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One Proposition about Dynamic Portfolio Selection in an Open Economy and International Diversification

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January 2008

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Abstract

This paper describes one proposition about dynamic Markowitz portfolio selection in an open economy. Here it is proved that, assuming that two countries in an open economy share the same risk absolute aversion coefficient and the same information set with some conditions, the portfolio each country holds always attains the same rate of return, regardless of the characteristics of each country's risky asset market, of the proportion in each country's personal asset holdings, of the characteristics of the exchange rate price process, or of the risk free rate in each country. One basic implication of this proposition is that, when two countries share the common information set, each country might be, under these non-general conditions, indifferent, regarding the allocation of home/foreign risky assets, to the diffusion of exchange rate price process. Finally, I discuss another implication of this proposition in the relation with international portfolio diversification and so called "the home bias puzzle".

1. Model and proposition

Organization of an open economy

We assume an open economy with two countries, two distinct risk-free assets, and two distinct, risky assets, one of which represents respectively one of two countries. Each risk-free asset is regarded as each country 's own currency, and is also exchangeable with each other. (So, needless to say, each country may hold the other country's assets.) Figure 1 shows the organization of this economic system. Assume the discrete time flow, t=0,1,2,…. Also, we denote each country, a representative economic agent, by k=1,2. In addition, the proportion of each country in its personal asset holdings as of time t ($A_{k,t}$), being p for

country 1 and $1-p$ for country 2.[1]

Price processes

In country 1 there exist one risk-free asset, denoted by fa, and one risky asset, rb, while in country 2 one risk-free asset, fc, and one risky asset, rd, do. Defining the allocation of each asset (fa, rb, fc, rd) for country k (k=1,2) respectively as $z_{k,a,t}$, $z_{k,b,t}$, $z_{k,c,t}$ and $z_{k,d,t}$, then the price process of each asset (fa, rb, fc, rd) as of time t are assumed to be represented as following:[2] [3]

$$\Delta \ln S_{a,t} = r_a \Delta t$$

$$\Delta \ln S_{b,t} = v_b \Delta t + g_{b,t-1}{}^T \gamma_b + \sigma_b \Delta B_{b,t} + \eta_b \cdot \{ p \cdot \Delta z_{1,b,t} + (1-p) \cdot \Delta z_{2,b,t} \}$$

$$\Delta \ln S_{c,t} = r_c \Delta t$$

$$\Delta \ln S_{d,t} = v_d \Delta t + g_{d,t-1}{}^T \gamma_d + \sigma_d \Delta B_{d,t} + \eta_d \cdot \{ p \cdot \Delta z_{1,d,t} + (1-p) \cdot \Delta z_{2,d,t} \} \quad (1.1)$$

Here, η_b and η_d are the demand elasticities of the market prices, $g_{b,t-1}{}^T$ and $g_{d,t-1}{}^T$ are the row vectors containing complete information sets regarding the price processes, $\Delta B_{b,t}$ and $\Delta B_{d,t}$ are the standard Brownian motions, v_b and v_d are the drifts, σ_b and σ_d are the diffusions, for risky asset rb and rd, respectively. Let $\{\Delta \ln S_{ex,t}\}_{t \to \infty}$ denote the exchange rate price process of the risk-free asset in country 1 (fa) to the risk-free asset in country 2 (fc). Then,

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[1] As seen later, the self-financing and utility maximizing portfolio selection strategy keeps the asset proportion of each country (p) automatically constant under some general assumptions of the proposition.

[2] $z_{k,a,t} + z_{k,b,t} + z_{k,c,t} + z_{k,d,t} = 1$ (for k=1,2 and for all t.) .

[3] The notation, Δ, represents the difference of time t+1 from time t.

$$\Delta \ln S_{ex,t} = v_{ex}\Delta t + \sigma_{ex}\Delta B_{ex,t} + \eta_{ex}\{-p\cdot[\Delta z_{1,c,t} + \Delta z_{1,d,t}] + (1-p)\cdot[\Delta z_{2,a,t} + \Delta z_{2,b,t}]\}\ (1.2)\ ^4$$

Estimation and forecast

Assume that the price process of a risky asset (denoted by r=rb, rd, say), $\{\Delta \ln S_{r,t}\}_{t\to\infty}$ is already revealed until time t-1. Then define its past history and the explanatory variable (information set) matrix as following.

$$y_{t-1} \equiv \begin{bmatrix} \Delta \ln S_{r,t-1} \\ \Delta \ln S_{r,t-2} \\ \vdots \\ \Delta \ln S_{r,t-T_e} \end{bmatrix} \quad \text{(T}_e\text{*1 column vector of the dependent variable.)}$$

$$X_{t-1} \equiv \begin{bmatrix} x_{t-1}^T \\ x_{t-2}^T \\ \vdots \\ x_{t-T_e}^T \end{bmatrix} \equiv \begin{bmatrix} x_{1,t-1},\cdots,x_{J,t-1} \\ x_{1,t-2},\cdots,x_{J,t-2} \\ \vdots \\ x_{1,t-T_e},\cdots,x_{J,t-T_e} \end{bmatrix} \quad \text{(T}_e\text{*J explanatory variable matrix.)}$$

$$(1.3)$$

Here, T_e (t-1, t-2,..., t- T_e) denotes the past periods on which the estimate is based, and X_{t-1} is a common explanatory variable matrix for each country, k=1,2. The first column vector of X_{t-1} is a summer vector, all elements of which are 1's. For one example, the first row vector of X_{t-1}, $x_{t-1}^T \equiv [x_{1,t-1},\cdots,x_{J,t-1}]$, contains 1, $g_{b,t-1}^T$, $g_{d,t-1}^T$, $p\cdot\Delta z_{1,b,t}$, $p\cdot\Delta z_{1,c,t}$, $p\cdot\Delta z_{1,d,t}$, $(1-p)\cdot\Delta z_{2,a,t}$, $(1-p)\cdot\Delta z_{2,b,t}$ and $(1-p)\cdot\Delta z_{2,d,t}$. We also define the above matrix and apply them to the following Neoclassical Regression Model.

$$y_{t-1} \sim N(X_{t-1}\beta_r, \sigma_t^2 I) \tag{1.4}$$

where X_t is non-stochastic and rank(X_{t-1})=J, and I is an identity matrix.

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[4] The no-arbitrage condition imposes an additional restriction, $v_{ex} = -(r_c - r_a)$.

We assume that each country, k=1,2, believes that this model applies. However, it does not have to be true. Then, defining $Q_{t-1} \equiv X_{t-1}{}^T X_{t-1}$, $A_{t-1} \equiv Q_{t-1}{}^{-1} X_{t-1}{}^T$, $N_{t-1} \equiv X_{t-1} A_{t-1}$, $M_{t-1} \equiv I - N_{t-1}$ and $e_{t-1} \equiv M_{t-1} y_{t-1}$, the estimated value of the variance of return $(\hat{\sigma}_t^2)$ can be represented as follows.

$$\hat{\sigma}_t^2 = \frac{e_{t-1}{}^T e_{t-1}}{T_e - J} \tag{1.5}$$

The forecast value of the expected rate of return as of time t $(\hat{v}_t)$ can be represented as follows.

$$\hat{v}_t = x_t{}^T Q_{t-1}{}^{-1} X_{t-1}{}^T y_{t-1} \tag{1.6}$$

where $x_t^T \equiv \left[x_{1,t}, \cdots, x_{J,t}\right]$ is the first row vector of the explanatory variable matrix, X_t.

Expected utility

Each country's expected utility function at time t-1 is a function of the rate of return ($\Delta \ln A_{k,t}$), and takes the following form.

$$E_{t-1} u_k^t = E_{t-1}[-\exp(-\alpha_k (1 + \Delta \ln A_{k,t}))] \tag{1.7}$$

where $A_{k,t}$ is the total asset value of the country k(=1,2) at time t, and $-u_k^t{}'' / u_k^t{}' = \alpha_k$.

This utility function possesses a constant coefficient of absolute risk aversion (α_k), and is upwardly concave over $A_{k,t}$'s entire domain (including where $A_{k,t}$ is negative). Each country holds a portfolio consisting of a risk-free asset and risky assets. The future expected rate of return and variance for each risky asset are estimated based on the past history of explanatory variables, and the portfolio allocation, overall expected rate of return, and

overall variance and determined so as to maximize expected utility for the following period. For example, consider the case in which a country has to determine the allocation (z_t) of risky assets (r, say) for the next time t (and hence the allocation ($1 - z_t$) of the risk-free asset (f, say)) as follows.

$$z_t = \arg\max E_{t-1}(-\exp(-\alpha(1 + \Delta \ln A_t)))$$ (1.8)

Since the (excess) rate of return follows a normal distribution, and we denote the allocation of the risky asset by z_t and therefore that of a risk-free asset by $1 - z_t$, then the allocation z_t, which maximizes the expected utility for the next time (t), is calculated, as a well-known result, in the following formula.

$$z_t = \frac{\hat{v}_t - r_f}{\alpha \cdot \hat{\sigma}_t^{\,2}}$$ (1.9)

where α is an absolute risk aversion coefficient.

Now we claim the following proposition.

Proposition . Assume that the absolute risk aversion coefficients are the same in both countries ($\alpha_1 = \alpha_2$). Then, the allocation of risky assets (rb in country 1 and rd in country 2) are always the same for both countries for all t. That is:

$$z_{1,b,t} = z_{2,b,t}, \quad z_{1,d,t} = z_{2,d,t} \quad \text{for all t}$$

In addition, the rates of return in the personal asset holdings are always the same for both countries for all t.

$$\Delta \ln A_{1,t} = \Delta \ln A_{2,t} \quad \text{for all t}$$

Note that these relationships should hold regardless of the values of r_a, r_c, v_b, v_d, v_{ex}, σ_b, σ_d, σ_{ex}, η_b, η_d, η_{ex}, γ_b, γ_d or p. The proof is shown in Section 3.

2. Method for allocating assets in an open economy

Asset allocation for country 1

For Country 1, three assets (rb, fc, rd) are risky assets.

$$y_{1,b,t-1} \equiv \begin{bmatrix} \Delta \ln S_{b,t-1} \\ \Delta \ln S_{b,t-2} \\ \vdots \\ \Delta \ln S_{b,t-T_e} \end{bmatrix} \qquad y_{1,c,t-1} \equiv \begin{bmatrix} r_c + \Delta \ln S_{ex,t-1} \\ r_c + \Delta \ln S_{ex,t-2} \\ \vdots \\ r_c + \Delta \ln S_{ex,t-T_e} \end{bmatrix}$$

$$y_{1,d,t-1} \equiv \begin{bmatrix} \Delta \ln S_{d,t-1} + \Delta \ln S_{ex,t-1} \\ \Delta \ln S_{d,t-2} + \Delta \ln S_{ex,t-2} \\ \vdots \\ \Delta \ln S_{d,t-T_e} + \Delta \ln S_{ex,t-T_e} \end{bmatrix} \qquad (2.1)$$

Defining the above matrix, $y_{1,b,t-1}, y_{1,c,t-1}, y_{1,d,t-1}$ are respectively the histories of the rates of return of three assets (rb, fc, rd) which are observed from country 1.

The expected rate and the variance of return at time t can be estimated as follows. Defining, as in Section 1, $Q_{t-1} \equiv X_{t-1}^T X_{t-1}$, $A_{t-1} \equiv Q_{t-1}^{-1} X_{t-1}^T$, $N_{t-1} \equiv X_{t-1} A_{t-1}$, $M_{t-1} \equiv I - N_{t-1}$, $e_{1,b,t-1} \equiv M_{t-1} y_{1,b,t-1}$, $e_{1,c,t-1} \equiv M_{t-1} y_{1,c,t-1}$ and $e_{1,d,t-1} \equiv M_{t-1} y_{1,d,t-1}$, then the variance and the covariance of return at time t can be estimated as:

$$\hat{\sigma}_{1,b,t}^2 = \frac{e_{1,b,t-1}^T e_{1,b,t-1}}{T_e - J}, \quad \hat{\sigma}_{1,c,t}^2 = \frac{e_{1,c,t-1}^T e_{1,c,t-1}}{T_e - J}, \quad \hat{\sigma}_{1,d,t}^2 = \frac{e_{1,d,t-1}^T e_{1,d,t-1}}{T_e - J},$$

$$\hat{\sigma}_{1,bc,t} = \hat{\sigma}_{1,cb,t} = \frac{e_{1,b,t-1}^T e_{1,c,t-1}}{T_e - J}, \quad \hat{\sigma}_{1,bd,t} = \hat{\sigma}_{1,db,t} = \frac{e_{1,b,t-1}^T e_{1,d,t-1}}{T_e - J},$$

$$\hat{\sigma}_{1,cd,t} = \hat{\sigma}_{1,dc,t} = \frac{e_{1,c,t-1}^T e_{1,d,t-1}}{T_e - J} \qquad (2.2)$$

The expected *excess* rate of return at time t can be forecast as:

$$\hat{v}_{1,b,t} - r_a = x_t^T Q_{t-1}^{-1} X_{t-1}^T (y_{1,b,t-1} - r_a * \underline{1}_{T_e})$$

$$\hat{v}_{1,c,t} - r_a = x_t^T Q_{t-1}^{-1} X_{t-1}^T (y_{1,c,t-1} - r_a * \underline{1}_{T_e})$$

$$\hat{v}_{1,d,t} - r_a = x_t^T Q_{t-1}^{-1} X_{t-1}^T (y_{1,d,t-1} - r_a * \underline{1}_{T_e}) \qquad (2.3)$$

where $\underline{1}_{T_e}$ is T$_e$ *1 summer column vector, all elements of which are 1's.

The estimated vector and the estimated covariance matrix of the excess return can be constructed as following.

$$r_{1,t} = \begin{bmatrix} \hat{v}_{1,c,t} - r_a \\ \hat{v}_{1,b,t} - r_a \\ \hat{v}_{1,d,t} - r_a \end{bmatrix}, \; V_{1,t} = \begin{bmatrix} \hat{\sigma}_{1,c,t}^2 & \hat{\sigma}_{1,cb,t} & \hat{\sigma}_{1,cd,t} \\ \hat{\sigma}_{1,bc,t} & \hat{\sigma}_{1,b,t}^2 & \hat{\sigma}_{1,bd,t} \\ \hat{\sigma}_{1,dc,t} & \hat{\sigma}_{1,db,t} & \hat{\sigma}_{1,d,t}^2 \end{bmatrix} \qquad (2.4)$$

Then the allocation vector of a tangency portfolio which consists of three assets (rb, fc, rd) is to be defined as:

$$w_{1,\tan,t} = (V_{1,t}^{-1} r_{1,t}) / (\underline{1}_3^T V_{1,t}^{-1} r_{1,t}) \qquad (2.5)$$

where $\underline{1}_3$ is 3*1 summer column vector, all elements of which are 1's.

The estimated value of the expected excess return of this tangency portfolio is calculated as:

$$\hat{r}_{1,\tan,t} = (r_{1,t}^T V_{1,t}^{-1} r_{1,t}) / (\underline{1}_3^T V_{1,t}^{-1} r_{1,t}) \qquad (2.6)$$

The estimated value of the variance of the excess return of this tangency portfolio is also calculated as:

$$\hat{\sigma}_{1,\tan,t}^2 = (r_{1,t}^T V_{1,t}^{-1} r_{1,t}) / (\underline{1}_3^T V_{1,t}^{-1} r_{1,t})^2 \qquad (2.7)[5]$$

We also assume that the excess return of this tangency portfolio follows a normal distribution, and denote the allocation by $z_{1,\tan,t}$ and therefore that of a risk-free asset (fa) by $1 - z_{1,\tan,t}$. Then the allocation $z_{1,\tan,t}$, which maximizes the expected utility for the next

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[5] (2.5), (2.6) and (2.7) are also given by, for example, Constantinides et al. (1995).

time (t), is calculated as following.

$$z_{1,\tan,t} = \frac{\hat{r}_{1,\tan,t}}{\alpha_1 \cdot \hat{\sigma}_{1,\tan,t}^2} \tag{2.8}$$

where α_1 is a absolute risk aversion coefficient for country 1.

Let $w_{1,\tan,t}(h)$ denote the h th element of the allocation vector $w_{1,\tan,t}$ (3*1 column vector).

Then,

$$z_{1,a,t} = 1 - z_{1,\tan,t}$$

$$z_{1,b,t} = z_{1,\tan,t} * w_{1,\tan,t}(2)$$

$$z_{1,c,t} = z_{1,\tan,t} * w_{1,\tan,t}(1)$$

$$z_{1,d,t} = z_{1,\tan,t} * w_{1,\tan,t}(3) \tag{2.9}$$

Asset allocation for country 2

For country 2, three assets (fa, rb, rd) are risky assets.

$$y_{2,a,t-1} \equiv \begin{bmatrix} r_a - \Delta \ln S_{ex,t-1} \\ r_a - \Delta \ln S_{ex,t-2} \\ \vdots \\ r_a - \Delta \ln S_{ex,t-T_e} \end{bmatrix} \qquad y_{2,b,t-1} \equiv \begin{bmatrix} \Delta \ln S_{b,t-1} - \Delta \ln S_{ex,t-1} \\ \Delta \ln S_{b,t-2} - \Delta \ln S_{ex,t-2} \\ \vdots \\ \Delta \ln S_{b,t-T_e} - \Delta \ln S_{ex,t-T_e} \end{bmatrix}$$

$$y_{2,d,t-1} \equiv \begin{bmatrix} \Delta \ln S_{d,t-1} \\ \Delta \ln S_{d,t-2} \\ \vdots \\ \Delta \ln S_{d,t-T_e} \end{bmatrix} \tag{2.10}$$

Here, $y_{2,a,t-1}, y_{2,c,t-1}, y_{2,d,t-1}$ are respectively the histories of the rates of return of three assets (fa, rb, rd) which are observed from country 2. Then, the remaining calculation is quite similar as in country 1. Finally, we get:

$$z_{2,a,t} = z_{2,\tan,t} * w_{2,\tan,t}(1)$$

$$z_{2,b,t} = z_{2,\tan,t} * w_{2,\tan,t}(2)$$

$$z_{2,c,t} = 1 - z_{2,\tan,t}$$

$$z_{2,d,t} = z_{2,\tan,t} * w_{2,\tan,t} \quad (3) \tag{2.11}$$

3. Proof of the proposition

In this section, we prove the proposition described in Section 1. The condition we use is just $\alpha_1 = \alpha_2 \equiv \alpha$. (3.1) The histories of the excess rates of return of three "risky" assets (rb, fc, rd) are observed from country 1 as follows.

$$r_{1,b,t-1} \equiv y_{1,b,t-1} - r_a * \underline{1}_{T_e} = \begin{bmatrix} \Delta \ln S_{b,t-1} - r_a \\ \Delta \ln S_{b,t-2} - r_a \\ \vdots \\ \Delta \ln S_{b,t-T_e} - r_a \end{bmatrix}$$

$$r_{1,c,t-1} \equiv y_{1,c,t-1} - r_a * \underline{1}_{T_e} = \begin{bmatrix} r_c + \Delta \ln S_{ex,t-1} - r_a \\ r_c + \Delta \ln S_{ex,t-2} - r_a \\ \vdots \\ r_c + \Delta \ln S_{ex,t-T_e} - r_a \end{bmatrix} = \begin{bmatrix} \Delta \ln S_{ex,t-1} + d \\ \Delta \ln S_{ex,t-2} + d \\ \vdots \\ \Delta \ln S_{ex,t-T_e} + d \end{bmatrix}$$

$$r_{1,d,t-1} \equiv y_{1,d,t-1} - r_a * \underline{1}_{T_e} = \begin{bmatrix} \Delta \ln S_{d,t-1} - r_a + \Delta \ln S_{ex,t-1} \\ \Delta \ln S_{d,t-2} - r_a + \Delta \ln S_{ex,t-2} \\ \vdots \\ \Delta \ln S_{d,t-T_e} - r_a + \Delta \ln S_{ex,t-T_e} \end{bmatrix} \tag{3.2}$$

where $\underline{1}_{T_e}$ is a T$_e$*1 summer column vector all elements of which are 1's, and

$$d \equiv r_c - r_a .$$

For simplicity, define the following vectors.

$$R_{ex} \equiv \begin{bmatrix} \Delta \ln S_{ex,t-1} \\ \Delta \ln S_{ex,t-2} \\ \vdots \\ \Delta \ln S_{ex,t-T_e} \end{bmatrix} \qquad R_b \equiv r_{1,b,t-1} \equiv \begin{bmatrix} \Delta \ln S_{b,t-1} - r_a \\ \Delta \ln S_{b,t-2} - r_a \\ \vdots \\ \Delta \ln S_{b,t-T_e} - r_a \end{bmatrix}$$

$$R_d \equiv \begin{bmatrix} \Delta \ln S_{d,t-1} - r_c \\ \Delta \ln S_{d,t-2} - r_c \\ \vdots \\ \Delta \ln S_{d,t-T_e} - r_c \end{bmatrix} \quad D \equiv \begin{bmatrix} d \\ d \\ \vdots \\ d \end{bmatrix} = d * 1_{T_e} \tag{3.3}$$

Furthermore, define R_1 as:

$$R_1 \equiv \begin{bmatrix} r_{1.c.t-1} & r_{1,b,t-1} & r_{1,d,t-1} \end{bmatrix} = \begin{bmatrix} R_{ex} + D & R_b & R_d + R_{ex} + D \end{bmatrix} \tag{3.4}$$

where R_1 is a $T_e * 3$ matrix.

Then, from the definition in Section 3, we have:

$$V_{1,t} = \frac{R_1^T M_{t-1} R_1}{T_e - J} \tag{3.5}$$

Here M_{t-1} is an idempotent matrix. For simplicity, we replace M_{t-1} with M, defining $M \equiv M_{t-1}$. Then,

$$
\begin{aligned}
z_{1,\tan,t} * w_{1,\tan,t} &= \frac{\hat{r}_{1,\tan,t}}{\alpha_1 \cdot \hat{\sigma}_{1,\tan,t}^2} * w_{1,\tan,t} \\
&= \frac{(r_{1,t}^T V_{1,t}^{-1} r_{1,t})/(1_{T_e}^T V_{1,t}^{-1} r_{1,t})}{\alpha_1 (r_{1,t}^T V_{1,t}^{-1} r_{1,t})/(1_{T_e}^T V_{1,t}^{-1} r_{1,t})^2} (V_{1,t}^{-1} r_{1,t})/(1_3^T V_{1,t}^{-1} r_{1,t}) \\
&= \frac{1}{\alpha_1} (V_{1,t}^{-1} r_{1,t})
\end{aligned}
\tag{3.6}
$$

The first, second and third element of the vector $z_{1,\tan,t} * w_{1,\tan,t}$ respectively represent the allocation of each asset (fc, rb, rd). Next, we calculate $\frac{1}{\alpha_1}(V_{1,t}^{-1} r_{1,t})$. For simplicity, we replace $V_{1,t}$ with V_1, defining $V_1 \equiv V_{1,t}$.

Defining $R_0 \equiv \begin{bmatrix} R_{ex} & R_b & R_d \end{bmatrix}$ (3.7),

$$R_1 \equiv R_0 J_1 + DH_1^T, \text{ where } J_1 \equiv \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } H_1 \equiv \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \tag{3.8}$$

Furthermore, $r_{1,t} \equiv \begin{bmatrix} \hat{v}_{1,c,t} - r_a \\ \hat{v}_{1,b,t} - r_a \\ \hat{v}_{1,d,t} - r_a \end{bmatrix} = \begin{bmatrix} Z(R_{ex} + D) \\ ZR_B \\ Z(R_D + R_{ex} + D) \end{bmatrix} = (ZR_1)^T$ (3.9)

$$= J_1^T R_0^T Z^T + H_1(ZD)^T = J_1^T R_0^T Z^T + H_1 d$$

where $Z = x_t^T Q_{t-1}^{-1} X_{t-1}^T = x_t^T A_{t-1}$.

Here we used the equality, $ZD = d$, for any arbitrary x_t^T and X_{t-1}.

Then, $V_1 = \dfrac{R_1^T M R_1}{T_e - J} = \dfrac{J_1^T R_0^T M R_0 J_1}{T_e - J} = J_1^T \left(\dfrac{R_0^T M R_0}{T_e - J} \right) J_1 = J_1^T V_0 J_1$ (3.10)

where $V_0 \equiv \dfrac{R_0^T M R_0}{T_e - J}$

Then, $V_1^{-1} = J_1^{-1} V_0^{-1} (J_1^T)^{-1}$ (3.11), and,

$$z_{1,\tan,t} * w_{1,\tan,t} = \frac{1}{\alpha_1} (V_{1,t}^{-1} r_{1,t})$$

$$= \frac{1}{\alpha_1} J_1^{-1} V_0^{-1} (J_1^T)^{-1} (J_1^T R_0^T Z^T + H_1 d)$$

$$= \frac{1}{\alpha_1} J_1^{-1} V_0^{-1} R_0^T Z^T + \frac{1}{\alpha_1} J_1^{-1} V_0^{-1} (J_1^T)^{-1} H_1 d \qquad (3.12)$$

$$= J_1^{-1} \left(\frac{1}{\alpha} V_0^{-1} R_0^T Z^T \right) + J_1^{-1} \left(\frac{d}{\alpha} V_0^{-1} \right) (J_1^T)^{-1} H_1$$

$$= J_1^{-1} S + J_1^{-1} W (J_1^T)^{-1} H_1$$

Here $S \equiv \dfrac{1}{\alpha} V_0^{-1} R_0^T Z^T$ is a 3*1 column vector, and $W \equiv \dfrac{d}{\alpha} V_0^{-1}$ is a 3*3 symmetric

matrix.

Similarly, the histories of the excess rates of return of three "risky" assets (fa, rb, rd) are

observed from country 2 as follows.

$$r_{2,a,t-1} \equiv y_{2,a,t-1} - r_c * \underline{1}_{T_e} = \begin{bmatrix} r_a - \Delta \ln S_{ex,t-1} - r_c \\ r_a - \Delta \ln S_{ex,t-2} - r_c \\ \vdots \\ r_a - \Delta \ln S_{ex,t-T_e} - r_c \end{bmatrix} = \begin{bmatrix} -\Delta \ln S_{ex,t-1} - d \\ -\Delta \ln S_{ex,t-2} - d \\ \vdots \\ -\Delta \ln S_{ex,t-T_e} - d \end{bmatrix}$$

$$r_{2,b,t-1} \equiv y_{2,b,t-1} - r_c * \underline{1}_{T_e} = \begin{bmatrix} \Delta \ln S_{b,t-1} - \Delta \ln S_{ex,t-1} - r_c \\ \Delta \ln S_{b,t-2} - \Delta \ln S_{ex,t-2} - r_c \\ \vdots \\ \Delta \ln S_{b,t-T_e} - \Delta \ln S_{ex,t-T_e} - r_c \end{bmatrix}$$

$$r_{2,d,t-1} \equiv y_{2,d,t-1} - r_c * \underline{1}_{T_e} = \begin{bmatrix} \Delta \ln S_{d,t-1} - r_c \\ \Delta \ln S_{d,t-2} - r_c \\ \vdots \\ \Delta \ln S_{d,t-T_e} - r_c \end{bmatrix} \tag{3.13}$$

Therefore, defining $R_2 \equiv \begin{bmatrix} r_{2.a.t-1} & r_{2,b,t-1} & r_{2,d,t-1} \end{bmatrix} = \begin{bmatrix} -R_{ex} - D & R_b - R_{ex} - D & R_d \end{bmatrix}$ (3.14)

(R_1 is a T_e*3 matrix),

$$R_2 \equiv R_0 J_2 + D H_2^{\,T}, \text{ where } J_2 \equiv \begin{bmatrix} -1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } H_2 \equiv \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} \tag{3.15}$$

Calculating similarly as in country 1,

$$z_{2,\tan,t} * w_{2,\tan,t} = \frac{1}{\alpha_2}(V_{2,t}^{-1} r_{2,t})$$

$$= \frac{1}{\alpha_2} J_2^{-1} V_0^{-1} (J_2^{\,T})^{-1}(J_2^{\,T} R_0^{\,T} Z^T + H_2 d)$$

$$= J_2^{-1}\left(\frac{1}{\alpha} V_0^{-1} R_0^{\,T} Z^T\right) + J_2^{-1}\left(\frac{d}{\alpha} V_0^{-1}\right)(J_2^{\,T})^{-1} H_2 \tag{3.16}$$

$$= J_2^{-1} S + J_2^{-1} W (J_2^{\,T})^{-1} H_2$$

Since $S \equiv \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix}, W \equiv \begin{bmatrix} w_{11} & w_{12} & w_{13} \\ w_{21} & w_{22} & w_{23} \\ w_{31} & w_{32} & w_{33} \end{bmatrix}, J_1^{-1} \equiv \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, J_2^{-1} \equiv \begin{bmatrix} -1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$

we have $J_1^{-1}S = \begin{bmatrix} s_1 - s_3 \\ s_2 \\ s_3 \end{bmatrix}$, $J_2^{-1}S = \begin{bmatrix} -s_1 - s_2 \\ s_2 \\ s_3 \end{bmatrix}$ and

$$J_1^{-1}W(J_1^T)^{-1}H_1 = \begin{bmatrix} w_{11} - w_{31} \\ w_{21} \\ w_{31} \end{bmatrix}, \quad J_2^{-1}W(J_2^T)^{-1}H_2 = \begin{bmatrix} -w_{11} - w_{21} \\ w_{21} \\ w_{31} \end{bmatrix}. \tag{3.17}$$

Summarizing all the above results, the allocation of all four assets (fa, rb, fc, rd) for country

1 is calculated as follows.

$$z_{1,a,t} = 1 - z_{1,\tan,t} = 1 - (s_1 + s_2) - (w_{11} + w_{21}) = 1 + z_{2,a,t}$$

$$z_{1,b,t} = z_{1,\tan,t} * w_{1,\tan,t}(2) = s_2 + w_{21} = z_{2,b,t}$$

$$z_{1,c,t} = z_{1,\tan,t} * w_{1,\tan,t}(1) = s_1 - s_3 + (w_{11} - w_{31}) = z_{2,c,t} - 1$$

$$z_{1,d,t} = z_{1,\tan,t} * w_{1,\tan,t}(3) = s_3 + w_{31} = z_{2,d,t} \tag{3.18}$$

Similarly, the allocation of all four assets (fa, rb, fc, rd) for country 2 is calculated as follows.

$$z_{2,a,t} = z_{2,\tan,t} * w_{2,\tan,t}(1) = -(s_1 + s_2) - (w_{11} + w_{21}) = z_{1,a,t} - 1$$

$$z_{2,b,t} = z_{2,\tan,t} * w_{2,\tan,t}(2) = s_2 + w_{21} = z_{1,b,t}$$

$$z_{2,c,t} = 1 - z_{2,\tan,t} = 1 + (s_1 - s_3) + (w_{11} - w_{31}) = 1 + z_{1,c,t}$$

$$z_{2,d,t} = z_{2,\tan,t} * w_{2,\tan,t}(3) = s_3 + w_{31} = z_{1,d,t} \tag{3.19}$$

Using these results, we prove $\Delta \ln A_{1,t} = \Delta \ln A_{2,t}$. We have:

$$\Delta \ln A_{1,t} = z_{1,a,t+1}\Delta \ln S_{a,t} + z_{1,b,t+1}\Delta \ln S_{b,t} + z_{1,c,t+1}(\Delta \ln S_{c,t} + \Delta \ln S_{ex,t}) + z_{1,d,t+1}(\Delta \ln S_{d,t} + \Delta \ln S_{ex,t})$$

$$\Delta \ln A_{2,t} = z_{2,a,t+1}(\Delta \ln S_{a,t} - \Delta \ln S_{ex,t}) + z_{2,b,t+1}(\Delta \ln S_{b,t} - \Delta \ln S_{ex,t}) + z_{2,c,t+1}\Delta \ln S_{c,t} + z_{2,d,t+1}\Delta \ln S_{d,t}$$

$$\tag{3.20}$$

Plugging (3.18) and (3.19) into (3.20), now the proof is done.

4. Some debates- Home bias puzzle in an open economy

The proposition presented in this paper has an interesting economic implication in the sense that it always holds, (1) regardless of the characteristics of each country's risky asset market, including the demand elasticity of market prices, and (2) regardless of the value of the proportion in each country's personal asset holdings (p for country 1 and 1-p for country 2), and (3) regardless of the characteristics of the exchange rate price process (so, whether each country's riskless asset, represented mainly by its currency, is devaluating or revaluating against the other country's), and (4) regardless of the risk free rate in each country (that is, $r_a = r_c \equiv r$ is not required.), which is supposed to be fixed as a result of each country's monetary policy. Instead, we just need that each country shares the same risk aversion coefficient ($\alpha_1 = \alpha_2 \equiv \alpha$), and the same explanatory variables (the same information set), X_{t-1}.[6]

This proposition is not a result which can be necessarily derived from conventional general/partial equilibrium theories, but that obtained from a formal coincidence in mathematical formulation combined between the risk-minimizing (risky) portfolio selection and the utility maximizing allocation of risky/riskfree assets under the CARA utility and normality assumptions, as shown in equation ((2.8) and) (3.6). For example, ((2.8) and) (3.6) easily breaks, if we abandon the CARA utility or the normality assumption, therefore, in these cases, the proposition does not hold. Instead, it always holds regardless of the market formation in each country (for example, perfectly/imperfectly competitive, or etc). Also, it always holds even if the underlying economic system is not in equilibrium, because we do

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[6] Of course, the assumption of this proposition, which implicitly takes account of cross sectional dependences among panel units, is only a possibility among many.

not assume any market clearing condition here, and in addition, because no-arbitrage condition in the exchange rate process between two countries, $v_{ex} = -(r_c - r_a)$ as in (1.2), is not needed at all, and all of these aspects show that this proposition is not a derivative result from general/partial equilibrium theories, but one peculiar mathematical aspect derived from a peculiar matching in formulation within an open economy framework. For example, the allocation for riskfree assets in each country may not be, in general, the same for any period. ($z_{1,a,t} \neq z_{2,a,t}$ and $z_{1,c,t} \neq z_{2,c,t}$, while $z_{1,a,t} + z_{1,c,t} = z_{2,a,t} + z_{2,c,t}$.)

Now consider the simplest case where two countries are in *symmetry*. For example, assume $r_a = r_c$, $v_b = v_d$, $\sigma_b = \sigma_d$, $\eta_b = \eta_d$, $v_{ex} = 0$, $p = 0.5$. Also, $\Delta B_{b,t}$, $\Delta B_{d,t}$ and $\Delta B_{ex,t}$ do not have to be independent with each other, but the correlations need to be symmetric between two countries.[7] Then, from symmetry in the economic system together with (3.18) and (3.19), each country is supposed to hold the same portion of two risky assets (rb and rd) in an *ergodic* sense, that is:[8] [9]

$$\bar{z}_{1,b} = \bar{z}_{1,d} = \bar{z}_{2,b} = \bar{z}_{2,d} \equiv \bar{z}_r \qquad (4.1)$$

where the upper bar denotes the mean over t.

Also, from (4.1) and (3.18/19), we have for country 1:

$$\bar{z}_{1,d} = -\bar{z}_{1,c} (= \bar{z}_{1,b} = \bar{z}_r), \quad \bar{z}_{1,a} = 1 - \bar{z}_{1,b} (= 1 - \bar{z}_r) \equiv \bar{z}_f \qquad (4.2)$$

where $\bar{z}_r + \bar{z}_f = 1$

(4.1) and (4.2) are equivalent, using the elements of matrix S, with:[10]

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[7] Also, assume that the information sets, $g_{b,t-1}^T$ and $g_{d,t-1}^T$ are generated from symmetric nature (the same functional form of stochastic distribution or etc.).

[8] In general, $z_{1,b,t} \neq z_{1,d,t}$, $z_{2,b,t} \neq z_{2,d,t}$.

[9] The symmetry assumption requires, $\bar{z}_{1,b} = \bar{z}_{2,d}$, $\bar{z}_{1,c} = \bar{z}_{2,a}$ and $\bar{z}_{1,d} = \bar{z}_{2,b}$.

[10] Since $r_a = r_c$ $(d = 0)$, all the elements of W are zeros.

$$\bar{s}_1 = 0 \quad \text{and} \quad \bar{s}_2 = \bar{s}_3 \tag{4.3}$$

where the upper bar denotes the mean over t.

This is somewhat counter intuitive, because, for example for country 1, clearly country 2's risky asset (rd) is more "risky" than its own risky asset (rb) because of the exchange rate volatility. But we easily see, in the first equation of (4.2), that all the part of volatility in foreign risky asset rd, attributed to exchange rate process, is completely hedged by holding the same amount of liability in foreign riskfree asset fc.[11] [12] Therefore, each symmetric country is, in the long run (in an ergodic sense), perfectly neutral against the exchange rate volatility (σ_{ex}). The allocation of home/foreign risky assets (rb and rd) itself is not only equated between them, as shown in (4.1), but also each country does really respond to the increase in the volatility of exchange rate process rather by keeping the total portion of home/foreign riskfree assets ($\bar{z}_{1,a} + \bar{z}_{1,c} (= \bar{z}_{2,a} + \bar{z}_{2,c})$) constant. Also, the long-run mean proportion of risky asset in domestic financial markets ($\bar{z}_{1,b} / (\bar{z}_{1,a} + \bar{z}_{1,b}) = \bar{z}_{1,b}$) does not change either with the larger volatility of exchange rate (σ_{ex}). One basic implication of this proposition under this symmetry assumption is that, *when two countries share the common information set, each country might be, under some non-general conditions, indifferent, between its home/foreign risky asset holdings (rb and rd), to the diffusion (volatility) of exchange rate price process, which is completely hedged by balancing the debt/holding of foreign riskfree/risky assets (fc and rd), so that each country always equates the allocation for each risky asset (rb and rd).* This aspect strongly supports each country's considerable

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[11] This is also counter intuitive and is not rigorously true for each respective period t, because the information set X_{t-1} is just stochastically generated. It holds only in an ergodic sense.

[12] As a matter of fact, the country can attain the same return (but the larger volatility, therefore the smaller utility gain) in a closed economy framework, dealing with only domestic assets fa and rb where:

$$\bar{z}_{1,a}{}' = 2\bar{z}_f - 1, \quad \bar{z}_{1,b}{}' = 2\bar{z}_r$$

gain from international portfolio diversification, as emphasized in Grauer and Hakansson (1987). On the other hand, under asymmetry assumption, for example where $v_{ex} \neq -(r_c - r_a)$, in general (4.1) and (4.2) do not hold, so that $\bar{z}_{1,d} \neq -\bar{z}_{1,c}$.[13] This inequality implies that the exchange rate volatility is not completely, but only partially canceled off, therefore the larger σ_{ex} surely causes the increase in home riskfree asset fa, while the sign of change in home risky asset rb is still uncertain. Even in this (asymmetry) case, each country is still indifferent to the exchange rate volatility in the sense that both countries hold the same share in risky asset rb and rd respectively, as shown in (3.18/19). In other words, the existence of exchange rate volatility does not actually make each country discriminate regarding the relative allocation in home/foreign risky assets (rb and rd).

Finally, within the framework of this model, I state some points, apart from some important explanations including "non-traded goods or small utility gains to diversification", in the relation with so called "the home bias puzzle", which is known as "the contradiction between the obvious benefits of holding a globally dispersed set of equities and the apparent reluctance to do so" (Obstfelt and Rogoff. (1996), Tesar and Werner (1995)). Also, Coakely, Fuertes and Smith (2006) and Drine and Rault (2007) are the examples in both econometric (panel data) and computational analysis of international finance including PPP puzzle.

1. Missing information and mis-specified model

As easily imagined, in case that, for example, one country does not hold some information attributed to the other country (for example, $g_{d,t-1}{}^T$, $\Delta z_{2,c,t}$ and $\Delta z_{2,d,t}$ for country 1), the regression model is made under mis-specification, which necessarily causes a

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[13] $v_{ex} \neq -(r_c - r_a)$ implies that the exchange rate process is in disequilibrium and does not meet no-arbitrage condition.

larger estimated variance and smaller allocation of foreign assets (fc and rd for country 1).[14]

This, the most plausibly, explains the home bias puzzle, but I exclude it from my focus, since

it is not a novel illustration. Also, it is noteworthy that even under misspecification, if two

countries share the same information set X_{t-1}, then still the proposition holds and the

home bias should disappear.

2. Borrowing constraints

Again, consider country 1 asset holdings in the symmetric case. Then in (4.2), the

allocation of foreign riskfree asset (fc) is just negative of that of foreign risky asset (rd), and

the total portion for foreign assets is just zero $(\bar{z}_{1,d} + \bar{z}_{1,c} = 0)$. The key point is that, *in order*

to be risk neutral against exchange rate volatility and fully to enjoy the utility gain from

international portfolio diversification, the investors must borrow the same amount of foreign

riskfree asset as the foreign risky asset holding. If this violates the borrowing constraint,

which the authority might impose for foreign investors, then necessarily we have $\bar{z}_{1,d} < \bar{z}_{1,b}$,

which is also one persuasive illustration of the home bias puzzle.[15]

3. CRRA (constant relative risk aversion) utility

4. Non-normality of disturbance terms

Another possibility is that the above two points might also explain the home bias

puzzle, that is, too excessive, empirically observed emphasis on the risky asset in home

country. Under a constant relative risk aversion (decreasing in CARA) or an upwardly

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[14] Also, the missing information about demand/supply gap of each assets ($\Delta z_{1,b,t}$, $\Delta z_{1,c,t}$, $\Delta z_{1,d,t}$, $\Delta z_{2,a,t}$, $\Delta z_{2,b,t}$ and $\Delta z_{2,d,t}$) induces the estimated residual of each price process to be correlated over time t.

[15] For foreign investors, as is country 1 investor in country 2, the borrowing constraint might be naturally $z_{1,d,t} \geq 0$, $z_{1,c,t} \geq 0$, while for domestic investors, as is country 2 investor in country 2, it might be, more loosely, $z_{2,d,t} + z_{2,c,t} \geq 0$. The condition $\bar{z}_{1,d} + \bar{z}_{1,c} = 0$ does not actually wipe out the high possibility of default in next period, that is $z_{1,d,t+1} + z_{1,c,t+1} < 0$.

deviated asymmetric distribution of disturbance terms, a smaller risk premium and a larger

incentive for holding home country risky asset might be induced for each country than under

CARA or a symmetric normal distribution.[16] In other words, in these circumstances the

common information sharing might not ensure that the diffusion of exchange rate can be

perfectly hedged by adjusting the total amount of home/foreign riskfree assets. One

hypothesis, for example, is: *under CRRA or upwardly tailed disturbance, the foreign risky*

asset holding is not neutral regarding the drift of exchange rate process ν_{ex} *, and the home*

country asset holding is not neutral regarding the diffusion of exchange rate process σ_{ex} *.*

Therefore, it might be worth doing simulation, which cannot be replaced with analytical

work, under a careful specification of the parameter values and nonparametric estimation of

distributions and, to examine this hypothesis within the framework of the model.

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[16] CRRA utility is defined as $u_k^t = (1 + \Delta \ln A_{k,t})^{1-\alpha_k} / (1 - \alpha_k)$ (1.7a),

where $-u_k^t{}''(1 + \Delta \ln A_{k,t}) / u_k^t{}' = \alpha_k$.

Country 1   Country 2

Risky asset
rb

Risky asset
rd

Risk-free
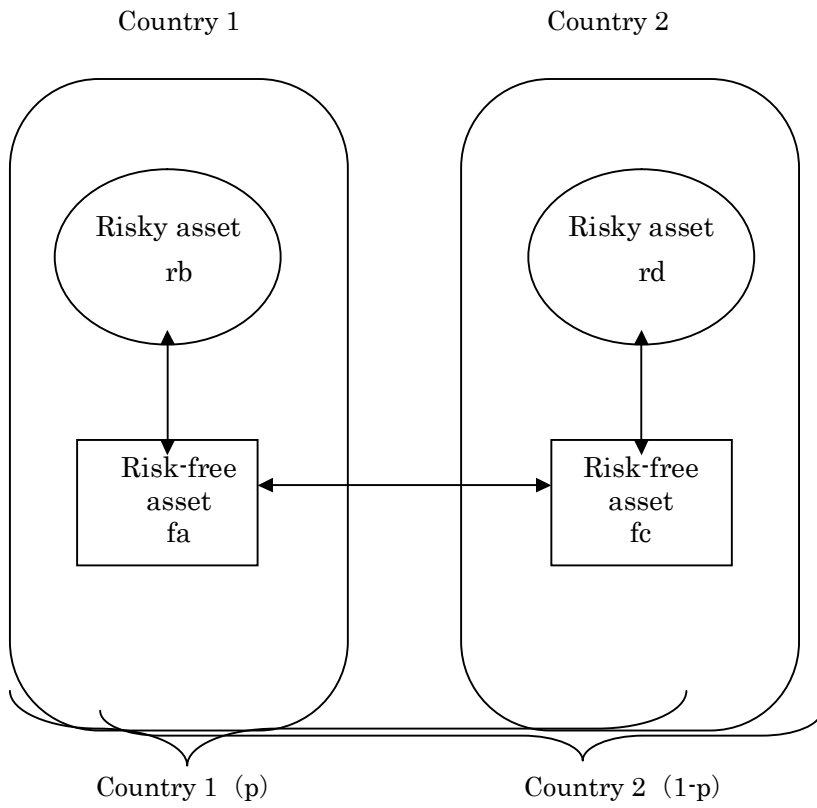asset
fa

Risk-free
asset
fc

Country 1（p）   Country 2（1-p)

Figure 1 The Organization of an Open Exchangeable
Economy