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A Model of Anticipated Regret and Endogenous Beliefs

Raghu Suryanarayanan

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DIPARTIMENTO DI SCIENZE ECONOMICHE - UNIVERSITÀ DEGLI STUDI DI SALERNO
Via Ponte Don Melillo - 84084 FISCIANO (SA)
Tel. 089-96 3167/3168 - Fax 089-96 3169 – e-mail: csef@unisa.it

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Abstract

This paper clarifies and extends the model of anticipated regret and endogenous beliefs based on the Savage (1951) Minmax Regret Criterion developped in Suryanarayanan (2006a). A decision maker chooses an action with state contingent consequences but cannot precisely assess the true probability distribution of the state. She distrusts her prior about the true distribution and surrounds it with a set of alternative but plausible probability distributions. The decision maker minimizes the worst *expected regret* over all plausible probability distributions and alternative actions, where regret is the loss experienced when the decision maker compares an action to a counterfactual feasible alternative for a given realization of the state. Preliminary theoretical results provide a systematic algorithm to find the solution to the decision problem and show how models of Minmax Regret differs from models of ambiguity aversion and expected utility. In particular, the solution to the decision problem can *always* be represented as a *saddle point* solution to an equivalent zerosum game problem. This new problem jointly produces the solution to the Anticipated Regret problem and the endogenous belief. We then use the endogenous belief to define the implicit certainty equivalent and to build an infinite horizon and time consistent problem for a decision maker minimizing her lifetime worst expected regrets.

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* University of Chicago and CSEF

Contents

1 Introduction

Modern Macroeconomics assumes market structures in which the competitive equilibrium price of any tradeable commodity is equal to the expected value of its discounted payoffs. Commodity payoffs are contingent upon the realization of an exogenous state of nature and expectations are measured with respect to the probability distribution of the state. Economically plausible discount factors are interpreted as the outcome of interactions between infinitely lived agents in the market. Assumptions about agents' preferences, constraints and beliefs about the probability distribution of the state of nature are then crucial in characterizing efficient allocations of macroeconomic risks and equilibrium prices.

The Rational Expectations hypothesis (Muth (1964)) is commonly used to model agents' anticipations of the future state and assumes that agents know the true probability distribution of the state. In this working paper, we are interested in departures from rational expectations in situations where agents cannot precisely assess the true probability distribution of the state of nature and we develop instead a model of endogenous beliefs based on a version of the Savage (1951) Minmax Regret criterion.

We build upon the framework of Sections 2 and 3 in Suryanarayanan (2006a) and extend the theoretical results. Consider the problem of a decision maker choosing an action with state contingent consequences when she cannot precisely assess the true probability distribution of the state. In the spirit of Hansen's and Sargent's applications of Robust Control theory, we assume that she distrusts her prior about the true distribution and surrounds it with a set of alternative but plausible probability distributions. One possible rationale to doubt about the prior is a concern for misspecifications (see Hansen and Sargent (2006)) and more generally a concern with the fact that a prior is an approximation. For each realization of the state and feasible action, the decision maker may feel regret for not choosing an alternative plan from which she may derive greater utility for that particular realization. We define regret as the loss experienced when the decision maker compares an action to a counterfactual feasible alternative for a given realization of the state. The decision maker anticipates her future regrets and minimizes the worst *expected regret* over all plausible probability distributions and alternative actions. Unlike Maxmin models of ambiguity aversion or Robust Control models, the decision maker anticipating her future regrets cares about the worst expected *regret* payoff, not the worst expected payoff. Thus, the decision maker is less pessimistic compared to an ambiguity averse decision maker since she also focuses on optimistic scenarios, and yet more cautious than a standard expected utility maximizer who would only favor one particular scenario.

We provide a theorem which gives general conditions that garantee the existence of a unique solution to the decision problem and show how general minmax regret models fundamentally differ from minmax and expected utility. Indeed, we show that the solution is also the unique saddle point solution of an equivalent zero-sum game problem. The saddle point property is what distinguishes anticipated regret from expected utility and is only satisfied in some cases for general minmax models, depending on the choice of the probability set or the constraint set. This property is also the one which delivers the most interesting implications of models of ambiguity aversion for asset allocation problems and equilibrium asset prices. For such applications, the theorem advocates the use of Minmax Regret type models instead.

The new zero-sum game problem also delivers the expost probability distribution, i.e. the distribution with respect to which an expected utility maximizer would also choose the solution to the ancitipated regret problem. We interpret it as the decision maker's implicit belief. Since it is jointly obtained with optimal decision, we say that the implicit belief is endogenous to the decision problem. The expost probability interpretation of the regret problem then allows us to define a notion of certainty equivalent which will be useful in formulating a recursive and time consistent problem for an infinitely lived decision maker anticipating her future regrets.

The next Sections are organized as follows. Section 2 defines and studies the one-period Anticipated Regret problem, shows how to derive the expost distribution interpreted as the endogenous belief and compares Anticipated Regret with the standard Expected Utility and the Ambiguity Aversion (Minmax) problems. Section 3 develops a recursive formulation of the decision problem for an infinitely lived decision maker anticipating her future regrets, compares the infinite horizon model with Epstein-Zin recursive utility, and suggests alternative formulations. Section 4 concludes.

2 The static model

In this Section, we define the choice environment and the one-period decision problem of Anticipated Regret (R), a version of the Savage Minmax Regret criterion. We recall and extend the equivalent zero-sum representation of the decision problem (R) and construct the expost probability distribution consistent with the decision problem which we interpret as the endogenous implicit belief of the decision maker. Indeed, an expected utility maximizer with a prior equal to the expost distribution would also choose the optimal solution to the anticipated regret problem.

2.1 The decision problem

2.1.1 The choice environment

Let Z be the set of states, a compact metric space with a Borel $\sigma-$algebra $B(Z)$. We identify Z with the set of all the possible realizations of an exogenous state of nature. We denote by $\Pi(Z)$ the set of all Borel probability measures on Z. Under the weak-convergence topology, $\Pi(Z)$ is also a compact metric space.

We denote by $\Gamma(Z)$ the set of all continuous and $B(Z)-$measurable functions mapping Z to the real line $\Re$, and $\Gamma^+(Z)$ the set of consumption plans, the cone of all positive functions in $\Gamma(Z)$.

2.1.2 The experience of anticipated counterfactual thinking

Consider a decision maker who faces the problem of choosing a consumption plan $(c(z))_{z \in Z}$ within a set of constraints summarized by C, a subset of $\Gamma^+(Z)$ which does not separate consumption plans across states. An example would be a typical budget constraint. Thus, the decision maker cannot choose to allocate her consumption independently for each state. Unless she has a perfect forecast upon the realization of the state of nature or unless she turns out to have chosen the expost optimal action, the decision maker is likely to feel regret for having made the inappropriate choice. Regret is commonly defined to be the nagging feeling of having made the wrong choice compared to a better alternative (Olson and Roese, 1995) and is the prominent form of counterfactual thinking, the comparison between the "what might have been" alternative choice, the counterfactual, and the "what has effectively been" choice.

Since the decision maker knows she will eventually learn about the true realization of the state of nature, she may anticipate her future regrets. The anticipation of counterfactual thoughts is shown to be innate and to arise naturally (Sirigu et al. (2004)[1] and Mandel et al. (2005)) in situations when the decision maker knows that there will be a direct and observable feedback subsequent to her decision.

We assume that the decision maker evaluates her experience of anticipated counterfactual thinking in state z for having chosen c instead of the counterfactual c^* through the function ψ :

$$\begin{array}{rcl} \Re^+ \times \Re^+ \times Z & \rightarrow & \Re \\ (c(z), c^*(z), z) & \rightarrow & \psi(c(z), c^*(z), z) \end{array}$$

where the quantity $-\psi(c(z), c^*(z), z)$ will measure anticipated regret. We make the following assumptions on ψ :

- **Assumption** $A1$: for all $z \in Z$, $\psi(\cdot, \cdot, z)$ is continuous on $\Re^+ \times \Re^+$, strictly increasing, strictly concave and differentiable in its first argument.

- **Assumption** $A2$: for all $z \in Z$, $\psi(\cdot, \cdot, z)$ is antisymmetric on $\Re^+ \times \Re^+$

$$\begin{array}{rcl} \psi(c(z), c^*(z), z) & = & -\psi(c^*(z), c(z), z) \\ \text{for all } (c, c^*) & \in & C \times C \end{array}$$

Assumption $A1$ is analogous to usual assumptions that are made for utility functions and decision problems in economics. The strict concavity property will be important for the main results that will follow.

Assumption $A2$ states that regret is antisymmetric. Regretting x for x^* is experienced as rejoicing x^* for x. This property was first mentioned in Fishburn (1982, 1984) who studied utility representations without the transitivity axiom

[1] Sirigu et. al (Science, May (2004)) identify the orbito prefrontal cortex, which is active in reward evaluation and comparison, as a fundamental human cerebral structure in mediating the experience of regret and the anticipation of counterfactual thoughts.

and derived a representation where the decision maker compares each possible pair of alternatives using the functional ψ. Experiments in Loomes and Sugden (1982, 1986) showed that $A2$ was an important property to capture the experience of regret. Note that by assumption $A2$, $\psi(\cdot,\cdot,z)$ is strictly convex and differentiable in its second argument.

2.1.3 Minimizing the worst regret

We assume that the decision maker cannot precisely assess the true probability distribution of the state. In the spirit of Hansen's and Sargent's application of Robust Control theory, she distrusts[2] her prior distribution p^* which she acknowledges to be an approximation to the true probability distribution and surrounds p^* with a set of alternative but plausible distributions P, a subset of $\Pi(Z)$. The following Section gives examples of specifications for P.

We define the decision function v with which the decision maker ranks consumption plans:

$$\begin{aligned} v &: \Gamma^+(Z) \to \Re \\ v(c) &= \min_{\pi\in P}\min_{c^*\in C} E_\pi \psi(c(z), c^*(z), z) \end{aligned}$$

and the associated decision problem:

$$(R) : \max_{c\in C} v(c)$$

where we make the following assumptions on the probability set P and the constraint set C :

- **Assumption** $A3$: P is a non-empty, convex, compact subset of $\Pi(Z)$
- **Assumption** $A4$: C is a non-empty, convex, compact subset of $\Gamma^+(Z)$

The interpretation is the following. The decision maker solves (R) to choose consumption plans that minimize the worst expected regret[3]. The quantity $-v(c)$ measures the worst expected regret that the decision maker could possibly experience if she decides to choose the consumption plan c. Indeed, first she selects for each possible probability measure π the best counterfactual plan she could choose if she did not doubt about π. In other words, given π this best counterfactual plan would give her the worst expected regret :

$$\max_{c^*\in C} E_\pi \left(-\psi(c(z), c^*(z), z)\right)$$

[2] Although the theoretical resutls in this paper do not depend on the particular interpretation of the multiple priors set P, we favor Hansen's and Sargent's interpretation in terms of model uncertainty when we want to study the observable implications for dynamic economies, and in particular for the application to asset pricing in Section 4.

[3] In Sirigu et. al (2004), patients without lesions in their orbito prefrontal cortex are shown to make decisions in anticipation of their counterfactual thoughts that try to avoid future regrets.

She then computes the maximum regret she could experience across all plausible probabilities:

$$\max_{\pi \in P} \max_{c^* \in C} E_\pi \left(-\psi(c(z), c^*(z), z)\right)$$

which can be rewritten as:

$$\begin{aligned} & -\min_{\pi \in P} \min_{c^* \in C} E_\pi \psi(c(z), c^*(z), z) \\ = \quad & -v(c) \end{aligned}$$

Switching the sign of $-\psi(c(z), c^*(z), z)$ to $+\psi(c(z), c^*(z), z)$ changes the $\max_{\pi \in P} \max_{c^* \in C}$ operators into $-\min_{\pi \in P} \min_{c^* \in C}$ and we see that $-v(c)$ measures the decision maker's worst expected regret.

2.1.4 Examples

- **The function** ψ

In applications, we will use the additive specification for ψ :

$$\psi(c(z), c^*(z), z) = u(c(z)) - u(c^*(z))$$

where u is a standard VNM utility function. The decision maker compares the utility derived from plan c to that derived from a counterfactual alternative plan c^*.

This enables to better compare our model with the expected utility model. Indeed, we see that when the probability set P is reduced to a singleton $\{\pi^*\}$, the anticipated regret model reduces to expected utility with prior π^*.

However, when P is not reduced to a singleton, i.e when ambiguity prevails, the anticipation of counterfactual alternatives affects the choice of the decision maker through the minimization over alternative plausible probabilities in P.

- **The constraint set** C

In most applications, C will be a typical budget constraint. For a standard consumer problem, where the exogenous state takes only two values, L and H, we may define C as

$$C = \left\{(c_L, c_H) \in \Re^+ \times \Re^+ \mid 0 \leq p_L c_L + p_H c_H \leq y\right\}$$

where y is the consumer's income and p_L and p_H are the spot prices in states L and H.

Alternatively, we could also adapt C for a one-period investment problem:

$$C = \left\{ \begin{array}{c} c \in \Re^+ \times \Re^+ \text{ and } \theta \in \Re^+ \times \Re^+ \mid \\ c_z = \theta R_z + (y - \theta) R^f, \ z \in \{L, H\} \text{ and } 0 \leq \theta \leq y \end{array} \right\}$$

where θ is the amount invested in the asset which pays R_z units of consumption in state z and $(y - \theta)$ is the amount invested in the risk-free asset which pays always R^f units of consumption.

Note that the convexity assumption rules out fixed costs. In particular, in asset pricing models that assume a participation constraint in the form of fixed costs, the results presented in this Section will not apply. However, this assumption can handle most other types of constraints used in asset pricing and macroeconomics such as short-selling constraints and solvency constraints.

- **The probability set** P

Specifications of sets of probability measures abound in the statistics literature. We refer to Epstein and Wang (1994) and references therein for a few insightful examples. In particular, the choice of $\varepsilon-$contamination sets is particularly appealing:

$$P = \{p \mid p = \varepsilon m + (1-\varepsilon)p^*,\ m \in M\}$$

where ε is random variable on Z with values in $[0,1]$, p^* is the prior of the decision maker about which she expresses doubts and M is a closed and convex subset of $\Pi(Z)$. When ε is constant and equal to 0, we have that $P = \{p^*\}$ and there is no ambiguity. In general, the set P then "contaminates" the measure p^* with a set of alternative measures $\varepsilon m + (1-\varepsilon)p^*$.

Hansen and Sargent use the Kullback-Leibler distance to specify the probability set P as a neighborhood of absolutely continuous measures around p^* for Robust Control problems:

$$\begin{aligned} P &= \{p \in \Pi(Z) \mid d_{KL}(p,p^*) \leq \varepsilon\} \\ d_{KL}(p,p^*) &= \int_{z\in Z} \frac{dp}{dp^*} \log\left(\frac{dp}{dp^*}\right) dp^*(z) \end{aligned}$$

where $\frac{dp}{dp^*}$ is the Radon-Nikodym derivative of p with respect to p^*. In this case, the decision maker is concerned with small deviations from p^* that are absolutely continuous with respect to p^* which she is not able to detect.

Alternatively, one could use a neighborhood induced by the Prohorov metric (as in Bergemann and Schlag[4] (2005) who study another version of the Savage Minmax Regret criterion within the context of monopoly pricing) to define P :

$$\begin{aligned} P &= \{p \in \Pi(Z) \mid d(p,p^*) \leq \varepsilon\} \\ d(p,p^*) &= \inf_{A\in B(Z)} \{\eta \mid p(A) \leq p^*(A^\eta) + \eta\} \\ A^\eta &= \{x \in Z \mid d_Z(x,A) \leq \eta\} \end{aligned}$$

[4] They define the anticipated regret problem by:

$$\max_{c\in C} \min_{\pi\in P} E_\pi \left(\min_{a\in C^*(z)} \psi(c(z),a,z) \right)$$

the best counterfactual is chosen state by state in the innermost minimization problem and $C^*(z)$ is the state z section of the constraint C. This problem is equivalent to the one studied in this paper in special cases ($Z = \{L,H\}$, $P = [0,1]$), but not in general when P is not the set of all possible measures on Z.

where d_Z is a given metric on Z. The Prohorov metric or metric of the weak convergence (convergence in distribution for probabilities) allows for both large deviations in probabilities within small neighborhoods and large neighborhoods with small deviations in probabilities. This could be particularly useful in equilibrium asset pricing where large deviations in prices from fundamentals and state dependent volatility in asset returns are hard to explain with existing models.

We conclude this paragraph with the two-state case $Z = \{L, H\}$. Any convex and compact subset of $\Pi(Z)$ is then identified with a closed and bounded interval $\left[\pi^{\inf}, \pi^{\sup}\right]$ included in $[0, 1]$ giving the range of plausible values for the probability of state L.

2.1.5 Savage's Minmax Regret

The decision problem (R) is an extension of Savage's (1951) Minmax Regret Criterion which can be expressed as follows. A decision maker derives utility $u(c(z), z)$ in state z from choosing the plan c and does not know the true realization of the state at the time of decision. Savage suggested to apply the worst case scenario criterion to a modified reward, which he terms the "loss", defined, for each plan c and state z, as the difference between the maximum utility obtainable for state z and the utility derived from plan c:

$$\min_{c \in C} \max_{z \in Z} \max_{c^* \in C} \left(u(c^*(z), z) - u(c(z), z)\right)$$

The minimization over c^* takes place for each state of the world z and defines the best action that can be taken if the decision maker knew that state z were the true state. Thus, the term $\max_{c^* \in C} \left(u(c^*(z), z) - u(c(z), z)\right)$ expresses the decision maker's regret in utility loss for taking action c in state z instead of the best action c^* in state z. The decision maker then minimizes the worst regret over all actions and states.

Savage's formulation of the problem is equivalent to problem (R) when the set of alternative distributions P is equal to the set of all possible measures $\Pi(Z)$. The first extension to Savage's criterion is thus to allow the decision maker to bound the plausible probabilities she considers with a set P which would not necessarily equal the set of all possible measures over Z. The second extension is to allow for a more general form of loss function ψ to measure anticipated regret.

In order to understand how introducing probability bounds changes the nature of the problem, consider the case when $Z = \{L, H\}$ and L represents the event of a serious stock market crash while H represents the event of a good performance of the stock market. Let us suppose that the utility payoffs $(u(c, z))$ of an investor who must choose between a risky stock R and a riskless asset R^f are given by:

	L	H
R^f	1	1
R	-10	5

that is, in the event of a serious stock market crash, the regret loss of investing in the risky stock R is very high (11) compared to the regret loss of investing in the safe asset R^f (4) in the good state H. When applying Savage's criterion to this decision problem, we see that the worst regret occurs when choosing the stock in the event of a crash. Thus, the investor would always choose the safe asset R^f. However, such a decision is not satisfactory. In reality, the event of a serious stock market crash is an extreme event and is very unlikely. A decision maker with this intuition in mind may, for example, bound the probability π of the market crash to be no more than 0.25 and solve the problem:

$$\min_{c\in\{Rf,R\}} \max_{\pi\in[0,0.25]} \max_{c^*\in\{Rf,R\}} \{E_\pi(u(c^*,z)-u(c,z))\}$$

where

$$\begin{aligned} E_\pi(u(R^f,z)) &= 1 \\ E_\pi(u(R,z)) &= -15\pi+5 \end{aligned}$$

In this case the worst expected regrets are respectively

$$\begin{aligned} -v(R) &= \max_{\pi\in[0,0.25]} \max_{c^*\in\{Rf,R\}} \{E_\pi(u(c^*,z)-u(R,z))\} \\ &= 15/4-4 \\ -v(R^f) &= \max_{\pi\in[0,0.25]} \max_{c^*\in\{Rf,R\}} \left\{E_\pi(u(c^*,z)-u(R^f,z))\right\} \\ &= 4 \end{aligned}$$

so that the investor minimizing the worst expected regret chooses to invest in the stock $(v(R)>v(R^f))$.

2.2 Equivalent zero-sum game formulation and main results

The following Section characterizes the solution of the decision problem (R) with Theorem 1 and shows with Theorem 3 that the decision problem (R) has an equivalent Minmax representation (B). Theorem 3 is an extension to Theorem 3 in Suryanarayanan (2006a). In particular the solution to (R) can be represented as the saddle point solution of the equivalent zero-sum game (B). In turn this delivers the "expost bayesian" interpretation of anticipated regret and allows us to define the implicit endogenous belief for a decision maker solving problem (R). We then provide a geometrical illustration of the results adapting Ferguson's (1967) representations of Bayesian and Minmax problems, which will also allow us to further compare Anticipated Regret with Expected Utility and Maxmin Expected Utility. In particular, interesting implications of Ambiguity Aversion models in Economics hinge upon the fact that the equivalent zero-sum game representation has a saddle-point solution. While this property is not always satisfied for general Maxmin problems, Theorem 1 and Theorem 3 show that it is always true for general Maxmin Regret models like (R). As we will see, the results presented in this Section will carry crucial implications for defining the infinite horizon extension of the decision problem and for characterizing shadow equilibrium asset prices.

2.2.1 Unique solution but multiple minimizing probabilities

Recall the decision problem (R) :

$$\begin{aligned}(R) \quad &: \quad \max_{c \in C} v(c) \\ &= \quad \max_{c \in C} \min_{\pi \in P} \min_{c^* \in C} E_\pi \psi(c(z), c^*(z), z)\end{aligned}$$

For each alternative c in C and for each probability measure π in P, define $c^*(\pi, c)$ the solution to the innermost minimization problem in (R) :

$$c^*(\pi, c) = \arg \min_{c^* \in C} E_\pi \psi(c(z), c^*(z), z)$$

and denote with $M(c)$ the subset of P containing the minimizing probabilities in the decision problem (R) :

$$M(c) = \left\{ \begin{array}{c} \pi \in P \text{ such that} \\ E_\pi \psi(c(z), c^*(c, \pi)(z), z) = \min_{\pi \in P} E_\pi \psi(c(z), c^*(c, \pi)(z), z) \\ \end{array} \right\}$$

The following theorem states that under $A1 - A4$, there exists a unique solution to the decision problem but there are multiple minimizing probability distributions whenever there is ambiguity, i.e whenever the set of plausible distributions P is not a singleton. As a consequence, the decision function v is not Gateaux-diffentiable[5] at the optimum. The theorem then characterizes the set of minimizing probability measures as well as the optimal solution which equalizes expected regrets across all minimizing measures.

Theorem 1 *Assume that assumptions $A1 - A4$ hold. Then,*

(i) $c^*(\pi, c)$ and v are well defined, and (R) has a unique solution in c, denoted by c^{opt}

(ii) If P is not a singleton, then $M(c^{opt})$ is not a singleton either, i.e (R) has multiple solutions in π. Moreover $M(c^{opt})$ is contained in P^e, the set of extremal points[6] of P

(iii) At the optimum the worst regrets $E_\pi \psi(c^{opt}(z), c^*(\pi, c^{opt})(z), z)$ are equalized across all minimizing probabilities $\pi \in M(c^{opt})$

[5] A function $f : C \to \Re$ is Gateaux-differentiable at c if the limit $\lim_{\alpha \to 0} \frac{f(c+\alpha h)-f(c)}{\alpha}$ exists for all $h \in C$ and is a linear function of h.

[6] π is an extremal point of P if $P \backslash \{\pi\}$ is convex. The Krein-Millman Theorem (see Aubin, p. 101) states that a convex and compact set P has at least one extremal point and is equal to the convex hull of the set P^e of all its extremal points.

The proof is given in Suryanarayanan (2006a).

A Corollary of point (ii) of the theorem is that the decision function v is not Gateaux-differentiable at the optimum when P is not a singleton. Because it is continuous and concave however, v has well defined right and left directional[7] derivatives.

Corollary 2 *If P is not a singleton, the decision function v is not Gateaux-differentiable at the optimum but admits right and left directional derivatives given by:*

$$\begin{aligned} dv^{+}(c^{opt})\cdot h &= \min_{\pi\in M(c^{opt})} E_{\pi}\left(\left(\psi_c^{'}(c^{opt}(z),c^{*}(\pi,c^{opt})(z),z)\right)h(z)\right) \\ dv^{-}(c^{opt})\cdot h &= \max_{\pi\in M(c^{opt})} E_{\pi}\left(\left(\psi_c^{\prime}(c^{opt}(z),c^{*}(\pi,c^{opt})(z),z)\right)h(z)\right) \end{aligned}$$

for all $h\in C$, where $\psi_c^{'}(\cdot,\cdot,z)$ denotes the derivative of $\psi(\cdot,\cdot,z)$ with respect to its first argument.

The proof is given in Suryanarayanan (2006 a).

Before further discussing Theorem 1, we formulate the equivalent zero-sum game representation of the decision problem (R) which helps to interpret and to better understand point (iii).

2.2.2 An equivalent zero-sum game problem

Recall the decision problem (R) :

$$(R):\max_{c\in C}\min_{\pi\in P}\min_{c^{*}\in C}E_{\pi}\psi(c(z),c^{*}(z),z)$$

and recall that the set of minimizing probabilities at the optimum is $M(c^{opt})$ and is included in the set of extremal probability measures P^{e}. Note that $M(c^{opt})$ is a compact and metric subset of $\Pi(Z)$ (Aubin, Proposition 11, p. 82) but non-convex. We then consider $B(M(c^{opt}))$ the Borel $\sigma-$ algebra for the induced metric of $\Pi(Z)$ on $M(c^{opt})$ and $\Lambda(c^{opt})$ the set of all Borel-probability measures on $M(c^{opt})$.

Define the modified regret function $\psi^{*}(c,\pi)$ associated with ψ which gives the expected regret for an alternative c in C and an extremal probability measure π in $M(c^{opt})$ as

$$\begin{aligned} C\times M(c^{opt}) &\rightarrow \Re \\ (c,\pi) &\rightarrow \psi^{*}(c,\pi)=E_{\pi}\psi(c(z),c^{*}(\pi,c),z) \\ \text{where } c^{*}(\pi,c) &= \arg\min_{c^{*}\in C}E_{\pi}\psi(c(z),c^{*}(z),z) \end{aligned}$$

[7] A function $f:C\rightarrow\Re$ admits right (resp. left) directional derivatives at c if the limit $\lim_{\alpha\downarrow 0}\frac{f(c+\alpha h)-f(c)}{\alpha}$ (resp. $\lim_{\alpha\uparrow 0}\frac{f(c+\alpha h)-f(c)}{\alpha}$) exists for all $h\in C$.

and consider the following zero-sum game problem:

$$(B) : \min_{\lambda\in\Lambda} \max_{c\in C} \int_{\pi\in M(c^{opt})} \psi^*(c,\pi) d\lambda(\pi)$$

Theorem 3 below states that under the assumptions $A1 - A4$, (B) has a unique saddle point solution $(\lambda^{\exp}, c^{opt})$ where c^{opt} is the solution to the original problem (R) and that problems (B) and (R) have the same value:

Theorem 3 *Under assumptions* $A1 - A5$,
(i) (B) *has a unique saddle point solution* $(\lambda^{\exp}, c^{opt})$
(ii) c^{opt} *is the solution to the problem* (R)
(iii) Problems (B) *and* (R) *have the same value:*

$$\int_{\pi\in P^e} \psi^*(c^{opt},\pi) d\lambda^{\exp}(\pi) = \min_{\pi\in M(c^{opt})} E_\pi \psi(c^{opt}(z), c^*(\pi, c), z)$$

The proof is an extension to the one given in Suryanarayanan (2006a) generalizing to the case where the set of minimizing probabilities is not necessarily the set of all extremal measures.

The new problem (B) has the flavor of a game against a malevolent nature as in the Robust Control problem studied by Hansen and Sargent. The decision maker computes her worst expected regret for all the extremal probability measures in $M(c^{opt})$, and for a given distribution λ over $M(c^{opt})$, she wants to minimize her worst regrets. A malevolent nature then picks the distribution $\lambda^{\exp}$ forcing the decision maker to choose the alternative that equalizes her expected regrets across all extremal probability measures as in point (iii) of Theorem 1.

Bergemann and Schlag (2005) also consider a zero-sum game formulation for their version of the Savage Minmax Regret. Because the innermost minimization to find the best counterfactual *plan* c^* involves solving an expected utility problem for each probability distribution π, problem (R) introduces strict concavity in the function $\left(\pi \to \min_{c^*\in C} E_\pi \psi(c(z), c^*(z), z)\right)$ and makes the set of minimizing measures $M(c^{opt})$ non-convex. Thus, we cannot simply invert the outermost "max" and "min" operators as in Bergemann and Schlag or in standard Minmax problems and we need to define a more subtle zero-sum game problem (B) consistent with the initial problem (R). Theorem 3 then states that it is possible to invert provided we consider the equivalent problem (B) instead and probabilities over $M(c^{opt})$.

We now provide a geometrical interpretation to illustrate the results of Theorem 1 and Theorem 3.

2.2.3 The geometrical interpretation

It will be helpful in this paragraph to relate to $Fig.$ 1 through $Fig.$ 4.

Let us consider the case of a finite state space with two elements $\{H, L\}$. The set of probability measures can be identified with a real interval of the type

$[\pi_l, \pi_h]$ contained in $[0,1]$. Each π in $[\pi_l, \pi_h]$ defines the probability of state L and $(1-\pi)$ the probability of state H. The set of feasible plans C will be identified with a convex and compact subset of the positive orthant $\Re^2_+$ of $\Re^2$.

We specialize to the case where the function ψ is of the form:

$$\psi(c(z), c^*(z), z) = u(c(z)) - u(c^*(z))$$

where u is continuous, strictly increasing and strictly concave. The decision-maker solves the problem:

$$\max_{c \in C} \min_{\pi \in [\pi_l, \pi_h]} \min_{c^* \in C} E_\pi(u(c(z), z) - u(c^*(z), z))$$

We now show how to geometrically represent the decision problem following Ferguson (1967).

- **The risk-set**

In the state-space, let us define the risk-set W:

$$W = \{(w_H, w_L) \mid \text{there exists } c \in C \text{ such that } w_z = -u(c(z), z)\}$$

The risk-set W is represented on $Fig.$ 1 as a convex subset of the state space. It fully embeds the properties of the objective function ψ and the constraint C in the decision problem and we can readily reformulate the original decision problem of finding an optimal plan c as the problem of finding an optimal point in the risk-set W by solving:

$$\min_{w \in W} \max_{\pi \in [\pi_l, \pi_h]} \max_{w^* \in W} \{\pi \cdot (w - w^*)\}$$

where $\pi \cdot w = \pi w_L + (1-\pi) w_H$ denotes the expected risk of w with respect to probability π.

- **The geometrical construction of the solution**

This problem is actually equivalent to a standard minmax problem associated with a distorted risk-set that we call the regret-risk set. To find the solution, we follow the following steps:

1. For the risk set W, compute the two Bayesian solutions $w^*(\pi_l)$ and $w^*(\pi_h)$ with respect to the two extreme measures π_l and π_h, that is solve the problems of minimizing the expected risk in W with respect to probabilities π_l and π_h (see Fig 1.):

$$\min_{w^* \in W} \{\pi_l \cdot w^*\}$$
$$\min_{w^* \in W} \{\pi_h \cdot w^*\}$$

2. Construct the regret-risk set R on the space of "extreme probabilities" $\{\pi_l, \pi_h\}$ (see $Fig.$ 2 and $Fig.$3)

$$R = \left\{ \begin{array}{c} r = (r_{\pi_l}, r_{\pi_h}) \text{ such that there exists } w \in W \text{ such that} \\ r_{\pi_l} = \pi_l \cdot (w - w^*(\pi_l)) \\ r_{\pi_h} = \pi_h \cdot (w - w^*(\pi_h)) \end{array} \right\}$$

In our case, since we only have two extreme probabilities, we identify the extreme probabilities space with the state space (see $Fig.$ 3).

3. The solution to the decision problem is given by the minmax solution for the regret-risk set R

$$\max_{\lambda \in [0,1]} \min_{r \in R} \{\lambda r_{\pi_l} + (1 - \lambda) r_{\pi_h}\}$$

Note that we may define an "expost" optimal probability measure $\pi^{\exp}$ supporting the solution of the anticipated regret problem in the Bayesian sense by

$$\pi^{\exp} = \lambda^{\exp} \pi_l + (1 - \lambda^{\exp}) \pi_h$$

where $\lambda^{\exp}$ solves the inverted Minmax problem for the regret-risk set:

$$\max_{\lambda \in [0,1]} \min_{r \in R} \{\lambda r_{\pi_l} + (1 - \lambda) r_{\pi_h}\}$$

We interpret $\pi^{\exp}$ as an expost bayesian probability measure in the sense that the Bayesian solution to the risk-set W associated with $\pi^{\exp}$, $w^*(\pi^{\exp})$, is actually equal to the solution to the original problem (see $Fig.$ 4). It will also play a central role in the recursive formulation of a dynamic extension of the decision problem.

2.2.4 Discussion

- **Minmax Regret versus Minmax**

Theorem 3 and the geometrical illustration show that there is *always* a saddle-point representation of the solution for general Minmax Regret problems like (R). In the two-state case of the geometrical illustration, this implies that the solution always lies on the 45 degree line of the Ferguson (1967) representation. This result only holds in some cases in general Minmax problems. In particular, when Z contains only two states, this will depend on the constraint C. For more complex state spaces, the choice of the probability set P will also be crucial.

The most interesting applications of ambiguity aversion require this property of the optimal solution. For example Dow and Werlang (1992) apply the result for portfolio choice inertia, Epstein and Wang (1994) for equilibrium asset pricing, Wen-Fang (1998) for risk-sharing in heterogeneous economies and Routledge and Zin (2000) for liquidity crisis. For these applications, Theorem 3 then advocates the use of Minmax Regret models instead of models of ambiguity aversion.

- **Attitude towards risk and uncertainty**

Although the decision criterion $v(c)$ does not define a preference relation in the sense of standard utility decision theory because it implicitly depends on the constraint set C, we may define indifference level curves for v. Fig. 5 represents the indifference curves in the state space when $Z = \{L, H\}$ and illustrates Corollary 2 and Theorem 3. Indeed, the indifference curves are *always* kinked at the optimum c^{opt} since v is never Gateaux-differentiable at c^{opt}.

In turn, this result shows that the decision maker displays an attitude towards risk similar to first-order risk-aversion[8] around the optimum. That is, the decision maker is highly sensitive to departures from the optimum. This property is only true for Maxmin preferences when the optimum is a saddle-point solution of the equivalent zero-sum game representation. As stated in the previous paragraph, this only happens in some cases.

First-order risk aversion has useful implications regarding the equity premium (see Epstein and Zin (1990)) as it makes departures from certain investments more costly and an investor would require a higher premium for holding risky assets. The implied risk-premium for small risky holdings can be shown to be proportional to the standard deviation of returns instead of the variance as it is the case for an expected utility investor. As pointed by Routledge and Zin (2004) however, in equilibrium asset pricing models, what would be needed are first-order risk aversion effects *away* from certainty since consumption and equity dividends are risky. This in turn would enable to lower the implied certainty equivalent and maintain a low risk-free rate. In this sense, as shown in Suryanarayanan (2006a), Anticipated Regret may have useful implications for equilibrium asset pricing, not only to deliver higher equity premium but also higher volatility.

- **Strict concavity in the regret loss function ψ**

The strict concavity property ψ was important to garanty the unicity of the optimal solution in c in Theorem 1 but also to garanty that problem (B) always has a unique saddle point solution in Theorem 3. However, when ψ is linear, these results could still hold depending on the constraints C and P.

For example consider the two-state portfolio problem of choosing the fraction α to be invested in a risky asset with uncertain returns $\{R_L, R_H\}$ $(R_L < R_H)$, the fraction $(1-\alpha)$ being allocated to a risk-free asset with return R^f $(R_L < R^f < R_H)$:

$$\max_{\alpha\in[0,1]} \min_{\pi\in[\pi_l,\pi_h]} \min_{\alpha^*\in[0,1]} E_\pi(R - R^f)(\alpha - \alpha^*)$$

where the possible probability distributions over the states $\{L, H\}$ can be identified with the interval $[\pi_l, \pi_h]$, the range of possible values for the probability of state L.

[8] The formal definition of first-order risk aversion in Segal and Spivak (1990) requires that the certainty equivalent be well defined, which is not the case for the decision criterion $v(c)$ which does not satisfy the Chew and Dekkel Betweenness property.

When $E_\pi R > R^f$ for all probabilities (or when $E_\pi R < R^f$ for all probabilities) in $[\pi_l, \pi_h]$, the optimal choice of the counterfactual portfolio α^* is always equal to 1 (or 0) regardless of the probabilities. The minimizing probability will be unique and be equal to one of the extreme points π_l or π_h.

The bigger the range $[\pi_l, \pi_h]$, the more difficult it will be to rank the expected returns relative to the risky return unambiguously for all probabilities. For some probabilities π, it will be greater ($E_\pi R > R^f$) and it will be lower for others ($E_\pi R < R^f$). This is in particular the case for the range $[0, 1]$. In that case the minimizing counterfactual portfolio α^* is probability dependent (equal to 1 when $E_\pi R > R^f$, to 0 when $E_\pi R < R^f$) and introduces strict concavity in the function $\pi \to \min_{\alpha^* \in [0,1]} E_\pi (R - R^f)(\alpha - \alpha^*)$. In such case Theorem 1 applies and the minimizing probability will be non-determined and the optimal solution α will be such that the expected regrets are equalized across the minimizing probabilities:

$$(R_L - R^f)\alpha = (R_H - R^f)(\alpha - 1)$$

which yields the strictly interior solution $\alpha = (R_H - R^f)/(R_H - R_L)$.

- **Endogenous reference point**

Last we relate the model of Anticipated Regret to Bewley's Inertia Preferences. Bewley (1986) consider incomplete preferences with the inertia assumption. Incomplete preferences can be represented with a set of probability measures P and a VNM utility function such that $c \in C$ is preferred to $c' \in C$ if and only if for every π in P :

$$\int u(c(z))d\pi(z) \;\geq \int u(c'(z))d\pi(z)$$

The inertia assumption states that any consumption point c can be made comparable to the inertia or reference point $\omega \in C$:

$$\begin{gathered}\text{Either choose } \omega \text{ or choose } c \text{ if} \\ \text{for all } \pi \in P, \int u(c(z))d\pi(z) \;\geq \int u(\omega(z))d\pi(z)\end{gathered}$$

This assumption makes the choice criterion revealed preferred.

Bewley's preferences can be shown to be observationally equivalent to the following type of preferences (Reference Point Utility) discussed in Suryanarayanan (2004) defined by:

$$U^\omega(c) = \min_{\pi \in P} \int (u(c(z)) - u(\omega(z)))d\pi(z)$$

Indeed, a decision maker with preferences defined by U^ω only chooses $c \neq \omega$ if and only if she derives greater expected utility for all probability measures in P. Otherwise the decision maker chooses to stay with the reference consumption ω. In Bewley's preferences and the Reference Point Utility, the reference point ω is given, interpreted as the default consumption as discussed in Rabin and Kosegi (2005), the consumption one would choose with the least cognitive effort. Of course, this interpretation of the reference point is rather vague, and it is hard to give insightful justifications to it. Introducing asymmetry between gains and losses relative to the reference point would be possibility and would lead back to Loss Aversion utility[9]. Alternatively, one could interpret it as a form of regret. Here the reference point is a counterfactual target with respect to which the decision maker measures her regret and chooses the consumption to avoid the worst regret.

We then can see the Anticipated Regret model as endogenizing the reference point ω. At the optimum, the decision maker is in fact at her reference point which is defined as the unique consumption point which equalizes the decision maker's expected regret across the minimizing probabilities. The reference point interpretation will prove particularly useful to develop evolutionary psychology foundations for the Anticipated Regret model in the sense of Rayo and Becker (2006).

2.2.5 The implicit and endogenous belief

Theorem 3 and the geometrical illustration deliver the expost bayesian interpretation of the decision problem (R). We define the expost distribution as

$$\pi^{\exp} = \int_{\pi \in P^e} \pi d\lambda^{\exp}(\pi)$$

We interpret $\pi^{\exp}$ as the implicit belief of the decision maker. This interpretation becomes more clear when we consider the case where ψ takes the additive form:

$$\psi(c(z), c^*(z), z) = u(c(z), z) - u(c^*(z), z)$$

In this case, we can show that an expected utility maximizer solving:

$$\max_{c \in C} E_{\pi^{\exp}} u(c(z), z)$$

would choose the worst expected regret minimizing solution c^{opt}. An interesting feature of the expost bayesian interpretation is that we may characterize the optimal solution c^{opt} by the familiar first-order conditions obtained for an expected utility maximizer:

$$E_{\pi^{\exp}} u'(c(z), z) = 0$$

except that we measure expectations with respect to the expost distribution $\pi^{\exp}$.

It is important and crucial to keep in mind however that $\pi^{\exp}$ will depend on the characteristics of the constraint set C as it reflects the counterfactual decisions and will change as C changes. This is not in general true for a subjective

[9] Kahneman and Tversky first used the term "Theory of Regret" to name their first sketches of Prospect Theory.

expected utility maximizer, her subjective probability distribution is fixed. To make this point more precise, recall that $(\lambda^{\exp}, c^{opt})$ is a saddle point solution of (B). This means that we cannot solve for c^{opt} independently from solving $\lambda^{\exp}$. Therefore, $\lambda^{\exp}$ is implicitly a function of the optimal solution c^{opt} and assigns strictly positive weights to all the extreme probability distributions in P^e. This implies that $\pi^{\exp}$ is interior to P and that there is also an implicit dependence of $\pi^{\exp}$ on the optimal solution c^{opt}. In this sense, we say that $\pi^{\exp}$ is endogenous to the decision problem.

We focus on the expost Bayesian interpretation of the anticipated regret problem (R) for two main reasons. First, in applications to equilibrium asset-pricing, the fact that there are always multiple minimizing probabilities to problem (R) in equilibrium implies that asset prices are always indetermined in equilibrium. This is a fundamental departure from a rational expectations equilibrium. We may assess this departure by using the expost probability distribution to define a shadow asset price, a particular candidate price among the multiple prices, as well as a shadow market price of ambiguity. Moreover, the constraint set C will be a typical budget constraint, therefore the expost distribution $\pi^{\exp}$ will always depend on prices. Even though we will show that the shadow equilibrium asset prices will satisfy a modified Euler pricing equation where expectations are measured with respect to $\pi^{\exp}$ instead of an exogenous probability distribution in standard expected utility, we need to keep in mind that $\pi^{\exp}$ itself will depend on the equilibrium prices. We immediately see that the anticipated regret model may have potentially and radically different implications for asset prices compared to standard expected utility preferences, even in the most simplest set-up. In particular, shadow asset returns may explain observed returns because they both embed a premium for risk as well as an additional premium for ambiguity.

Secondly, the implicit and endogenous belief enables to define a notion of a certainty equivalent for a decision maker minimizing her worst expected regrets. For simplicity, assume that the utility function u is not state dependent. For an expected utility maximizer with prior q, the certainty equivalent of the risky consumption plan $(c(z))_{z\in Z}$ is the certain consumption μ which would yield her the same utility:

$$u(\mu) = E_q u(c(z))$$

For the worst expected regret minimizer who distrusts the prior q, we may define a similar notion of certainty equivalence at the optimum c^{opt}, replacing q by the expost distribution $\pi^{\exp}$:

$$u(\mu^{opt}) = E_{\pi^{\exp}} u(c^{opt}(z))$$

Assume now that Z is discrete. We may then rewrite the implied certainty equivalent in the spirit of Hansen and Sargent as:

$$\mu^{opt} = u^{-1}\left(E_q\left(\frac{\pi^{\exp}(z)}{q(z)}\right)u(c^{opt}(z))\right)$$

In particular, assuming that u is strictly increasing, we see how distortions in the probabilities relative to the prior q measured by the ratio[10] $\frac{\pi^{\exp}}{q}$ may lower the implied certainty equivalent of the risky optimum c^{opt} compared to the expected utility case. The certainty equivalent plays a crucial role in formulating time consistent and recursive versions of infinite horizon decision problems, as in Epstein and Zin (1989) . As we will see, the expost distribution $\pi^{\exp}$ will play a similar role in formulating the problem of an infinitely lived decision maker minimizing his worst expected regrets over her entire lifetime.

3 Minimizing lifetime expected regrets

In this Section, we develop an infinite horizon extension of the anticipated regret problem seen in the previous Section. In particular, we show how to use the expost probability distribution to formulate a recursive and time consistent decision problem.

3.1 The infinite horizon decision problem

3.1.1 The choice environment

- **States, plans and controls**

We build upon the notation of the previous Sections and borrow from Epstein and Wang (1994) to describe the choice environment. In the infinite horizon economy, the driver of uncertainty is a time homogeneous and Markovian exogenous state process (z_t) which takes its values in Z, a compact metric space with Borel $\sigma-$algebra $B(Z)$. We denote by $\Pi(Z)$ the set of all Borel probability measures on Z. Under the weak-convergence topology, $\Pi(Z)$ is also a compact metric space.

We extend the state space Z to Z^{∞} the product space $\left(\prod_{t=1}^{+\infty}(Z)\right)$ with associated Borel $\sigma-$algebra $B(Z^{\infty})$. Let z^t be the vector of histories of the realizations of the exogenous state in period t, an element of the product set Z^t with Borel $\sigma-$algebra $B\left(Z^t\right)$, induced by $B(Z^{\infty})$ on Z^t. A consumption plan is a real-valued process $\left(c(z^t)\right)_t$ which is positive, $B\left(Z^t\right)-$adapted and continuous. Likewise, an investment plan is an $\Re^m$ valued process $\left(a\left(z^t\right)\right)_t$, $B\left(Z^t\right)-$adapted and continous. We combine both the consumption $\left(c(z^t)\right)_t$ and the investment plans $\left(a\left(z^t\right)\right)_t$ into a single control variable $\left(d(z^t)\right)_t = \left(c(z^t), a\left(z^t\right)\right)_t$. We assume that the control $d\left(z^t\right)$ always lies in a convex subset $\bar{D}$ of $\Re^{m+1}$.

We now also add an endogenous state space X, a convex subset of the real line $\Re$ and define an endogenous state x_t evolving according to the law of motion:

$$x_{t+1} = F(x_t, d(z^t), z_t, z_{t+1})$$

[10] When Z is continuous, if we assume that $\pi^{\exp}$ is $B(Z)-$absolutely continuous with respect to q, we may measure the distortions in the probabilities by the Radon-Nikodym derivative $\frac{d\pi^{\exp}}{dq}$ of $\pi^{\exp}$ with respect to q.

where F is the state transition function, mapping $X \times \bar{D} \times Z \times Z$ into the real line. We assume that F is linear (not necessarily jointly) in x_t, $c(z^t)$ and $a(z^t)$ and increasing in x_t. In each period, x_t together with the realization of the exogenous state z_t will fully summarize the economic environment. Thus, the control variable d will then only be a function of the pair (x_t, z_t). We will interchangeably use the notation $d(x_t, z_t)$ and $d(z^t)$.

Last, we define the constraint set D for the controls as a continuous correspondence which maps the endogenous and exogenous states to a convex and compact subset of $\Re^{m+1}$:

$$\begin{aligned} X \times Z &\rightarrow 2^{\bar{D}} \\ (x, z) &\rightarrow D(x, z) \end{aligned}$$

- **Ambiguity**

The decision maker cannot precisely assess the true probability distribution of the state (z_t). She doubts about her prior π^* about the Markovian exogenous state (z_t) and surrounds her one period ahead probability distribution π_z^* conditional on state z with the set $P(z)$, where P is a convex, compact valued and continuous correspondence mapping the exogenous state z into a subset of $\Pi(Z)$:

$$\begin{aligned} Z &\rightarrow 2^{\Pi(Z)} \\ z &\rightarrow P(z) \end{aligned}$$

P is the belief correspondence used in Epstein and Wang (1994). In the asset pricing application where the exogenous state can only take two values L and H, respectively representing the event of a recession and a boom, we will assume that the decision maker perceives more ambiguity in recessions relative to booms, differentiating $P(L)$ from $P(H)$.

For given realizations of the exogenous state until period t, $(z_0, z_1, ...z_t)$, plausible values for the probability $p^t(z^t)$ of history z^t can be expressed in the form of a product of conditional probabilities

$$p^t(z^t) = p_{t-1}(z_t) \times p_{t-2}(z_{t-1})... \times p_0(z_1)$$

where $p_s(\cdot)$ (for $s = 0, ..., t-1$) lies in $P(z_s)$. We will therefore identify a plausible probability p^t distribution over Z^t given z^{t-1} by the $t-$dimensional vector of conditionals $(p_0, ...p_{t-2}, p_{t-1})$. As in Section 2, we define for any compact subset M of $P(z)$ with the induced metric, the associated Borel σ-algebra $B(M)$ and $\Lambda_M(z)$ the set of all possible Borel probability measures on M.

3.1.2 The decision problem

We adopt a different approach from Suryanarayanan (2006a) to define the infinite horizon decision problem for a decision maker who wish to minimize her worst expected lifetime regrets. We start by defining the current period utility

u, a continuous, strictly increasing and strictly concave mapping from $\Re^+$ to $\Re$ with which the decision maker ranks current consumption in each period. We then define the intertemporal decision function recursively as follows.

Let us consider the sequence of functions v_t, continuous on the space of consumption plans and $B(Z_t)$-measurable, and each recursively defined for a consumption plan c and a history of states z^t as:

$$(RV)\ \ v_t(c, z^t) = u(c_t) + \beta E_{\pi_t} v_{t+1}(c, (z^t, z_{t+1}))$$

where

$$\begin{aligned}
\beta &< 1 \text{ and } \pi_t = \int_{\pi \in M_t} \pi d\lambda_t(\pi) \\
M_t &= \left\{ \pi \in P(z_t) \mid \psi_t(c, \pi) = \arg \min_{\pi \in P(z_t)} \psi_t(c, \pi) \right\} \\
\psi_t(c, \pi) &= u(c_t) - u(c_t^{*\pi}) + \beta E_\pi (v_{t+1}(c, z^{t+1}) - v_{t+1}(c^{*\pi}, z^{t+1})) \\
d^{*\pi} &= \arg \min_{\substack{(d_t^*)_t \\ d_t^* \in D(x_t, z_t),\ d_s^* \in D(x_s^*, z_s) \text{ for } s>t,\ x_s^* = F(x_{s-1}^*, d_{s-1}^*, z_{s-1}, z_s) \text{ and } x_t^* = x_t}} \\
&\quad \{u(c_t) - u(c_t^{*\pi}) + \beta E_\pi (v_{t+1}(c, z^{t+1}) - v_{t+1}(c^{*\pi}, z^{t+1}))\} \\
\lambda_t &= \arg \min_{\lambda \in \Lambda_{M_t}(z_t)} \max_{\substack{(d_t)_t \\ d_t \in D(x_t, z_t)}} \int_{\pi \in M_t} \psi_t(c, \pi) d\lambda(\pi)
\end{aligned}$$

v_t will be interpreted as the time t decision criterion with which the decision maker ranks the stream of future consumption ${}^t c = (c_t, c_{t+1}, ...)$ conditional on the past history z^{t-1}.

The following proposition shows that there exists a unique sequence $(v_t)_t$ satisfying the above recursion.

Proposition 4 *(i) There exists a unique sequence $(v_t(c, z^t))_t$ of $B(Z_t)$-measurable and continuous functions satisfying the recursion (RV)*

(ii) In particular, $v(c, z_0) = v_0(c, z^0)$ satisfies

$$\begin{aligned}
v_t(c, z^t) &= v({}^t c, z^t) \\
v({}^t c, z_t) &= u(c_t) + \beta E_{\pi_t} v({}^{t+1} c, z_{t+1})
\end{aligned}$$

(iii) $v(c, z_0)$ is continuous, strictly increasing and concave in c

The proof is an extension of that given in Suryanarayanan (2006a) (Appendix to Section 4.2) for the case when the set of minimizing measures is not always equal to the set of extremal measures.

We name v_t the recursive decision function and v the intertemporal decision function. The associated intertemporal decision problem is then:

$$(DR) : \max_{\substack{(d) \\ d_t \in D(x_t, z_t)}} v(c, z_0)$$

3.2 Time consistency

The formulation of the decision problem we consider is one way to define a time consistent problem for a decision maker who wishes to minimize her lifetime expected regrets. The intertemporal decision function $v(c, z_0)$ is the shadow discounted future utility flows in state z_0 and likewise $v({}^t c, z_t)$ measures the discounted future utility flows at date t and state z_t conditional on the past history z^{t-1}.

The interpretation to the decision problem (DR) is then the following. In each period t and for each possible one-period ahead conditional probability measure π in $P(z_t)$, the decision maker compares the discounted future utility flows derived from plan c and that derived from the counterfactual plan c^* and forms the intertemporal regret $\psi_t(c, \pi)$ for the best possible counterfactual alternative given the probability measure π. The decision maker then minimizes her worst regret over all possible plans and conditional measures in $P(z_t)$. Thanks to the recursive definition of v, such a decision protocol is time consistent in the sense that choosing the optimal plan at each time t by maximizing $v({}^t c, z_t)$ conditional on history z^{t-1} is equivalent to choosing the lifetime plan c at time 0 by solving (DR).

We further discuss the time consistency of the decision problem by characterizing the decision problem (DR) in an alternative way. For each period t and for each possible probability distribution p_t for the future realizations of the exogenous state, we define the intertemporal regret of the decision maker as the loss experienced when the decision maker compares the expected present value of utility flows derived from a consumption plan $(c_s(z^s))_{s\geq t}$ to that derived from an alternative counterfactual plan $(c_s^*(z^s))_{s\geq t}$:

$$\begin{aligned} & w_t^{p_t}((c_s(z^s))_{s\geq t}, (c_s^*(z^s))_{s\geq t}, z^t) \\ = & \sum_{s=t}^{+\infty} \beta^{s-t} \int_{z^s \in Z^s} (u(c_s^*(z^s)) - u(c_s(z^s)))dp_t^s(z^s) \\ \text{where } p_t^s = & \; p_{s-1}^s(z_s) \times p_{s-2}^s(z_{s-1}) ... \times p_t^s(z_{t+1}) \text{ and } dp_t^t(z^t) = 1 \end{aligned}$$

Let (DR_t) be the associated decision problem, given (x_t, z_t):

$$(DR_t): \min_{(d_s)_{s\geq t}} \max_{p} \max_{(d_s^*)_{s\geq t}} w_t^{p_t}((c_s(z^s))_{s\geq t}, (c_s^*(z^s))_{s\geq t}, z^t)$$

Subject to:

1. $d_s \in D(x_s, z_s)$ and $d_s^* \in D(x_s^*, z_s)$ for $s \geq t$
2. $x_s = F(x_{s-1}, d_{s-1}, z_{s-1}, z_s)$, for all $s \geq t$ and x_t is a given state in X
3. $x_t^* = x_t$ and $x_s^* = F(x_{s-1}^*, d_{s-1}^*, z_{s-1}, z_s)$, for all $s \geq t$
4. $p_{s-k}^s \in P(z_{s-k})$ for all $s \geq t$ and $k \in \{0, ..., s-t\}$

In the above decision problem, note that the decision maker considers all possible future paths for the counterfactual endogenous state x_s^* given that the choice environment in the current period is characterized by (x_t, z_t). In order to relate the sequence of problems (DR_t) to the original decision problem (DR), we need to define the notion of time consistency within our context:

Definition 5 *Let $\overline{d}^t$ be the solution of problem (DR_t). We say that $\overline{d}^t$ is time consistent if $\left(\overline{d}_l^t\right)_{l\geq s}$, the continuation of $\overline{d}^t$ from period s is solution to the problem (DR_s) for all s greater than t.*

The problem (DR) is then equivalent to the following problem (see the Appendix in Suryanarayanan (2006 a)):

$$(DR_0): \min_{(d_s)_{s\geq 0}} \max_{p_0} \max_{(d_s^*)_{s\geq 0}} w_0^{p_0}((c_s(z^s))_{s\geq 0}, (c_s^*(z^s))_{s\geq 0}, z_0)$$

Subject to:

1. $d_s \in D(x_s, z_s)$ and $d_s^* \in D(x_s^*, z_s)$ for $s \geq 0$
2. $x_s = F(x_{s-1}, d_{s-1}, z_{s-1}, z_s)$, for all $s \geq 0$ and x_0 is a given state in X
3. $x_0^* = x_0$ and $x_s^* = F(x_{s-1}^*, d_{s-1}^*, z_{s-1}, z_s)$, for all $s \geq 0$
4. $p_{s-k}^s \in P(z_{s-k})$ for all $s \geq 0$ and $k \in \{0, ..., s\}$
5. Time consistency: $(d_s)_{s\geq t}$ solves (DR_t) for all t

With the time consistency requirement 5, we see that the intertemporal decision problem will only involve the endogenous state variable x_s besides the exogenous state z_s. This is not the case when we take each problem (DR_t) separately, where both x_s and the counterfactual endogenous state x_s^* are involved. The decision maker keeps no memories of the past realizations of the counterfactual endogenous state and only takes into account the realization of the actual endogenous state x_t in each period t to form future counterfactuals for the subsequent periods.

We show how to formulate the decision problem (DR) recursively and how to construct a *shadow* present value of future utility flows $V(x, z)$ consistent with the decision problem. While in Suyanarayanan (2006a), we make the assumption that the set of minimizing one-period ahead conditional probabilities in each period t in (DR) is the set of all extremal measures $P^e(z_t)$ of $P(z_t)$, we develop the recursive formulation without this restriction.

3.3 The recursive formulation

Let $\Gamma(X, Z)$ be the set of all continuous and $B(\Re \times Z)-$measurable functions mapping $X \times Z$ to the real line $\Re$. For any function J in $\Gamma(X, Z)$, consider the problems $(R1)$ and $(R2)$

$$\begin{aligned} & (R1): \max_{d\in D(x,z)} \min_{p\in P(z)} \min_{d^*\in D(x,z)} \\ & \{u(c)-u(c^*)+\beta E_p\left(J(x',z')-J(x^{*\prime},z')\right)\} \\ \text{where } x' &= F(x,d,z,z'),\ x^{*\prime}=F(x,d^*,z,z') \end{aligned}$$

$$\begin{aligned} & (R2): \min_{\lambda\in\Lambda_M(z)} \max_{d\in D(x,z)} \int_{\pi\in M} R^*(\pi,d,x,z)d\lambda(\pi) \\ & R^*(\pi,d,x,z)=u(c)-u(c^*_\pi)+\beta E_\pi\left(J(x',z')-J(x^{*\prime}_\pi,z')\right) \\ M &= \left\{\pi\in P(z)\mid R^*(\pi,d,x,z)=\arg\min_{p\in P(z)} R^*(p,d,x,z)\right\} \end{aligned}$$

where $x'=F(x,d,z,z')$ and $x^{*\prime}_\pi=F(x,d^*_\pi,z,z')$, and for each π in M, $d^*_\pi=(c^*_\pi,a^*_\pi)$ solves:

$$\max_{d^*\in D(x,z)} \{u(c^*)+\beta E_\pi J(x^{*\prime},z')\}$$

Problem $(R1)$ is a special case of the one-period anticipated regret problem and $(R2)$ is the associated zero-sum game problem seen in Section 2. The function J will refer to a measure of present value of future utility flows for the decision maker. When solving $(R1)$, the decision maker minimizes her worst intertemporal regrets in state (x,z). These two problems are the building blocs for defining the recursive formulation of the decision problem (DR).

We define Ψ the largest subset of functions J in $\Gamma(X,Z)$ for which $(R1)$ is well defined and has a unique solution in d, $(R2)$ is equally well defined, has a unique saddle point solution $(d,\lambda^{\exp})$ where d is also the solution to $(R1)$, and yields the unique expost probability distribution $\pi^{\exp}(x,z)$ associated with $(R1)$:

$$\pi^{\exp}(x,z)=\int_{\pi\in P^e(z)} \pi d\lambda^{\exp}(\pi)$$

According to Section 2, the set Ψ includes the set of functions J that are strictly increasing and concave in their first argument. For functions J in Ψ, denote with $\overline{d}$ the common solution to $(R1)$ and $(R2)$ and define $G(J)$ such that:

$$\begin{aligned} G(J)(x,z) &= u(\overline{c}(x,z))+\beta E_{\pi^{\exp}(x,z)}J\left(\overline{x}',z'\right) \\ \text{where } \overline{x}' &= F(x,\overline{d}(x,z),z,z') \end{aligned}$$

We name G the "intertemporal regret regulator" operator.

The following proposition states that G has a unique fixed point V and that solving problem $(R1)$ for V gives the solution to the intertemporal regret problem (DR).

Proposition 6 (i) *Any solution to the intertemporal regret problem* (DR) *is a fixed point of the operator* G.

(ii) G has a unique fixed point V. Solving $(R1)$ for V gives the value and policy functions to the infinite horizon regret problem (DR) :

$$V(x,z) = \max_{\substack{(d) \\ x_0=x,\ z_0=z,\ d_t\in D(x_t,z_t)}} v(c,z)$$

The proof of this proposition uses standard results for discounted dynamic programming and is an extension of the Proposition 4 in Suryanarayanan (2006a).

The operator G enables to compute the solution to problem (DR) recursively as follows:

1. Start with an initial guess $V(x,z)$
2. Solve $(R2)$ which yields $\left(\overline{d}, \pi^{\exp}\right)$
3. Replace V with: $G(V)(x,z) = u(\overline{c}(x,z)) + \beta E_{\pi^{\exp}(x,z)} V(\overline{x}', z')$ where $\overline{x}' = F(x, \overline{d}, z, z')$
4. Iterate the procedure until convergence

We may then summarize the recursive decision problem as:

$$\begin{aligned} R(x,z) &= -\max_{d\in D(x,z)} \min_{p\in P(z)} \min_{d^*\in D(x,z)} \\ &\quad \{u(c) - u(c^*) + \beta E_p\left(V(x',z') - V(x^{*\prime},z')\right)\} \\ \text{where } x' &= F(x,d,z,z'),\ x^{*\prime} = F(x,d^*,z,z') \end{aligned}$$

$$\begin{aligned} V(x,z) &= u(\overline{c}(x,z)) + \beta E_{\pi^{\exp}(x,z)} V\left(\overline{x}', z'\right) \\ \text{where } \overline{x}' &= F(x, \overline{d}(x,z), z, z') \end{aligned}$$

where $R(x,z)$ is the intertemporal regret value and $V(x,z)$ is a measure of present value of future utility flows for the decision maker.

3.4 Discussion

3.4.1 The shadow value function $V(x,z)$

We define the shadow value function $V(x,z)$ as:

$$\begin{aligned} V(x,z) &= u(\overline{c}(x,z)) + \beta E_{\pi^{\exp}(x,z)} V\left(\overline{x}', z'\right) \\ \text{where } \overline{x}' &= F(x, \overline{d}(x,z), z, z') \end{aligned}$$

where $\overline{d}$ is the optimal solution to the intertemporal regret problem (DR). Iterating on the above recursive relation, we interpret $V(x,z)$ as a measure of present value of future utility flows for the decision maker. Note how the expost probability distribution $\pi^{\exp}(x,z)$ plays the central role in defining $V(x,z)$ as it enables to bridge the current period utility with the subsequent period utilities, like the certainty equivalent operator in standard dynamic programming with time additive expected utility, or more generally as in Epstein and Zin (1989). This in turn enables the optimal policy $\overline{d}$ to satisfy the time consistency requirement in problem (DR).

Thus, $V(x_t, z_t)$ embeds all current and subsequent decisions from period t onwards when the current state is (x_t, z_t). In order to be consistent with her future decisions, the decision maker evaluates her expected present value of future utility flows for plausible one-period ahead conditional probabilities $p \in P(z_t)$ and current period choice $d(x_t, z_t)$ as:

$$u(c(x_t, z_t)) + \beta E_p V(x_{t+1}, z_{t+1})$$

where $x_{t+1} = F(x_t, d(x_t, z_t), z_t, z_{t+1})$, and her intertemporal regret for choosing $d(x_t.z_t)$ instead of the counterfactual alternative d^* as:

$$u(c^*) - u(c(x_t, z_t)) + \beta E_p \left(V(x^*_{t+1}, z_{t+1}) - V(x_{t+1}, z_{t+1}) \right)$$

where $x^*_{t+1} = F(x_t, d^*, z_t, z_{t+1})$. As in the three-period example, x^*_{t+1} is the counterfactual future endogenous state in period $t+1$ if d^* were chosen instead of d and z_{t+1} is realized. The decision maker then minimizes her worst intertemporal regrets over all plausible one-period ahead conditional distributions p in $P(z_t)$ and counterfactual alternatives d^* in each period t and each state (x_t, z_t).

3.4.2 The endogenous belief $\pi^{\exp}(x, z)$

As in the static problem in Section 2, we interpret the expost one-period ahead conditional distribution $\pi^{\exp}(x, z)$ as the implicit endogenous and conditional belief of the decision maker in state (x, z). The expost distribution is endogenous in the sense that it will depend implicitly on the current policy functions and it embeds the shadow value function $V(x, z)$. In turn, this means that the implied certainty equivalent of a risky stream of consumption is sensitive to the present value of future utilities derived from the stream. As we will see, this property will be crucial in applications to equilibrium asset pricing.

3.4.3 Alternative dynamic extensions

In a finite horizon 3 period setting, we discuss two possible alternative dynamic extensions of the one-period Anticipated Regret problem studied in Section 2.

A first alternative formulation would be to consider only the problem (DR_0). The decision maker evaluates her worst expected regrets at time 0 considering all possible consequences of choosing alternative plans d^*_0 in $D(x_0, z_0)$ and d^*_1 in $D(x^*_1, z_1)$ where x^*_1 is the would-be endogenous state if d^*_0 is chosen. Problem (DR) adds the additional time consistency requirement to problem (DR_0). The decision maker has then no memory of consequences of past counterfactual alternatives and only cares about the future consequences of future alternatives and counterfactuals given the choice environment characterized by the endogenous and exogenous state in each period. This feature differentiates (DR) from (DR_0). In particular, a recursive formulation of problem (DR_0) would involve the two endogenous states x_t and x_t^* whereas we only need one endogenous state x_t to define the recursive version of (DR).

The second alternative dynamic extension would have the decision maker only consider future counterfactuals for the actual realizations of the endogenous state and may be expressed as:

$$(D)\text{:}\ \max_{(d_t)} \min_{p \in P^3} \min_{(d_t^*)_t} \sum_{t=0}^{2} \beta^t \sum_{z^t \in Z^t} p(z^t)(u(c_t(z^t)) - u(c_t^*(z^t)))$$

Subject to:

1. $d_t \in D(x_t, z_t)$ and $d_t^* \in D(x_t, z_t)$ for $t = 0, 1$
2. $c_2 = F(x_1, d_1, z_1, z_2)$, $c_2^* = F(x_1, d_1^*, z_1, z_2)$ and $a_2 = a_2^* = 0$
3. $x_t = F(x_{t-1}, d_{t-1}, z_{t-1}, z_t)$, for all $t \geq 1$ and x_0 is a given state in X

Problem (D) is closer in spirit to our recursive formulation as only one endogenous state x_t is involved in the decision problem. However, problem (DR) requires intertemporal regrets to be equalized across extreme probability distributions in both periods 0 and 1 whereas this will be only the case in period 0 for (D). In particular, the expost distribution associated with problem (D) will in general differ from the product of one-period ahead expost conditional distributions $\pi_0^{\exp}(x_0, z_0)\times\ \pi_1^{\exp}(x_1, z_1)$ associated with problem (DR).

An interesting feature in the two alternative dynamic extensions (DR_0) and (D) is that one can define a time zero objective function for the decision maker independently of the constraints as a dynamic extension of the utility function in Fishburn (1982) and define a worst regret intertemporal decision function by solving the two inner minimization problems like the decision function v in the static problem. Instead, we formulate problem (DR) via the (indirect) value function $V(x_0, z_0)$ and the recursive decision criterion v.

4 Concluding remarks

This paper develops a solid framework for applying the model of Anticipated Regret and Endogenous Beliefs by further clarifying and generalizing the results in Suryanarayanan (2006a).

Preliminary theoretical results provide a systematic algorithm to find the solution to the decision problem and show how models of Minmax Regret differs from models of ambiguity aversion and expected utility. In particular, the solution to the decision problem can *always* be represented as a *saddle point* solution to an equivalent zero-sum game problem. This new problem jointly produces the solution to the Anticipated Regret problem and the endogenous belief. We then use the endogenous belief to define the implicit certainty equivalent and to build an infinite horizon and time consistent problem for a decision maker minimizing her lifetime worst expected regrets.

The model of Anticipated Regret carries important implications both from a positive and normative viewpoint. Positive investigations include Suryanarayanan (2006a) which shows how endogenous distortions beliefs helps to generate higher and time-varying market prices of risk and volatility, as well as to match the pricing kernel implied by observing the historical mean and volatility of aggregate stock market indices and treasury bills. Further research should focus on the judicious choice of probability sets that would preserve tractability in continuous state spaces in applications to asset pricing in order to better assess the performance of the model. An interesting line of research would be to investigate the potential of the model to generate lower Euler equations pricing errors. As the Anticipated Regret directly lowers the implied certainty equivalent and dampens the sensitivity of policy rules to returns to investment, the model also carries important implications for the investment and savings decision problems, both for households (see Chamberlain and Wilson (1984)) and for firms. Other positive investigations include Bergemann and Schlag (2005) and more recently Gallice (2006) who conducts lab experiments of one-shot two-player games where players cannot forecast their opponent's move. She shows that a typical player's strategy is best approximated by the one implied by the version of Savage's minmax regret used in Bergemann and Schlag (2005). She further notes that the Nash equilibrium strategy is not a relevant approximation of players' behavior.

From a normative viewpoint, Regret theory has been used by econometricians (see Sawa and Hiromatsu (1973) to compute endogenous critical points defining test regions in regression analysis. The use of the Savage Minmax Regret criterion in this context improves the performance of the test in terms of square error loss, crucial for model selection. Droge (1998) relates Regret theory to Stein's shrinkage estimator. OLS estimators are the best linear unbiased estimators (BLUE) but they are not the estimators which minimize squared error losses. In many problems, especially in large scale estimations of asset returns and associated asset allocation problems (see Brandt (2004)), a significant gain in improving efficiency of estimators is worth a minor loss in their unbiasedness. Droge shows within a simple example that efficient estimators in the sense of minimizing the quadratic regret loss improves significantly the efficiency of the estimator with only a minor loss in unbiasedness. The theoretical results presented in this paper pertaining to Statistical Decision Theory will help to give solid foundations to model selection problems and large scale estimation problems as in Droge (1984).

Manski (2004) focuses on policy problems and considers an utilitarian social planner who must select an optimal treatment rule using the sample data generated by a classical randomized experiment. When only sample data on treatment response are available, Manski investigates the use of a version of the Minmax Regret Criterion as the objective function of the social planner. He shows in particular that the use of available covariate information is only optimal if the sample size is sufficiently large enough. More generally, a policy maker may wish to look for robust policy rules in situations when, as stated by Manski (2005), full information on the future effect of a policy on population behavior is not available. By generalizing Manski (2004) for intertemporal problems, the Anticipated Regret model may be one way to provide such rules.

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Fig. 1 Risk-set and construction of reference points

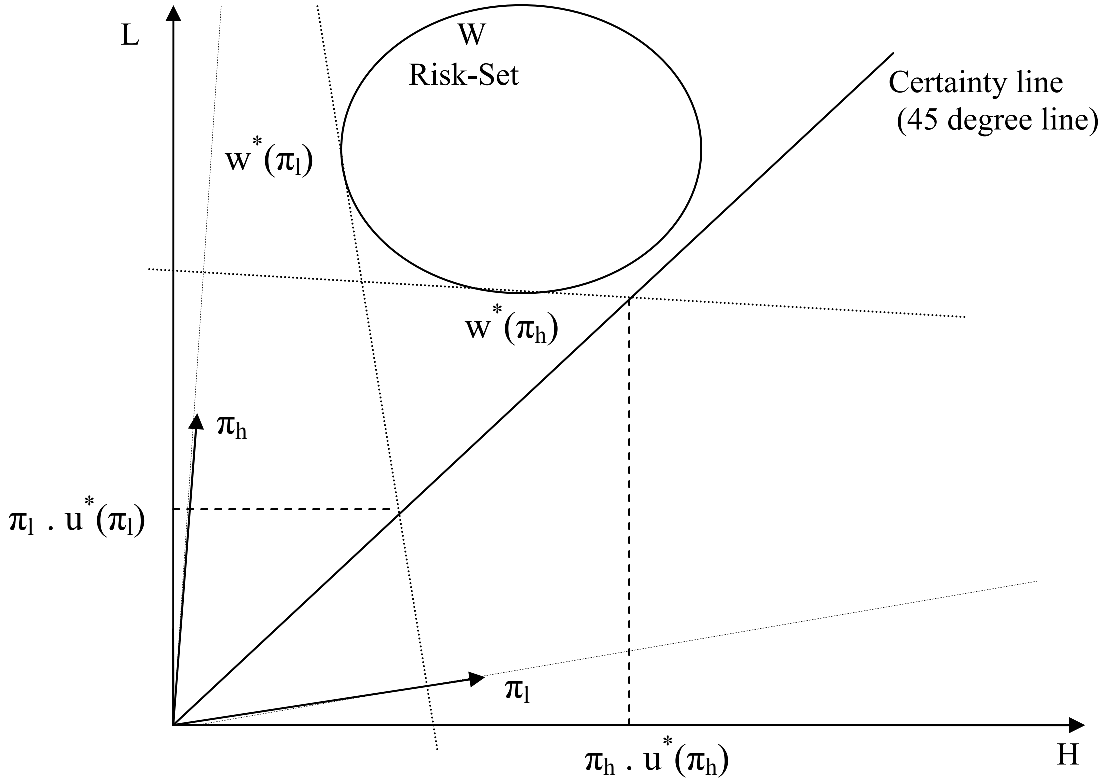


Fig. 2 Construction of the regret-risk set

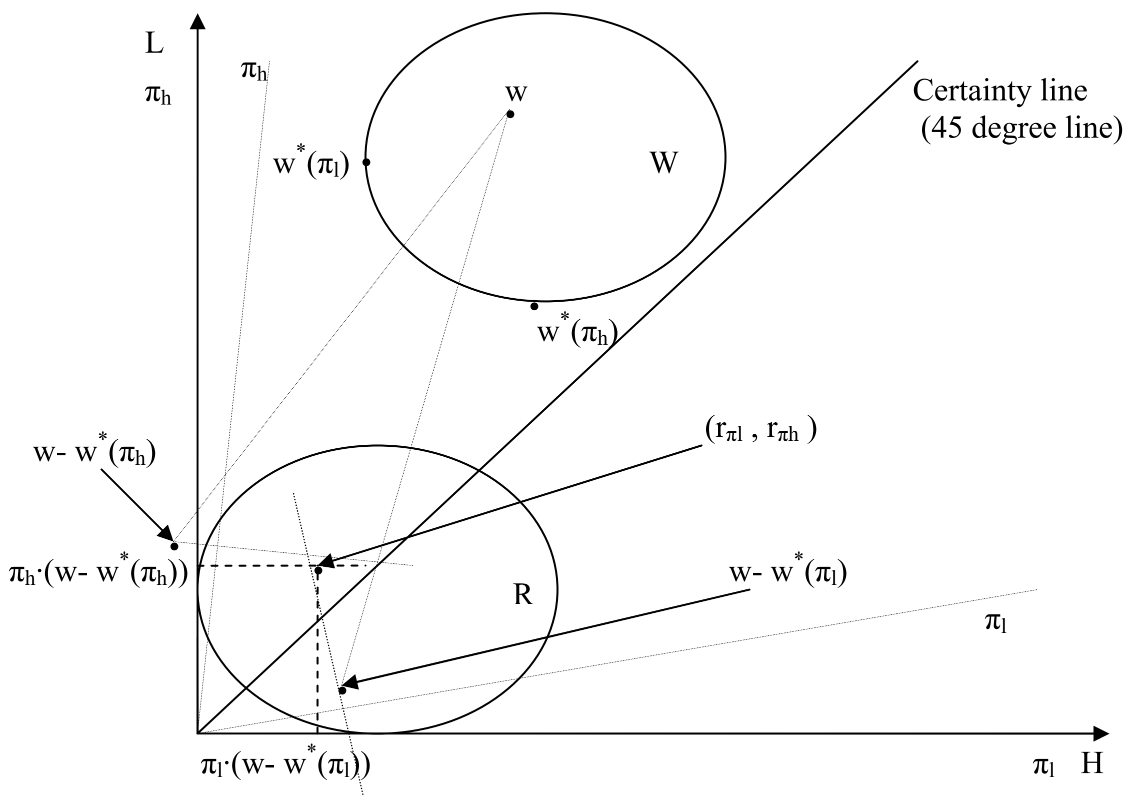


Fig. 3 Construction of the solution

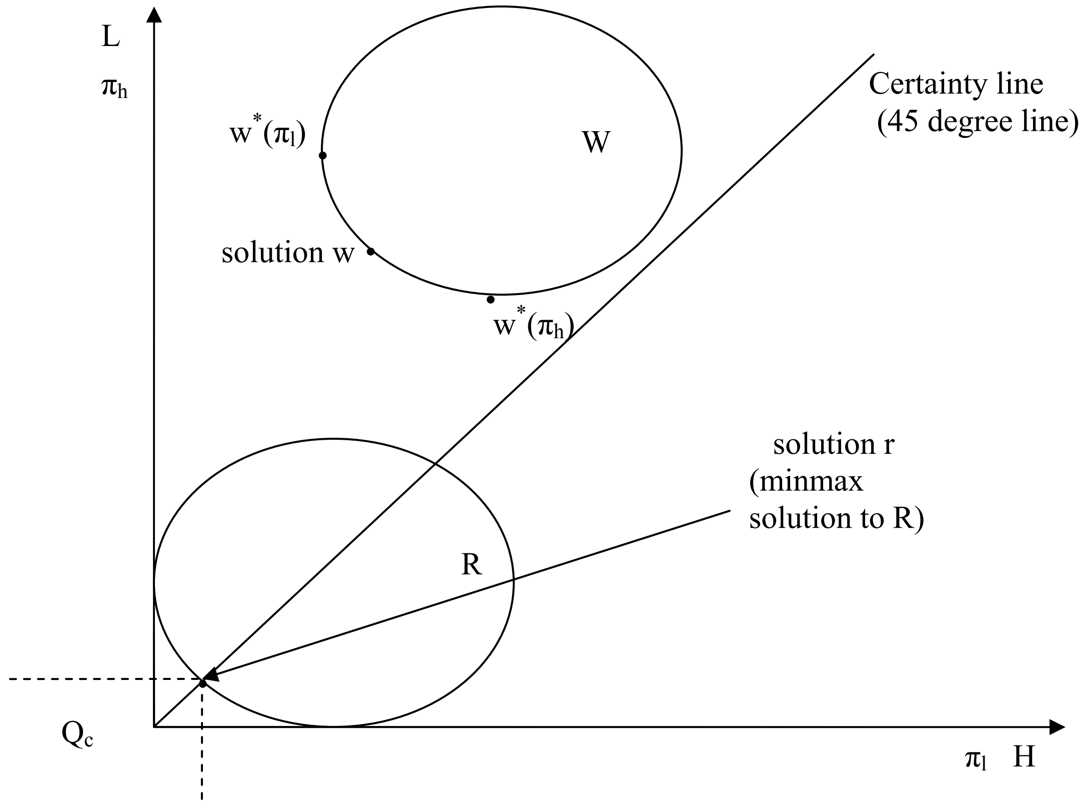
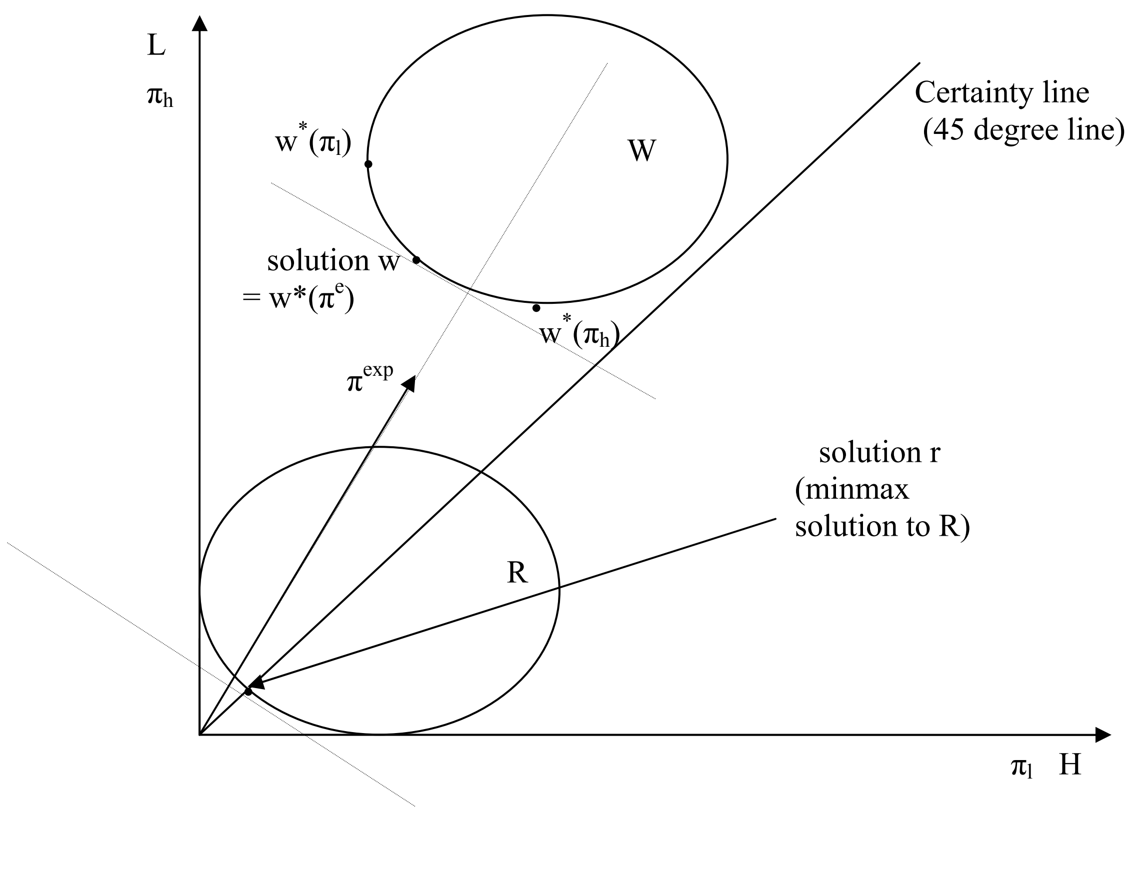


Fig. 4 Expost Bayesian interpretation



Indifference curve