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Abstract: In a pure exchange smooth economy with fixed total resources, we define the length between two regular equilibria belonging to the equilibrium manifold as the number of intersection points of the evolution path connecting them with the set of critical equilibria. We show that there exists a minimal path according to this definition of length.

Keywords: Equilibrium manifold, regular economies, critical equilibria, catastrophes, Jordan-Brouwer separation theorem.

JEL Classification: D50, D51, D52, D80.

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1 Introduction

The equilibrium manifold (E henceforth) is defined as the set of pairs of price vectors and endowments such that the aggregate excess demand function is equal to zero. The global and local topological structure of the set E has been deeply investigated by Balasko (see [1] and his monograph [4]). One of the global topological properties of this set, which we are concerned with in our analysis, is the arc-connectedness property. This property has a straightforward economic meaning: let ω and ω' be two m-tuples of initial endowments and let p and p' be two equilibrium price vectors associated with ω and ω', respectively. Then there exists a continuous modification $(\omega_1(t), \dots, \omega_m(t))$, $t \in [0, 1]$, from the m-tuple of initial endowments ω to ω' such that for every t there is an equilibrium price vector $p(t)$ associated with $(\omega_1(t), \dots, \omega_m(t))$ and $p(t)$ is a continuous function, $p(0) = p$ and $p(1) = p'$. Observe that there exist infinite trajectories connecting two given equilibria. Furthermore, the space E also enjoys the property of being simply connected, i.e., it is always possible to deform continuously a continuous trajectory linking two equilibria, (p, ω) and (p', ω'), to another one linking the same equilibria (see Section 4 in [1]) and still lying on E.

In his monograph ([4] p. 69) Balasko observed that, under the assumption that the initial and final equilibria (p, ω) and (p', ω') are exogenously determined, a continuous evolution path from the initial to the final state (p', ω') should be considered preferable, from an economic point of view, to any discontinuous one, being discontinuity synonymous of catastrophes. The motivation of our analysis is based on the natural question raised by Balasko ([4] p.70) of choosing one path to follow in the equilibrium manifold E to move from (p, ω) to (p', ω'). In this paper we tackle this question. As Balasko observed, one of the most natural ideas would be to minimize length (i.e. to choose the shortest path). But then a new problem arises: according to which metric can one define a distance on the equilibrium manifold? In order to avoid the complication arising from the definition of a metric economically meaningful (see [10, 9]), it can be observed that the same argument which has led to prefer a continuous evolution path also suggests us to choose "regular" paths, namely paths which "avoid" as much as possible the set of critical equilibria. In fact we recall that only the set of regular equilibria is characterized by the desirable economic properties of local uniqueness of equilibria, of the continuity of the equilibrium price correspondence and of the possibility of comparative statics analysis (see [6]). If the path should cross the set of critical equilibria, all these properties could be lost giving rise to *catastrophes* (see [2]). Therefore we can consider as a rough, but economically relevant, measure of the length of a path the number of intersection points of the path with the set of critical equilibria. We recall that the distance is defined as the infimum of the length of all curves connecting two points. Therefore, the minimal path (i.e., which minimizes distance) is identified with the one less catastrophic.

This consideration also suggests us to be concerned with the codimension one stratum of critical equilibria (S_1 henceforth). The reason is that S_1 is the only stratum of critical equilibria which can disconnect $E(r)$. This means that if the evolution path should link two regular equilibria belonging to two disconnected components of $E(r) \setminus S_1$, it would cross S_1. On the contrary, it is evident that there always exists a path connecting two regular equilibria belonging to the same connected component of $E(r) \setminus S_1$ which does not intersect the set of critical equilibria. A natural aspiration, when dealing with equilibria belonging to different connected components, would be to find a path containing at most one critical equilibrium. Unfortunately this event crucially depends on the structure of the set of critical equilibria $E_c(r)$ (see Theorem 2.1). In the more general case, being the set S_1 composed by a union of closed smooth manifolds, the equilibrium manifold is divided into several connected components. According to our definition of length, we can say that two equilibria belonging to the same connected component have distance zero, two points separated by only one stratum of critical equilibria have distance one, and so on...

In this paper we show that there exists a path connecting the equilibria which realizes the distance (see Theorem 3.7). As we will see, the possibility of defining the above distance deeply relies on the Jordan-Brouwer separation theorem (see Theorem 3.2), a standard result in differential topology.

Some features that characterize our analysis deserve a few comments. First, total resources are assumed to be fixed. This means that $\sum_{i=1}^{m} \omega_i = r$, where $r \in R^l$ is a fixed vector. Then the equilibrium manifold is defined as

$$E(r) = \{(p, \omega) \in S \times \Omega(r) | \sum_{i=1}^{m} f_i(p, p \cdot \omega_i) = r\},$$

where S is the set of normalized prices, $\Omega(r)$ is the space of economies and $\sum_{i=1}^{m} f_i(p, p \cdot \omega_i)$ is the aggregate demand function (see Section 2 below). Second, in our construction we heavily rely on the very nice topological properties enjoyed by the set of critical equilibria $E_c(r)$, this set being a finite and disjoint union of closed smooth submanifolds $\mathcal{S}_i$ of $E(r)$ (see [3, 4] or Theorem 2.1 below).

This paper is organized as follows. Section 2 recalls the economic setting. Section 3 contains our definitions and main results.

2 The economic setting

We consider a pure exchange economy with l goods and m consumers. Let $S = \{p = (p_1, \dots p_l) \mid p_j > 0, j = 1, \dots l, p_l = 1\}$ be the set of normalized prices. Let

$\Omega = (\mathbb{R}^l)^m$ denote the space of endowments $\omega = (\omega_1, \ldots, \omega_m)$, $\omega_i \in \mathbb{R}^l$. We assume that the standard assumptions of smooth consumer's theory are satisfied (see [4] Chapter 2). The problem of maximizing the smooth utility function $u_i : \mathbb{R}^l \to \mathbb{R}$ subject to the budget constraint $p \cdot \omega_i = w_i$ gives the unique solution $f_i(p, w_i)$, i.e. consumer's i demand. Let E be the closed set consisting of pairs $(p, \omega) \in S \times \Omega$ satisfying the following equations:

$$\sum_{i=1}^{m} f_i(p, p \cdot \omega_i) = \sum_{i=1}^{m} \omega_i.$$

The set E is a smooth submanifold of $S \times \Omega$ globally diffeomorphic to $(\mathbb{R}^l)^m$ (see [4]). Let $\pi : E \to \Omega$ be the *natural projection*, i.e. the smooth map defined by the restriction to E of $(p, \omega) \mapsto \omega$. Let E_c be the set of critical equilibria, namely the pairs $(p, \omega) \in E$ such that the derivative of π at (p, ω) is not onto. We now analyze the fixed total resources setting. Let $r \in \mathbb{R}^l$ denote the vector representing the total resources of the economy and $\Omega(r)$ denote the space of economies associated with the fixed total resources, i.e., $\Omega(r) = \{\omega \in (\mathbb{R}^l)^m | \sum_{i=1}^m \omega_i = r\}$. Define

$$E(r) = \{(p, \omega) \in S \times \Omega(r) | \sum_{i=1}^{m} f_i(p, p \cdot \omega_i) = r\},$$

denote by $\pi : E(r) \to \Omega(r)$ the restriction of the natural projection to $E(r)$ and by $E_c(r)$ the set of critical points of π. It can be shown (see [4]) that (p, ω) is a critical equilibrium with respect to $\pi : E(r) \to \Omega(r)$ if and only if it is a critical equilibrium with respect to $\pi : E \to \Omega$. The set of critical equilibria when total resources are fixed is denoted $E_c(r)$. The structure of $E(r)$ and $E_c(r)$ is described in the following theorem where we summarize some results due to Balasko (see [3] and [4]).

Theorem 2.1 (Balasko) *$E(r)$ is a smooth manifold globally diffeomorphic to $\mathbb{R}^{l(m-1)}$ and $E_c(r)$ is a disjoint union of closed smooth submanifolds $\mathcal{S}_i$, $i = 1, \ldots, inf(l-1, m-1)$ of $E(r)$. The manifold $\mathcal{S}_i$ has dimension $l(m-1) - i^2$ and $\mathcal{S}_i = \emptyset$ for $i > inf(l-1, m-1)$.*

We will refer to the decomposition $E_c(r) = \cup_i \mathcal{S}_i$, given by the previous theorem, as *Balasko's stratification* of the set of critical equilibria $E_c(r)$. The closed manifolds $\mathcal{S}_i$ will be called the *strata* of this stratification. Observe that for a fixed i the manifold $\mathcal{S}_i$ could not be connected. Denote by $\mathcal{S}_i^{j_i}$ its connected components, where the index j_i is varying in a countable (possible infinite) set. It is a challenging problem to study the topology of these strata. In particular, it is still an open question to understand when the strata $\mathcal{S}_i$ are a *finite* union of their connected components (i.e. the index j_i is varying in a finite set).

3 Main results

In the sequel we identify, without further comments, the equilibrium manifold $E(r)$ with the Euclidean space $\mathbb{R}^n, n = l(m-1)$, via Balasko's theorem 2.1. Observe also that the structure of $E_c(r)$, as defined by the same Theorem 2.1, entitles us to define a smooth path $\gamma : [0,1] \to E(r)$, connecting two regular equilibria in $E(r)$, *transversal* to $E_c(r)$ if it intersects each stratum S_i transversally.

We refer the reader to Section 5 of Chapter 1 and to Chapter 2 in [7] for the standard material about transversality theory. Actually, the only results about this theory used in this paper are the following three facts and the Lemma 3.1 below.

(i) Two submanifolds X and Z of $\mathbb{R}^n$ are said to be transversal, if for every point $x \in X \cap Z$ the following holds (see Section 5, Chapter 1, in [7]):

$$T_x X + T_x Z = T_x \mathbb{R}^n = \mathbb{R}^n;$$

(ii) If X and Z are two transversal submanifolds of $\mathbb{R}^n$, then $X \cap Z$ is a submanifold of $\mathbb{R}^n$. Moreover the codimension of $X \cap Z$ in $\mathbb{R}^n$ equals the codimension of X in $\mathbb{R}^n$ plus the codimension of Z in $\mathbb{R}^n$ (see theorem on page 30 in [7]);

(iii) (Transversality Theorem) Let X and Z be two closed submanifolds of $\mathbb{R}^n$. Then there exists an arbitrary small vector $s \in \mathbb{R}^n$ (i.e. the Euclidean norm of s can be taken arbitrary small) such that the manifold

$$X + s = \{y \in \mathbb{R}^n |\ y = x + s, x \in X\}$$

intersects Z transversally. In particular (by (ii)) $(X+s) \cap Z$ is a submanifold of $\mathbb{R}^n$ (see Section 3, Chapter 2, and the discussion on page 69 in [7]).

In this paper we are only concerned with the intersection between a stratum $\mathcal{S}_i$ (i.e. a submanifold of $E(r)$ of codimension i^2) and the image $\operatorname{Im}\gamma = \gamma([0,1])$ of a path $\gamma : [0,1] \to E(r)$ (i.e. a submanifold of $E(r)$ of codimension $n-1$). With a slight abuse of language, we will say that γ intersects (transversally) the stratum $\mathcal{S}_i$ if $\operatorname{Im}\gamma$ intersects (transversally) $\mathcal{S}_i$. By (i) above, γ is transversal to $\mathcal{S}_i$, for fixed i, if for every intersection point $x_0 = \gamma(t_0), t_0 \in [0,1]$, between $\operatorname{Im}\gamma$ and $\mathcal{S}_i$ the tangent vector $\gamma^{'}(t_0)$ of $\gamma(t)$ together with $T_{x_0}\mathcal{S}_i$ generate a n-dimensional space, being $n = l(m-1)$ the dimension of $E(r)$. Hence a path γ is transversal to $\mathcal{S}_i$ in the following two cases:

1. the codimension of $\mathcal{S}_i$ is greater or equal to 1 (i.e. $i \geq 1$) and $\operatorname{Im}\gamma$ does not intersect $\mathcal{S}_i$;

2. the codimension of $\mathcal{S}_i$ is one, (i.e. $i = 1$), γ intersects $\mathcal{S}_1$ in a finite number of points and for each of these points, say $x_0 = \gamma(t_0), t_0 \in [0,1]$, the tangent vector $\gamma'(t_0)$ does not belong to $T_{x_0}\mathcal{S}_1$. In this second case $\operatorname{Im}\gamma \cap \mathcal{S}_1$ consists of a finite number of points since, by (ii) above, $\operatorname{Im}\gamma \cap \mathcal{S}_1$ is a zero dimensional compact manifold.

The following lemma applies the previous results of transversality theory to a path joining two regular equilibria on the equilibrium manifold.

Lemma 3.1 *Given two regular equilibria $x, y \in E(r)$, there exists a smooth path γ joining them and intersecting $E_c(r)$ transversally. Such a path does not intersect $\mathcal{S}_i$ for $i > 1$ and intersects a finite number, say $\mathcal{S}_1^1, \dots, \mathcal{S}_1^k$, of the connected components of $\mathcal{S}_1$, each one in a finite number of points.*

Proof: Take any smooth path $\sigma : [0,1] \to E(r)$ joining x with y (σ exists since $E(r)$ is connected) and let $p = \inf(l-1, m-1)$ (cf. Theorem 2.1). We will prove the theorem by an induction argument on p. Let $p = 1$. By applying the Transversality Theorem to $X = \operatorname{Im}\sigma$ and $Z = \mathcal{S}_1$ we can find a vector s in $E(r)$ such that $\operatorname{Im}\sigma + s$ intersects $\mathcal{S}_1$ transversally. By taking s sufficiently small, we can assume that $x + s$ and $y + s$ are regular and there exist smooth paths σ_1 and σ_2 joining x with $x + s$ and $y + s$ with y, respectively, which do not intersect $\mathcal{S}_1$. The path γ is obtained by suitably smoothing the path $\sigma_1 \cup (\sigma + s) \cup \sigma_2$ around the points $x + s$ and $y + s$, where $\sigma + s : [0,1] \to E(r)$ is the path defined as $(\sigma + s)(t) = \sigma(t) + s$. Assume by induction we have found a path $\tilde{\gamma} : [0,1] \to E(r)$ joining x and y and intersecting each $\mathcal{S}_1, \dots \mathcal{S}_{p-1}$ transversally (in a finite number of points). Let $x_1 = \tilde{\gamma}(t_1)$ (resp. $x_2 = \tilde{\gamma}(t_2)$) be the first point (resp. the last point) where $\tilde{\gamma}$ intersects the stratum $\mathcal{S}_p$. Since the strata of Balasko's stratification are disjoint and closed we can find a sufficiently small δ such that $x_\delta = \tilde{\gamma}(t_1 - \delta)$ and $y_\delta = \tilde{\gamma}(t_2 + \delta)$ are regular and $\tilde{\gamma}$ restricted to $[t_1 - \delta, t_2 + \delta]$ does not intersect $\cup_{j=1}^{p-1}\mathcal{S}_j$. Call this restriction β. By applying again the Transversality Theorem, to $\operatorname{Im}\beta$ and $\mathcal{S}_p$ we can find a vector $s \in \mathbb{R}^n$ such that $\operatorname{Im}\beta + s$ is transversal to $\mathcal{S}_p$. By taking this vector sufficiently small, we can assume that $\operatorname{Im}\beta + s$ does not intersect $\cup_{j=1}^{p-1}\mathcal{S}_j$ and we can find smooth paths β_1 and β_2 joining x_δ with $x_\delta + s$ and $y_\delta + s$ with y_δ, respectively, which do not intersect $E_c(r) = \cup_{j=1}^{p}\mathcal{S}_j$. Consider the continuous path $\tilde{\gamma}_1 \cup \beta_1 \cup (\beta + s) \cup \beta_2 \cup \tilde{\gamma}_2$, where $\beta + s : [0,1] \to E(r)$ is the path defined as $(\beta + s)(t) = \beta(t) + s$ and $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ denote the restriction of $\tilde{\gamma}$ to the interval $[0, t_1 - \delta]$ and $[t_2 + \delta, 1]$. Finally, by suitably smoothing this path around the points $x_\delta, x_\delta + s, y_\delta + s, y_\delta$, we get the desired path γ. The last part of the lemma follows by 2.) above. $\square$

Our analysis deeply relies on the Jordan-Brouwer separation theorem that we recall.

Theorem 3.2 (Jordan-Brouwer) *Let S be a closed, connected, codimension one submanifold of $\mathbb{R}^n$. Then $\mathbb{R}^n \setminus S$ consists of two disjoint connected open sets, R_1 and R_2, which have S as common boundary.*

The proof of the previous theorem can be found in Section 5 of Chapter 2 in [7] under the stronger assumption that the manifold S is compact. The proof of Theorem 3.2, namely when $S \subset \mathbb{R}^n$ is only assumed to be closed (not necessarily bounded), is available in Lima's paper [8] (the authors do not know any text-book where this standard differential topology theorem is proved in the closed case). Actually Lima first proved the Jordan–Brouwer separation theorem for compact manifolds $S \subset \mathbb{R}^n$ but in the concluding remark on page 41 of [8], he indicates a way to prove this theorem under the weaker assumption that S is closed in $\mathbb{R}^n$.

Let $S \subset \mathbb{R}^n$ be as in the previous theorem and let $\gamma : I \to \mathbb{R}^n$ be a smooth curve connecting x and y in $\mathbb{R}^n \setminus S$, namely $\gamma(0) = x$ and $\gamma(1) = y$. Assume that γ intersects S transversally. Then $\operatorname{Im}\gamma \cap S$ is a (possibly empty) zero dimensional manifold and hence it consists of isolated points which are forced to be finite for the compactness of $\operatorname{Im}\gamma \cap S$. Observe, by Theorem 3.2, that the parity of $\operatorname{Im}\gamma \cap S$ depends on the location of x and y. More precisely, we have the following straightforward corollary of Theorem 3.2.

Corollary 3.3 *Let $S \subset \mathbb{R}^n$, R_1 and R_2 be as in Theorem 3.2 and let $\gamma : [0,1] \to \mathbb{R}^n$ be a smooth path joining x and y in $\mathbb{R}^n \setminus S$ and intersecting S transversally. Then the parity of $Im\gamma \cap S$ is even if and only if x and y both belong to R_1 or they both belong to R_2. In particular, if such a path intersects S in a single point, then x and y belong to different connected components of $\mathbb{R}^n \setminus S$.*

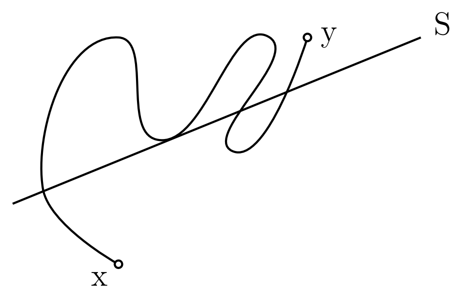

Figure 1: Violation of the transversality property

Figure 1 depicts a situation in which the transversality assumption is violated. In this case, the parity of $\operatorname{Im}\gamma \cap S$ does not give any information on the location of the points x and y.

A tool used in the following lemma, which represents a technical result needed to prove our main theorem, is the existence of a smooth unitary normal vector field $n : S \to \mathbb{R}^n$ on a closed and connected codimension one submanifold $S \subset \mathbb{R}^n$, namely a smooth $\mathbb{R}^n$-valued function on S such that $n(s)$ is perpendicular to T_sS and $|n(s)| = 1, \forall s \in S$. It can be proved that the existence of n is equivalent to the *orientability* of S (see Section 2, Chapter 3, in [7]). Observe that, once a smooth unitary vector field n on S is given, then $-n : S \to \mathbb{R}^n$ defines another unitary smooth vector field on S and, by Jordan-Brouwer separation theorem, the vectors $n(s)$(resp. $-n(s)$) point towards either R_1 (resp. R_2) or R_2 (resp. R_1), where R_1 and R_2 are the connected components of $\mathbb{R}^n \setminus S$ as in Theorem 3.2. When S is compact, the proof of its orientability can be found in [7], Ex. 13 and 18, on page 104 and 106, respectively. We refer the reader to [11] for an elegant and concise proof of the orientability of S in the closed case.

Lemma 3.4 *Let $S \subset \mathbb{R}^n$ be a closed and connected codimension one submanifold of $\mathbb{R}^n$, let C be a closed set such that $S \cap C = \emptyset$ and let σ be a smooth curve joining two points in the complement of $S \cup C$ which does not intersect C and which intersects S transversally. Then there exists a smooth curve γ joining the given points which does not intersect C and which intersects S transversally in at most one point.*

Proof: Let $x_1 = \sigma(t_1), \dots, x_k = \sigma(t_k)$, with $t_1 < \dots < t_k$, be the intersection points of σ with S. Assume $k \geq 2$, otherwise there is nothing to prove. We will show that there exists a smooth curve, $\tilde{\sigma}$, joining x and y, disjoint from C, and intersecting S transversally in $x_3, \dots, x_k$ (actually $\tilde{\sigma}$ will be constructed in such a way to coincide with σ in $[0, t_1 - \delta]$ and $[t_2 + \delta, 1]$, for a suitable chosen δ). By iterating this procedure, this yields the existence of the desired path γ and, hence, the proof of the lemma. In order to construct $\tilde{\sigma}$, let $\beta : [0, 1] \to S$ be any smooth curve on S joining $x_1 = \beta(0)$ and $x_2 = \beta(1)$ (β exists since S is connected). Take a smooth unitary normal vector field $n : S \to \mathbb{R}^n$ on S pointing toward R_1, where R_1 is the connected component of $\mathbb{R}^n \setminus S$ where x belongs to. Take a positive real number λ, $0 < \lambda < r$, where r denotes the distance between $\operatorname{Im} \beta$ and C. Then the curve $\lambda n(\beta(t))$ is a smooth curve on R_1 not intersecting C such that $\lambda n(\beta(0)) = \lambda n(x_1)$ and $\lambda n(\beta(1)) = \lambda n(x_2)$. For δ and ϵ sufficiently small positive real numbers, the distance between $x_\delta = \sigma(t_1 - \delta)$ and $\epsilon\lambda n(x_1)$ (resp. $y_\delta = \sigma(t_2 + \delta)$ and $\epsilon\lambda n(x_2)$) is arbitrary small (indeed, these distances go to zero as $\delta, \epsilon \to 0$). Hence we can connect x_δ with $\epsilon\lambda n(x_1)$ and $\epsilon\lambda n(x_2)$ with y_δ with line segments β_1 and β_2, respectively, in such way that $\operatorname{Im} \beta_1$ and $\operatorname{Im} \beta_2$ belong to R_1 and do not intersect C. Then the continuous path $\alpha = \sigma_1 \cup \beta_1 \cup (\epsilon\lambda n(\beta)) \cup \beta_2 \cup \sigma_2$, where σ_1 and σ_2 are the restrictions of σ to the intervals $[0, t_1 - \delta]$ and $[t_2 + \delta, 1]$, does not intersect C and intersects S transversally in $x_3, \dots, x_k$. Finally, by suitably smoothing the path α around the points $x_\delta, \epsilon\lambda n(x_1), \epsilon\lambda n(x_2), y_\delta$, we get the desired path $\tilde{\sigma}$. $\square$

In our economic setting, let $\gamma : [0,1] \to E(r)$ be a smooth path joining two regular equilibria x and y in $E(r) \setminus E_c(r)$. The *length* of γ, denoted by $\ell(\gamma)$, is defined as the number of intersection points of $\operatorname{Im}\gamma$ with $E_c(r)$. Observe that if γ intersects $E_c(r)$ transversally, then $\ell(\gamma)$ is a non negative integer which is zero iff $\operatorname{Im}\gamma \cap E_c(r) = \emptyset$.

We define the *distance*, $d(x,y)$, between two regular equilibria $x, y \in E(r) \setminus E_c(r)$ as the infimum of $\ell(\gamma)$, when γ is varying amongst all the smooth curves joining x and y. We define a path γ *minimal* if $\ell(\gamma) = d(x,y)$. By the above considerations, $d(x,y)$ is always a nonnegative integer.

Remark 3.5 It is worth pointing out that $d(x,y)$ does not define a metric space structure on $E(r) \setminus E_c(r)$. The distance between two regular equilibria x, y belonging to the same connected component of $E(r) \setminus E_c(r)$ is zero even if $x \neq y$. On the other hand, one can define an equivalence relation $\sim$ on $X = E(r) \setminus E_c(r)$, by defining $x \sim y$ iff x and y belong to the same connected component of X. We denote with $X/_\sim$ the quotient space and with $[x]$ the equivalence class of x. The function

$$\widetilde{d} : X/_\sim \times X/_\sim \to \mathbb{N} \subset \mathbb{R},$$

defined by $\widetilde{d}([x],[y]) = d(x_0, y_0)$, where x_0 and y_0 are any two regular equilibria in the class of $[x]$ and $[y]$, respectively, is well defined, namely it does not depend on the choice of $x_0 \in [x]$ and $y_0 \in [y]$. Moreover it is immediate to verify that $(X, \widetilde{d})$ is a metric space.

The following proposition describes a sufficient condition for a path γ in $E(r)$ to be minimal. Let $\gamma : [0,1] \to E(r)$ be a smooth curve joining two regular equilibria $x, y \in E(r) \setminus E_c(r)$ and assume that γ is transversal to $E_c(r)$, namely $\operatorname{Im}\gamma$ does not intersect the codimension > 1 strata and intersects transversally a finite number of the connected components of S_1, say $S_1^j, j = 1, \dots k$, in a finite number of points (see Lemma 3.1).

Proposition 3.6 *Take γ as above and assume it intersects transversally each $S_1^j, j = 1, \dots k$, in at most one point. Then γ is a minimal path.*

Proof: If $\ell(\gamma) = 0$ there is nothing to prove. Let $l = \ell(\gamma) > 0$ and let $\mathcal{S}_1^{j_1}, \dots, \mathcal{S}_1^{j_l}$ be the connected submanifolds of $\mathcal{S}_1$ intersected transversally exactly in one point by γ. Assume by contradiction that there exists another path in $E(r)$, say σ, joining x and y and such that $\ell(\sigma) < \ell(\gamma)$. Then it must exist an index $j_0, 1 \leq j_0 \leq l$, such that $\operatorname{Im}\sigma$ does not intersect $S_1^{j_0}$. This implies that x and y belong to the same connected components of $E(r) \setminus S_1^{j_0}$. On the other hand γ is a curve joining

x and y and intersecting $S_1^{j_0}$ transversally only in one point. Hence, Corollary 3.3, applied to the closed and connected codimension one submanifold $S = S_1^{j_0} \subset E(r) = \mathbb{R}^n$, implies that x and y belong to the two different connected components of $E(r) \setminus S_1^{j_0}$, which is the desired contradiction. $\square$

Our main result is the following theorem which asserts the existence of a minimal path joining two given regular equilibria.

Theorem 3.7 *Given two regular equilibria x and y, there exists a smooth curve $\gamma : [0, 1] \to E(r)$, $\gamma(0) = x$ and $\gamma(1) = y$, such that $\ell(\gamma) = d(x, y)$.*

Proof: Take any curve $\tilde{\sigma}$ joining x and y and intersecting $E_c(r)$ transversally, whose existence is guaranteed by Lemma 3.1, and let $S_1^1, \ldots, S_1^k$ be the number of connected components of S_1 intersected by $\operatorname{Im}\tilde{\sigma}$ (observe that $\operatorname{Im}\tilde{\sigma} \cap \mathcal{S}_i$ is empty for $i > 1$). From Proposition 3.6 the theorem will be proved if there exists a path γ which intersects each of the manifolds $S_1^1, \ldots, S_1^k$ transversally in at most one point and does not intersect $E_c(r)$ in any other point. We construct such a path by an induction argument on the number k of the previous submanifolds. The case $k = 1$ follows by applying Lemma 3.4 to $S = S_1^1$ and to the closed set $C = E_c(r) \setminus S_1^1$. Assume by induction we have found a path σ which intersects $S_1^1, \ldots, S_1^{k-1}$ transversally in at most one point. If $\operatorname{Im}\sigma$ does not intersect $S_1^1 \cup \ldots \cup S_1^{k-1}$, the proof of the theorem follows by applying again Lemma 3.4 to the path $\tilde{\sigma}$, $S = S_1^k$ and $C = E_c(r) \setminus S_1^k$. Otherwise, let S_1^j, $j \leq k - 1$, be the last manifold intersected transversally exactly in one point, say $\bar{x} = \sigma(\bar{t})$, by $\operatorname{Im}\sigma$. For small $\delta \in \mathbb{R}$, the point $\bar{x_\delta} = \sigma(\bar{t} + \delta)$ is regular. By applying Lemma 3.4 to the path σ restricted to $[\bar{t} + \delta, 1]$, $C = E_c(r) \setminus S_1^k$, $S = S_1^k$, we get a path $\bar{\gamma}$ which does not intersect C, which intersects S_1^k transversally in at most one point and which connects x_δ with y. Consider the continuous path $\bar{\sigma} \cup \bar{\gamma}$, where $\bar{\sigma}$ is the restriction of σ to $[0, \bar{t} + \delta]$. This path intersects $E_c(r)$ only in the submanifolds $S_1^1, \ldots, S_1^k$ and each of them transversally in at most one point. By suitably smoothing this path around the point x_δ, we get the desired path γ. $\square$

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