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Abstract: The standard two-sector growth model with physical and human capital characterizes a process of material accumulation involving simple dynamics; constant long run growth is observable when assuming conventional Cobb-Douglas production functions in both sectors. This framework is developed under a central planner scenario: it is a representative agent that chooses between consumption and capital accumulation, on one hand, and between allocating human capital to each one of the two sectors, on the other. We concentrate in this second choice and we argue that the outcome of the aggregate model is incompatible with a scenario where individual agents, acting in a market economy, are free to decide, in each time moment, how to allocate their human capital in order to produce goods or to create additional skills. Combining individual incentives, the effort of a central planner (i.e., government) to approximate the decentralized outcome to the optimal result and a discrete choice rule that governs the decisions of individual agents, we propose a growth framework able to generate a significant variety of long term dynamic results, including endogenous fluctuations.

Keywords: Endogenous growth, Human capital, Endogenous business cycles, Discrete choice, Nonlinear dynamics, Chaos.

JEL classification: O41, E32, C61.

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1. Introduction

In the last two decades, the literature on economic growth has produced a large amount of discussion around the two-sector model with physical and human capital that was initially proposed by Uzawa (1965) and Lucas (1988) and that can be considered a fundamental landmark with respect to the theory of endogenous growth.

Under a general formulation, namely considering generic production functions, this model may produce complicated dynamics. For example, Caballé and Santos (1993) emphasize the presence of multiple steady states, meaning that economies with different initial endowments of physical and human capital will never converge in the absence of any exogenous disturbance. Other features are highlighted, for instance, by Boldrin and Montrucchio (1986), Nishimura, Shigoka and Yano (1998), Boldrin, Nishimura, Shigoka and Yano (2001) and related literature, who address the possibility of endogenous cycles in the competitive two-sector growth model; their main conclusion is that strange dynamics only occur in one of two circumstances: unrealistic values of parameters (e.g., a too low discount factor) or departures from the standard convex formulation of preferences and technology.

Nevertheless, if one sticks to an utility function with the conventional CIES property and standard constant returns to scale production functions (more specifically, a two inputs Cobb-Douglas production function for final goods and a linear one input production function on the human capital sector), then a unique steady state is determined and the dynamics of the model are characterized by a saddle-path stable equilibrium, which is accomplished independently of the values of parameters [see Barro and Sala-i-Martin (1995), chapter 5]. The independence of the dynamic result relatively to the values of technology indexes, depreciation rates, the discount rate or the output-capital elasticity implies that the analysis of the Jacobian matrix of the system underlying the optimal problem does not allow to find any bifurcation and, thus, no qualitative change in dynamics will occur. We might say that the standard Uzawa-Lucas model without production externalities or peculiar features affecting aggregate consumption or the processes of capital accumulation, allows for a unique type of dynamic result, which is robust to changes in preferences or technological conditions.

Further insights about the transitional dynamics and equilibrium properties of the two-sector endogenous growth model with human capital can be found in the work of Mulligan and Sala-i-Martin (1993), Bond, Wang and Yip (1996), Ladrón-de-Guevara, Ortigueira and Santos (1997, 1999) and Ortigueira (2000). Other work on the Uzawa-Lucas model stresses particular features like the role of fiscal policy in guaranteeing an optimal decentralized equilibrium [García-Castrillo and Sanso (2000), Gómez (2003, 2005)] and the efficiency of the competitive equilibrium under sector specific externalities associated to human capital in the goods sector [Gómez (2004, 2006)]. An additional note goes to the work of Restrepo-Ochoa and Vázquez (2004), who adopt a stochastic version of the two-sector model in order to compare results on fluctuations with the benchmark Real Business Cycles setup.

The systematic debate around the two-sector growth model means essentially that although it is a powerful benchmark structure to address some of the most relevant questions around the theme of material wealth accumulation, it also leaves several unanswered interrogations. One of the most striking points of debate respects to the validity of the notion of central planner. The representative agent that chooses an optimal path of consumption in order to maximize a stylized utility function that should reflect the preferences of the average consumer, is not an uncontroversial notion, as Krussell and Smith (1998) and Kirman (1992, 2004), for instance, emphasize.

In the words of Onozaki, Sieg and Yokoo (2003),

"*A representative agent typifies preferences and technologies as well as rational behaviour of the whole society of agents. One possible argument in favour of simplifying a model by assuming a 'representative rational agent' apparatus is that all the different behaviour as already died out and only the representative agent survives (…) Different types of behaviour can survive simultaneously.*

Another possible defence of assuming a representative agent is that a majority of the agents behave in the same way and their behaviour determines the dynamics of the market. (…) The 'representative rational agent' is a theoretical apparatus that works with certainty only when all agents behave in the same way." pages 1917-1918.

Also Gomes (2006*a*) emphasizes a similar idea:

"*(…) collective action cannot be reduced to the behaviour of some average or representative agent. Macro behaviours, i.e., aggregate phenomena, are intrinsically complex because social interaction of boundedly rational agents implies features that are not observable at the level of the individual. Herd behaviour, bubbles or persistent* *excess volatility are determined by two simultaneous phenomena: coexistence of different simple behavioural rules and interaction.*

Emphasis must be placed on the idea that aggregate behaviour is not the behaviour of a representative individual. It is a fundamental mistake to identify aggregation with the selection of the average behaviour." page 454.

The previous remarks build an important argument against the main economic theoretical paradigm which relies on the concept of representative consumer.

In the two-sector growth model, even harder to accept than the notion of representative consumer is the idea that a representative agent optimally chooses how much of her human capital endowment is allocated to each one of the two productive processes. First, the notion that this agent may be the government has little empirical support (authorities can influence private sector decisions about how skills are allocated, but they cannot certainly impose a given share of resources to each one of the sectors unless one considers a completely centralized economy).

Second, if the choice of human capital allocation is left to private agents under a scenario of perfect rationality, then a corner solution is likely to end up by dominating. In an economy populated by a large number of individual agents, each one taking private isolated decisions, they will all select the best immediate option which is, in each time moment, given by the strategy that allows for withdrawing the best one of the productive rewards (which correspond, in a competitive setup, to marginal productivities). Since all agents are assumed as identical (each unit of human capital is equally productive in each sector), they will choose to concentrate their human capital in just one sector, the one with a higher marginal productivity. As a consequence, purely decentralized human capital allocation decisions produce inefficiency because they tend to an agglomeration of human capital in one sector and this does not allow for taking advantage of productive complementarities: income will be equal to zero because human capital fully concentrates in the goods sector (no human capital is produced and therefore it does not exist to assist on producing goods), or it fully concentrates in the education sector (the accumulation of human capital rises but there is no participation of this input to generate output in the goods sector).

The previous arguments can be associated to the notion of tragedy of the commons: there is a social optimal result (the one that can be found by solving the conventional two-sector optimal growth model), but there are no private incentives for each individual agent to act according to the common good. If at a given moment the agent has to choose between the action that allows for a better long term social outcome and the action that gives an immediate better result, she will prefer the second, because she has no means to know what the action of all the other agents will be. Instantaneous private results are always preferred to long run collective results, and this is the reason why the two-sector optimal growth model gives a wrong idea of the actual growth process: the allocation of human capital in a market economy cannot be modelled as an intertemporal problem solved collectively; instead, it is the result of multiple individual decisions that, because they cannot be coordinated, are taken one at a time.

Individual decisions in a market economy are therefore inconsistent with an intertemporal representative choice.

The aim of this paper is to furnish, within a two-sector growth framework, a plausible mechanism of human capital allocation among sectors that allows for decentralized decisions triggered by private incentives but that simultaneously gives the possibility of an interior solution. This is done by adding two additional features to the model: first, agents are assumed as boundedly rational; second, public authorities (the central planner) act with the goal of influencing private decisions, in order to approximate them to the optimal result on human capital allocation.

The notion of bounded rationality that we adopt is similar to the one in the rational routes to randomness literature, developed among others by Brock and Hommes (1997, 1998), Barucci (1999), Gaunersdorfer (2000), Chiarella and He (2002), Chiarella, Gallegati, Leombruni and Palestrini (2003), Negroni (2003), Westerhoff (2004, 2005) and Gomes (2005), which relies on the adoption of a discrete choice rule. An intensity of choice parameter regulates the degree in which individual agents change their behaviour in face of better alternative results. A high intensity of choice reveals that the agent will be inclined to switch to another possible rule of behaviour if this seems to perform better.

Relatively to the role of the government, in our analysis this is reduced to an income reallocation role that does not produce any kind of efficiency loss. Public authorities will distribute income from the agents that decide to allocate their human capital to the sector that already has a share of human capital above the social optimal level, to the agents that remain in the sector where human capital is used in a percentage below the socially optimal. Therefore, the government exerts no direct influence over production and growth; there is only an indirect impact of changing the way individual agents perceive rewards regarding human capital allocation.

The sluggish adjustment mechanism associated with the discrete choice rule, combined with the systematic effort of the authorities in order to make individual agents take actions in the direction of a socially optimal distribution of human capital among productive sectors, will allow for a new analytical structure, more complex but simultaneously better equipped to justify a large set of possible long term results, which vary with changes in some of the parameter values, like the intensity of choice or some policy parameter.

The dynamic properties of the model are addressed both through a local and a global perspectives. In each case, general meaningful results are hard to withdraw (for instance, explicit steady state expressions are not feasible to obtain); nevertheless, some representative examples give a clear picture of the underlying dynamics. The local analysis is destined to identify points of bifurcation; we know that in a two-equation system three qualitative results are identifiable in the neighbourhood of a steady state point: fixed-point stability (two eigenvalues of the Jacobian matrix inside the unit circle); saddle-path stability (one eigenvalue inside the unit circle and the other one outside); and instability (two eigenvalues outside the unit circle). Bifurcation points constitute the border points between two regions with different local properties. For different examples, we inquire about possible bifurcations; in particular, we observe the occurrence of a flip bifurcation in most of the cases; this will separate a region of fixed-point stability from another region where saddle-path stability is evidenced.

The study of global dynamics can only be undertaken with explicit functional forms for the utility function and for the production functions and taking an array of concrete parameter values. Various examples allow to confirm the presence of bifurcation points, that in most cases signal the transition from a stability region into a region of cycles (first, two-period cycles, and, eventually, through a process of period doubling or other route to chaos, cycles of higher order or even chaotic motion), before instability sets in. These bifurcations are found when varying the values of several of the parameters underlying the theoretical structure.

With the proposed framework, one intends to accomplish two outcomes: first, the main goal is to turn the two-sector growth model more realistic, in the sense that the allocation of human capital cannot be conceived, under a competitive environment, as the pure result of collectively imposed conditions; second, introducing the new features, the model is able to reproduce, for admissible parameter values, the presence of business cycles. Therefore, the model can be thought alongside with the recent literature on endogenous business cycles [see Christiano and Harrison (1999), Schmitt-

Grohé (2000), Guo and Lansing (2002), Cellarier (2006), Dosi, Fagiolo and Roventini (2006), Gomes (2006*b*, 2006*c*), Lloyd-Braga, Nourry and Venditti (2006), among many others].

The advantage of the presented framework relatively to the bulk of the referred work is that we do not need to introduce any market inefficiency [mainly, final goods production externalities that imply an increasing returns to scale technology] in order to produce cycles. It is precisely by making the model more in line with reality, without departing from the competitive structure, that we are able to identify cycles.

The remainder of the paper is organized as follows. Section 2 reviews the main features of the benchmark endogenous growth model with two sectors. Section 3 formalizes the main assumptions regarding the decentralized allocation of human capital among sectors. Section 4 addresses some of the steady state properties. In section 5, local dynamics are explored in search for conditions under which bifurcations may occur. Section 6 addresses global dynamics and identifies circumstances in which cycles and chaos are observable for admissible economic conditions. Finally, section 7 presents some final remarks.

2. The Standard Two-Sector Growth Model

Consider an economy where a social planner solves an infinite horizon optimal control problem; the social planner maximizes the utility of consumption, given two state constraints that characterize the accumulation of physical and human capital. The utility function, $U(c_t)$, has the usual properties: it is an increasing and concave function for all $c_t \geq 0$, with c_t the level of real per capita consumption. Moreover, one assumes that the utility function is characterized by a constant intertemporal elasticity of substitution and that $\lim_{c\to 0} U' = \infty$ and $\lim_{c\to +\infty} U' = 0$. Consumption utility is discounted in time; $\beta \in (0,1)$ is the discount factor. The maximization problem takes the form: $Max \sum_{t=0}^{+\infty} U(c_t)\cdot\beta^t$.

The two constraints respect to the accumulation over time of per capita physical capital, $k_t \geq 0$, and per capita human capital, $h_t \geq 0$.[1] These constraints are the standard ones.

The accumulation of physical capital is given by difference equation (1),

[1] To simplify the presentation, one assumes that population does not grow.

$$k_{t+1} - k_t = f(k_t, u_t h_t) - c_t - \delta k_t, \quad k_0 \text{ given} \tag{1}$$

In (1), $\delta \geq 0$ is a depreciation rate and $u_t \in [0,1]$ corresponds to the share of human capital that is allocated to the production of final goods rather than to the education sector. The production function obeys to the following properties:

i) Function f is continuous and has continuous partial derivatives of second-order (it is a C^2 function);

ii) Function f exhibits constant returns to scale (the function is linearly homogeneous) and decreasing marginal returns to each input;

iii) There is factor complementarity: $f(0,u_t h_t)=0$ and $f(k_t,0)=0$;

iv) The Inada conditions are satisfied: $\lim_{k \to \infty} f_k = \infty$; $\lim_{h \to \infty} f_h = \infty$.

Basically, function f is a neoclassical production function. Since we are working with a competitive environment, the wage rate that rewards each unit of human capital in the final goods sector corresponds to the marginal productivity of human capital, that is, it is given by the partial derivative f_{uh}.

The human capital sector is characterized by the following dynamic equation:

$$h_{t+1} - h_t = g((1-u_t) \cdot h_t) - \delta h_t, \quad h_0 \text{ given} \tag{2}$$

To simplify the analysis, we assume that human capital is subject to the same depreciation rate as physical capital. Production function g is a C^2 function and it is linearly homogeneous. Given that g has a single argument (the amount of human capital available to produce additional human capital), the homogeneity property implies constant marginal returns to human capital. We assume that this is an increasing function, $g'>0$, for which $g(0)=0$.

Definition 1: The standard two-sector growth model corresponds to the infinite horizon discounted problem of intertemporal consumption utility maximization, subject to constraints (1) and (2). An optimal solution of this problem is a set of paths $\{k_t, h_t, c_t, u_t\}_{t=0}^{+\infty}$ that solve the optimization problem. Per capita consumption and share u_t constitute the control variables of the central planner. In this problem, a balanced growth path is defined as the long term locus in which k_t, h_t and c_t grow at constant rates and u_t is constant.

Note that according to the description of balanced growth in definition 1 [that follows directly the one in Caballé and Santos (1993)], we can define the steady state only with regard to constant long run values. Defining $\omega_t \equiv k_t / h_t$ and $\psi_t \equiv c_t / k_t$, the following is a new version of the previous notion,

Definition 2: The steady state / balanced growth path of the standard two-sector model is a set of constant values $\{\omega^*, \psi^*, u^*\}$, that is obtained for $\omega_{t+1}=\omega_t$, $\psi_{t+1}=\psi_t$ and $u_{t+1}=u_t$.

The aim of the present analysis is not to fully characterize the dynamics of the problem in definition 1. This is well known from the literature and, unless one departs from the production function and utility properties depicted above, a unique steady state is generally found and a saddle path relation describes the dynamics in the steady state vicinity. Convergence to the balanced growth path is only guaranteed if the initial values of c_t and u_t chosen by the central planner are already on the saddle trajectories. The convergence to the steady state is generally described by a movement of opposite direction of each one of the control variables (ψ_t and u_t) relatively to the state variable (ω_t), that is, saddle paths are negatively sloped [see Barro and Sala-i-Martin (1995), chapter 5].

In order to proceed with the analysis, it is convenient to present the equation of motion for the ratio ω_t, and to compute the steady state for a version of the model with explicit functional forms of the utility and the production functions.

The capital ratio equation of motion is obtained in a straightforward way:

$$\omega_{t+1} = \frac{f(u_t / \omega_t) - \psi_t + 1 - \delta}{g(1-u_t) + 1 - \delta} \cdot \omega_t \tag{3}$$

To present (3) we have resorted to the homogeneity property, so that $f(k_t, u_t h_t)/k_t = f(1, u_t h_t / k_t)$ and $g((1-u_t)\cdot h_t)/h_t = g(1-u_t)$.

Assume now a simple utility function $U(c_t) = \ln c_t$; a Cobb-Douglas production function for the final goods sector, $y_t = Ak_t^{\alpha} \cdot (u_t h_t)^{1-\alpha}$, with A>0 a technology index and $\alpha \in (0,1)$ the output-capital elasticity; and $y_t^h = B \cdot (1-u_t) \cdot h_t$ the production function of the human capital sector, with B>0 reflecting the level of technology on education. With these functional forms, we solve the optimal control problem in order to gain access to an explicit steady state. Through the computation of first order conditions we can add to (3) two relations that are satisfied under optimality, which are,

$$\psi_{t+1} = \frac{\beta \cdot \left[\alpha A \cdot (u_{t+1} / \omega_{t+1})^{1-\alpha} + 1 - \delta\right]}{A \cdot (u_t / \omega_t)^{1-\alpha} - \psi_t + 1 - \delta} \cdot \psi_t \tag{4}$$

$$\chi_{t+1} = \frac{1 + B - \delta}{\alpha A \cdot (u_t / \omega_t)^{1-\alpha} + 1 - \delta} \cdot \chi_t \tag{5}$$

with $\chi_t \equiv p_t / q_t$ the ratio between the shadow-prices of physical capital (p_t) and human capital (q_t). Another optimality condition is given by the relation $\chi_{t+1} = \frac{B}{(1-\alpha) \cdot A} \cdot \left(\frac{u_t}{\omega_t}\right)^{\alpha}$; this relation allows to state that, according to the definition of balanced growth path, the prices' ratio possesses a constant value in the steady state. Therefore, one derives steady state results by solving the system $\omega_{t+1}=\omega_t$, $\psi_{t+1}=\psi_t$ and $\chi_{t+1}=\chi_t$. The results are expressed in proposition 1.

Proposition 1: Representative agent model steady state. *The two-sector growth model in definition 1 with logarithmic utility and Cobb-Douglas production functions has a unique steady state, which is:*

$$\begin{pmatrix} \omega^* \\ \psi^* \\ u^* \end{pmatrix} = \begin{bmatrix} \left(\frac{\alpha A}{B}\right)^{1/(1-\alpha)} \cdot (1-\beta) \cdot \left(1 + \frac{1-\delta}{B}\right) \\ (1/\alpha - \beta) \cdot B + (1-\beta) \cdot (1-\delta) \\ (1-\beta) \cdot \left(1 + \frac{1-\delta}{B}\right) \end{bmatrix}.$$

To guarantee an interior balanced growth path, we must secure that $0 < u^* < 1$; given the constraints upon parameters, this condition is satisfied under $B > (1-\delta) \cdot \frac{1-\beta}{\beta}$. The steady state values allow as well to deliver the growth rate at which aggregates k_t, h_t, c_t, y_t and y_t^h grow in the long run. Replacing u^* in the human capital constraint, it is straightforward to obtain $\gamma = \beta \cdot (B - \delta) - (1-\beta)$. The long term growth of the economy is positively related to the technological capabilities on the education sector and negatively related with the discount rate of future consumption and with the depreciation rate.

In the next section, we modify the benchmark two-sector growth model by assuming that u_t is no longer determined by a representative agent.

3. Decentralized Decisions on the Allocation of Human Capital

The assumed economy is endowed, in each time moment, with h_t units of human capital. The human capital is allocated to a large number of individual agents, in varying shares; the individual agents have to decide if they will participate in the production of additional skills or, instead, if they allocate their effort to the production of goods.

The notion of a representative agent is hardly compatible with a competitive environment where multiple agents interact. For the notion of representative agent to hold it would be necessary to conceive an economy working as a community, where everyone would trust in one another so that each agent could allocate part of their human capital to each one of the possible uses, knowing that in this way in the long run the optimal growth rate would be accomplished. Because this guarantee is not possible, every single agent has the advantage of behaving under an egoistic way, that is, neglecting any social commitment that eventually could arise to resemble a social planner.

Thus, the point is that in an economy of atomistic agents there is no room for individual intertemporal choices concerning the allocation of the labour force. Each agent will just choose in each moment the option that gives place to a higher reward. This is not to say that individuals have the ability to change sector in every moment; there are certainly several types of costs involved and these are captured by the discrete choice mechanism that will be introduced. This mechanism allows the individual worker to make choices given accumulated potential rewards.

As stated, if no interference by the government is present, agents will just react to the private reward of participating on each one of the productive processes. The direct measure of this reward is the wage rate, that is, the marginal productivity of the human capital allocated to each sector. Thus, the private value functions respecting each sector are $v^k(u_t,\omega_t) = f_{uh}$ and $v^h(u_t,\omega_t) = g_{(1-u)\cdot h}$. If agents gave value only to present rewards and not to past rewards as well, it is clear that a corner solution would dominate: agents would concentrate on the goods sector ($f_{uh}>g_{(1-u)\cdot h}$) or, otherwise, in the education sector.

Besides the private evaluation of rewards, individual agents are subject to a redistribution policy by the government. In our model, the single function of the government is to create incentives to reach a long term outcome in terms of human capital allocation that approaches the optimal result derived in the previous section. To influence the private allocation of skills, the government redistributes income from the workers that are attached to the sector with excess of human capital (i.e., human capital above the long term optimum) to the other sector. The mechanism will work as follows:

- If $u_t < u^*$, then an amount of resources $z(u_t - u^*)\cdot(1-u_t)\cdot h_t$ will be transferred from the households that allocate human capital to the education sector to the households on the other sector. The tax in the education sector will be $z(u_t - u^*)$ per unit of human capital, while the subsidy per unit of human capital in the goods sector comes $z(u_t - u^*)\cdot\frac{1-u_t}{u_t}$. Note that $z(\cdot)$ must be a positive function for all u_t and $z'>0$ in order to guarantee that the amount of the transfer rises with the distance between the observed share and the long term optimal one.
- If $u_t > u^*$, then the transfer process follows the opposite direction. A tax imposed over the human capital in the goods sector reduces the individual benefit in $z(u_t - u^*)$ [the total amount of the transfer is, in this case, $z(u_t - u^*)\cdot u_t h_t$], and rises the income of the ones engaged in generating additional knowledge, by an amount of $z(u_t - u^*)\cdot\frac{u_t}{1-u_t}$ per human capital unit.

With this income transfer process, the value functions relating each sector become:

- If $u_t < u^*$, then:

$$v^k(u_t,\omega_t) = f_{uh} + z(u_t - u^*)\cdot\frac{1-u_t}{u_t} \tag{6}$$

and

$$v^h(u_t,\omega_t) = g_{(1-u)\cdot h} - z(u_t - u^*) \tag{7}$$

- If $u_t > u^*$, then:

$$v^k(u_t, \omega_t) = f_{uh} - z(u_t - u^*) \tag{8}$$

and

$$v^h(u_t, \omega_t) = g_{(1-u)\cdot h} + z(u_t - u^*) \cdot \frac{u_t}{1-u_t} \tag{9}$$

Synthesizing, when evaluating which is the best way to allocate their productive human capital, individual agents face two realities: the strictly private reward given by their marginal productivity and the income distribution policy of the government intended to maintain the share of human capital allocated to each sector near the optimal level.

The shape of function $z(\cdot)$ is important in terms of the dynamic analysis. Later, we will work with a functional form of $z(\cdot)$ that attributes a larger weight to larger differences between u_t and u^*; specifically, we take $z(u_t - u^*) = z \cdot (u_t - u^*)^2$, with z a positive parameter, that reflects the highest possible transfer rate.

Households will not make an instantaneous evaluation of value functions as presented in expressions (6) to (9). They will worry about the flow of value functions, which they accumulate in variables v_t^k and v_t^h. These performance measures are subject to a process of memory loss, according to which far in the past results have less influence in allocation decisions than close to the present outcomes. Thus, letting ρ>0 be the rate of memory loss, the accumulated performance measures will evolve in time according to rules (10) and (11),

$$v_{t+1}^k - v_t^k = v^k(u_t, \omega_t) - \rho v_t^k, \quad v_0^k \text{ given} \tag{10}$$

$$v_{t+1}^h - v_t^h = v^h(u_t, \omega_t) - \rho v_t^h, \quad v_0^h \text{ given} \tag{11}$$

Each agent will compare v_t^k and v_t^h and choose the sector to which they will offer their productive skills. The mechanism that avoids all individuals making the same choice in each moment of time is given by a discrete choice rule. Agents are boundedly rational and thus they can stay in one sector even though the reward is higher in the other sector; the discrete choice rule introduces sluggishness in the switching across sectors, and thus different behaviours will coexist.[2] The referred rule takes the form

$$u_t = \frac{\exp(bv_t^k)}{\exp(bv_t^k) + \exp(bv_t^h)} \tag{12}$$

In expression (12), parameter b represents the intensity of choice, that is, it is the parameter that indicates the degree of reaction of the individuals to the accumulated performance measure. In the limit cases, $b=0$ implies that the agents will never switch behaviour, while $b\to\infty$ means that the allocation of human capital to the goods sector will be 100% certain if $v_t^k>v_t^h$ (and 0% if the opposite condition is observed). Under the proposed setup, u_t becomes not only the share of the economy's human capital allocated to the goods sector, but also the probability of an individual choosing to allocate her human capital to this first sector.

Some algebra applied over (10), (11) and (12), allows for finding a state constraint attached to the evolution of share u_t over time, which is

$$u_{t+1} = 1 \Bigg/ \left\{ 1 + \left(\frac{1-u_t}{u_t}\right)^{1-\rho} \cdot \exp\left[b \cdot \left(v^h(u_t,\omega_t) - v^k(u_t,\omega_t)\right)\right] \right\} \tag{13}$$

In the following sections, we propose to analyze system (3)-(13), with a consumption – physical capital ratio that remains constant in the steady state: ψ^*. This system will characterize the evolution over time of variables ω_t and u_t in an economy where human capital allocation is determined by decentralized household decisions, where the government has an important redistributive role in trying to influence the allocation of human capital, and where individual agents are boundedly rational.

Definition 3: The dynamics of the two-sector growth model with decentralized decisions regarding human capital allocation and a constant consumption – physical capital ratio are given by the system of two deterministic difference equations (3)-(13).

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[2] We stress that the boundedly rational hypothesis is sufficient for heterogeneous behaviour among homogeneous agents. Recall that agents are identical in the sense they respond exactly to the same incentives.

4. Properties of the Steady State

The analysis of local dynamics collides with an important obstacle. Even if we consider the assumed explicit functional forms for the utility function, for the production functions and for the government transfer function, one is unable to find steady state values for the endogenous variables neither we are able to demonstrate which are the exact conditions of uniqueness of the balanced growth path.

Our main result regarding the general properties of the steady state is given in proposition 2.

Proposition 2: Steady state existence. *Consider the decentralized human capital allocation growth model with the specified utility, production and government transfer functions. In this problem, at least one interior balanced growth path (* $\overline{\varpi},\overline{u}$ *) is observable.*

Proof: appendix 1.

The previous proposition has an important corollary regarding government policy. In particular, steady state conditions as depicted in the proof of the proposition evidence that a public policy that completely fulfils the goal of transforming the long run share of human capital allocated to the production of physical goods into an optimal share is not feasible.

Corollary of proposition 2: Impossibility of applying an optimal steady state transfer policy. *Under the steady state result of proposition 1, public authorities are unable to follow an optimal resource transfer policy (that is, a policy that allows to obtain* $\overline{u}=u^*$ *). This would lead to an infinite resource re-allocation.*

Proof: appendix 2.

Unique steady state results can be derived under particular conditions of our system. Proposition 3 proposes one of these cases,

Proposition 3: Unique steady state in a particular case. *In the decentralized human capital allocation growth model (under the same functional forms as in* *proposition 2),* $\bar{u} = u^{*} + \frac{1}{\alpha} - (1-\alpha)^{(1-\alpha)/\alpha} \cdot \left(\frac{A}{B}\right)^{1/\alpha}$ *is a unique steady state point (assuming that* $0 < \bar{u} < 1$ *holds).*

Proof: appendix 3.

In the next section, the analysis of local dynamics will be pursued taking basically a class of specific steady state results, in the case the ones for which $\bar{u}$ is dependent upon the parameters of the optimal representative agent growth problem but not on the parameters of the decentralized version, which are the intensity of choice, the memory loss rate and the income transfer policy parameter. Proposition 4 indicates in which conditions such class of steady state results are feasible.

Proposition 4: The steady state for $\bar{u}(\alpha, A, B, u^{*})$. *Assuming that* $\bar{u}$ *is dependent on the several parameters of the representative agent model (i.e.,* α*, A, B,* β *and* δ*), but not on the parameters introduced with the decentralized version of the model (b,* ρ*, z), one can state that* $\bar{u} = \tilde{u} \in (0,1)$ *represents an admissible growth path if none of the following sets of conditions is satisfied,*

$$\tilde{u} \leq 0.5 \wedge \tilde{u} \leq u^{*} \wedge \tilde{u} \leq u^{*} + \frac{1}{\alpha} - (1-\alpha)^{(1-\alpha)/\alpha} \cdot \left(\frac{A}{B}\right)^{1/\alpha}$$

or

$$\tilde{u} \geq 0.5 \wedge \tilde{u} \geq u^{*} \wedge \tilde{u} \geq u^{*} + \frac{1}{\alpha} - (1-\alpha)^{(1-\alpha)/\alpha} \cdot \left(\frac{A}{B}\right)^{1/\alpha}$$

Proof: appendix 4.

For the suggested class of balanced growth paths, we can identify a peculiar case for $\bar{u} = 0.5$. In this case, we can clearly solve the steady state condition $u_{t+1}=u_t$ and solve it in order to *z*, to present the following corollary,

Corollary of proposition 4: The special case $\bar{u}(\alpha, A, B, u^{*})$***=0.5****. When* $\bar{u} = 0.5$ *is not a result dependent on the values of b,* ρ *and z, one is able to obtain the transfer policy parameter z solely as a function of the values of parameters on the representative agent model:*

$$z=\begin{cases} B\cdot\left[1-(1-\alpha)\cdot\left(\frac{A}{B}\right)^{1/(1-\alpha)}\cdot\left(\frac{\alpha}{1-\alpha\cdot(0.5-u^*)}\right)^{\alpha/(1-\alpha)}\right]\Big/\left[2\cdot(0.5-u^*)^2\right] \text{ if } u^*>0.5 \\ B\cdot\left[(1-\alpha)\cdot\left(\frac{A}{B}\right)^{1/(1-\alpha)}\cdot\left(\frac{\alpha}{1-\alpha\cdot(0.5-u^*)}\right)^{\alpha/(1-\alpha)}-1\right]\Big/\left[2\cdot(0.5-u^*)^2\right] \text{ if } u^*<0.5 \end{cases}$$

5. The Analysis of Bifurcations

Assume that a balanced growth path ($\overline{\varpi},\overline{u}$) exists for system (3)-(13), under the previously proposed logarithmic utility function, Cobb-Douglas final goods production function, linear education production function and quadratic transfer policy function. We linearize this system in the neighbourhood of the steady state, $\begin{bmatrix}\omega_{t+1}-\overline{\varpi}\\ u_{t+1}-\overline{u}\end{bmatrix}=J\cdot\begin{bmatrix}\omega_t-\overline{\varpi}\\ u_t-\overline{u}\end{bmatrix}$, with the Jacobian matrix given by

$$J=\begin{bmatrix} 1-\frac{(1-\alpha)\cdot A\cdot(\overline{u}/\overline{\varpi})^{1-\alpha}}{B\cdot(1-\overline{u})+1-\delta} & \frac{(1-\alpha)\cdot A\cdot(\overline{\varpi}/\overline{u})^{\alpha}+B\overline{\varpi}}{B\cdot(1-\overline{u})+1-\delta} \\ \alpha\cdot(1-\alpha)\cdot Ab\cdot(\overline{u}/\overline{\varpi})^{1-\alpha}\cdot(1-\overline{u}) & 1-\rho-\alpha\cdot(1-\alpha)\cdot Ab\cdot(\overline{\varpi}/\overline{u})^{\alpha}\cdot(1-\overline{u})+\Xi \end{bmatrix},$$

where $\Xi\equiv bz\cdot\frac{1-\overline{u}}{\overline{u}}\cdot(\overline{u}^2-u^{*2})$ if $\overline{u}<u^*$ and $\Xi\equiv-bz\cdot\frac{\overline{u}}{1-\overline{u}}\cdot\left[\overline{u}\cdot(2-\overline{u})-u^*\cdot(2-u^*)\right]$ if $\overline{u}>u^*$.

To study local dynamics, we consider the particular case of the previous section where $\overline{u}(\alpha,A,B,u^*)$ and analyze three numerical examples. We begin by defining a vector of concrete values for the parameters that we encounter both in the optimal problem and in the proposed setup. These values are reasonable in the sense that they allow for optimal steady state results that can be associated to the reality observed in most developed countries. The vector is: [A α β δ B]=[1 0.25 0.96 0.05 0.133]. With these values, we get an optimal growth rate γ=0.04 (the economy grows at a rate of 4% per period), and the following optimal steady state values: $\omega^*=0.756$, $\psi^*=0.442$ and $u^*=0.326$. The representative agent problem furnishes a balanced growth path where, for the selected parameters' vector, the accumulated physical capital corresponds to 75.6% of the stock of human capital, the level of consumption is 44.2% of the level of physical capital, and where 32.6% of the available human capital is allocated to the production of physical goods.

The analysis is restricted to cases where combinations of parameters *b*, ρ and *z* are such that $\bar{u}$ is constant, and we look at three examples: $\bar{u} = 0.25$; $\bar{u} = 0.5$; and $\bar{u} = 0.75$. Each specific case is studied separately.

Case 1: $\bar{u} = 0.25$. In this case, $\bar{u} < u^*$; thus, we are taking in consideration value functions (6) and (7). This means we need to solve equation $\bar{u} = 1 \Big/ \left\{ 1 + \left(\frac{1-\bar{u}}{\bar{u}} \right)^{1-\rho} \cdot \exp\left[b \cdot \left(B - A \cdot (1-\alpha) \cdot (\overline{\omega} / \bar{u})^{\alpha} - z \cdot (\bar{u} - u^*)^2 / \bar{u} \right] \right\}$ to obtain the relation between parameters *b*, ρ and *z* that turns this case feasible. This relation is, for our values' vector, $1.099\rho + 0.787b + 0.023bz = 0$. Since the three parameters are all nonnegative values, this condition never holds, and thus $\bar{u} = 0.25$ is a result that cannot be obtained at all. No stability conditions can be discussed in this particular case.

Case 2: $\bar{u} = 0.5$. Regarding that now $\bar{u} > u^*$, we solve $\bar{u} = 1 \Big/ \left\{ 1 + \left(\frac{1-\bar{u}}{\bar{u}} \right)^{1-\rho} \cdot \exp\left[b \cdot \left(B - A \cdot (1-\alpha) \cdot (\overline{\omega} / \bar{u})^{\alpha} + z \cdot (\bar{u} - u^*)^2 / (1-\bar{u}) \right] \right\}$. From this expression we derive the constraint *z*=13.45 (see the corollary of proposition 4 in the previous section). This case is feasible and it establishes a direct relation between the constant result for the human capital share and the highest possible transfer value.

Matrix *J* is in this particular case the following: $J = \begin{bmatrix} 0.625 & 1.086 \\ 0.048b & 1 - \rho - 2.864b \end{bmatrix}$. The corresponding trace and determinant are: $Tr(J) = 1.625 - \rho - 2.864b$ and $Det(J) = 0.625 - 0.625\rho - 1.842b$.

From our definition of parameters, we have naturally imposed the boundaries *b*>0 and ρ>0. Applying these conditions over the previous expressions for the trace and the determinant, one immediately restricts the dynamic analysis of the model to the set $-0.419 + 0.643 \cdot Tr(J) < Det(J) < -0.391 + 0.625 \cdot Tr(J)$. This region can be represented graphically in a diagram relating the trace and the determinant, as shown in figure 1.

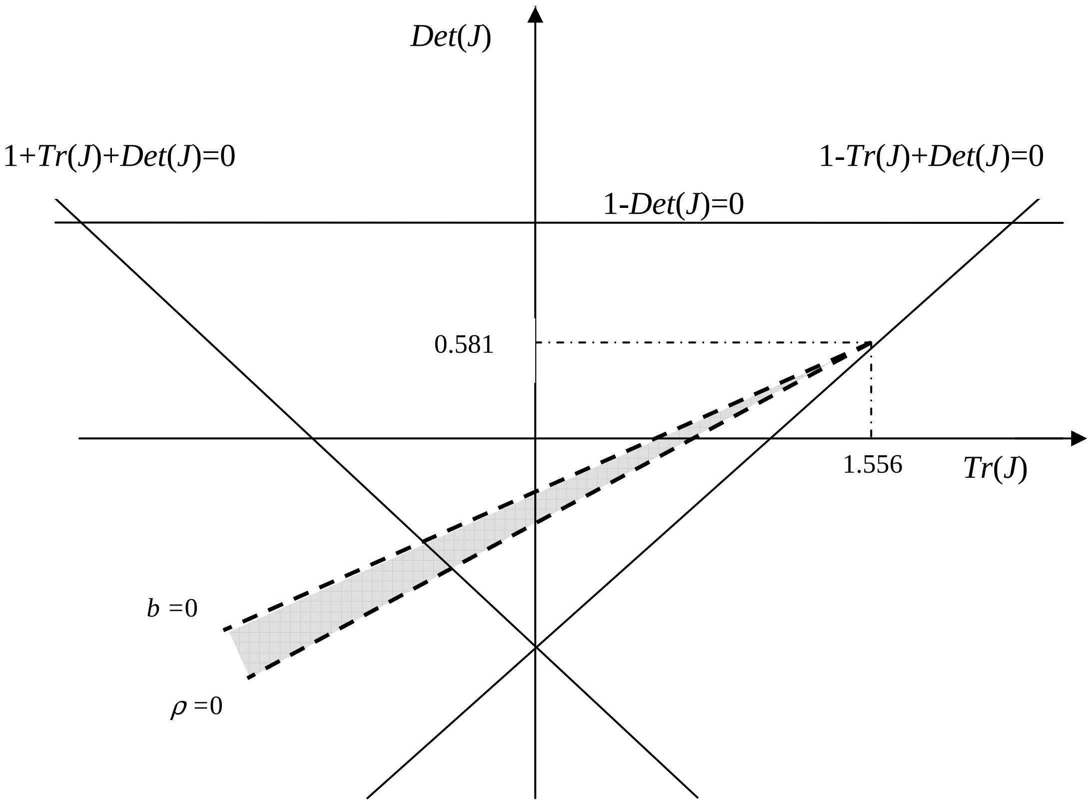

Figure 1 – Local dynamics in the case $\bar{u} = 0.5$.

With figure 1, we realize that under positive parameter values the possible local dynamic results become constrained to a relatively small area of the Trace – Determinant space. The lines b=0 and ρ=0 intersect in point [$Tr(J)$,$Det(J)$]=(1.556;0.581); the intensity of choice is positive to the right of the line b=0, while a positive memory loss parameter can be found in the area above (to the left of) ρ=0. The intersection of the areas where parameters assume positive values is given by the grey region. Therefore, only two different qualitative dynamic outcomes are admissible, and these are separated by a flip bifurcation line.

In the graphic of figure 1, three bifurcation lines are presented (each line reflects a different kind of bifurcation). Inside the inverted triangle formed by the bifurcation lines one has fixed-point stability (i.e., both eigenvalues of the Jacobian matrix rest inside the unit circle). Hence, for some combinations of b and ρ, stability is found.

The bifurcation line 1+$Tr(J)$+$Det(J)$=0 will represent a flip bifurcation (over this line, one of the eigenvalues remains inside the unit circle and the other will become equal to -1). The other possible qualitative result (to the left of this bifurcation line) corresponds to saddle-path stability (one eigenvalue inside the unit circle and the other one outside); thus, there are pairs of positive values b and ρ that can only guarantee stability, from a local analysis point of view, if the one dimensional saddle trajectory is followed from a given initial state. Note that any other possible bifurcation and qualitative result in the steady state neighbourhood is not admissible, as one immediately understands by observing figure 1.

Some computation allows withdrawing information about the flip bifurcation. When b=0, the bifurcation occurs at point [$Tr(J)$,$Det(J)$]=(-0.646;-0.375); in this point ρ=1.951. When ρ=0, the bifurcation takes place for [$Tr(J)$,$Det(J)$]=(-0.625;-0.354); in this case, b=0.65. In general terms, the flip bifurcation line can be presented as a relation between the two parameters: b=0.691-0.345ρ, as long as these parameters stay inside the following bounds: 0<b<0.691 and 0<ρ<2.003. Thus, fixed-point stability requires b<0.691-0.345ρ, while saddle-path stability is evidenced for b>0.691-0.345ρ.

To better understand local dynamics, consider now a reasonable value for the memory loss rate, e.g. ρ=0.1. In this specific case, the relation between the trace and the determinant is: $Det(J) = -0.417 + 0.643Tr(J)$. The above expressions for the trace and the determinant allow us to understand that when b=0 in this particular example, the system is located in point [$Tr(J)$,$Det(J)$]=(1.525;0.563); as b increases, we follow the above relation between trace and determinant from the right to the left; the bifurcation point is reached when b=0.657, that is, fixed-point stability is obtained for b<0.657; for b higher than this value, saddle-path stability is observed. Therefore, in this case it is straightforward to conclude that the lower the intensity of choice concerning the allocation of human capital to alternative sectors, the higher is the probability of the system to converge to the steady state.

Another example may be assumed for a changing ρ and, e.g., b=0.5. In this case, $Det(J) = -0.417 + 0.625Tr(J)$ is the line that defines local dynamics. Now, ρ=0 implies considering the following point: [$Tr(J)$,$Det(J)$]=(0.139;-0.296). As the memory loss value rises, we will move from the right to the left over the derived relation between trace and determinant; again, a first segment of the line relates to fixed-point stability (low values of ρ). Once the bifurcation is passed (for relatively high values of the memory loss parameter), one observes, once again, saddle-path stability. The exact value of the parameter to which the bifurcation occurs is ρ=0.554. Accordingly to previous reasoning, ρ<0.554 implies fixed-point stability.

The two examples, for concrete values of the two parameters that we have focused on, allow to conclude that in the case under consideration ($\bar{u} = 0.5$), for low values of the referred parameters stability holds, but as any of these values becomes relatively high saddle-path stability will tend to prevail. This can be confirmed by looking at the graphical representation of the referred lines in figure 2.

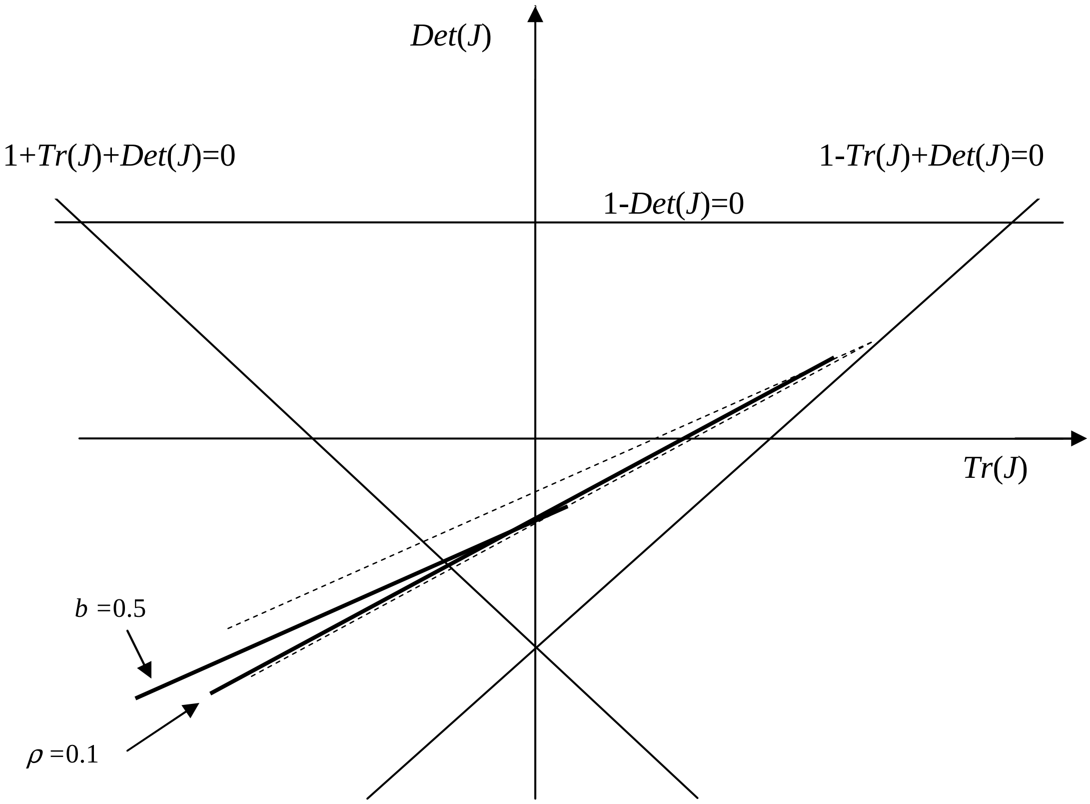

Figure 2 – Local dynamics in the case $\bar{u} = 0.5$; two particular examples (ρ=0.1, *b*=0.5).

Figure 2 indicates that any specific case where only one parameter varies is a case where low values of such parameter implies stability until the bifurcation occurs; then, we will have a result that locally is identified as saddle-path stability, but as we shall see in the global dynamic analysis corresponds to a region where a period doubling bifurcation leads to cycles of several periodicities and eventually chaos, before instability becomes dominant.

Case 3: $\bar{u} = 0.75$. As in the second case, $\bar{u} > u^*$, but now the condition that leads to the proposed value of the steady state human capital share is not so straightforward as before. By solving the same equation as in case 2, we obtain the relation $1.099\rho - 0.828b + 0.719bz = 0$, that is, the transfer parameter must be such that $z = 1.152 - 1.528\rho / b$. For parameter *z* to be positive, we have to guarantee the veracity of condition $b > 0.754\rho$.

The Jacobian matrix will be, in the present case, $J=\begin{bmatrix} 0.638 & 1.251 \\ 0.022b & 1+0.881\rho-1.478b \end{bmatrix}$. The trace and determinant come: $Tr(J)=1.638+0.881\rho-1.478b$ and $Det(J)=0.638+0.562\rho-0.97b$.

In the present case three constraints upon parameters are relevant: $b>0 \Rightarrow Det(J)<-0.407+0.638Tr(J)$; $\rho>0 \Rightarrow Det(J)<-0.437+0.657Tr(J)$; and $b>0.754\rho \Rightarrow Det(J)>-0.55+0.725Tr(J)$. Note that $b=0$, $\rho=0$ and $b=0.754\rho$ imply $[Tr(J),Det(J)]=(1.638;0.638)$, a point that is over the bifurcation line $1-Tr(J)+Det(J)=0$; thus, given the above inequality conditions, feasible dynamic results can be encountered only for values of the trace and the determinant of *J* to the left of such bifurcation line. Therefore, one identifies a same type of dynamics as in the previous example, i.e., for some values of *b* and ρ both eigenvalues of the Jacobian matrix lie inside the unit circle; a flip bifurcation line is eventually crossed, and afterwards one of the eigenvalues assumes a value lower than one with the other staying inside the unit circle. Figure 3 sketches the underlying dynamics.

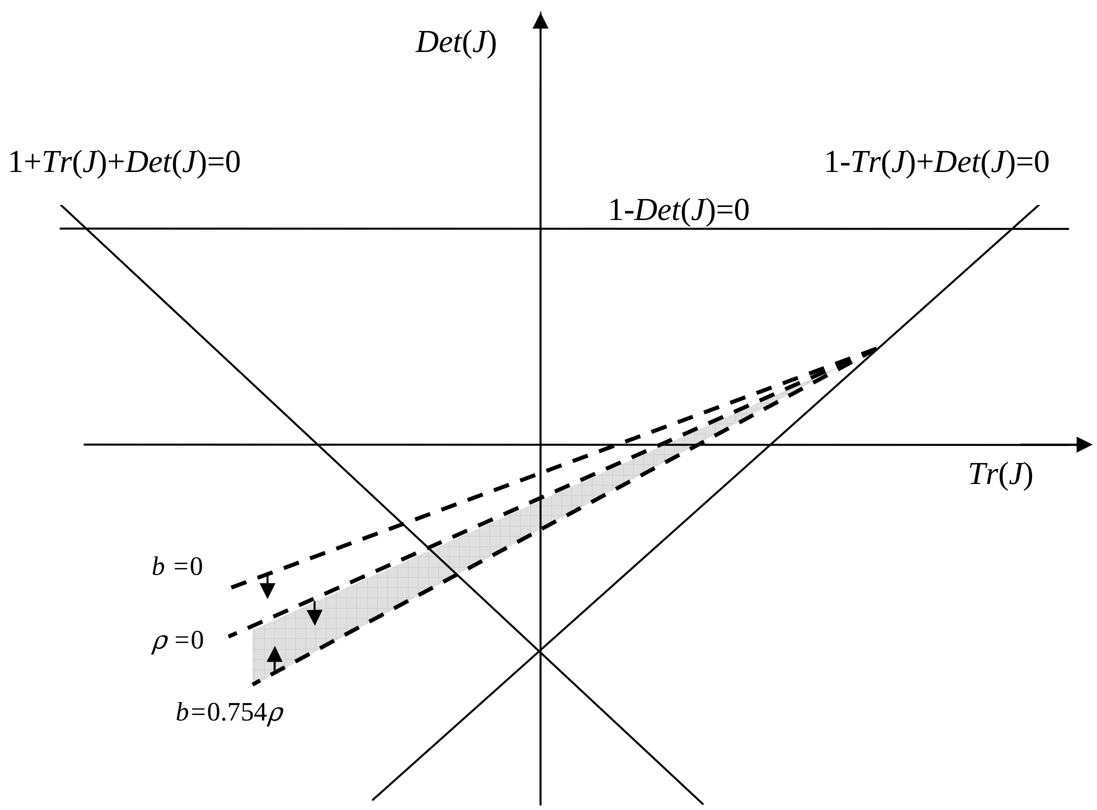

Figure 3 – Local dynamics in the case $\bar{u}=0.75$.

In figure 3, the grey area is the intersection of the three inequalities presented above. As one observes, the condition $b>0$ becomes irrelevant, and the area of admissible dynamic results will restrict to the one confined by the other two conditions.

The flip bifurcation line will correspond to the following relation: $b=1.338+0.589\rho$, which is obtained by replacing the expressions of the trace and the determinant as functions of the two parameters in the equality $1+Tr(J)+Det(J)=0$. Fixed-point stability requires $b<1.338+0.589\rho$ and saddle-path stability is evidenced for $b>1.338+0.589\rho$. Note that the referred bifurcation line is relevant only for $1.338<b<6.114$ and $0<\rho<8.109$.

As in case 2, we reemphasize the dynamic results presenting the line that defines the dynamic behaviour of the system for two feasible examples: $\rho=0.1$ and $b=1.5$. Assuming $\rho=0.1$, the local dynamics in the trace-determinant referential are given by expression $Det(J)=-0.439+0.657Tr(J)$; assuming $b=1.5$, dynamics are characterized by $Det(J)=-0.469+0.638Tr(J)$. These relations do not differ qualitatively much from what one as seen in case 2 ($\bar{u}=0.5$): the first line corresponds to the possible values that b may assume (it starts inside the unit circle, when the parameter's value is zero and it crosses the flip bifurcation when $b=1.397$); the second line refers to the dynamics of a changing ρ (in this case, the bifurcation point is $\rho=0.275$, but now the dynamics flow in the opposite direction: $\rho=0$ corresponds to point $[Tr(J),Det(J)]=(-1.684;-1.68)$, that is for low values of the memory parameter saddle-path stability holds, while for $0.275<\rho<1.989$, fixed-point stability prevails.

Figure 4 presents the two discussed lines.

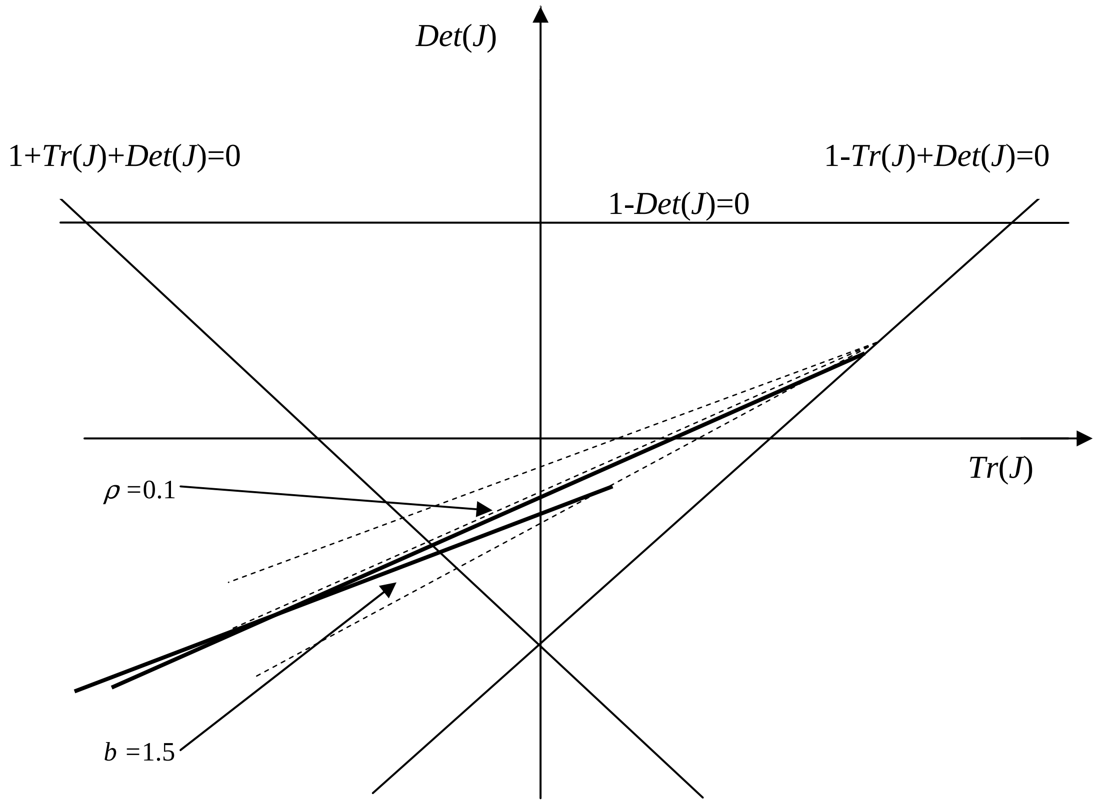

Figure 4 – Local dynamics in the case $\bar{u} = 0.75$; two particular examples (ρ=0.1, *b*=0.5).

The third case differs relatively to the second because now the policy parameter *z* is not a constant but a value that relates to the other two assumed parameters. To better understand the role of this parameter (the only one that the government may manipulate directly) we transform the previous graphical analysis in an analysis that focus on the space of parameters.

Figure 5 displays the three bifurcation lines [1+*Tr*(*J*)+*Det*(*J*)=0 $\Rightarrow$ b=1.338+0.589ρ; 1-*Tr*(*J*)+*Det*(*J*)=0 $\Rightarrow$ b=0.628ρ; 1-*Det*(*J*)=0 $\Rightarrow$ b=-0.373+0.579ρ] for admissible values of the parameters *b*, ρ and *z*. As we have referred previously, condition 1-*Det*(*J*)=0 is outside the bounds of the parameter's values, condition 1-*Tr*(*J*)+*Det*(*J*)=0 just establishes a border, and therefore, the only true bifurcation is the one triggered by relation 1+*Tr*(*J*)+*Det*(*J*)=0. The essential point in figure 5 is that as we increase the value of the policy parameter (i.e., as the public authorities increase the amount of transfer between agents that allocate their human capital to different sectors), the eventuality of getting a fixed-point stable dynamic outcome decreases for identical values of the other two parameters.

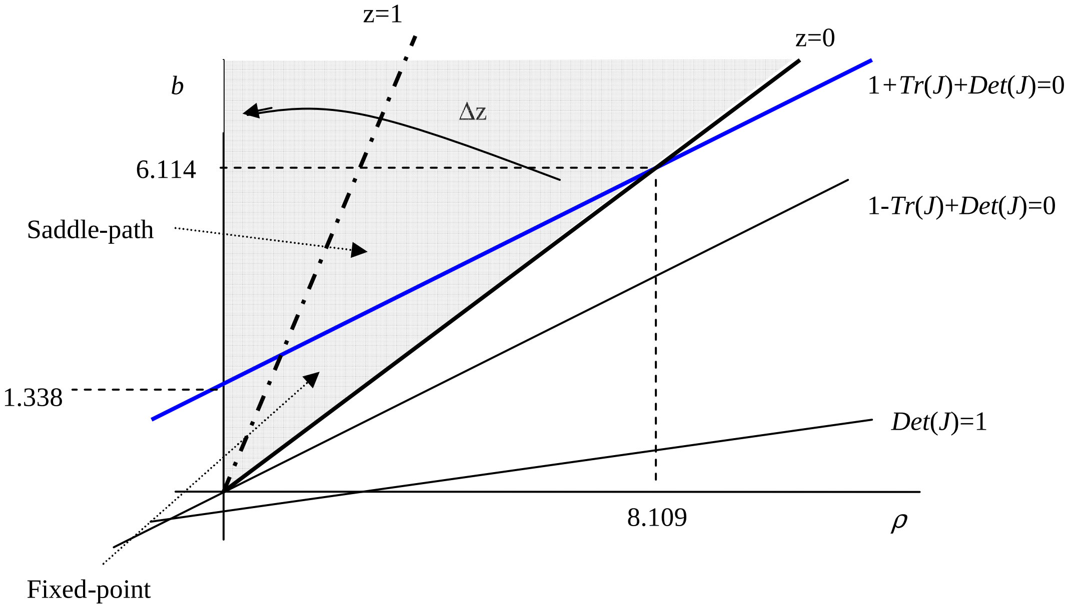

Figure 5 – Local dynamics in the case $\bar{u} = 0.75$; parameters' dynamics.

6. Global Dynamics

We can address global dynamics only through numerical examples. We consider the same set of parameter values as in the previous section and explore three examples. The first two intend to analyze further cases 2 and 3 of the last section. The third example does not impose any constraint on the relation between parameters ρ, b and z and thus the steady state value of share u_t can be a function of these parameters. For each example, we make essentially a graphical analysis, presenting bifurcation diagrams, the basin of attraction and attractors for the cases in which strange dynamics occur. As stated earlier, the bifurcation implies the passage of a stability result to a different dynamic outcome, that locally corresponds to a saddle-path equilibrium but that one will realize through the global dynamic analysis to correspond to a zone of period doubling cycles.

Example 1: $\bar{u} = 0.5$. We have seen that this case implies z=13.45. The result in figure 1 must now be compared with the bifurcation diagrams in figures 6 and 7.[3] We restrict the analysis to a varying b. Figures 6 and 7 consider two extreme cases for the

[3] These figures, and all the following, are drawn using IDMC software (interactive Dynamical Model Calculator). This is a free software program available at www.dss.uniud.it/nonlinear, and copyright of Marji Lines and Alfredo Medio.

memory loss parameter: the boundary case, ρ=0, and ρ=1, which represents a memory loss of 100%.

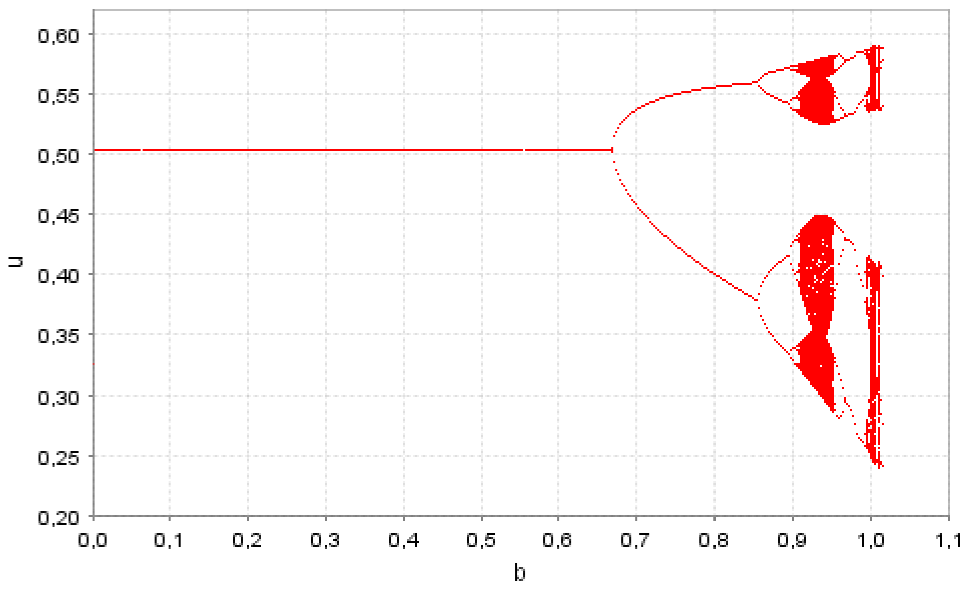

Figure 6 – Bifurcation diagram (b,u_t) for ρ=0 and z=13.45.

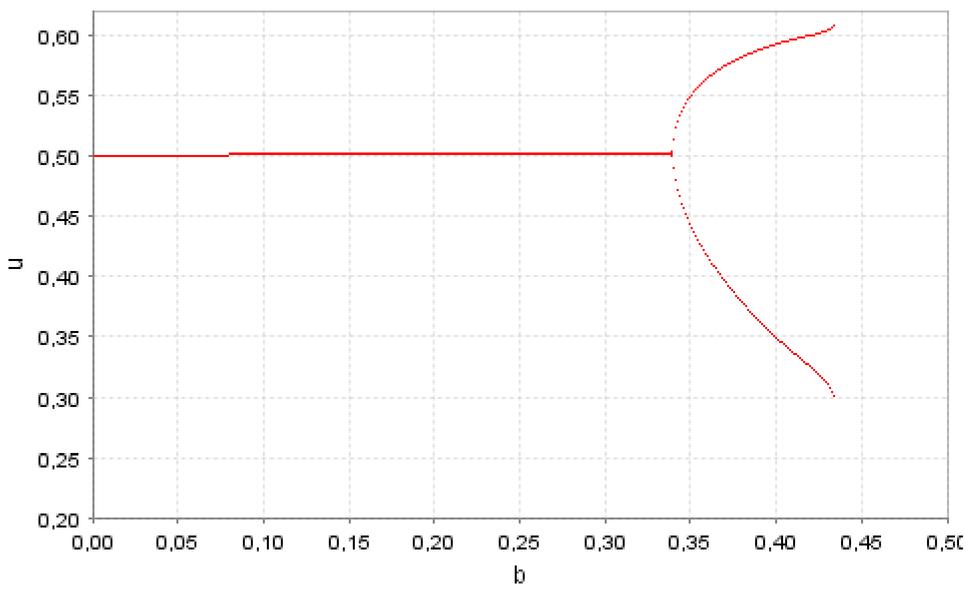

Figure 7 – Bifurcation diagram (b,u_t) for ρ=1 and z=13.45.

In both figures, 6 and 7, we confirm the fixed-point stability result that one has found in the local analysis for low values of b (lower than the value that originates a bifurcation); recall that under the study of local dynamics, the bifurcation produced the transition to a saddle-path stable outcome, meaning that unless we choose initial values that are exactly located over the saddle trajectory, instability will prevail. The global analysis indicates that the stable area is the same, but after the bifurcation we will not have immediately instability; before one reaches the unstable outcome a period doubling process of bifurcations may lead to strange dynamics (chaotic motion), in the case if we consider low values of ρ. For high levels of the memory loss parameter the bifurcation only produces a period two cycle, while low levels of ρ imply higher periodicity and even regions of chaotic motion (we only display two extreme cases, but one finds the possibility of cycles of multiple periodicities for several values of ρ; below we present an example for a memory loss rate of 10%). Chaos can be associated with sensitive dependence on initial conditions (SDIC); SDIC characterizes a deterministic system where nearby orbits tend to diverge exponentially, producing long run strange attractors.

Figure 8 presents a strange attractor concerning the system under analysis for a combination of parameters where chaotic motion can be identified; in the particular case, we consider ρ=0.1 and b=0.97 (we maintain z=13.45).

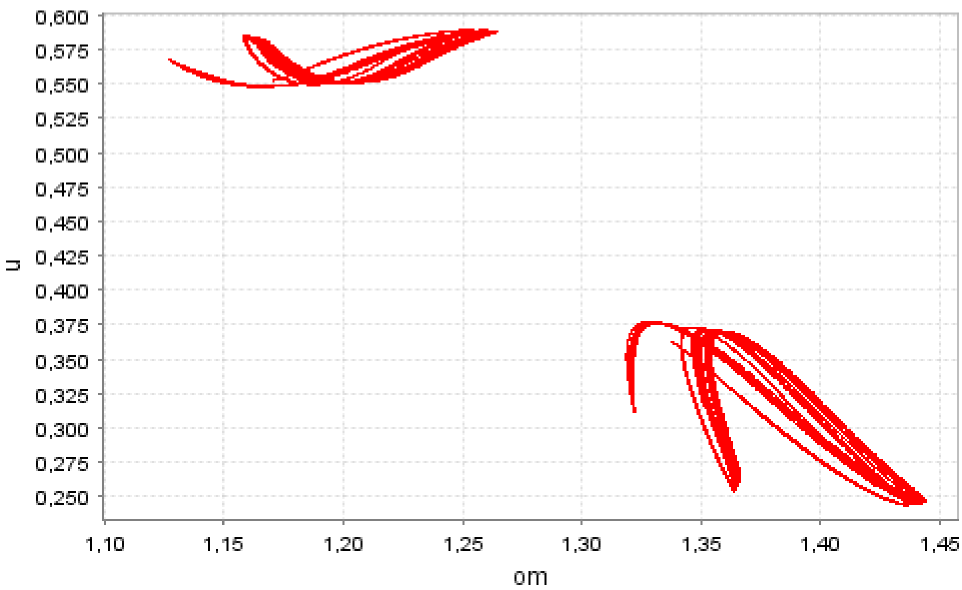

Figure 8 – Strange attractor (ω_t,u_t) for ρ=0.1, b=0.97 and z=13.45.

To confirm the chaotic nature of the system under the selected combination of parameters, we present the long term time trajectories of both endogenous variables (figures 9 and 10). We conclude that there is no convergence to a periodic long run path; this is indeed chaotic.

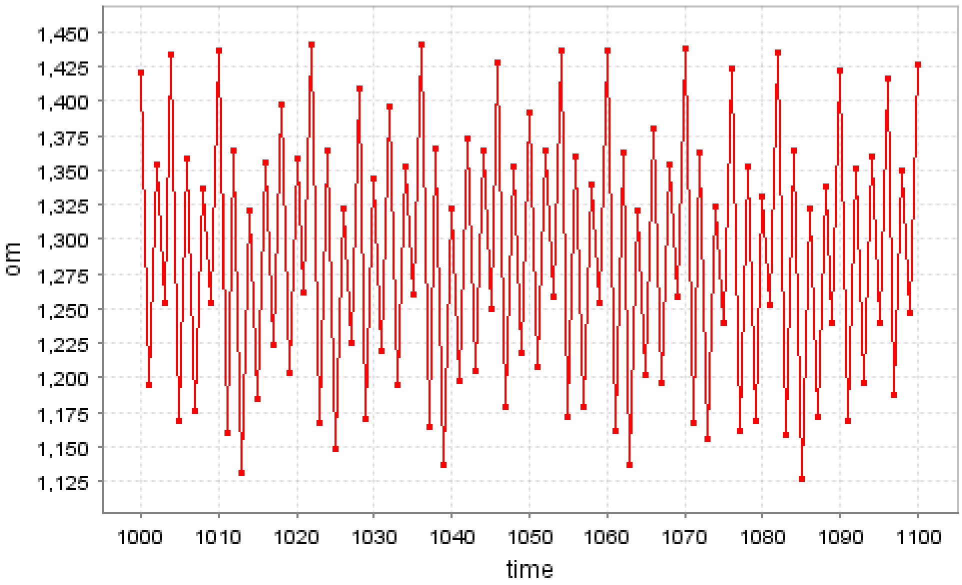

Figure 9 – Time path of ω_t for ρ=0.1, b=0.97 and z=13.45.

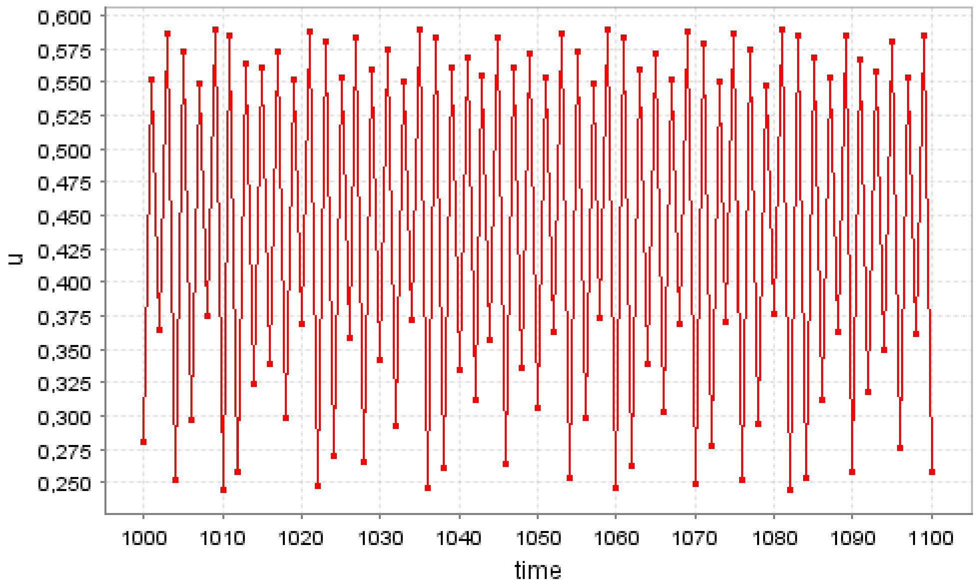

Figure 10 – Time path of u_t for ρ=0.1, b=0.97 and z=13.45.

The graphical illustration of this case ends with the presentation of the basin of attraction (figure 11), that is, the set of initial points (ω_0,u_0) from which the convergence to the chaotic attractor is possible. With this figure, one realizes that not all pairs (ω_0,u_0) allow to converge to the attractor; however, the values that we have chosen, (ω_0,u_0)=(0.756;0.326) are inside the basin of attraction.

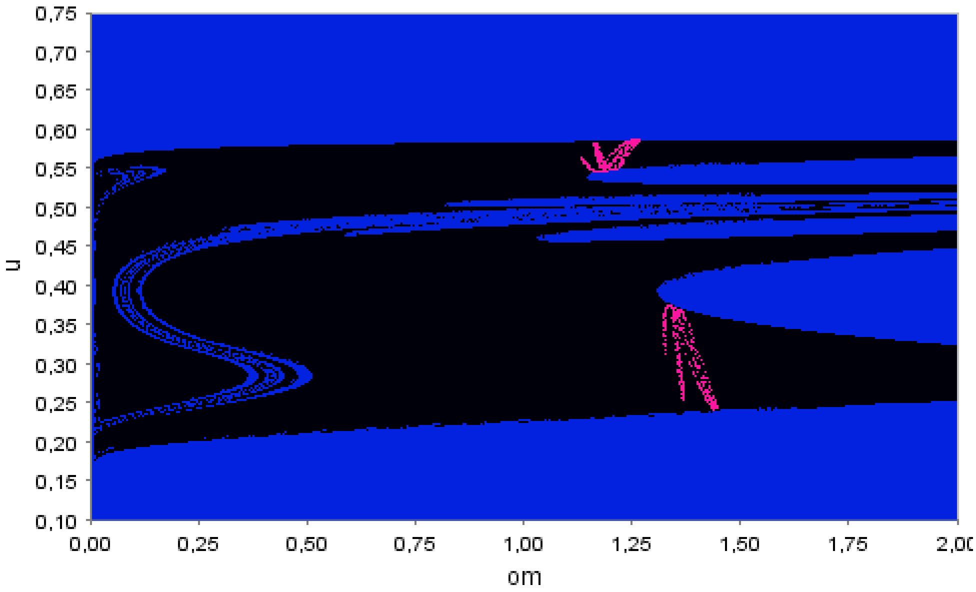

Figure 11 – Basin of attraction (ρ=0.1, *b*=0.97 and *z*=13.45).

This first example was helpful in illustrating how the bifurcation produces changes in the qualitative behaviour of the system. The system undergoes a flip bifurcation, and as a result endogenous fluctuations arise; these fluctuations can take exclusively the form of period two cycles, but for low memory loss values, the fluctuations take many eventual periodicities including complete a-periodicity. From an economic point of view, the main implication is that the two-sector endogenous growth model exhibits endogenous cycles and therefore we are in the presence of a framework where decentralized decisions in a market economy generate business cycles without the need of assuming external stochastic shocks, as in the RBC literature.

Example 2: $\bar{u} = 0.75$. For this case, one has imposed the constraint $z = 1.152 - 1.528\rho / b$. Now, we characterize the global dynamics of the system for this constraint, letting b be the bifurcation parameter. As in the previous example, we present bifurcation diagrams and a strange attractor for a given combination of parameters. Since the results are qualitatively close to the ones in the previous case, we omit the presentation of the time series and of the basin of attraction.

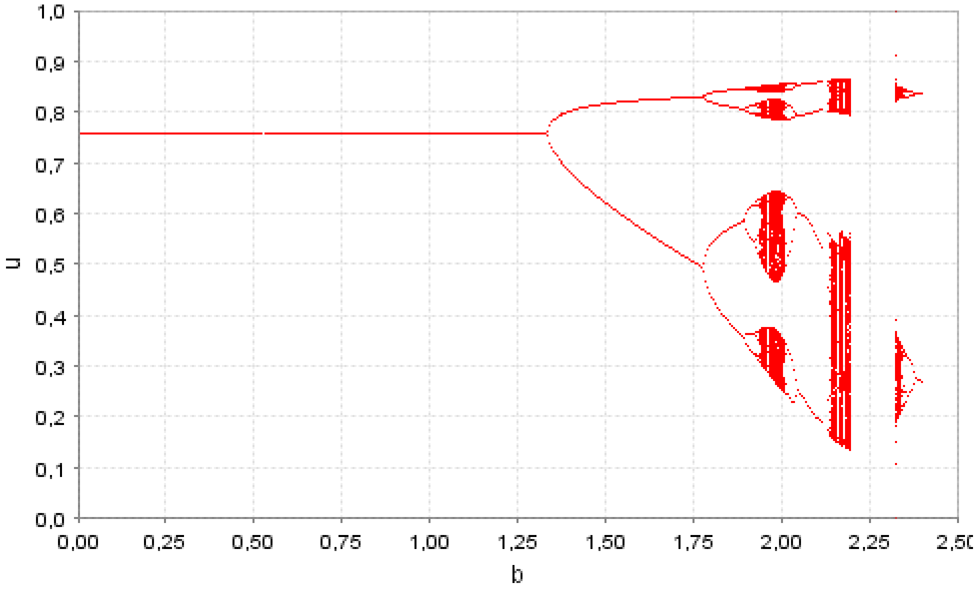

Figure 12 – Bifurcation diagram (b,u_t) for ρ=0 and $z = 1.152 - 1.528\rho / b$.

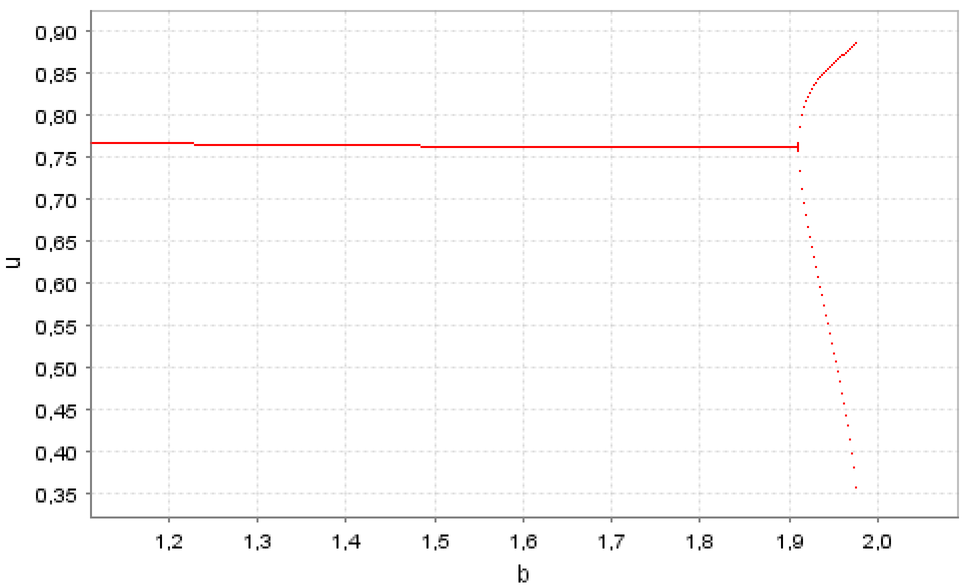

Figure 13 – Bifurcation diagram (b,u_t) for ρ=1 and $z = 1.152 - 1.528\rho / b$.

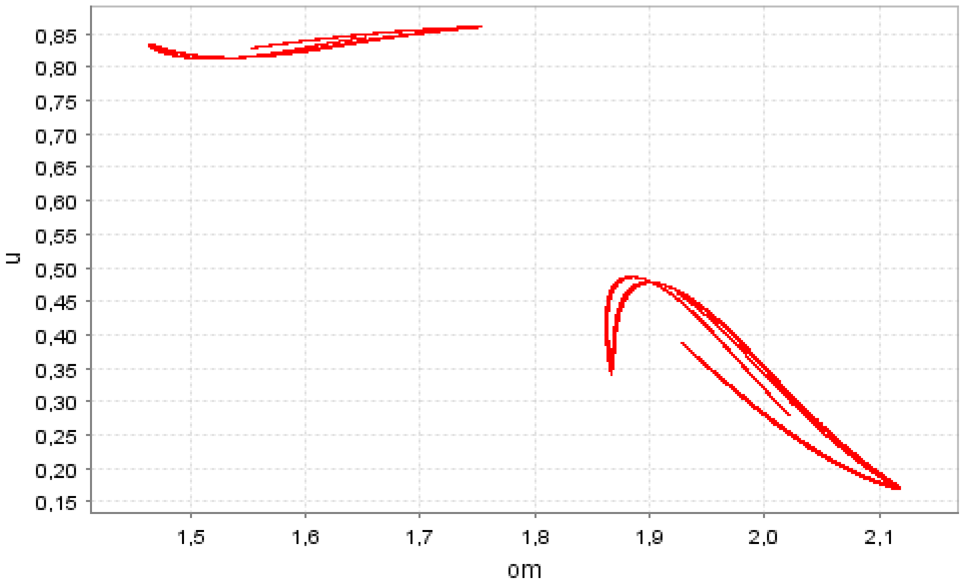

Figure 14 – Strange attractor (ω_t,u_t) for ρ=0.05, b=2.17 and $z=1.152-1.528\rho/b$.

Qualitatively, this second example does not differ significantly from the first. Even the attractors in figures 8 and 14 are similar. Once again, we observe that low levels of the memory loss value can lead to endogenous fluctuations of multiple types.

Example 3: No constraint on the relation between parameters ρ, b and z. This is a general case that the local analysis was unable to capture. Just to present an example, that, however, is not qualitatively too different from the previous ones, we consider ρ=0.1 and z=1.5, to present the bifurcation diagram in figure 15.

In this situation, the bifurcation occurs around b=1.27, and this gives place once again to cycles of various periodicities. Chaotic motion exists for instance when b=2, as one can confirm looking at the attractor in figure 16.

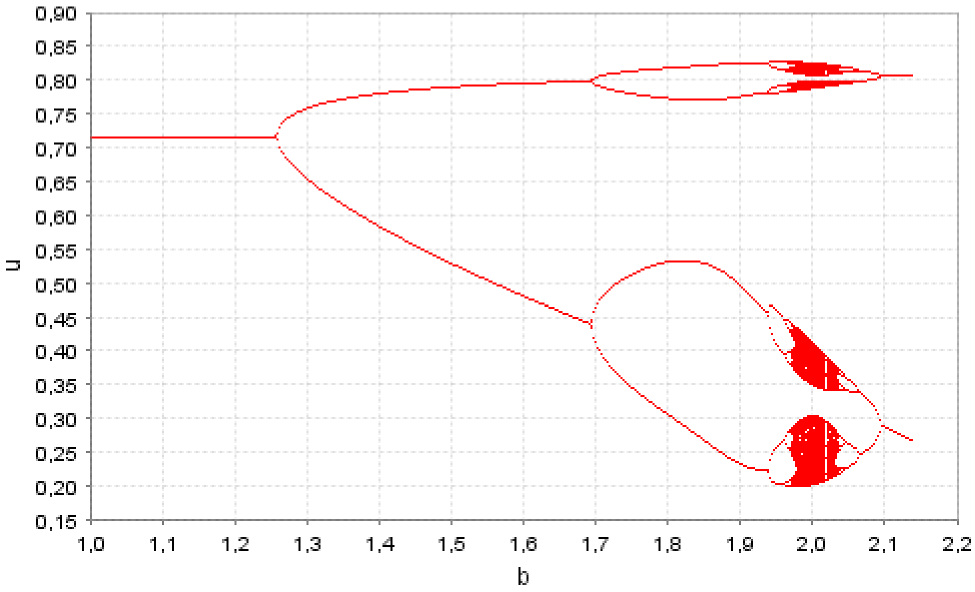

Figure 15 – Bifurcation diagram (b,u_t) for ρ=0.1 and z=1.5 (unconstrained case).

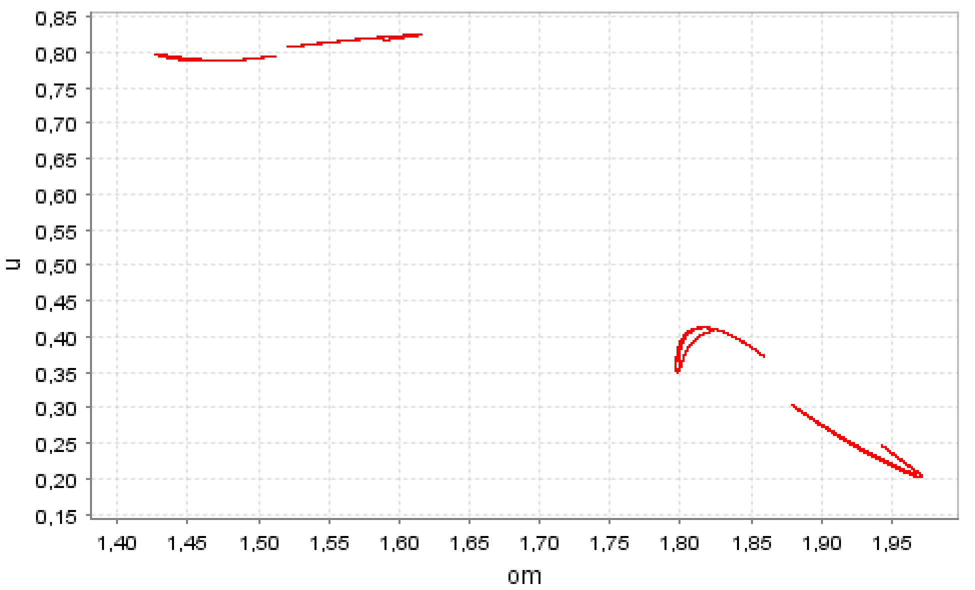

Figure 16 – Strange attractor (ω,u_t) for ρ=0.1, b=2 and z=1.5.

Implications over growth: The three examples that we have just presented share a same qualitative behaviour (flip bifurcation, and possibility of cycles in the region locally identified as saddle-path stable); hence, we will take just one example (the third) to graphically address long term growth.

When presenting the representative agent optimal growth model in section 2, we have stated that, given the state constraints, all the per capita aggregates would grow in the long run at an identical rate, which is $\gamma = B \cdot (1-\bar{u}) - \delta$. In the representative agent setup this would always be a constant growth rate because no endogenous fluctuations could be observed. Now, for example with the values of parameters used to draw figure

16, the long term economy's growth rate is not constant (recall that it was 4%, for our parameter values), but fluctuates around this value; economic growth is subject to business cycles generated endogenously; furthermore, the model does not lose its property of endogenous growth: growth continues to be positive in average in the long run. Figure 17 presents the time path of the growth rate.

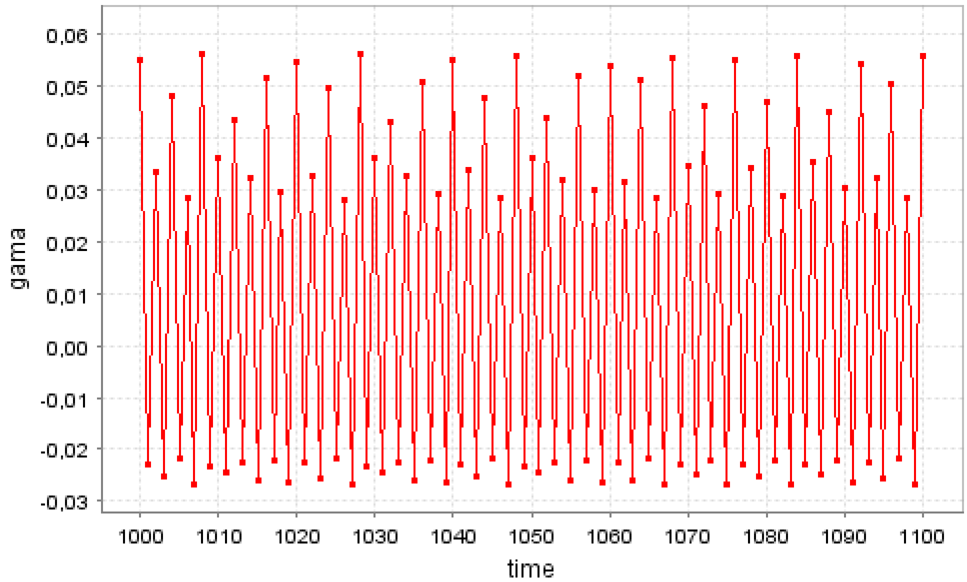

Figure 17 – Long term economy's per capita growth rate for ρ=0.1, b=2 and z=1.5.

7. Final Comments

In this paper, we have proposed a model of endogenous growth and endogenous business cycles. The starting point was the two-sector Uzawa-Lucas growth model with physical and human capital produced with distinct technologies. We have argued that individual agents do not think or act collectively and, thus, it is incorrect to take a representative agent perspective on the choice of the agents concerning the allocation of their skills to the production of final goods or additional skills.

The proposed environment considers three forces that determine the allocation of human capital in a decentralized framework. First, individuals respond to market rewards (i.e., they will choose to work in the sector that pays them more); second, a central planner (i.e., the government) redistributes income in order to obtain a social result that is closer to the optimal social outcome; third, private agents are boundedly rational, what in the present setup means that they are sluggish to adjust: they can take into account some other non modelled factors that make them stay in one sector even if the other sector attributes better productive rewards (cultural and social factors may imply a low intensity of choice, and therefore workers are not stimulated only through strict economic incentives).

The new features introduce strong nonlinearities in the growth setup, and thus a generic dynamic analysis is not feasible. Through a group of suggestive numerical examples we have derived a result that seems to hold in most circumstances: taking the intensity of choice as the bifurcation parameter, a low intensity of choice seems to guarantee stability (the system converges to a fixed point result); stability breaks down with a flip bifurcation that gives place to cycles. Depending on the values of other parameters, these cycles can be only of periodicity two, but one has observed that if the memory loss parameter is close to zero high periodicity and chaotic motion arise.

The main conclusion is then the following: through an assumption about human capital allocation that seems more realistic than the one underlying the representative agent setup, we have generated a dynamic system that allows for multiple types of results concerning the dynamic behaviour of the endogenous variables. The presence of business cycles affecting the long term growth rate of the economy should be particularly stressed.

Furthermore, the results can be addressed in a stabilization policy point of view; according to the found dynamics, a public policy aimed at reducing the periodicity of cycles or aimed at eliminating the deterministic component of cycles will be a policy that influences agents to lower their intensity of choice; this seems intuitively correct since a low intensity of choice makes individuals to stay a longer period attached to a sector, what reduces changes in the economy, and thus economic instability. Note, as well, that the policy parameter that authorities directly control is parameter z; by changing the amount of transfers, stabilization can also be subject to analysis.

Appendices

Appendix 1 – Proof of proposition 1.

Defining the steady state as the pair $(\overline{\omega},\overline{u})$ for which $\omega_{t+1}=\omega_t$ and $u_{t+1}=u_t$, equation (3) allows us to establish the relation $\overline{\omega}=\left[\frac{A}{\psi^*+B\cdot(1-\overline{u})}\right]^{1/(1-\alpha)}\cdot\overline{u}$, which is equivalent to $\overline{\omega}=\left[\frac{A}{B}\cdot\frac{\alpha}{1-\alpha\cdot(\overline{u}-u^*)}\right]^{1/(1-\alpha)}\cdot\overline{u}$. Thus, if $\overline{u}$ exists, then $\overline{\omega}$ will exist in precisely the same number. Replacing the above expression in (13), one obtains the following steady state result,

$$\bar{u} = 1 \Bigg/ \left\{ 1 + \left(\frac{1-\bar{u}}{\bar{u}} \right)^{1-\rho} \cdot \exp\left[b \cdot \left(B - (1-\alpha) \cdot A^{1/(1-\alpha)} \cdot B^{-\alpha/(1-\alpha)} \cdot \left[\frac{\alpha}{1-\alpha \cdot (\bar{u}-u^*)} \right]^{\alpha/(1-\alpha)} - z \cdot \frac{(\bar{u}-u^*)^2}{\bar{u}} \right) \right] \right\}$$

, if $\bar{u} < u^*$; and,

$$\bar{u} = 1 \Bigg/ \left\{ 1 + \left(\frac{1-\bar{u}}{\bar{u}} \right)^{1-\rho} \cdot \exp\left[b \cdot \left(B - (1-\alpha) \cdot A^{1/(1-\alpha)} \cdot B^{-\alpha/(1-\alpha)} \cdot \left[\frac{\alpha}{1-\alpha \cdot (\bar{u}-u^*)} \right]^{\alpha/(1-\alpha)} + z \cdot \frac{(\bar{u}-u^*)^2}{1-\bar{u}} \right) \right] \right\},$$

if $\bar{u} > u^*$.

Let

$$f^{I}(\bar{u}) = \ln\left(\frac{1-\bar{u}}{\bar{u}} \right) \cdot \rho,$$

$$f^{II}(\bar{u}) = bB \cdot \left[1 - (1-\alpha) \cdot \left(\frac{A}{B} \right)^{1/(1-\alpha)} \cdot \left(\frac{\alpha}{1-\alpha \cdot (\bar{u}-u^*)} \right)^{\alpha/(1-\alpha)} \right],$$

$$f^{III}{}_{(-)}(\bar{u}) = \frac{(\bar{u}-u^*)^2}{\bar{u}} \cdot bz,$$

$$f^{III}{}_{(+)}(\bar{u}) = -\frac{(\bar{u}-u^*)^2}{1-\bar{u}} \cdot bz.$$

We can write the balanced growth expression as $f^{I}(\bar{u}) + f^{III}(\bar{u}) = f^{II}(\bar{u})$, with $f^{III}(\bar{u}) = f^{III}{}_{(-)}(\bar{u})$ if $\bar{u} < u^*$ and $f^{III}(\bar{u}) = f^{III}{}_{(+)}(\bar{u})$ if $\bar{u} > u^*$. Relatively to each one of the defined functions, the following derivatives are computed:

$$f^{I}(\bar{u})' = -\frac{1}{\bar{u} \cdot (1-\bar{u})} \cdot \rho \quad [<0, \forall \bar{u} \in (0,1)];$$

$$f^{I}(\bar{u})'' = -\frac{2 \cdot \bar{u} - 1}{\bar{u} \cdot (1-\bar{u})} \cdot \rho \quad [>0, \text{if } \bar{u} < 0.5;\ <0, \text{if } \bar{u} > 0.5];$$

$$f^{II}(\bar{u})' = -b \cdot \alpha^{\frac{2-\alpha}{1-\alpha}} \cdot B^{\frac{\alpha}{1-\alpha}} \cdot A^{\frac{1}{1-\alpha}} \cdot \left[1 - \alpha \cdot (\bar{u}-u^*) \right]^{-\frac{1}{1-\alpha}} \quad [<0, \forall \bar{u} \in (0,1)];$$

$$f^{II}(\bar{u})'' = -b \cdot \frac{\alpha^{\frac{3-2\alpha}{1-\alpha}}}{1-\alpha} \cdot B^{\frac{\alpha}{1-\alpha}} \cdot A^{\frac{1}{1-\alpha}} \cdot \left[1 - \alpha \cdot (\bar{u}-u^*) \right]^{-\frac{2-\alpha}{1-\alpha}} \quad [>0, \forall \bar{u} \in (0,1)];$$

$$f^{III}{}_{(-)}(\bar{u})' = \frac{\bar{u}^2 - u^{*2}}{\bar{u}} \cdot bz \quad [<0, \forall \bar{u} \in (0, u^*)];$$

$$f^{III}{}_{(+)}(\bar{u})' = -\frac{(\bar{u}-u^*) \cdot (2-\bar{u}-u^*)}{(1-\bar{u})^2} \cdot bz \quad [<0, \forall \bar{u} \in (u^*,1)];$$

$$f^{III}{}_{(-)}(\bar{u})'' = \frac{2u^{*2}}{\bar{u}^3}\cdot bz \;\; [>0, \forall \bar{u} \in (0,u^*)];$$

$$f^{III}{}_{(+)}(\bar{u})'' = -2\cdot\frac{(1-\bar{u})^2 + (\bar{u}-u^*)\cdot(2-\bar{u}-u^*)}{(1-\bar{u})^3}\cdot bz \;\; [<0, \forall \bar{u} \in (u^*,1)];$$

Given the previous computation, $f^I(\bar{u})$ is a decreasing function for all $\bar{u} \in (0,1)$, convex if $\bar{u} \in (0,0.5)$ and concave for $\bar{u} \in (0.5,1)$; note, as well, that the function assumes positive values for $\bar{u} \in (0,0.5)$ and negative values for $\bar{u} \in (0.5,1)$. Function $f^{II}(\bar{u})$ is a decreasing and convex function. Finally, function $f^{III}(\bar{u})$ is positive and convex in the case $\bar{u} < u^*$, and negative and concave otherwise; for all admissible values of $\bar{u}$ this is a decreasing function.

Furthermore, note that $\lim_{\bar{u}\to 0} f^I(\bar{u}) = +\infty$ and $\lim_{\bar{u}\to 1} f^I(\bar{u}) = -\infty$, and, likewise, $\lim_{\bar{u}\to 0} f^{III}(\bar{u}) = +\infty$ and $\lim_{\bar{u}\to 1} f^{III}(\bar{u}) = -\infty$. Relatively to $f^{II}(\bar{u})$, one can establish finite boundary values

$$f^{II}(0) = bB\cdot\left[1-(1-\alpha)\cdot\left(\frac{A}{B}\right)^{1/(1-\alpha)}\cdot\left(\frac{\alpha}{1+\alpha\cdot u^*}\right)^{\alpha/(1-\alpha)}\right]$$

$$f^{II}(1) = bB\cdot\left[1-(1-\alpha)\cdot\left(\frac{A}{B}\right)^{1/(1-\alpha)}\cdot\left(\frac{\alpha}{1-\alpha\cdot(1-u^*)}\right)^{\alpha/(1-\alpha)}\right]$$

Because $f^{II}(\bar{u})$ is a decreasing function, $f^{II}(0) > f^{II}(1)$.

If one takes the sum of $f^I(\bar{u})$ with $f^{III}(\bar{u})$, one obtains a decreasing function for all $\bar{u}$ that goes from +∞ to -∞. The first segment of this function is convex (at least until one reaches $\bar{u} = u^*$ or $\bar{u} = 0.5$, whatever comes first), and the second segment (starting somewhere between the reference values $\bar{u} = u^*$ and $\bar{u} = 0.5$) is concave. Recall the steady state is given by the point (or points) where function $f^I(\bar{u}) + f^{III}(\bar{u})$ intersects $f^{II}(\bar{u})$; given the properties presented above for each function, at least one intersection point will always be obtained. Situations where more than one steady state is obtained will require a large slope of $f^{II}(\bar{u})$. To illustrate this result, the following set of pictures depicts two possible situations: the first relating to the case of a unique steady state, the second for a three balanced growth paths outcome.

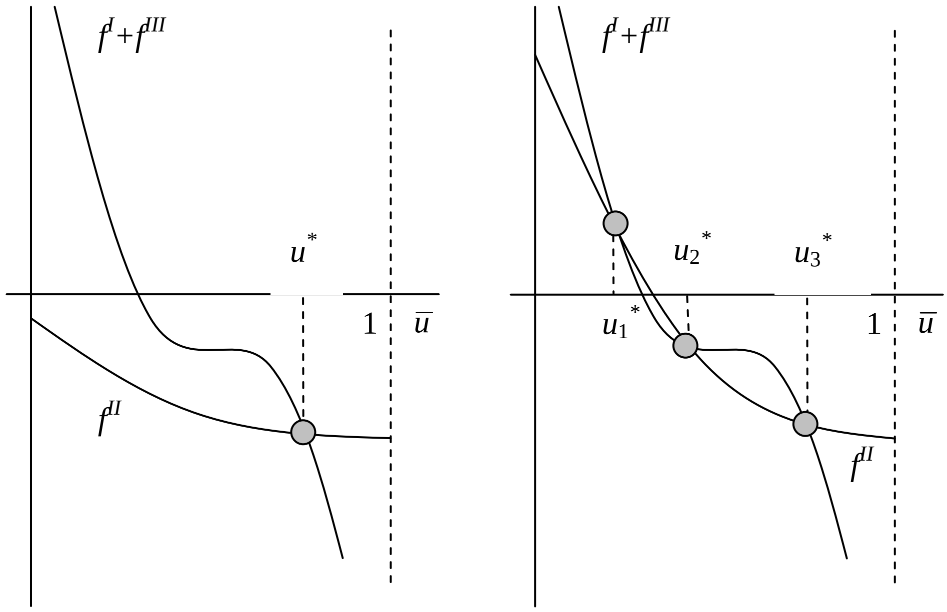

Figure A1 – Determination of the steady state■

Appendix 2 – Proof of the corollary of proposition 2.

One can present the steady state relation $f^I(\bar{u}) + f^{III}(\bar{u}) = f^{II}(\bar{u})$ in a way to highlight the role of the policy parameter. In particular, we have

$$z = \begin{cases} \dfrac{f^{II}(\bar{u}) - f^I(\bar{u})}{(\bar{u} - u^*)^2} \cdot \dfrac{\bar{u}}{b}, & \bar{u} < u^* \\ \dfrac{f^I(\bar{u}) - f^{II}(\bar{u})}{(\bar{u} - u^*)^2} \cdot \dfrac{1 - \bar{u}}{b}, & \bar{u} > u^* \end{cases}$$

We notice with the previous expression that as the government makes an effort to approximate $\bar{u}$ to u^*, it will have to increase z further and further. In the limit, $\lim_{\bar{u} \to u^{*+}} z = \lim_{\bar{u} \to u^{*-}} z = +\infty$ ■

Appendix 3 – Proof of proposition 3.

Replacing $\bar{u}$ by the value given in the proposition, function $f^{II}(\bar{u})$ defined in the proof of proposition 2 will be equal to zero for all $\bar{u} \in (0,1)$. Therefore, the steady state becomes the result of solving the equation $f^I(\bar{u}) = -f^{III}(\bar{u})$ (these functions were also defined in the proof of proposition 1). Knowing that $f^I(\bar{u})$ and $f^{III}(\bar{u})$ are both decreasing functions that tend to infinite values as $\bar{u}$ approaches 0 and 1, then $f^I(\bar{u})$ and $-f^{III}(\bar{u})$ can only intersect in one point, which defines the balanced growth solution $\bar{u}$ ■

Appendix 4 – Proof of proposition 4.

For $\bar{u} = \tilde{u}$, the steady state condition can be written as $\theta^{I} \cdot \rho + \theta^{II} \cdot b + \theta^{III} \cdot bz = 0$, with

$$\theta^{I} = \ln\left(\frac{1-\tilde{u}}{\tilde{u}}\right),$$

$$\theta^{II} = B \cdot \left[(1-\alpha) \cdot \left(\frac{A}{B}\right)^{1/(1-\alpha)} \cdot \left(\frac{\alpha}{1-\alpha \cdot (\tilde{u}-u^{*})}\right)^{\alpha/(1-\alpha)} - 1\right],$$

$$\theta^{III} = \begin{cases} \frac{(\tilde{u}-u^{*})^{2}}{\tilde{u}}, & \tilde{u} < u^{*} \\ -\frac{(\tilde{u}-u^{*})^{2}}{1-\tilde{u}}, & \tilde{u} > u^{*} \end{cases},$$

As it is evident, given the constraint on parameters $\rho, b, z > 0$, we cannot have θ^{I}, θ^{II} and θ^{III} with the same sign, otherwise the steady state condition cannot hold. This is precisely what the conditions in the proposition state■

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