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A NOTE ON CHOOSING THE ALTERNATIVE WITH THE BEST MEDIAN EVALUATION

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Abstract

The voting rule proposed by Basset and Persky (*Public Choice* 99: 299-310) picks the alternative with best median evaluation. The present note shows that this MaxMed principle is equivalent to ask the social planner to apply the MaxMin principle allowing him to discard half of the population. In one-dimensional, single-peaked domains, the paper compares this rule with majority rule and the utilitarian criterion. The MaxMed outcome is rejected by a majority of voters in favor of outcomes which are also utilitarian improvements.

1 Introduction

When different individuals evaluate one object on some common scale, taking as a summary of the various evaluations their median is a natural idea, often used in Statistics. Using this idea for collective choice among several alternative objects is thus occasionally proposed: the suggestion is that an alternative with the largest median evaluation is to be chosen. I will call this method the *MaxMed* voting scheme.

In the modern literature this was (up to my knowledge) proposed by Basset and Persky, 1999 [2]. Properties of the best median can be expressed in the language of utility theory (Bossert and Weymark 2004 [3]): the informational basis is the ordinal and inter-individuallly comparable framework. In terms of utilities, the utility levels attached by the different individuals to a given alternative are compared (inter-individual comparability), and the median is stable under any strictly increasing transformation of utility provided that the transformation is the same for all individuals (ordinality). This is the same informational basis as for the MaxMin, or Rawls principle. As will be shown, instead of maximizing the satisfaction of the least favored individual in the whole society, as the MaxMin does, MaxMed maximizes the satisfaction of the least favored individual *when half of the population can be discarded.*

As to strategic aspects, supporters of this method claim that it is relatively immune to individual misrepresentation of evaluations. The reason for this claim is that the median (contrary to the mean) is "robust" in the statistical sense. For instance, in most cases, over-evaluating an object in order to push up its median has simply no consequence. This argument was put forward by Basset and Persky [2], who used the term "Robust Voting" to describe this method. But, in fact, voting rules generally share the property that one voter is rarely pivotal so that, in practice, median voting does not appear to be less, nor more, manipulable than other voting rules (Gerhlein and Lepelley 2003 [6]).

Some properties of the MaxMed method are collected by Warren Smith at the Center for Range Voting [10]. Using a limited set of grades, several candidates usually end up with the same grade. Balinski and Laraki, 2007 [1] have proposed several, more or less complicated, ways to choose among them, and have ellaborated on the question of robustness to manipulation. Felsenthal and Machover, 2008 [4] is a discussion of these issues.

In this paper I will leave aside the strategic questions and assume that voters vote sincerely. I also leave aside the problem of ties associated with the use of MaxMed voting in practice, by assuming that an evaluation can be any real number. I will attempt to compare the outcomes of different choice principles, including MaxMed, in a setting which is standard in Economic and Political Theory and is relevant for the applications. I consider one-dimensional, single-peaked utility profiles with distributions of voters' ideal points which are skewed (without loss of generality: skewed to the left).

From the point of view of Social Choice Theory, one-dimensional, single peaked profiles are very specific profiles because they avoid Condorcet cycles, but they nevertheless constitute an interesting benchmark case. In Political Economy, this assumption is so common that it is often not even mentioned. The idea that the distribution of voters with respect to the relevant parameter is not uniform or symmetric but skewed is an empirical observation: it seems to be a general rule that socioeconomic relevant parameters are "skewed to the left," the paradigmatic example being income distributions: most people earn less than the average.

In this setting, Condorcet-consistent voting rules are very simple, and all alike: all of them chose the median of the voters' ideal points, which is a *Condorcet* winner in virtue of the celebrated Median Voter Theorem. This is the outcome of majority voting. By comparison, the utilitarian choice (the efficient alternative in the sense of maximizing the sum of individual utilities, I will call it the *Bentham* winner) tends, in the same setting, to produce choice which are favored by richer people. Although this observation is not as clear-cut as the Median Voter Theorem is, it matches the economic intuition and can be stated formally if one makes the (standard) assumption of quadratic utility. In that case the utilitarian optimum is simply the mean of the distribution of ideal points. For left-skewed distribution, the mean is larger than the median.

Then, where is the alternative with the best median evaluation located? Is it close to the utilitarian optimum or not, and if not, in which direction does it diverges from the optimum: to the left, in the direction of the ideal points of the majority, or to the right, in the direction of the rich minority? How does it compare to the outcome of Majority rule, the Condorcet winner? Is it more or less efficient than this outcome?

In order to answer this question, I present one very simple analytical example, plus computer simulations under various hypothesis which will be described in the sequel. The reached conclusion is always the same: the best median is located on the wrong side of the Condorcet winner. The Bentham-innefficiency of the best median choice is of the same kind but worse than the Bentham-innefficiency of majority voting.

The example maybe useful in order to understand what one does when comparing medians. The usual argument for rejecting an alternative A in favor of another alternative B when a majority of individuals prefer A to B is that members of the relatively small population who gain in this move gain a lot while members of the losing majority incur a relatively small loss. This is a typical Benthamite, utilitarian argument. Conversely, the Bentham-innefficiency of majority voting derives from the democratic power of the (many) poor[1]. This inefficiency can be justified by normative political arguments in favor of the principle of majority rule. It can also often be justified by invoking the argument of decreasing individual marginal utility. In the one-dimensional settings under scrutiny, the best median choice goes further away from efficiency, for reasons that are easily understood from the mathematical point of view (the example in section 5 will explain this point) but which have no normative or political appeal: apply Rawls principle to the most homogeneous half-population, with no regard for the other half.

2 Interpretation of the MaxMed

In order to explain what one does when chosing the alternative with the best possible median evaluation, it is useful to introduce some definitions and pieces of notation.

Definition 1 *Consider a population P of individuals and a set X of alternatives. The* Rawlsian evaluation of $x \in X$ by P *is:*

$$R(P,x) = \min_{i \in P} u_i(x)$$

and the Rawlsian satisfaction of P *is:*

$$R(P) = \max_{x \in X} R(P,x).$$

Notice that this vocabulary may not be exactly true to John Rawls' ideas (Rawls 1971 [8]), but it is customary in Economics to call Rawls' principle the idea of giving full priority to the worst-off individuals (Kolm 1972 [7], Fleurbaey and Hammond 2004 [5]).

[1] At the limit one can wonder why, under majority rule, "the poor do not expropriate the rich"; see Roemer, 1998 [9].

Definition 2 *Let I, with measure 1, be the (infinite) set of inviduals and let $\mathcal{H}$ be the set of all sub-populations of measure $1/2$. Suppose that, for an alternative x, the evaluation $u_i(x)$ of x by i is continuously distributed in I. Then we denote by* $\text{med}_{i\in I}u_i(x)$ *the* median evaluation *of x, uniquely defined by the fact that for some half-population $H \in \mathcal{H}$,*

$$u_j(x) \leq \text{med}_{i\in I}u_i(x) \leq u_h(x)$$

for all $h \in H$ and all $j \in I \setminus H$.

Notice that for a finite population, if the number of individual is odd then there are no subpopulation of measure $1/2$, and that if the number of individuals is odd then the median is generally not unique For simplicity I consider from now an infinite population in which the median evaluation of each alternative is well defined, and I shall indicate how the findings extend to the finite case.

Let x_{MaxMed} be an alternative with the largest median evaluation and let u_{MaxMed} be this evaluation. By definition those are obtained by the following algorithm:

1. For each alternative $x \in X$, compute $u_{\text{med}}(x) = \text{med}_i u_i(x)$ the median of the evaluations of x by all individuals.

2. Find the largest of these numbers $u_{\text{MaxMed}} = \max_x u_{\text{med}}(x)$ and an associated alternative x_{MaxMed} which achieves this maximum.

The next lemma will show that the following definition is equivalent:

1. For each half-population H of P, compute its Rawlsian satisfaction $R(H)$.

2. Find the largest of these numbers $u_{\text{MaxMed}} = \max_H R(H)$ and an associated alternative x_{MaxMed} which achieves this maximum.

Lemma 1

$$\max_{x\in X} \ \text{med}_{i\in I} \ u_i(x) = \max_{H\in\mathcal{H}} \ R(H)$$

Proof. Write $u^* = \max_{x\in X} \text{med}_{i\in I}u_i(x)$ and $r^* = \max_{H\in\mathcal{H}} R(H)$. To see that $u^* \leq r^*$, take any $x \in X$. By definition of the median there exists a half-population $H \in \mathcal{H}$ such that $\forall j \in H$, $\text{med}_{i\in I}u_i(x) \leq u_j(x)$. For this H, $R(H, x) = \min_{j\in H} u_j(x) \geq \text{med}_{i\in I}u_i(x)$, which implies that $R(H) \geq \text{med}_{i\in I}u_i(x)$. Thus $r^* \geq \text{med}_{i\in I}u_i(x)$, hence $r^* \geq u^*$.

Conversely, suppose $u^* > r^*$. Then there exists $\varepsilon > 0$ and there exists $x \in X$ such that $\text{med}_{i\in I}u_i(x) > r^* + \varepsilon$. For that x:

$$\forall\ H \in \mathcal{H},\ \text{med}_{i\in I}u_i(x) > R(H) + \varepsilon.$$

By definition of the median there exists H_0 such that $u_h(x) \geq \text{med}_{i\in I}u_i(x)$ for all $h \in H_0$, hence, for this H_0, $u_h(x) > R(H_0) + \varepsilon$. But $R(H_0) = \max_{y\in X} R(H_0, y)$ thus, for $y = x$, we have that

$$\forall\ h \in H_0,\ u_h > R(H_0, x) + \varepsilon.$$

Taking the min on h, one obtains the contradiction $R(H_0, x) \geq R(H_0, x) + \varepsilon$. **Q.E.D.**

For a finite population of odd size n, the same result holds when considering for $\mathcal{H}$ the subpopulations of size $(n+1)/2$. Here is, for illustration, a simple example with three voters and two alternatives:

$u_i(x)$	$i=1$	$i=2$	$i=3$	$\text{med}_i u_i(x)$
$x=a$	6	7	8	**7**
$x=b$	4	3	9	4

Table 1: best median evaluation.

$R(H,x)$	$H=\{1,2\}$	$H=\{1,3\}$	$H=\{2,3\}$
$x=a$	6	6	7
$x=b$	3	4	3
$R(H)$	6	6	**7**

Table 2: Most favored half population

3 The one-dimensional model

The set of available alternatives is the set of positive real numbers $X = [0, +\infty[$. To each individual i is attached an ideal point $x_i \in X$. The utility of i for an alternative $y \in X$ is denoted $u_i(y)$. Under the quadratic utility assumption one has:

$$u_i(y) = -\left(y - x_i\right)^2.$$

(Variants of this assumption will be considered later.)

The society is then described by the distribution of ideal points. Let F denotes the cumulative distribution: for any $x \in X$, $F(x)$ is the proportion of individuals i such that $x_i < x$. We suppose that F is continuous and has a mean. The Condorcet winner is then well-defined and unique, it is the median of the distribution, that is the alternative, denoted x_{Cond} such that:

$$F(x_{\text{Cond}}) = 1/2.$$

The alternative which maximizes the total utility $W(y) = \int_X -(y-x)^2 \mathrm{d}F(x)$ is the average of the ideal points. We call this point the Bentham optimum and denote it by x_{Bentham}:

$$x_{\text{Bentham}} = \int_x x \; \mathrm{d}F(x).$$

For any alternative y, the utility levels obtained by the various individuals induce a probability distribution that we denote by E_y : for any utility level $\overline{u} \leq 0$, $E_y(\overline{u})$ is the proportion of individuals in the society whose utility for

y is less than $\overline{u}$. Given our assumption about utilities, these individuals are precisely those whose ideal points are at a distance larger than $\sqrt{-\overline{u}}$ from y:

$$\begin{aligned} u_i(y) < \overline{u} &\iff -(y - x_i)^2 < \overline{u} \\ &\iff x_i \notin \left[y - \sqrt{-\overline{u}}, y + \sqrt{-\overline{u}}\right] \end{aligned}$$

thus

$$E_y(\overline{u}) = 1 - F(y + \sqrt{-\overline{u}}) + F(y - \sqrt{-\overline{u}})$$

and the median evaluation of y, denoted $u_{\text{med}}(y)$ is such that $E_y(u_{\text{med}}(y)) = 1/2$, that is:

$$F(y + \sqrt{-u_{\text{med}}(y)}) - F(y - \sqrt{-u_{\text{med}}(y)}) = 1/2.$$

It may be more convenient to write this formula with $d(y) = \sqrt{-u_{\text{med}}(y)}$ as:

$$\int_{y-d(y)}^{y+d(y)} \mathrm{d}F(x) = 1/2.$$

The best median evaluation choice, the outcome of "median evaluation" is denoted by x_{MaxMed}, its is the point which maximizes u_{med} or, equivalently, which minimize d. In general it is difficult to compute this point, but the example in the next sections makes these computations very simple.

4 Two results for non-symetric distributions

Here we consider the case of continuous distribution of ideal points with a density that is continuous and monotonous on its support. Without loss of generality we take the density to be decreasing and the support to be $S = [0, b]$ or $S = [0, +\infty)$.

Theorem 1 *Suppose that the distribution F of voters' ideal points has a continuous density f on its support S and that f is decreasing on S. Suppose that voters have quadratic utilities $u_i(y) = -(y - x_i)^2$. Then $x_{\text{Bentham}} \geq x_{\text{Cond}}$.*

Proof. The fact that $x_{\text{Bentham}} \geq x_{\text{Cond}}$ is just the fact that, for such a distribution, the average is larger than the median. For a proof, consider G the distribution made of a mass m at 0 plus a uniform distribution on $[0, c]$. Routine computation shows that, for G, the median is smaller than the average. Take $G(x) = \frac{1}{2} + (x - x_{\text{Cond}}) \cdot f(x_{\text{Cond}})$, $c = x_{\text{Cond}} + \frac{1}{2f(x_{\text{Cond}})}$ and $m = 1/2 - x_{\text{Cond}} f(x_{\text{Cond}})$. The median of G is x_{Cond}. The cumulative function G is above F so F stochastically dominates G and thus the average of F is larger than the average of G. Therefore the average of F is larger than its median. **Q.E.D.**

Theorem 2 *Suppose that the distribution F of voters' ideal points has a continuous density f on its support S and that f is decreasing on S. Suppose that voters have utilities ordinally equivalent to quadratic utilities $u_i(y) = \psi\left(-(y-x_i)^2\right)$ for ψ strictly increasing. Then $x_{\text{MaxMed}} = \frac{1}{2} x_{\text{Cond}}$.*

Proof. Note that for any population P with support $[y, y+2d]$, the Rawlsian satisfaction is obtained for the choice $y+d$ and has value $R(P) = \psi(-d^2/4)$. Therefore, in view of Lemma 1, x_{MaxMed} is the value of $y+d$ that minimizes d under the constraint that

$$\int_y^{y+2d} \mathrm{d}F = 1/2$$

for some $y \geq 0$. This constraint can be written as

$$F(y+2d) = 1/2 + F(y)$$

or, equivalently:

$$d = (1/2)F^{-1}(1/2 + F(y)) \equiv \phi(y).$$

Given our hypotheses, the function $\phi(\cdot)$ is diferentiable for $y \geq 0$, and

$$2\phi'(y) = \frac{f(y)}{f(y+2\phi(y))} - 1.$$

Because f is decreasing, the ratio in this expression is smaller than 1, thus ϕ is decreasing too, and is maximized for the smallest value $y = 0$. It follows that $F(2d) = 1/2$, which means that $x_{\text{Cond}} = 2d$. Hence the result. **Q.E.D.**

5 A simple example

Suppose that the distribution of voters' ideal point has bounded support $[0, 1]$ with an affine density function parametrized by a skewness parameter δ, with $0 < \delta \leq 1$:

$$f_\delta(t) = 1 + \delta - 2\delta t.$$

For $\delta = 0$ the density is uniform on $[0, 1]$ and for $\delta > 0$ the density is skewed on the left. For $\delta = 1$ the density is triangular. Figure 1 is drawn for $\delta = 1/3$. Then for $0 \leq x \leq 1$,

$$F_\delta(x) = \int_0^x (1 + \delta - 2\delta t) \; \mathrm{d}t = (1+\delta)x - \delta x^2.$$

Majority Voting: Condorcet winner. See point C in Figure 1. The median of the ideal points is such that $F_\delta(x_{\text{Cond}}) = 1/2$, that is $2\delta x^2 - 2(1+\delta)x + 1 = 0$, which gives:

$$x_{\text{Cond}} = \frac{1}{2\delta}\left(1 + \delta - \sqrt{1+\delta^2}\right).$$

	x_{Bentham}	x_{Cond}	x_{MaxMed}
$\delta \approx 0$	1/2	1/2	1/4
$\delta = 1/2$	.42	.38	.19
$\delta = 1$	.33	.29	.15

Table 1: Choices for an affine distribution with skewness δ

Some values are provided in Table 1. When the skewness δ tends to 0, x_{Cond} tends to 1/2.

Utilitarian evaluation: Bentham optimum. See point B in Figure 1. The average of the ideal points is $x_{\text{Bentham}} = \int_0^2 t f_\delta(t)\mathrm{d}t$ and computation shows:

$$x_{\text{Bentham}} = \frac{1}{2} - \frac{\delta}{6}.$$

Again, when the skewness δ tends to 0, x_{Bentham} tends to 1/2, like x_{Cond}.

Maximal Median evaluation: See point M in Figure 1. In order to compute the median evaluation at a point y one has to find a radius $d(y)$ such that half of the ideal points are located in the segment $[y - d(y), y + d(y)]$, and half are located outside. The best median is then obtained at the point y such that $d(y)$ is the smallest. In Figure 1, one can see that the best median is precisely the mid-point between 0 and the Condorcet winner x_{Cond}. Indeed, by definition of x_{Cond}, half of the population belongs to the segment $[0, x_{\text{Cond}}]$, and because the density is decreasing, any other segment of length smaller or equal will contain strictly less than half of the population. Therefore:

$$x_{\text{MaxMed}} = \frac{x_{\text{Cond}}}{2} = \frac{1}{4\delta}\left(1 + \delta - \sqrt{1 + \delta^2}\right).$$

When the skewness δ tends to 0, x_{MaxMed} tends to 1/4, unlike the two other choices.

One can see on this simple example the result announced in the introduction:

Proposition 1 *For any affine distribution f_δ of skewness $\delta > 0$:*

$$x_{\text{MaxMed}} < x_{\text{Cond}} < x_{\text{Bentham}}.$$

Proof. This follows from the theorems above, but a direct proof is easy: For any $\delta \in (0, 1]$, the inequality $\frac{1}{2\delta}\left(1 + \delta - \sqrt{1 + \delta^2}\right) < \frac{1}{2} - \frac{\delta}{6}$ is shown algebraically by letting $u = \sqrt{1 + \delta^2}$, and the result follows. **Q.E.D.**

Notice that the phenomenon, which holds for any value of the skewness parameter δ, is all the most striking for very small vallues of δ, which coresponds to distributions which are *almost not* skewed. In that case, letting δ tend to 0, the

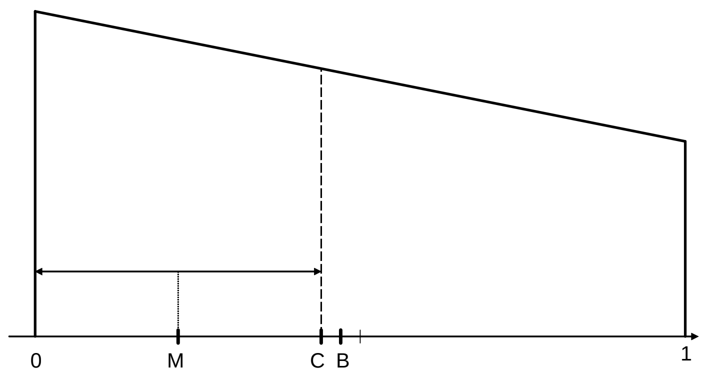

Figure 1: Three choices from a skewed distribution of ideal points

Condorcet winner and the utilitarian optimum both tend to 1/2, the center of the distribution: for almost symmetric distributions, there is no conflict between the utilitarian and the majoritarian principles: both chose a consensual outcome in the middle of the distribution. But things are totally different with MaxMed, whose choice tends to 1/4. As we have seen this choice is due to the fact that the MaxMed calculus allows to leave aside half of the population.

The graphical representation makes transparent the explanation of the phenomenon. Half of the population lies on each side of the Condorcet winner $x_{\text{Cond}} = C$, but the right side (between C and 1) is wider than the left one (between 0 and C). This should be an argument in favor of a collective choice larger than C, like the utilitarian optimum $x_{\text{Bentham}} = B$, because individuals at the left of C are relatively close to C while those at the right of C are relatively far from C. This is a typical utilitarian argument, that weights numbers of individuals and intensity of preferences. The reasoning of the MaxMed argument, that leads to the choice $x_{\text{MaxMed}} = M$, is reversed: the window being narrower on the left and wider on the right, chosing a point in the middle of the left window will satisfy the left half of the population and will reach for the — already quite satisfied — members of this group a relatively high level of satisfaction because this group is not too diverse. On the contrary, choosing a point on the right of M, such as C or B, the satisfied half of the population would be spread over a larger segment and it will be more difficult to reach the same level of satisfaction for this half population, because these people are more diverse.

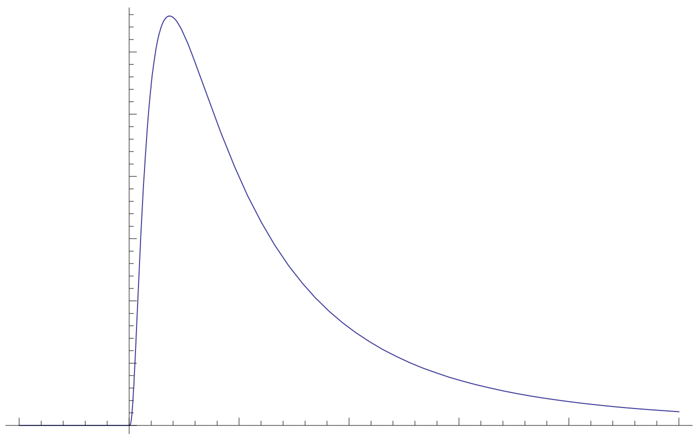

Figure 2: Log-Normal distribution

The choice of M is dictated by the level of satisfaction obtained by some half of the population. By definition of the MaxMed, this level is the largest than can be obtained by any half of the population, thanks to inter-individual comparability. But, doing so, it neglects the (maybe very low) level of satisfaction obtained by the other half of the population, thanks to ordinality. There is no compromise here: find the half of the population which would be better off if they were in power ! This "majoritarian" logic is flawed, as one can see on the example, because the result is that moves away from M in the direction of the center simultaneously (i) satisfy the majority criterion: more losers than winners, and (ii) satisfy the utilitarian criterion: the losers loose less than what the winners win.

6 Simulated examples with Log-Normal distribution

In the simulations, I compute the Bentham optimum, the outcome of Majority Rule (the Concorcet winner), the outcome of the Borda rule (the Borda winner) and the point with the best median evaluation. I work with a Log-Normal distribution, which is typically the kind of distribution met in Social Sciences (Figure 2) rather than affine distributions as in the previous section. The theoretical mean of the Log-Normal distribution of parameters 0 and 1 is 1.649 and the standard deviation is 2.161. The distribution vanishes quickly after $x = 5$ and, when needed, I restrict attention to the segment $[0, 5]$.[2] I pick at random 999 ideal points according to this distribution. Then, in a first example, the individual

[2] This has no impact on the value of the Condorcet winner nor on the MaxMed. It leads to a slight underestimation of the Bentham optimum and the Borda winner.

candidates	n_k	$u_i(y)$	**Bentham**	**Borda**	**Condorcet**	**MV**
uniform	11	$-d^2$	2	1	1	.5
		$-d$	1	1	1	.5
		$-d^{1/2}$	1	1	1	.5
	49	$-d^2$	1.77	1.15	1.04	.62
		$-d$	1.04	1.15	1.04	.62
		$-d^{1/2}$	.83	1.15	1.04	.62
repres.	11	$-d^2$	1.68	.92	.92	.56
		$-d$	.92	.92	.92	.56
		$-d^{1/2}$	.83	.92	.92	.56
	49	$-d^2$	1.89	.93	1.04	.56
		$-d$	1.04	.93	1.04	.56
		$-d^{1/2}$	.85	.93	1.04	.56

Table 2: Various specifications and choices with a Log-Normal Distribution

utility functions are *quadratic*: $u_i(y) = -(y - x_i)^2$, and there are 11 candidates evenly spread between 0 and 5 (at points 0, .5, 1, 1.5, 2, 2.5, ..., 5). In that case, the Bentham optimum is located at $x_{\text{Bentham}} = 2$, the Borda winner and the Condorcet winner are both located at $x_{\text{Cond}} = x_{\text{Borda}} = 1$, and the best median evaluation is obtained at 0.5, thus: $x_{\text{MaxMed}} < x_{\text{Cond}} = x_{\text{Borda}} < x_{\text{Bentham}}$

An alternative specification is that the utility is decreasing linearly with the distance: $u_i(y) = -|y - x_i|$. I label this case *linear* utility. Going from one specification to the other does not change the Condorcet or Borda winner nor does it change the median evaluations, because this is a strictly increasing transformation common to all individuals. But it changes the utilitarian optimum, which is now equal to the Condorcet winner. (Recall that the solution to the problem $\min_y \sum_i |y - x_i|$ is the median of the x_is.). Thus in that case, one has: $x_{\text{MaxMed}} < x_{\text{Cond}} = x_{\text{Borda}} = x_{\text{Bentham}}$.

Although economists usually work with concave utility functions (used as VNM utilities, these function are risk-adverse), it also makes sense in Politics to consider that the marginal utility is decreasing with the distance to the ideal point and to use utilities of the form $u_i(y) = -\sqrt{|y - x_i|}$. We label this case *root* utility. In the example, simulation shows: $x_{\text{MaxMed}} < x_{\text{Cond}} = x_{\text{Borda}} = x_{\text{Bentham}}$.

Table 2 reports the results of simulations in various cases. Quadratic utility is used for the lines labelled $-d^2$, linear utility for the lines $-d$, and root utility for the lines $-d^{1/2}$. The number of candidates, n_k, is 11 or 49. Candidates are either evenly spaced from 0 to 5 ("uniform" case), or chosen according to the same probability distribution of the voters ("representative" case).

The simulations reported in Table 2 are not averaged. Randomness comes from the choice of the 999 individual ideal points and (in the "representative" setting) of the choice of the candidate positions. This last point is important for $n_k = 11$. In order to check the robustness of the results I replicated some of the above experiences. For instance here are the results obtained during 100

(100 simulations)	**Bentham**	**Borda**	**Condorcet**	**MV**
mean value	1.56	.87	1.01	.65
standard deviation	.28	.17	.13	.12

Table 3: Robustness of the results for 11 representative candidates and quadratic utility

(100 simulations)	**Bentham-Condor**	**Condor-Borda**	**Borda-MV**
mean value	.55	.14	.22
standard deviation	.29	.22	.19

Table 4: Robustness of the results for 11 representative candidates and quadratic utility

simulations for the representative case with $n_k = 11$ candidates and quadratic utilities.

Out of 100 simulations, the usual ranking is: $x_{\text{MaxMed}} < x_{\text{Borda}} \leq x_{\text{Cond}} < x_{\text{Bentham}}$. More exactly:

- the strict inequality $x_{\text{MaxMed}} < x_{\text{Borda}}$ occurs 82 times, $x_{\text{MaxMed}} = x_{\text{Cond}}$ occurs 17 times, and $x_{\text{MaxMed}} > x_{\text{Cond}}$ occurs only once;

- the strict inequality $x_{\text{Borda}} < x_{\text{Cond}}$ occurs 50 times, $x_{\text{Borda}} = x_{\text{Cond}}$ occurs 46 times, and $x_{\text{Borda}} > x_{\text{Cond}}$ occurs 4 times;

- the strict inequality $x_{\text{Cond}} < x_{\text{Bentham}}$ occurs 95 times, $x_{\text{Cond}} = x_D$ occurs 5 times, and $x_{\text{Cond}} > x_{\text{Bentham}}$ does not occur.

Tables 3 and 4 provide further details about this robustness analysis. In this model, Borda and Condorcet are not well distinguished, but the ranking $x_{\text{MaxMed}} < x_{\text{Cond}} < x_{\text{Bentham}}$ appears to be robust. This indicates that the observations that were made in the analytical sections are robust.

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