of the option.
Remark. In the next example we wish to highlight the fact that the face value of the bond is a fixed value. Let $t = t_0 < T_{op} < T = T_{op} + 1$ and B (t , T) = \$0.9. Assume that B (T_{op} , T) equal to \$0.9 or \$0.99 with probabilities p > 0 and (1 – p) correspondingly. The strike price of the call option assume to be \$0.9 and the settlement date of the option contract is $T = T_{op} + 1$. If the option payoff realized in cash then the scenario B (T_{op} , T) = \$0.99 seems better than scenario B (T_{op} , T) = \$0.9. Nevertheless either the cash settlement or bond delivery both lead to the same future value at $T = T_{op} + 1$ in which bond face value is \$1. Thus the better scenario in the bond does not actually leads to higher profit. Nevertheless the advantage may be found in extended market.

Bellow we introduce other form of interest rate option. It deals with the interest rates with constant maturity. In this case the underlying security is the interest rates itself. Let $\lambda > 0$ be a fixed parameter representing time period measured in days and let B (t , t + λ) denotes the value of the security at t with payoff \$1 at the maturity t + λ. In the real world B (t , t + λ) is defined for the finite number moments t , λ. At these dates a new set of the T-treasury securities with various maturities come up on the market. Denote i_d (t) = i_d (t , λ) the one day discount rate corresponding to the bond B (t , t + λ) , i.e.

$$[\,1 - i_d(t)\,\lambda/360\,] = B(t, t+\lambda)$$

Investors at date t can buy discount yield $i_d(t)$ over λ period for the price $B(t, t+\lambda)$ and sell it next day for $B(t^1, t^1+\lambda)$, where $t^1 = t + 1$.

Consider European call option written on discount yield. At maturity date T_{op} the new discount rate on λ period $i_d(T_{op}) = i_d(T_{op}, \lambda)$ is traded on the market with the newest issued bond. The price of this discount rate $B(T_{op}, T_{op}+\lambda)$ is unknown at date t, and therefore can be interpreted as a random variable. The call option payoff at T_{op} is by definition

$$C(T_{op}, i_d(T_{op})) = [\,B(T_{op}, T_{op}+\lambda) - K\,]\ \chi\{\,i_d(T_{op}) < i_{dK}\,\} =$$

$$= [\,i_{dK}(T_{op}) - i_d(T_{op})\,]\,\lambda/360\ \ \chi\{\,i_d(T_{op}) < i_{dK}(T_{op})\,\}$$

Here i_{dK} denotes the discount rate implied by the strike price K. That is $i_{dK}(T_{op})$ is a solution of the equation

$$K = 1 - i_{dK}(T_{op})\,\lambda/360$$

From the equation (27) it follows that the value of the call option at date t is

$$C(t, i_d(t)) = \frac{1 - i_d(t)\dfrac{\lambda}{360}}{1 - i_d(T_{op})\dfrac{\lambda}{360}}\,[\,i_{dK}(T_{op}) - i_d(T_{op})\,]\frac{\lambda}{360}\,\chi\{\,i_{dK}(T_{op}) > i_d(T_{op})\}$$

The put option payoff is defined as

$$P(T_{op}, i_d(T_{op})) = [\,K - B(T_{op}, T_{op}+\lambda)\,]\ \chi\{\,i_d(T_{op}) > i_{dK}\,\} =$$

$$= [\,i_d(T_{op}) - i_{dK}(T_{op})\,]\,\lambda/360\ \ \chi\{\,i_d(T_{op}) > i_{dK}(T_{op})\,\}$$

and the date t price of the put option is then

$$P(t, i_d(t)) = \frac{1 - i_d(t)\dfrac{\lambda}{360}}{1 - i_d(T_{op})\dfrac{\lambda}{360}}\,[\,i_d(T_{op}) - i_{dK}(T_{op})\,]\frac{\lambda}{360}\,\chi\{\,i_{dK}(T_{op}) < i_d(T_{op})\}$$

Using formula (3) we can extend the option pricing formulas on coupon bond. In this case it will be convenient to use the bond price rather then accompanying interest rate. In this case basic pricing equation can be written in the form

$$\frac{B_c(T_{op}, T)}{B_c(t, T)}\,\chi\{B_c(T_{op}, T) > K\} = \frac{\max\{\,B_c(T_{op}, T) - K, 0\,\}}{C(t, B_c(t, T), T_{op})}$$

Thus

$$C(t, B_c(t, T), T_{op}) = \frac{B_c(t, T)}{B_c(T_{op}, T)} \max\{ B_c(T_{op}, T) - K, 0 \} =$$

$$= \frac{\sum_{j=1}^{N} cB(t_j, T) + F}{\sum_{j, t_j \geq T_{op}}^{N} cB(t_j, T) + F} \max\{ \sum_{j, t_j \geq T_{op}}^{N} cB(t_j, T) + F - K, 0 \}$$

REFERENCES

1. Brownlee , K.A. (1960). Statistical Theory and Methodology in Science and Engineering. John Wiley & Sons, Inc.
2. Gikhman , Il.I.(2004). On Black-Scholes Equation. ICFAI J.of Applied Finance v.10, No.4, p.47-74.
3. Gikhman, Il.I.(2005). Some basic remarks on options valuation. Actes de la 3Xme Conférence Internationale de Finance - IFC 3, Formalisation et Modélisation, Hors Série N°3, IFC3 - 3-5 March, 2005, Hammamet, Tunisia, p.121-136
4. Gikhman, Il.I.(2006). http://papers.ssrn.com/sol3/papers.cfm?abstract_id=895679.
5. Gikhman, Il.I.(2006). http://papers.ssrn.com/sol3/papers.cfm?abstract_id=900111.
6. Gikhman, Il.I.(2006). http://papers.ssrn.com/sol3/papers.cfm?abstract_id=918873.
7. Hull, J. (2003). Options, Futures, and other Derivatives, Prentice Hall, Englewood Cliffs, NJ.
8. Hull, J. , White, A. (2005). The Perfect Copula.
9. Jarrow, R. , Turnbull, S. (2000). Derivatives Securities, 2nd ed, South-Western College Publishing.
10. Merton, R.C. (1974). On the pricing of corporate debt: The risk structure of interest rates. Journal of Finance 29, 449-470.
11. Vasicek, O.A. (1987). Probability of loss on loan portfolio, KMV Corporation.
12. Vasicek, O.A. (1991). Limiting Loan Loss Distribution, KMV Corporation.
13. Vasicek, O.A. (2002). The distribution of loan portfolio value. Risk.
14. Basel Committee on Banking Supervision, International Convergence of Capital Measurements and Capital Standard: A Revised Framework, (2004).