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OPTIMAL GROWTH WITH INTERMEDIATE GOODS INTERDEPENDENCE: A DIFFERENCE GAME APPROACH

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Abstract.
Two countries face a strategic interdependence in producing intermediate goods. Producing these intermediate goods requires both of domestic capital and another imported intermediate good. Individually they determine its balanced growth path by taking into account this interdependence. By allowing for strategic interactions in the analysis we adapted a two-agent dynamic setting and find an interior Markov Perfect Equilibrium (MPE) as well as an open-loop equilibrium. We find that main results resemble each other but growth rates will be higher when strategies are allowed to be revised dynamically.
JEL Classification System: C73, F15

1 INTRODUCTION

Interdependence among economies has increased due to globalization (Ventura, 1997; Jones, 2000). There exist many production and complementation agreements among multinational firms conjointly to trade liberalization processes among countries. For instance, a considerable part of the car production in any given economy requires importing spare parts and non locally produced components. This way, the manufacturing in diverse economic sectors is internationally complemented through local production and the importing of raw and intermediate inputs from abroad. This has been already noted by (Sanyal and Jones, 1982) who conclude that the majority of international trade is mainly composed by intermediate and raw goods which require local processing before reaching the final consumer. Trade agreements have fostered this process and helped to firms in diverse economies to specify their production plans based on free tariff inputs. In fact, local produced and foreign inputs usually compete in the domestic market (Chen et al., 2004). In these highly integrated economies, the production of intermediate inputs in one economy becomes crucial for production of final or intermediate goods in the associated economy[1].

We study a simple model of optimal economic growth in which two countries maintain an interdependence relation in intermediate goods production. Each economy produces a final good and an intermediate good that exports to its trade partner. Producing one unit of its own capital requires a non domestically produced input (intermediate good) that is imported from another economy. Both economies have liberated their middle product trade and intermediate goods and services are available to both economies free of any commercial restriction. The reasons for this trade liberalization can be associated to comparative advantages in production cost or extraction or maybe the result of a political trade agreement previously signed[2]. We assume that both economies acknowledge this mutual interdependence and plan its long-run growth paths assuming this interdependence[3].

We work with a two-economy model, where each economy is represented by an agent that maximizes consumption by producing two types of goods: productive capital and an intermediate good. The production of intermediate goods has strategic considerations: each economy requires intermediate goods from another economy to produce its own intermediate goods. The model is strongly based in (Dockner and Nishimura, 2004)'s analytical framework. It also draws upon game-theoretic differential game literature (Zeeuw and van der Ploeg, 1991; Fischer and Mirman, 1996), and on literature intermediate good trade (Ventura, 1997; Sanyal and Jones, 1982; Chen et al., 2004).

The main findings of the model are that equilibria exist under the open-loop and Markov strategies. When the internalization of the effect is updated period to period, economies grow faster. By internalizing the dependence of foreign intermediate products agents obtain greater rates of growth. For this to be observed, each economy must have an own technology of inter-

[1] As suggested earlier, many complementarities emerge from free trade agreements, and car industry is one clear example. For instance, the Mercosur treaty has imposed a reduction in tariffs among main car producers in member countries and the production of car parts is currently highly complementary. It is also observed in domestic personal computer production and assembling where each component often have different production origin. Finally, another example is the aircraft industry in Europe where parts or even complete sections of a fuselage are produced in several countries and assembled in one common factory.

[2] Other contributions named this process as strategic outsourcing (Chen et al., 2004).

[3] Another model of optimal growth and trade is (Ventura, 1997). However, he relies in a Ramsey model with trade that includes no strategic interactions and he focus on the international interdependence in the middle products sector and its influence in the growth process. (Peng et al., 2006) also model a two-economy (or region) model focus on the population agglomeration effects and differential wage incidence on growth and trade.

mediate goods production with strictly decreasing returns to scale.

The paper is organized as follows. Section 2 describes the model and definitions. Section 3 presents the results in the strategic growth model and its main outcomes. Section 4 ends the paper with the conclusions.

2 THE MODEL

Consider two economies, each of which is represented by a single agent indexed by $i \neq j$, that accumulates productive capital and produces intermediate products through capital investment and input imports and derives utility from consumption[4]. Each country tries to maximize economic growth observing this mutual interdependence. Output of one country's middle products no only depends on its capital stock and labor but also on the use of intermediate goods from the other country. Thus, each country faces an externality in its capital stock depending of the adopted behavior of the other economy.

The structure of the model is as follows. Let K_t^i, Z_t^i, Y_t^i, and C_t^i be the productive capital stock, intermediate-goods stock, output, and consumption of agent i in period *t*, respectively. For simplicity, we assume that total labor used in each country is constant and given by L^i. The production function of agent i is $Y_t^i = \pi_i \left(K_t^i\right)^{\alpha_i} \left(Z_t^i\right)^{\beta_i} \left(L^i\right)^{1-\alpha_i-\beta_i}$ where $\pi_i > 0$ is a total factor productivity index. We assume that the level of $Z_t^i = f\left(K_t^i, Z_t^j\right)$, with $j \neq i$, in a formulation of the type:

$$Z_t^i = \left(\frac{K_t^i}{L^i}\right)^{\frac{\varsigma_i}{\beta_i}} \left(\frac{Z_t^j}{L^j}\right)^{\frac{\varphi_j}{\beta_i}} L^i \tag{1}$$

where $\varsigma_i + \varphi_j = \beta_i < 1$. This function expresses that intermediate goods Z_t^i are produced as a combination of the stock of productive capital of economy i, K_t^i, and intermediate inputs from the economy j, Z_t^j.

Assuming constant rates of depreciation for both types of stock of capital, δ and χ respectively, and applying the income identity, i.e., output in the current period is used for consumption and investment, results in the following accumulation equation

$$Y_t^i = \pi_i \left(K_t^i\right)^{\alpha_i} \left(Z_t^i\right)^{\beta_i} \left(L^i\right)^{1-\alpha_i-\beta_i} - C_t^i - (1-\delta) K_t^i - (1-\chi) Z_t^i$$

If we redefine variables in terms of units per labor employed, i.e., $k_t^i \equiv K_t^i/L^i$, $z_t^i \equiv Z_t^i/L^i$, $c_t^i \equiv C_t^i/L^i$, and assume that there is full depreciation, δ and χ, we obtain the following

$$k_{t+1}^i = \pi_i \left(k_t^i\right)^{\alpha_i} \left(z_t^i\right)^{\beta_i} - c_t^i \tag{2}$$

The accumulation equation (2) gives rise to a model of strategic growth in which agents recognize that their respective capital stocks do influence each others and gives rise so that this is internalized emerging a strategic externality. The initial state dynamics is represented by $k_{t+1}^i = \pi_i \left(k_t^i\right)^{\alpha_i} \left(z_t^i\right)^{\beta_i} - c_t^i$. But for (1) we have that $z_t^i \equiv Z^i/L^i \equiv \left(k_t^i\right)^{\frac{\varsigma_i}{\beta_i}} \left(z_t^j\right)^{\frac{\varphi_j}{\beta_i}}$, and then replacing we obtain $k_{t+1}^i = \pi_i \left(k_t^i\right)^{\alpha_i} \left(k_t^i\right)^{\varsigma_i} \left(z_t^j\right)^{\varphi_i} - c_t^i$ this way, in case the interaction is fully internalized by the two agents the state dynamics becomes

$$k_{t+1}^i = \pi_i \left(k_t^i\right)^{\alpha_i+\varsigma_i} \left(z_t^j\right)^{\varphi_j} - c_t^i \tag{3}$$

[4] The model could also be interpreted as a two-region one, with different agents in each region.

and the resulting model can either be formulated as an open-loop (precommitment) or a Markov game without commitment (closed-loop)[5]. In case agents play a game in which they have access to Markov strategies they design their current actions as decision rules that depend on the current stocks of both countries summarized in the defense parity parameter θ, while in the case of an open-loop game agents choose their strategies as simple time paths that do not depend on the current level of the capital stocks.

We assume that the production function, $y_t^i = \pi_i \left(k_t^i\right)^{\alpha_i+\varsigma_i} \left(z_t^j\right)^{\varphi_j}$, satisfies the following parameter restrictions: π_i is strictly positive and constant and $0 < \alpha_i, \varsigma_i, \varphi_j < 1$ and $\alpha_i+\varsigma_i+\varphi_j 1$. Each agent derives utility in period t only from current consumption and the utility function, $u^i\left(c_t^i\right)$ is logarithmic, i.e. $u^i\left(c_i^t\right) = \ln c_t^i$. Each agent lives forever, i.e. $T = \infty$, and maximizes the discounted stream of utility with the discount rate given by $\rho = (1/(1+r))$, where r is the rate of interest satisfying $r > 0$. We consider only domestic consumption in this model. Finally, we assume that trade balance is always in equilibrium and does not require any capital flows compensation.

Each agent maximizes the discounted stream of utility given by

$$J^i = \sum_{i=0}^{\infty} \rho^i \ln c_t^i,$$

subject to

$$k_{t+1}^i = \pi_i \left(k_t^i\right)^{\alpha_i+\varsigma_i} \left(z_t^j\right)^{\varphi_j} - c_t^i$$

and given initial conditions $\left(k_0^i, z_0^j\right)$.

Specifically, this strategic growth game in reduced form looks as follows:

$$\max_{\left\{k_{t+1}^i\right\}_{t=0}^{\infty}} \left\{ J^i = \sum_{t=0}^{\infty} \rho^t \ln \left(\pi_i \left(k_t^i\right)^{\alpha_i+\varsigma_i} \left(z_t^j\right)^{\varphi_j} - k_{t+1}^i \right) \right\} \tag{4}$$

subject to the initial conditions $\left(k_0^i, z_0^j\right)$.

We follow with the derivation of equilibria of the two postulated models.

3 STRATEGIC GROWTH GAME EQUILIBRIA

We mentioned earlier that we have two possible equilibrium strategies for the agents. The first one that we are going to analyze is the open-loop solution that requires that agents commit to some initial strategy of capital augmentation during the complete period of analysis. En the second, agents used Markov strategies that reconsider in each period of time their strategy during the complete period of analysis.

3.1 The open-loop game

In open-loop games agents choose their consumption as simple time paths and commit themselves to stick to these time profiles during the entire game. It is clear that such a game cannot capture all the strategic interactions present in the dynamic game. It resembles many features of a one shot game and the equilibrium optimum could be analyzed as the Nash equilibrium of this game.

[5] Readers interested in this concept of solutions should consult (Dockner et al., 2000) for the differential games case and (Zeeuw and van der Ploeg, 1991) for the difference games case.

To characterize the open-loop equilibrium we apply the first order conditions, which are also sufficient given strict concavity, and get

$$\frac{-1}{\pi_i\left(k_t^i\right)^{\alpha_i+\varsigma_i}\left(z_t^j\right)^{\varphi_j}-k_{t+1}^i}+\rho\frac{\pi_i\left(\alpha_i+\varsigma_i\right)\left(k_{t+1}^i\right)^{\alpha_i+\varsigma_i-1}\left(z_{t+1}^j\right)^{\varphi_j}}{\pi_i\left(k_{t+1}^i\right)^{\alpha_i+\varsigma_i}\left(z_{t+1}^j\right)^{\varphi_j}-k_{t+2}^i}=0 \tag{5}$$

Rearranging terms and dividing (5) by $\left(k_{t+1}^i\right)^{\alpha_i+\varsigma_i}\left(z_{t+1}^j\right)^{\varphi_j}$ we obtain

$$\pi_i+\rho\alpha_i\pi_i-\rho\alpha_i\pi_i^2\frac{\left(k_t^i\right)^{\alpha_i+\varsigma_i}\left(z_t^j\right)^{\varphi_j}}{k_{t+1}}-\frac{k_{t+2}^i}{\left(k_{t+1}^i\right)^{\alpha_i+\varsigma_i}\left(z_{t+1}^j\right)^{\varphi_j}}=0$$

Now we create a variable $x_{t+1}^i=\frac{k_{t+1}^i}{\left(k_t^i\right)^{\alpha_i+\varsigma_i}\left(z_t^j\right)^{\varphi_j}}$ and we replace in the FOC and we get the dynamic equation:

$$\pi_i+\rho\alpha_i\pi_i-\rho\alpha_i\pi_i^2\frac{1}{x_{t+1}}-x_{t+2}^i=0 \tag{6}$$

Any growth process is governed by the dynamical system (6).

Lemma 1 *The followings results hold: There exists an open-loop equilibrium of the strategic growth model that results in consumption rules given by*

$$c^i\left(k_t^i,z_t^j\right)=\left(1-\alpha_i\rho\right)\pi_i\left(k_t^i\right)^{\alpha_i+\varsigma_i}\left(z_t^j\right)^{\varphi_j} \tag{7}$$

Departing from initial stocks $\left(k_0^i,z_0^j\right)$, equilibrium dynamics is governed by

$$k_{t+1}^i=\omega_i\left(k_t^i\right)^{\alpha_i+\varsigma_i}\left(z_t^j\right)^{\varphi_j} \tag{8}$$

$$k_{t+1}^j=\omega_j\left(k_t^i\right)^{\alpha_i+\varsigma_i}\left(z_t^j\right)^{\varphi_j} \tag{9}$$

where $\omega_i=\left(\alpha_i+\varsigma_i\right)\rho\pi_i$ and $\omega_j=\left(\alpha_j+\varsigma_j\right)\rho\pi_j$.

Proof: See Appendix.

The system develops governed by (8) and (9). For characterizing the system we propose the following theorem.

Lemma 2 *In case there are decreasing returns to scale, i.e. $\alpha_i+\varsigma_i+\varphi_j<1$, equilibrium dynamics admit a unique and stable steady state given by*

$$\left(\bar{k}_0^i,\bar{z}_0^j\right)=\left(\left[\left(\omega_i\right)\right]^{\frac{1-\alpha_j-\varsigma_j}{\Phi}}\left[\omega_j\right]^{\frac{\varphi_j}{\Phi}},\left[\omega_i\right]^{\frac{1-\alpha_i-\varsigma_i}{\Phi}}\left[\omega_j\right]^{\frac{\varphi_j}{\Phi}}\right) \tag{10}$$

where $\Phi=\left(1-\alpha_i-\varsigma_i\right)\left(1-\alpha_j-\varsigma_j\right)-\varphi_i\varphi_j$

Proof: See Appendix.

The case of decreasing returns of scale leads results directly interpretable. The case of constant returns to scale follows as another interesting case for analysis.

Lemma 3 *Assume that the technologies exhibit constant returns to scale, that is,* $\alpha_i + \varsigma_i + \varphi_j = 1$ *and that* π_i *is chosen in a way that* $\pi_i(\alpha_i + \varsigma_i)\rho > 1$ *and* $\pi_j(\alpha_j + \varsigma_j)\rho > 1$ *is verified. Then it verifies that there exists a balanced growth path factor given by*

$$\iota = [\omega_i]^{\frac{\varphi_j}{\varphi_i + \varphi_j}} [\omega_j]^{\frac{\varphi_i}{\varphi_i + \varphi_j}} \tag{11}$$

where $\omega_i = \pi_i(\alpha_i + \varsigma_i)\rho$, $\forall i, j$.

Proof: See Appendix.

These results also resemble the AK-model. For instances, if both countries begin with the same initial capital stocks, then the equilibrium dynamics is represented by:

$$k_{t+1} = \omega (k_t)^{\alpha + \varsigma + \varphi}$$

but if the available technology shows constant returns to scale we observe that the dynamics of balanced-growth path is represented by

$$k_{t+1} = \omega k_t = A k_t,$$

more familiarly known as a type of AK-growth model.

3.2 The closed-loop or Markov game

We follow with the derivation of a Markov Perfect Equilibrium (MPE) of the strategic growth model. A MPE is a Markov Nash Equilibrium that is at the same time a subgame perfect equilibrium (SPE). The issue of existence of MPE for difference games is not an easy test. This approach resembles many features of a repeated game where agents can revise their strategies and the equilibrium optimum could be analyzed as the subgame perfect equilibrium of this game.

Lemma 4 *The following results hold: A MPE equilibrium exists to the strategic growth game that results in consumption rules given by*

$$c^i\left(k_t^i, z_t^j\right) = \left(1 - \rho\left((\alpha_i + \varsigma_i) + \frac{\rho\varphi_i\varphi_j}{1 - (\alpha_i + \varsigma_i)\rho}\right)\right)\pi_i \left(k_t^i\right)^{\alpha_i + \varsigma_i}\left(z_t^j\right)^{\varphi_j} \tag{12}$$

Departing from initial stocks $\left(k_0^i, z_0^j\right)$, *equilibrium dynamics is governed by*

$$k_{t+1}^i = \psi_i \left(k_t^i\right)^{\alpha_i + \varsigma_i}\left(z_t^j\right)^{\varphi_j} \tag{13}$$

$$k_{t+1}^j = \psi_j \left(k_t^j\right)^{\alpha_j + \varsigma_j}\left(z_t^i\right)^{\varphi_i} \tag{14}$$

for $i \neq j$, *where*
$\psi_i = \pi_i\left((\alpha_i + \varsigma_i)\rho + \rho^2\frac{\varphi_i\varphi_j}{1-(\alpha_i+\varsigma_i)\rho}\right)$ *and* $\psi_j = \pi_j\left((\alpha_j + \varsigma_j)\rho + \rho^2\frac{\varphi_i\varphi_j}{1-(\alpha_j+\varsigma_j)\rho}\right)$.

In case there are decreasing returns to scale, i.e. $\alpha_i + \varsigma_i + \varphi_j < 1$, *equilibrium dynamics admit a unique steady state given by*

$$\left(\bar{k}_0^i, \bar{z}_0^j\right) = \left([\psi_i]^{\frac{1-\alpha_j-\varsigma_j}{\Phi}}[\psi_j]^{\frac{\varphi_i}{\Phi}}, [\psi_j]^{\frac{1-\alpha_i-\varsigma_i}{\Phi}}[\psi_i]^{\frac{\varphi_j}{\Phi}}\right) \tag{15}$$

where $\Phi = (1 - \alpha_i - \varsigma_i)(1 - \alpha_j - \varsigma_j) - \varphi_i\varphi_j$

In case of constant returns to scale, i.e. $\alpha_i + \beta_i + \nu_j = 1$*, there exists a balanced growth path with the growth factor given by*

$$\vartheta = \psi_i^{\frac{\varphi_j}{\varphi_i + \varphi_j}} \psi_j^{\frac{\varphi_i}{\varphi_i + \varphi_j}} \tag{16}$$

Proof: See Appendix.
Next section is dedicated to analyze the results of both games.

4 ANALYSIS AND CONCLUSIONS

Qualitatively, the results of both models resemble each other. The main differences arise when comparing the rate of growth. For instance, in the derivation process of the economies' rates of growth we obtained, as (Dockner and Nishimura, 2004) did, that $\psi > \omega$, i.e., the rate of growth of the economy in the case of the Markov game is higher than in the case of open-loop strategies. It seems that not only by internalizing but for dynamically reviewing the optimality of the investment decision at each time step requires a higher rate of economic growth.

Inputs and domestic capital produce the exportable intermediate good under decreasing scale returns. While this is consistent with the structure of the model, empirical evidence shows that international intermediate goods usually embody increasing scale returns associated with technological improvements (Ciccone, 2002). This is an issue that deserves further improvements.

Summarizing, we study a simple optimum economic growth model where agents interact in producing mutually traded intermediate goods. We apply a technique that take into account the mutual interdependence in a dynamic framework. The dynamics of interaction is focused in a direct response in present time. We confirm that economies growth faster when representative agents are able to update their dynamic growth path. As other contributions show, integration enhances economic growth.

5 APPENDIX

Following demonstrations are adapted from (Dockner and Nishimura, 2004).

Proof of Lemma 1 *The dynamical system (6) has two negative steady states given by*

$$\hat{x}^i = (\alpha_i + \varsigma_i)\, \rho \pi_i,$$

$$\tilde{x}^i = \pi_i$$

Since $\tilde{x}^i = \pi_i$ *implies zero consumption it is not an interesting equilibrium, since consumption becomes null. Even more, it is easy to show that* $\hat{x}^i = (\alpha_i + \varsigma_i)\, \rho \pi_i$*, is unstable. Now, we employ the transversality condition to derive an equilibrium.*

$$\lim_{t \to \infty} \rho^{t+1} \frac{\partial \ln \left(\pi_i \left(k_t^i\right)^{\alpha_i + \varsigma_i} \left(z_t^j\right)^{\varphi_j} - k_{t+1}^i \right)}{\partial k_{t+1}^i} k_{t+1}^i = 0$$

Where we follow that

$$\rho^{t+1} \frac{\partial \ln \left(\pi_i \left(k_t^i\right)^{\alpha_i + \varsigma_i} \left(z_t^j\right)^{\varphi_j} - k_{t+1}^i \right)}{\partial k_{t+1}^i} k_{t+1}^i = \frac{-\rho^{t+1} k_{t+1}^i}{\pi_i \left(k_t^i\right)^{\alpha_i + \varsigma_i} \left(z_t^j\right)^{\varphi_j} - k_{t+1}^i}$$

and the transversality condition becomes

$$\lim_{t\to\infty}\rho^{t+1}\frac{x^i_{t+1}}{\pi_i - x^i_{t+1}} = 0,$$

and it is satisfied if x^i_{t+1} is bounded away from π_i. However, since $\hat{x}^i$ is unstable, the path which does not start from $\hat{x}^i$ converges to zero or to π_i. Since this last result makes consumption goes to zero, we look forward for the path converging to zero. From (6) any path $\{x^i_t\}$ converging to π_i is monotone and satisfies

$$\frac{x^i_{t+1}}{\pi_i - x^i_{t+1}} = \frac{x^i_{t+1}}{\rho\left(\alpha_i + \varsigma_i\right)\pi_i}\left(\frac{x^i_t}{\pi_i - x^i_t}\right)$$

Iterating we obtain

$$\rho^{t+1}\frac{x^i_{t+1}}{\pi_i - x^i_{t+1}} = \frac{x^i_{t+1}x^i_t\cdots x^i_1}{\left(\left(\alpha_i + \varsigma_i\right)\pi_i\right)^{t+1}}\left(\frac{x^i_0}{\pi_i - x^i_0}\right)$$

As $x^i_{t+1}\to\pi_i$ the right hand side goes to infinity. Therefore, a path violates the transversality condition. The other possible solution, the steady state $\hat{x}^i = \left(\alpha_i - \varsigma_i\right)\rho\pi_i$ do not violate the transversality condition and hence it corresponds to an equilibrium path. This steady state results in the following optimum dynamics

$$k^i_{t+1} = \left(\alpha_i + \varsigma_i\right)\rho\pi_i\left(k^i_t\right)^{\alpha_i+\varsigma_i}\left(z^j_t\right)^{\varphi_j}$$

which implies, by (2), a consumption function equal to

$$c^i\left(k^i_t, z^j_t\right) = \pi_i\left(k^i_t\right)^{\alpha_i+\varsigma_i}\left(z^j_t\right)^{\varphi_j} - \left(\alpha_i + \varsigma_i\right)\rho\pi_i\left(k^i_t\right)^{\alpha_i+\varsigma_i}\left(z^j_t\right)^{\varphi_j}$$

$$c^i\left(k^i_t, z^j_t\right) = \left(1 - \left(\alpha_i + \varsigma_i\right)\rho\right)\pi_i\left(k^i_t\right)^{\alpha_i+\varsigma_i}\left(z^j_t\right)^{\varphi_j}$$

Proof of Lemma 2 *We begin for defining the Jacobian of the dynamical system for analyzing the characteristic equation. This would be the first step for dealing with the existence of a steady state in this system.*

$$\begin{bmatrix} \alpha_i + \varsigma_i & \varphi_i\pi_i\left(\alpha_i + \varsigma_i\right)\rho^{\frac{1-\alpha_i-\varsigma_i-\varphi_i}{\Phi}}\pi_j\left(\alpha_j + \varsigma_j\right)\rho^{\frac{\alpha_i+\varsigma_i+\varphi_i-1}{\Phi}} \\ \varphi_j\pi_i\left(\alpha_i + \varsigma_i\right)\rho^{\frac{\alpha_i+\varsigma_i+\varphi_i}{\Phi}}\pi_j\left(\alpha_j + \varsigma_j\right)\rho^{\frac{1-\alpha_i-\varsigma_i-\varphi_i}{\Phi}} & \alpha_j + \varsigma_j \end{bmatrix}$$

where $\Phi = \left(1 - \alpha_i - \varsigma_i\right)\left(1 - \alpha_j - \varsigma_j\right) - \varphi_i\varphi_j$.
By getting the characteristic equation of this system we obtain

$$f\left(\lambda\right) = \lambda^2 - \left(\alpha_i + \varsigma_i + \alpha_j + \varsigma_j\right)\lambda + \left(\left(\alpha_i + \varsigma_i\right)\left(\alpha_j + \varsigma_j\right)\right) - \varphi_i\varphi_j$$

Roots are real and we have that

$$f\left(1\right) = \left(1 - \alpha_i - \varsigma_i\right)\left(1 - \alpha_j - \varsigma_j\right) - \varphi_i\varphi_j > \varphi_i\varphi_j - \varphi_i\varphi_j = 0$$

and

$$f\left(-1\right) = \left(1 - \alpha_i - \varsigma_i\right)\left(1 - \alpha_j - \varsigma_j\right) - \varphi_i\varphi_j > 0$$

The function is strictly convex since $f'' = 2 > 0$ and its minimum is at $(\alpha_i + \varsigma_i + \alpha_j + \varsigma_j)/2 < 1$ existing two roots, $\lambda_i, i = 1, 2$, such that $|\lambda_i| < 1$. Both roots are positive if $(\alpha_i + \varsigma_i)(\alpha_j + \varsigma_j) > \varphi_i\varphi_j$; one is positive and the other is negative if $(\alpha_i + \varsigma_i)(\alpha_j + \varsigma_j) < \varphi_i\varphi_j$.

□

Proof of Lemma 3 *Assume that the technologies exhibit constant returns to scale, that is, $\alpha_i + \varsigma_i + \varphi_j = 1$ and that π_i is chosen in a way that $\pi_i(\alpha_i + \varsigma_i)\rho > 1$ and $\pi_j(\alpha_j + \varsigma_j)\rho > 1$ is verified.*

We get the following result:

The rate of growth for the economy in the open-loop case is giving by

$$\iota = \left[(\pi_i(\alpha_i + \varsigma_i)\rho)\right]^{\frac{\varphi_j}{\varphi_i+\varphi_j}} \left[\pi_j((\alpha_j + \varsigma_j)\rho)\right]^{\frac{\varphi_i}{\varphi_i+\varphi_j}}$$

Proof

Assuming that

$$k_t^i = \iota^t k_0^i \text{ and } k_t^j = \iota^t k_0^j \tag{17}$$

is verified. Then we get that $k_t^j/k_t^i = k_t^i/k_t^j$. Substituting (17) in (8) we get that

$$\iota^{t+1}k_0^i = ((\alpha_i + \varsigma_i)\rho\pi_i)\,\iota^{t\varphi_i}\iota^{t(\alpha_i+\varsigma_i)}\left(k_0^i\right)^{\alpha_i+\varsigma_i}\left(k_0^j\right)^{\varphi_i}$$

$$\iota^{t+1}k_0^i = ((\alpha_i + \varsigma_i)\rho\pi_i)\left(k_0^i\right)^{\alpha_i+\varsigma_i-1}\left(k_0^j\right)^{\varphi_i}$$

$$\iota = ((\alpha_i + \varsigma_i)\rho\pi_i)\left(\frac{k_0^j}{k_0^i}\right)^{\varphi_i}$$

An identical argument states that

$$\iota = ((\alpha_j + \varsigma_j)\rho\pi_j)\left(\frac{k_0^i}{k_0^j}\right)^{\varphi_j}$$

By joining both results we obtain

$$\iota = \left[\pi_i^{((\alpha_i+\varsigma_i)\rho\pi_i)}\right]^{\frac{\varphi_j}{\varphi_i+\varphi_j}} \left[\pi_j^{((\alpha_j+\varsigma_j)\rho\pi_j)}\right]^{\frac{\varphi_i}{\varphi_i+\varphi_j}}$$

□

Proof of Lemma 4 *Employing Markovian decision rules means that representative agents choose strategies of the type $c_t^i = c^i\left(k_t^i, z_t^j\right)$ when designing their actions. In order to derive a Markovian Perfect Equilibrium we therefore make use of dynamic programming techniques. Let us define the value function for agent i as*

$$V^i\left(k_t^i, z_t^j\right) \equiv \max \sum_{s=t}^{\infty} \rho^s \ln c_s^i \tag{18}$$

These value functions must satisfy the Bellman equations

$$V^i\left(k_t^i, z_t^j\right) \equiv \max_{c^i}\left\{\ln c^i + \rho V^i\left(\pi_i\left(k_t^i\right)^{\alpha_i+\varsigma_i}\left(z_t^j\right)^{\varphi_j} - c^i, \pi_j\left(k_t^j\right)^{\alpha_j+\varsigma_j}\left(z_t^i\right)^{\varphi_i} - c^j\right)\right\}. \tag{19}$$

By following (Dockner and Nishimura, 2004) and (Fischer and Mirman, 1996), we guess a candidate solution for the value function of the type:

$$V^i\left(k_t^i, z_t^j\right) \equiv A_{ii}\ln\left(\pi_i\left(k_t^i\right)^{\alpha_i+\varsigma_i}\left(z_t^j\right)^{\varphi_j}\right) + B_{ij}\ln\left(\pi_j\left(k_t^j\right)^{\alpha_j+\varsigma_j}\left(z_t^i\right)^{\varphi_i}\right) + D_i \tag{20}$$

where A_{ii}, B_{ij} and D_i are appropriately chosen constants. Based on these value functions we can guess a policy function for the consumption strategy of agent i by

$$c^i\left(k_t^i, z_t^j\right) = a_i\left(k_t^i\right)^{\alpha_i+\varsigma_i}\left(z_t^j\right)^{\varphi_j} \tag{21}$$

where a_i , in the same way, needs to be determined from the equilibrium conditions. Taking under consideration these specifications of the policy functions the corresponding dynamical system of the capital stock becomes

$$k_{t+1}^i = \left(\pi_i - a_i\right)\left(k_t^i\right)^{\alpha_i+\varsigma_i}\left(z_t^j\right)^{\varphi_j} \tag{22}$$

If we now substitute the policy functions (21) and the proposed value functions (20) into the Bellman equation (19) we get

$$\begin{aligned} &A_{ii}\ln\left(\left(k_t^i\right)^{\alpha_i+\varsigma_i}\left(z_t^j\right)^{\varphi_j}\right) + B_{ij}\ln\left(\left(k_t^j\right)^{\alpha_j+\varsigma_j}\left(z_t^i\right)^{\varphi_i}\right) + D_i = \ln\left(a_i\left(k_t^i\right)^{\alpha_i+\varsigma_i}\left(z_t^j\right)^{\varphi_j}\right) + \\ &+\rho\left[A_{ii}\ln\left(\pi_i\left(k_t^i\right)^{\alpha_i+\varsigma_i}\left(z_t^j\right)^{\varphi_j} - a_i\left(k_t^i\right)^{\alpha_i+\varsigma_i}\left(z_t^j\right)^{\varphi_j}\right)\right] \\ &+\rho\left[B_{ij}\ln\left(\pi_j\left(k_t^j\right)^{\alpha_j+\varsigma_j}\left(z_t^i\right)^{\varphi_i} - a_i\left(k_t^j\right)^{\alpha_j+\varsigma_j}\left(z_t^i\right)^{\varphi_i}\right) + D_i\right] \end{aligned}$$

where we see that if the constants A_{ii} and B_{ij} satisfy the equations

$$A_{ii} = 1 + \rho\left[\left(\alpha_i + \varsigma_i\right)A_{ii} + \varphi_j B_{ij}\right] \tag{23}$$

$$B_{ij} = \rho\left[\varphi_i A_{ii} + \left(\alpha_j + \varsigma_j\right)B_{ij}\right] \tag{24}$$

Then (20) and (21) solve (19). A solution to the system (23) and (24) is easily found and given by

$$A_{ii} = \frac{1 - \left(\alpha_j + \varsigma_j\right)\rho}{\left(1 - \left(\alpha_i + \varsigma_i\right)\rho\right)\left(1 - \rho\right) - \rho^2\varphi_j\varphi_i} \tag{25}$$

$$B_{ij} = \frac{\varphi_j\rho}{\left(1 - \left(\alpha_i + \varsigma_i\right)\rho\right)\left(1 - \rho\right) - \rho^2\varphi_i\varphi_j} \tag{26}$$

for $i \neq j$

Proving that the policy function (21) is in fact an equilibrium we need to show that the maximum of the right hand side of (19) given the specification (20).

The maximization gives:

$$\frac{1}{c^i} = \frac{\rho\left(\left(\alpha_i + \varsigma_i\right)A_{ii} + \varphi_i B_{ij}\right)}{\pi_i\left(k_t^j\right)^{\alpha_j+\varsigma_j}\left(z_t^i\right)^{\varphi_i} - c^i}$$

Solving for the optimal policy rule shows that the functional form given by (17) is attended, by using A_{ii} and B_{ij} as obtained earlier, i.e.,

$$c^i\left(k^i, z_t^j\right) = \frac{\pi_i \left(k_t^j\right)^{\alpha_j+\varsigma_j} \left(z_t^i\right)^{\varphi_i}}{1+\rho\left(\left(\alpha_i+\varsigma_i\right) A_{ii}+\varphi_i B_j\right)} = a_i \left(k_t^j\right)^{\alpha_j+\varsigma_j} \left(z_t^i\right)^{\varphi_i}$$

where $a_i = \pi_i/1+\rho\left(\left(\alpha_i+\varsigma_i\right) A_{ii}+\varphi_i B_j\right)$ *. Hence, the equilibrium dynamics becomes*

$$k_{t+1}^i = \psi_i \left(k_t^i\right)^{\alpha_i+\varsigma_i} \left(z_t^j\right)^{\varphi_j} \tag{27}$$

$$k_{t+1}^j = \psi_j \left(k_t^j\right)^{\alpha_j+\varsigma_j} \left(z_t^i\right)^{\varphi_i} \tag{28}$$

for $i \neq j$*, where*
$\psi_i = \pi_i \left(\left(\alpha_i+\varsigma_i\right)\rho+\rho^2 \frac{\varphi_i\varphi_j}{1-\left(\alpha_i+\varsigma_i\right)\rho}\right)$ *and* $\psi_j = \pi_j \left(\left(\alpha_j+\varsigma_j\right)\rho+\rho^2 \frac{\varphi_i\varphi_j}{1-\left(\alpha_j+\varsigma_j\right)\rho}\right)$

In case of constant returns to scale, the balanced growth path with the growth factor given by

$$\vartheta = \psi_i^{\frac{\varphi_i}{\varphi_i+\varphi_j}} \psi_j^{\frac{\varphi_j}{\varphi_i+\varphi_j}}$$

□

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