Wishart Stochastic Volatility : Asymptotic Smile and Numerical Framework

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Résumé

In this paper, a study of a stochastic volatility model for asset pricing is described. Originally presented in [7, 9], the Wishart volatility model identifies the volatility of the asset as the trace of a Wishart process. Contrary to a classic multifactor Heston model, this model allows to add degrees of freedom with regard to the stochastic correlation. Thanks to its flexibility, this model enables a better fit of market data than the Heston model. Besides, the Wishart volatility model keeps a clear interpretation of its parameters and conserves an efficient tractability.

Firstly, we recall the Wishart volatility model and we present a Monte Carlo simulation method in sight of the evaluation of complex options. Regarding stochastic volatility models, implied volatility surfaces of vanilla options have to be obtained for a short time. The aim of this article is to provide an accurate approximation method to deal with asymptotic smiles and to apply this procedure *to the Wishart volatility model in order to well understand it and to make its calibration easier. Inspired by the singular perturbations method introduced by J.P Fouque, G. Papanicolaou, R. Sircar and K. Solna [14, 15], we suggest an efficient procedure of perturbation for affine models that provides an approximation of the asymptotic smile (for short maturities and for a two-scale volatility). Thanks to the affine properties of the Wishart volatility model, the perturbation of the Riccati equations furnishes the expected approximations. The convergence and the robustness of the procedure are analyzed in practice but not in theory. The resulting approximations allow a study of the parameters influence and can also be used as a calibration tool for a range of parameters.

Key-words : Wishart processes, stochastic volatility models, stochastic correlation, singular perturbation, asymptotic smile, Monte Carlo simulation

The Black-Scholes model (1973) does not incorporate the observable phenomenon that implied volatility of derivative products is strike- and maturity- dependent. To reproduce some market conditions like the smile effect, various models have been introduced such as local volatility and stochastic volatility models. The first step was the introduction by Dupire (1994) of local volatility models where the underlying volatility $\sigma(t, S)$ depended on the level of the underlying S itself. The most famous one is the Constant Elasticity of Variance model (CEV) in which the volatility is proportional to S^α, where α is a positive constant. Then, stochastic volatility models appeared where volatility is assumed to be a stochastic process. Thus, models became more complex because of the market incompleteness which implies that traders can not hedge their products by dealing only with the asset. The Heston model (1992) presents a volatility with an effect of mean reversion, and is commonly used in financial markets because of its flexibility. However, these models cannot fit accurately market data for short or long maturities, and recent researchs have been carried out to improve this point.

A way to solve this problem is the introduction of a multifactor stochastic volatility model. C. Gourieroux and R. Sufana [19, 20] (2006) developed a multifactor version of the Heston model. Indeed, they assumed that volatility follows a Wishart process introduced by M.F. Bru [3] (1991) so that the model preserves its linear properties and consequently its tractability. J. Da Fonseca, M. Grasselli and C. Tebaldi [7](2005) have improved the initial modeling considering that the volatility of the asset is the trace of a Wishart process. That allows to take into account stochastic correlation between the underlying asset and the volatility process and provides a wealthy but complex model. The simplistic case where the matrix of mean reversion, the volatility of volatility matrix and the correlation matrix are diagonal matrices gives a small intuition of the model performance given that the diagonal components of the Wishart process are in fact Cox-Ingersoll-Ross processes : by considering the volatility as the trace of the Wishart process, the model is equivalent to a simple multifactor Heston model. However, one can see that a multifactor Heston model (like in [5]) is not flexible enough in regard to the stochastic correlation. Indeed, in a classic extension of a multifactor Heston model, the factors appearing in the stochastic correlation formulae are exactly the same as the volatility expression's ones. In order to extend this model, we will focus on a specification of the Wishart volatility model allowing to add freedom degrees concerning the stochastic correlation. Besides, we will consider the case where the correlation matrix is not diagonal and we will highlight the fact that this model differs from a standard multifactor Heston model given that the stochastic

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correlation depends on a new factor.

This model belongs to the class of affine models and is tractable in the way that it exists an explicit expression of the conditional charateristic function which allows the use of the FFT method introduced by P. Carr and D. Madan [4] (1999). Being factorial models, affine models have consequently intrinsic capabilities to integrate source of risk and are powerful tools of projection. Besides, with its mean-reversion effect, this Wishart volatility model is compatible with classic macroscopic phenomena.

Nevertheless, generating the smile of implied volatility is not fast enough, therefore the calibration remains a problem. Accurate approximations are needed in order to calibrate directly or to furnish a relevant initial set of parameters for initiating the calibration procedure. Moreover, for the users of this model such as traders, having instantly an approximation with a closed formulae for the smile could be very useful. Indeed, these explicit expressions give an intuition of the influence of the parameters like correlation, volatility of volatility and mean-reversion parameters. Many academicians and practitioners have tackled this challenge and have introduced specific methods. To cite some of them, there are the singular perturbation method based on the perturbation of the evaluation PDE [14, 15, 16], a procedure considering the link between the spot and the implied volatility in short maturities [13], a method based on geometric using the heat kernel expansion on a Riemann manifold [27], a method based on Mallivin calculus [26] and so on. These methods have not been studied and done for the Wishart volatility model but the standard singular perturbation method, allowing an approximation for a two-scale volatility, becomes quickly hard to apply (the second order is really difficult to obtain) [2].

This paper is organized as follows : we first present the definition and the framework of the Wishart volatility model where the volatility of the asset is given by the trace of a Wishart process. Then, we will also see some numerical ideas to simulate the Wishart volatility model by Monte Carlo methods. Indeed, there are many options that need a Monte Carlo evaluation but it is not easy to conserve the symmetry and the positiveness of the Wishart process. In Section 3, we will study the asymptotic behaviour of the smile for very short maturities and for two scales of maturities : those maturities represent "characteristic life times" for the volatility as detailed in the article. Inspired by the singular perturbations methods developed by Fouque and al. described in [14, 15, 16] to the Wishart volatility model, we have found a useful asymptotic approximation. This method uses the affine properties of the model through a perturbation of the Riccati equations and is easy to

carry on. Those simple expressions allow to obtain instantly the skew and the smile for some reasonable range of parameters. Moreover, the convergence and the robustness in relation to parameters of this approximation is analyzed in practice but not in theory. Finally, this approximation provides an adequate calibration tool for admissible parameters as well as an outline of the parameters' influence in the Wishart volatility model.

Notations

To begin with, let us introduce some notations on sets of matrices :
- $\mathcal{M}_{n,m}(\mathbb{R})$: set of real matrices $n \times m$.
- $GL_n(\mathbb{R})$: set of real invertible matrices $n \times n$.
- $\sigma(M)$: set of the eigenvalues of the matrix M, called spectrum of the matrix M.
- $S_n(\mathbb{R})$: set of real symmetric matrices.
- $S_n^-(\mathbb{R})$: set of nonpositive symmetric matrices i.e for $M \in S_n^-(\mathbb{R})$, $\forall \lambda \in \sigma(M)$, $\lambda \leq 0$.
- $S_n^+(\mathbb{R})$: set of nonnegative symmetric matrices i.e for $M \in S_n^+(\mathbb{R})$, $\forall \lambda \in \sigma(M)$, $\lambda \geq 0$.
- $\tilde{S}_n^+(\mathbb{R})$: set of strictly nonnegative symmetric matrices i.e for $M \in \tilde{S}_n^+(\mathbb{R})$, $\forall \lambda \in \sigma(M)$, $\lambda > 0$.
- $A^\top$ is the transposed matrix of A.
- For A a nonnegative symmetric matrix, $\sqrt{A}$ is the unique nonnegative symmetric matrix such as $\sqrt{A}\sqrt{A} = A$.
- For $A \in \mathcal{M}_{n,n}(\mathbb{R})$, $Tr(A)$ is the trace of the matrix A.

Let us consider a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ equipped with a filtration $\{\mathcal{F}_t\}$ satisfying usual conditions. The probability measure $\mathbb{Q}$ corresponds to a "risk-neutral" measure under which the price of any asset is the conditional expectation of its discounted future cash flows

1   The Wishart Process

The Wishart process which was originally studied by Bru [3] in 1991 was introduced in finance by Gourieroux and al. in [19]. Then, many authors have developed stochastic volatility models using the Wishart processes and this paper is focused on the "Wishart volatility model" presented by Da Fonseca and al. [7, 9]. Some specifications and properties

of the Wishart process are recalled for a better understanding.

1.1 Definition of the Wishart processes

1.1.1 Definition

The Wishart standard distribution is a multidimensional generalization of the χ^2 distribution and is very useful for the estimation of the covariance matrices in multivariate statistics [28]. Indeed, the Wishart distribution corresponds to the law of the covariance matrix estimator of a Gaussian vector sample : Let $X_1, ..., X_n$ be n independent Gaussian vectors in $\mathbb{R}^p$ such as $X_i \sim \mathcal{N}(0, \Sigma)$, $i = 1, ..., n$.
The law of the $p \times p$ random matrix $S = \sum_{i=1}^n X_i X_i^\top$ is called *Wishart distribution* noted $S \sim W(\Sigma, p, n)$. When $X_i \sim \mathcal{N}(\mu, \Sigma)$, $i = 1, ..., n$, the law of $S = \sum_{i=1}^n X_i X_i^\top$ is called *non central Wishart distribution* noted $S \sim W(\Sigma, p, n, \mu)$.

In order to model covariance matrices dynamics, we need to focus on process valued on nonnegative matrices. Then, the trace of such matrices may be considered for modelling positive stochastic volatility process. Consider now the classic definition of the Wishart process through its diffusion equation.

Definition 1 (Bru [3]). *Let $\{W_t, t \geq 0\}$ denote a $n \times n$ matrix-valued Brownian motion under the probability measure $\mathbb{Q}$. The matrix-valued process V is said to be a Wishart process if it satisfies the following diffusion equation*

$$dV_t = (\beta Q^\top Q + MV_t + V_t M^\top)dt + \sqrt{V_t}dW_t Q + Q^\top dW_t^\top \sqrt{V_t}, \qquad V_0 = v_0.$$

where $Q \in GL_n(\mathbb{R})$ is a $n \times n$ invertible matrix, M is a $n \times n$ nonpositive matrix, $v_0 \in \tilde{S}_n^+$ is a strictly nonnegative symmetric matrix and β a real such as $\beta > (n-1)$.

The condition $\beta > n - 1$ is introduced to ensure existence and unicity of the solution $V_t \in S_n^+$ of Equation 1.1. Moreover, as shown in [3, 7], for $\beta \geq n + 1$, eigenvalues of the solution are strictly nonnegative $\forall t \geq 0$ a.s $V_t \in \tilde{S}_n^+$

1.1.2 The Laplace transform

The diffusion satisfied by the Wishart process belongs to the class of affine diffusions [11]. Consequently, an explicit expression for the Laplace transform is available via the resolution of a Riccati system and this notion will be a key tool for future developments. This Laplace transform belongs to the family of Wishart distribution Laplace transforms as described for example in [28] what explains the designation "Wishart process".

Proposition 1 (Gourieroux & Sufana [19]). *For* $\Theta \in S_n^+$ *and* $t, h \geq 0$, *let* $\Psi_t(h, \Theta) = \mathbb{E}[\exp(-Tr(\Theta V_{t+h}))|V_t]$ *the Laplace transform of* V_{t+h} *given* V_t. *Then,*

$$\Psi_t(h, \Theta) \;=\; \frac{\exp - Tr[\Delta(h)^\top \Theta (I_n + 2\Sigma(h)\Theta)^{-1}\Delta(h)V_t]}{(\det[I_n + 2\Sigma(h)\Theta])^{\frac{\beta}{2}}}, \tag{1.1}$$

with

$$\Delta(h) \;=\; \exp(hM), \qquad \Sigma(h) \;=\; \int_0^h \Delta(s)Q^\top Q\Delta(s)^\top ds.$$

1.2 Examples

Let us present some useful and simple examples.

1.2.1 Multidimensional Cox-Ingersoll-Ross processes

Let us consider a Wishart process with diagonal matrices M, Q and R. It can be seen as a multidimensional CIR process. Indeed, the dynamics of the diagonal components are :

$$dV_t^{ii} = (\beta(Q^{ii})^2 + 2M^{ii}V_t^{ii})dt + 2Q^{ii}\sum_{k=1}^n \sqrt{V_t}^{ki} dW_t^{ki}.$$

Although their dynamics seem to depend on other components of the Wishart process, that is just an illusion. The Brownian motions can be concatenated through the following expression

$$d < V^{ii}, V^{ii} >_t \;=\; 4\left((Q^{ii})^2(\sum_{k=1}^n \sqrt{V_t}^{ki}\sqrt{V_t}^{ki})\right) dt = 4(Q^{ii})^2(\sqrt{V_t}\sqrt{V_t})^{ii} = 4(Q^{ii})^2 V_t^{ii}.$$

So, $B = (B^1, ..., B^n)$ defined by $dB_t^i = \sqrt{V_t^{ii}}^{-1}\sum_{k=1}^n (\sqrt{V_t^{ki}})dW_t^{ki}$ is a vector of n independent Brownian motions. Then,

$$dV_t^{ii} = \left(\beta(Q^{ii})^2 + 2M^{ii}V_t^{ii}\right) dt + 2Q^{ii}\sqrt{V_t^{ii}}dB_t^i.$$

Then, in this simple case where parameters are diagonal matrices, the diagonal components of the Wishart process are independent Cox-Ingersoll-Ross processes.

1.2.2 Link between Wishart processes and Ornstein Uhlembeck processes

A natural way to generate a "Wishart process" is to replace the Gaussian vectors X_i in the definition of a Wishart distribution (Section 1.1.1) with Ornstein-Uhlenbeck processes $X_{i,t}$. With this representation which implies that β is an integer, the Wishart volatility

model described in Section 2 appears as an example of Quadratic Gaussian models.

Considering β independent n-dimensional Ornstein-Uhlenbeck processes :

$$dX_{k,t} \; = \; MX_{k,t}dt + Q^\top dW_{k,t}, \qquad \text{and} \qquad V_t := \sum_{k=1}^{\beta} X_{k,t}X_{k,t}^\top$$

Then, the process (V_t) is a Wishart process with the following dynamics.

$$dV_t \; = \; (\beta Q^\top Q + MV_t + V_t M^\top)dt + \sqrt{V_t}dW_t Q + Q^\top dW_t^\top \sqrt{V_t},$$

with W a matrix-valued Brownian motion, determined by

$$\sqrt{V_t}dW_t \; = \; \sum_{k=1}^{\beta} X_{k,t}dW_{k,t}^\top.$$

In this framework, the matrix M can be seen as the mean-reversion parameter of the Wishart process and Q as the volatility parameter.

Consequently, the simulation Ornstein-Uhlenbeck processes allows the simulation of a Wishart process in the case where β is an integer. Therefore, concerning the simulation of Wishart processes, the first step is to consider the simple case where $\beta \geq n+1$ is an integer. For the initialization of the Ornstein-Uhlenbeck processes, on can notice that the initial matrix V_0 is real and symmetric and can be then diagonalized. Using the eigenvectors Φ_k and the eigenvalues λ_k, one can write :

$$V_0 \; = \; \sum_{k=1}^{n} \lambda_k \Phi_k \Phi_k^\top.$$

Then, the initialization of the Ornstein-Uhlenbeck processes can be done

$$\begin{cases} \bar{X}_{k,0} & = & \sqrt{\lambda_k}\Phi_k \;\; \text{if } 1 \leq k \leq n, \\ \bar{X}_{k,0} & = & 0 \;\; \text{if } n+1 \leq k \leq \beta \end{cases}$$

Finally, for $1 \leq i \leq N$, simulating $\{\bar{X}_{k,t_i}, 1 \leq k \leq \beta\}$ by one of the methods described in the next paragraph furnishes the discretization of this Wishart process

$$\bar{V}_{t_i} \; = \; \sum_{k=1}^{\beta} \bar{X}_{k,t_i}\bar{X}_{k,t_i}^\top.$$

This simulation procedure for a Wishart process requires the simulation of Ornstein-Uhlenbeck processes which is classic and can be done by an Euler scheme or an exact simulation using the knowledge of the conditional distribution (See Appendice B).

However, this way does not provide the most general class of the Wishart process and there is a very useful extension in practice by considering the same dynamics with a real β. As developed in Section 1.3.2, Girsanov theorem gives a change of probability measure allowing to move from β integer to β real. The simulation in the general case will be detailed after.

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1.3 Determinant dynamics and change of probability measure

In this part, the stochastic differential equation describing the determinant dynamics of Wishart process is presented and analyzed. At the end of this section, a change of probability measure is highlighted allowing a change of drift in the dynamics of Wishart process. Using determinant dynamics, the *likelihood ratio martingale* has an explicit functional form 1.3 and allows an efficient simulation procedure.

1.3.1 Determinant dynamics

The dynamics of determinant will be essential to handle this change of probability measure that will be very important for the simulation of the Wishart process in the general case, where β is not an integer. A useful theorem given in [3] has to be recalled (without proof) for the future understanding.

Theorem 1 (Bru [3]). *Let $\{\zeta_t = (\zeta_t^{ij})_{1 \leq i,j \leq n}\}$ be a continuous semimartingale process valued in the matrix set $S_n(\mathbb{R})$. Let $U_t \in O_n(\mathbb{R})$ such as $U_t^\top \zeta_t U_t = diag(\lambda_t^1, ..., \lambda_t^n)$ where $\{\lambda_t^i, 1 \leq i \leq n\}$ are the eigenvalues of ζ_t.*
Let $\{A_t = (A_t^{ij})_{1 \leq i,j \leq n}, t \geq 0\}$ be the process defined as

$$A_t = \int_0^t U_s^\top d\zeta_s U_s, \forall t \geq 0 \quad such\ as \quad d < A^{ij}, A^{ji} >= \Gamma_t^{ij} dt.$$

Then, the dynamics of eigenvalues can be written as

$$d\lambda_t^i = dM_t^i + dJ_t^i, \qquad dJ_t^i = \sum_{j \neq i} \frac{1}{\lambda_t^i - \lambda_t^j} \Gamma_t^{ij} dt + d\Theta_t^i.$$

where dM^i and $d\Theta^i$ are respectively the martingale part and the finite variation part of dA^{ii}.

Let us denote $\tilde{Q}_t = U_t^\top Q U_t$ and $\tilde{M}_t = U_t^\top M U_t$ and $Z_t = \int_0^t U_s^\top dW_s U_s$.
By applying the previous theorem, the dynamics of eigenvalues are characterized by the following stochastic differential equation.

$$d\lambda_t^i = [\beta(\tilde{Q}_t^\top \tilde{Q}_t)^{ii} + 2\lambda_t^i \tilde{M}_t^{ii} + \sum_{j \neq i} \frac{(\lambda_t^i \tilde{Q}_t^\top \tilde{Q}_t)^{jj} + (\lambda_t^j \tilde{Q}_t^\top \tilde{Q}_t)^{ii}}{\lambda_t^i - \lambda_t^j}] dt + 2\sqrt{\lambda_t^i} \sum_{k=1}^n \tilde{Q}_t^{ki} dZ_t^{ik}.$$

It is important to notice that all eigenvalues are independent and are non-colliding [3, 12]. Since $d < \lambda^i, \lambda^j >_t = 0$, the differentiation of the determinant yields $d(\det(V_t)) = \det(V_t) \sum_{i=1}^n \frac{d\lambda_t^i}{\lambda_t^i}$. Moreover, by using the trace invariance by a change of basis, the dynamics of determinant are obtained by

$$\frac{d(\det(V_t))}{\det(V_t)} = [(\beta - n + 1)Tr(V_t^{-1}Q^\top Q) + 2Tr(M)]dt + 2Tr(\sqrt{V_t^{-1}}dW_t Q).$$

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which implies that the

$$d \log(\det(V_t)) = [(\beta - n - 1)Tr(V_t^{-1}Q^\top Q) + 2Tr(M)]dt + 2Tr(\sqrt{V_t^{-1}}dW_t Q) \quad (1.2)$$

1.3.2 Change of the probability measure

From a mathematical point of view, it is really important to analyze the change of the probability measure allowing a change of the drift in the process dynamics. In practice, for a financial use like calibration of parameters on historic data, a Wishart process has to be simulated in its general form with $\beta \geq n+1$ and $\beta \in \mathbb{R}$. Considering the floor function allows to write $\beta = K + 2\nu$ with $K = [\beta] \geq n+1$ and ν a real number such as $0 \leq \nu \leq \frac{1}{2}$. The aim of this development is to find a change of probability measure in order to change the generalized Wishart diffusion into the simple one where K is an integer. This new probability measure is noted $\mathbb{Q}^*_{|\mathcal{F}_T}$ and can be expressed as follows

Theorem 2. *Let $q = K + \nu - n - 1$. If $f_T(\mathbb{Q}, \mathbb{Q}^*) = \frac{d\mathbb{Q}_{|\mathcal{F}_T}}{d\mathbb{Q}^*_{|\mathcal{F}_T}}$ defines the Radon-Nikodym derivative of $\mathbb{Q}_{|\mathcal{F}_T}$ with respect to $\mathbb{Q}^*_{|\mathcal{F}_T}$, then*

$$f_T(\mathbb{Q}, \mathbb{Q}^*) = \frac{\det(V_T)^{\frac{\nu}{2}}}{\det(V_0)} \exp[-\nu T\, Tr(M)] \exp\left[-\frac{\nu}{2}q \int_0^T Tr(V_s^{-1}Q^\top Q)ds\right]. \quad (1.3)$$

Proof: Like in [12], a probability measure $\mathbb{Q}^*$ can be specified through an exponential martingale

$$\frac{d\mathbb{Q}}{d\mathbb{Q}^*}\bigg|_{\mathcal{F}_T} = \exp\left[\nu \int_0^T Tr(\sqrt{V_s^{-1}}dW_s Q) - \frac{\nu^2}{2} \int_0^T Tr(V_s^{-1}Q^\top Q)ds\right].$$

Therefore, this expression suggests to define a new process W^* by

$$W_t^* = W_t + \nu \int_0^t \sqrt{V_s^{-1}}Q^\top ds.$$

By Girsanov theorem, it is easy to check that W^* is a matrix-valued Brownian motion under the probability measure $\mathbb{Q}^*$. Consequently, the dynamics of Wishart process under this probability measure $\mathbb{Q}^*$ can be rewritten as follows

$$dV_t = (KQ^\top Q + MV_t + V_t M^\top)dt + \sqrt{V_t}dW_t^* Q + Q^\top (dW_t^*)^\top \sqrt{V_t}.$$

The Radon-Nikodym derivative can be simplified using the determinant dynamics. Indeed, as noted in the first section, we have

$$\log\left[\frac{\det(V_t)}{\det(V_0)}\right] = 2T\, Tr(M) + (K - n - 1) \int_0^T Tr(V_s^{-1}Q^\top Q)ds + 2 \int_0^T Tr(\sqrt{V_s^{-1}}dW_s^* Q).$$

Finally, the change of the probability measure can be obtained by

$$d\mathbb{Q}_{|\mathcal{F}_T} = \frac{\det(V_T)^{\frac{\nu}{2}}}{\det(V_0)} \exp[-\nu T\, Tr(M)] \exp\left[-\frac{\nu}{2}(K + \nu - n - 1) \int_0^T Tr(V_s^{-1}Q^\top Q)ds\right].d\mathbb{Q}^*_{|\mathcal{F}_T}.$$

which completes the proof. $\qquad\square$

This change of probability measure allows to bring back the problem to the simple case considered before. It will be very useful for many applications and more particularly for evaluation by Monte Carlo method.

2 Wishart stochastic volatility model in the stock derivatives market

2.1 Presentation of the Wishart volatility model

In this part, the Wishart volatility model is presented as well as its essential characteristics. The properties inherent in this model will allow an accurate and an efficient consistency with the market. The underlying volatility is defined as the trace of a Wishart process. Therefore, the diagonal components of the Wishart matrix will be the factors guiding the dynamics of volatility. Under the risk neutral probability measure, the asset dynamics are given by the following expression

$$
\begin{aligned}
\frac{dS_t}{S_t} &= rdt + Tr[\sqrt{V_t}(dW_t R + dZ_t \sqrt{I_n - RR^\top})] & S_0 = x. \\
dV_t &= (\beta Q^\top Q + MV_t + V_t M^\top)dt + \sqrt{V_t}dW_t Q + Q^\top dW_t^\top \sqrt{V_t} & V_0 = v_0. \quad (2.1)
\end{aligned}
$$

where $\{V_t, t \geq 0\}$ is a Wishart matrix-valued process as introduced in the previous section, r is the interest rate considered constant, S_t the price of the asset at date t, R a matrix such as $\rho(R) = \max\{\lambda, \lambda \in \sigma(R)\} \leq 1$, W, Z are independent matrix-valued Brownian motions.

The matrix R describes to the correlation between the Brownian matrices of the asset and those of the Wishart process. The matrix M represents the matrix of mean reversion and the matrix Q is the volatility of the volatility.

The dynamics of log-price $Y_t = \log(S_t)$ are deduced easily

$$
\begin{aligned}
dY_t &= (r - \frac{Tr(V_t)}{2})dt + Tr[\sqrt{V_t}(dW_t R + dZ_t \sqrt{I_n - RR^\top})] & Y_0 = y. \\
dV_t &= (\beta Q^\top Q + MV_t + V_t M^\top)dt + \sqrt{V_t}dW_t Q + Q^\top dW_t^\top \sqrt{V_t} & V_0 = v_0 (2.2)
\end{aligned}
$$

It is conventional to write the log-price and its volatility dynamics as driven by two correlated Brownian motions B_s^Y and B_s^V. In fact, as it is done in [9] by using a concatenation of the Brownian motions like in Section 1.2, the asset dynamics can be rewritten as follows

$$
dY_t = (r - \frac{Tr(V_t)}{2})dt + \sqrt{Tr(V_t)}dB_t^Y. \tag{2.3}
$$

$$dTr(V_t) \quad = \quad (\beta Tr(Q^\top Q) + 2Tr(MV_t))dt + 2\sqrt{Tr(Q^\top QV_t)}(\rho_t dB_t^Y + \sqrt{1-\rho_t^2}dB_t^V)$$

where $d < B^Y, B^V >_t = \rho_t = \dfrac{Tr(R^\top QV_t)}{\sqrt{Tr(V_t)}\sqrt{Tr(Q^\top QV_t)}}$.

2.1.1 Fourier transform

In order to calculate price of options by FFT method, define the instantaneous realized mean variance $U_t = \int_0^t Tr(V_s)ds$, and consider the Laplace Transform $\Psi_t(T, \theta, \gamma, \delta) = \mathbb{E}[e^{Tr(\theta V_t) + \gamma Y_T + \delta U_T}|\mathcal{F}_t]$ of (V, Y, U).

This function has to be explicitly calculated in order to price many derivative products such as vanilla options or Variance Swap. Since the system (V, Y, U) defines an affine model, the Laplace Transform Ψ can be easily expressed with functions that are solutions of a Riccati system [11]. The resolution is possible for example by using the linearization method introduced by J. Da Fonseca and al.[7].

Theorem 3.

$$\Psi_t(T, \theta, \gamma, \delta) \quad = \quad \exp[Tr[A(T-t)V_t] + B(T-t)Y_t + C(T-t)U_t + D(T-t)],$$

where A, B, C and D are solutions of the following Riccati equations

$$A'(\tau) \quad = \quad \left(\frac{\gamma(\gamma-1)}{2} - \delta\right)I_n + A(\tau)M + (M^\top + 2\gamma R^\top Q)A(\tau) + 2A(\tau)Q^\top QA(\tau).$$

$$B(\tau) \quad = \quad \gamma, \qquad C(\tau) \quad = \quad \delta.$$

$$D'(\tau) \quad = \quad r\gamma + \beta Tr[Q^\top QA(\tau)]. \tag{2.4}$$

with the terminal conditions $A(0) = \theta$ and $D(0) = 0$.

By using the linearization method, the solutions for A and D are given by

$$A(\tau) \quad = \quad F(\tau)^{-1}G(\tau).$$

$$D(\tau) \quad = \quad \gamma\tau[r - \beta Tr(R^\top Q)] - \frac{\beta}{2}[Tr(M)\tau + \log(\det F(\tau))]. \tag{2.5}$$

with

$$[G(\tau) \ F(\tau)] \quad = \quad [0 \ I_n]\exp(\tau Z(\gamma, \delta))$$

$$Z(\gamma, \delta) \quad = \quad \begin{pmatrix} M & -2Q^\top Q \\ (\frac{\gamma(\gamma-1)}{2} - \delta)I_n & -(M^\top + 2\gamma R^\top Q) \end{pmatrix}.$$

Proof:

The function $f_t(T, V, Y, U) = \exp[Tr[A(T-t)V_t] + B(T-t)Y_t + C(T-t)U_t + D(T-t)]$ has to be a martingale. Thus, Girsanov theorem suggests to find a martingale M_t such as

$$\log[f_t(T, V, Y, U)] \quad = \quad M_t - \frac{1}{2} < M_t, M_t >. \tag{2.6}$$

Applying Ito formula to the affine function $f_t(T, V, Y, U)$ yields

$$
\begin{aligned}
d\log[f_t(T, V, Y, U)] \;=\; & -\left(Tr[A^{'}(T-t)V_t]dt + B^{'}(T-t)Y_t + C^{'}(T-t)U_t + D(T-t)\right)dt \\
& + \left(Tr[\beta Q^\top Q A(T-t)] + Tr[A(T-t)MV_t] + Tr[M^\top A(T-t)V_t]\right)dt \\
& + Tr[A(T-t)\sqrt{V_t}dW_t Q] + Tr[A(T-t)Q^\top dW_t^\top \sqrt{V_t}] \\
& + \left(B(T-t) - \frac{1}{2}Tr[B(T-t)V_t]\right)dt \\
& + Tr[B(T-t)\sqrt{V_t}dW_t R] + Tr[B(T-t)dZ_t\sqrt{I_n - RR^\top}] \\
& + C(T-t)Tr(V_t)dt. \qquad\qquad (2.7)
\end{aligned}
$$

This expression allows us to find the martingale M_t

$$
\begin{aligned}
dM_t \;=\; & Tr[A(T-t)\sqrt{V_t}dW_t Q] + Tr[A(T-t)Q^\top dW_t^\top \sqrt{V_t}] \\
& + Tr[B(T-t)\sqrt{V_t}dW_t R] + Tr[B(T-t)dZ_t\sqrt{I_n - RR^\top}].
\end{aligned}
$$

For the calculation of the quadratic variation of M_t, noted $< M_t, M_t >$, let us recall some useful expressions

$$
d < Tr(AdW_t B) >= Tr(BAA^\top B^\top)dt
$$

$$
d < Tr(AdW_t B), Tr(CdW_t D) >= Tr(DCA^\top B^\top)dt
$$

Then, the quadratic variation $< M_t, M_t >$ of the martingale is expressed as follows

$$
\begin{aligned}
d < M_t, M_t > \;=\; & \left(4Tr[A(T-t)Q^\top Q A(T-t)V_t] + B^2(T-t)Tr[V_t]\right)dt \\
& + \left(Tr[B(T-t)R^\top Q A(T-t)V_t]\right)dt.
\end{aligned}
$$

then, replacing this expression in 2.6 and identifying with 2.7 gives

$$
\begin{aligned}
0 \;=\; & \left(-2Tr[A(T-t)Q^\top Q A(T-t)V_t] - \frac{1}{2}B^2(T-t)Tr[V_t] - \frac{1}{2}Tr[B(T-t)R^\top Q A(T-t)V_t]\right) \\
& - \left(Tr[A^{'}(T-t)V_t]dt + B^{'}(T-t)Y_t + C^{'}(T-t)U_t + D(T-t)\right) \\
& + \left(Tr[\beta Q^\top Q A(T-t)] + Tr[A(T-t)MV_t] + Tr[M^\top A(T-t)V_t]\right) \\
& + \left(B(T-t) - \frac{1}{2}Tr[B(T-t)V_t]\right) + C(T-t)Tr(V_t)
\end{aligned}
$$

Finally, identifying the terms with V_t, Y_t and U_t gives

$$
\begin{aligned}
A^{'}(\tau) \;=\; & \left(\frac{\gamma(\gamma-1)}{2} - \delta\right)I_n + A(\tau)M + (M^\top + 2\gamma R^\top Q)A(\tau) + 2A(\tau)Q^\top Q A(\tau) \\
B(\tau) \;=\; & \gamma, \qquad C(\tau) \;=\; \delta \\
D^{'}(\tau) \;=\; & r\gamma + \beta Tr[Q^\top Q A(\tau)]
\end{aligned}
$$

Riccati equation linearization

Let us search A such as

$$A(\tau) = (F(\tau))^{-1}G(\tau), \quad F(0) = I_n, \quad G(0) = 0.$$

By replacing this expression in the first Riccati equation of 2.8

$$
\begin{aligned}
-(F(\tau))^{-1}F^{'}(\tau)A(\tau) + (F(\tau))^{-1}G^{'}(\tau) = {} & \left(\frac{\gamma(\gamma-1)}{2} - \delta\right)I_n + (F(\tau))^{-1}G(\tau)M, \\
& + (M^\top + 2\gamma R^\top Q)A(\tau) \\
& + 2(F(\tau))^{-1}G^{'}(\tau)Q^\top QA(\tau).
\end{aligned}
$$

Therefore, by multiplying on the left by $F(\tau)$ and identifying terms with A and without A, we obtain

$$
\begin{aligned}
G^{'}(\tau) &= G(\tau)M + \left(\frac{\gamma(\gamma-1)}{2} - \delta\right)F(\tau), \\
F^{'}(\tau) &= -G(\tau)Q^\top Q - F(\tau)(M^\top + 2\gamma R^\top Q).
\end{aligned}
\tag{2.8}
$$

This system of equations can be written as follows

$$[G^{'}(\tau) \ F^{'}(\tau)] = [G(\tau) \ F(\tau)]Z(\gamma, \delta),$$

with $Z(\gamma, \delta) \in \mathcal{M}_{2n}(\mathbb{R})$ defined by

$$Z(\gamma, \delta) = \begin{pmatrix} M & -2Q^\top Q \\ (\frac{\gamma(\gamma-1)}{2} - \delta)I_n & -(M^\top + 2\gamma R^\top Q) \end{pmatrix}.$$

Thus, the solution is given by

$$[G(\tau) \ F(\tau)] = [0 \ I_n]exp(\tau Z(\gamma, \delta)).$$

Resolution of the equation that D follows

By integrating the fourth equation of 2.8, we have

$$D(\tau) = \gamma r\tau + \beta Tr[\int_0^\tau Q^\top QA(s)ds].$$

On the other hand, by multiplying on the left left by $F(\tau)^{-1}$ the second equation of 2.8, integrating it between 0 and τ, and taking the trace, we find

$$\beta Tr[\int_0^\tau Q^\top QA(s)ds] = -\frac{\beta}{2}(Tr[\int_0^\tau F(s)^{-1}F^{'}(s)ds] + Tr(M^\top + 2\gamma R^\top Q)\tau).$$

Finally the solution is given by

$$D(\tau) = \gamma\tau[r - \beta Tr(R^\top Q)] - \frac{\beta}{2}[Tr(M)\tau + \log(detF(\tau))].$$

$$\square$$

In order to calculate the price of Call options with a resolution by FFT method, a closed formulae for the Fourier transform of log-price is needed. The expression of this Fourier transform is defined by $\Phi_t(T, \gamma) = \mathbb{E}[\exp(i\gamma Y_T)|\mathcal{F}_t] = \Psi_t(T, 0, i\gamma, 0)$ and an explicit formulae can be deduced easily. Another proof of this corrolary can be found in [7]

Corrolary 1.

$$\Phi_t(T, \gamma) = \exp[Tr[A(T - t)V_t] + B(T - t)Y_t + C(T - t)],$$

where A and C are solutions of the following Riccati equations

$$
\begin{aligned}
B(\tau) &= i\gamma. \\
A'(\tau) &= \frac{i\gamma(i\gamma - 1)}{2}I_n + A(\tau)M + (M^\top + 2i\gamma R^\top Q)A(\tau) + 2A(\tau)Q^\top QA(\tau). \\
C'(\tau) &= ir\gamma + \beta Tr[Q^\top QA(\tau)].
\end{aligned}
$$

The linearization method can be used and gives

$$
\begin{aligned}
A(\tau) &= F(\tau)^{-1}G(\tau). \\
C(\tau) &= i\gamma\tau[r - \beta Tr(R^\top Q)] - \frac{\beta}{2}[Tr(M)\tau + \log(\det F(\tau))].
\end{aligned}
\tag{2.9}
$$

with

$$
\begin{aligned}
[G(\tau)\ F(\tau)] &= [0\ I_n]\exp(\tau Z(\gamma)) \\
Z(\gamma) &= \begin{pmatrix} M & -2Q^\top Q \\ \frac{i\gamma(i\gamma - 1)}{2}I_n & -(M^\top + 2i\gamma R^\top Q) \end{pmatrix}.
\end{aligned}
$$

2.1.2 Flexibility of the Wishart volatility model

As seen in Section 1.2.1, the Wishart process is an extension of a multidimensional Cox-Ingersoll-Ross process. Thus, the Wishart volatility model will be also an extension of a multidimensional Heston model by considerind diagonal matrices. Indeed, there is a bijection concerning the volatility parameters between the Heston model and the Wishart volatility model with this restricted specification. Moreover, there is the same correlation structure for those models and the stochastic correlation is given by

$$\rho_t = \frac{\sum_{i=1}^n R^{ii}Q^{ii}V_t^{ii}}{\sqrt{\sum_{i=1}^n V_t^{ii}}\sqrt{\sum_{i=1}^n (Q^{ii})^2 V_t^{ii}}}.$$

This particular case underlines the fact that the Wishart volatility model extends the multifactor Heston model in terms of volatility and correlation. In practice, the model

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used will be a simple extension of the multifactor Heston model. Moreover, the Wishart volatility model in the general case has additional properties. Indeed, in a multidimensional Heston model, the correlation is stochastic but depends on factors that generates the volatility dynamics. In the case of the Wishart volatility model, the correlation depends on the volatility factors but depends also of other factors in formulae 2.4 (the non diagonal components of the Wishart process V_t). It is an important property allowing degrees of freedom for the correlation and consequently for the skew and the smile. For example, in the Heston model, the change of sign of the skew is constrained by the correlation coefficients and the volatility factors whereas in the Wishart volatility model, the change of sign does not have this constraint thanks to the additional independent factor [8].

To develop the intuition on the importance of the stochastic correlation, we refer to asymptotic results in short maturities from V. Durrleman [13] where the link between correlation and implied volatility skew at the money are explicit.

2.1.3 Asymptotic skew in the Wishart volatility model

The Wishart volatility model gives a stochastic correlation (for $n \geq 2$) which is an important property. The correlation ρ_t being stochastic allows the model to generate a smile with a wealthy structure. Indeed, the market is more and more complex and now the skew is considered by practitioners as stochastic. Following standard notations with $\Sigma_t(T, K)$ denotes the implied volatility of a Call with a maturity T and a strike K, the skew at the money can be calculated for short maturities by using asymptotic methods developed by Durrleman.

Theorem 4. (V. Durrleman [13])

Let us suppose that we are given adapted processes $a, \tilde{a}, b, c$ such that there exists a strictly positive solution to the following stochastic differential equation :

$$d\sigma_t^2 \quad = \quad (b_t + a_t^2 + \sigma_t^2(a_t + c_t))dt - 2\sigma_t(a_t dW_t + \tilde{a}_t d\tilde{W}_t).$$

where a given initial condition σ_0 and such that

$$d < a, W >_t = -\sigma_t \left(\frac{3c_t}{2} + \frac{3a_t^2}{4\sigma_t^2} - \frac{a_t^2}{\sigma_t^2} \right) dt$$

Further let

$$S_t = S_0 \exp \left(\int_0^t \sigma_s dW_s - \frac{1}{2} \int_0^t \sigma_s^2 ds \right)$$

be the stock process and $\Sigma_t(T, K)$ be the corresponding implied volatility, then

$$\Sigma_t(t, S_t) = \sigma_t \qquad \frac{\partial \Sigma_t}{\partial K}(t, S_t) = -\frac{a_t}{2S_t \Sigma_t(t, S_t)} \qquad \frac{\partial \Sigma_t}{\partial T}(t, S_t) = \frac{b_t}{4\Sigma_t(t, S_t)}$$

$$\frac{\partial^2 \Sigma_t}{\partial K^2}(t, S_t) = \frac{1}{2S_t^2 \Sigma_t(t, S_t)}\left(a_t + c_t - \frac{a_t^2}{2\Sigma_t(t, S_t)^2}\right) \qquad (2.10)$$

Therefore, the skew being strongly linked with the correlation represented by $\frac{a_t}{\sigma(t)}$, this correlation has to be stochastic. Then, the need of a model that can explain this phenomena becomes essential. The stochastic correlation appears to be one of the crucial notions describing the capacity of a model to generate smiles. As explained in the next section, in terms of correlation, the Wishart volatility model appears more flexible than a multidimensional Heston model. For the expression of the skew at the money $(K = S_t)$ in the Wishart volatility model, the application of this theorem with $a_t = -\frac{Tr(R^T QV_t)}{Tr(V_t)}$ provides the following expression

$$\frac{\partial \Sigma_t}{\partial K}(t, S_t) = \frac{Tr(R^\top QV_t)}{2S_t(Tr(V_t))^{\frac{3}{2}}}.$$

Comment 1. *It is important to notice that Theorem 3 also provides an expression of $\frac{\partial^2 \Sigma_t}{\partial K^2}(t, S_t)$ and $\frac{\partial \Sigma_t}{\partial T}(t, S_t)$ allowing to analyze the convexity and the slope at the money of the asymptotic smile, but it is not brought up in this development. We focus here on the skew at the money to bring to the light the stochastic correlation propertie and to give a benchmark for the approximation in the next section but we could have also described the convexity and the slope at the money of the implied volatility.*

2.2 Evaluation of options by Monte Carlo methods

There are many products for which the evaluation does not have any closed-form expression. Therefore, a Monte Carlo method has to be considered in order to price those contracts and the problem to simulate a Wishart process becomes essential.

The use of a standard Euler scheme is prohibited since it does not guarantee the positiveness of the simulated matrices. The other method consisting in taking the positive part of the eigenvalues would introduce a bias and is rejected. Then, an efficient method, based on a fundamental link between Wishart and Ornstein-Uhlenbeck processes, was highlighted in the previous section for the simulation of the Wishart processes in the case of β real. This method will naturally keep the positiveness of the matrix-valued Wishart process [17] [1]. Now, the question is to see how the complete Wishart volatility model can

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1. During the revision August 2009 - January 2010, a working paper has been posted on September 2009 by Pierre Gauthier and Dylan Possamai about the simulation of the Wishart model

be simulated and how the european options can be evaluated by Monte Carlo. Considering a general Wishart process, the underlying dynamics are expressed under the probability measure $\mathbb{Q}^*$ in order to use the simulation method developed in Section 1.2.2

$$
\begin{aligned}
\frac{dS_t}{S_t} &= [r - \nu Tr(Q^\top R)]dt + Tr[\sqrt{V_t}(dW_t^* R + dZ_t \sqrt{I_n - RR^\top})]. \\
dV_t &= (KQ^\top Q + MV_t + V_t M^\top)dt + \sqrt{V_t}dW_t^* Q + Q^\top (dW_t^*)^\top \sqrt{V_t}.
\end{aligned}
$$

Integrating this stochastic differential equation furnishes a useful form of the log-price discretization

$$
\begin{aligned}
\log(S_{t_i}) &= \log(S_{t_{i-1}}) + \left((r - \nu Tr(Q^\top R))\Delta t - \frac{1}{2}\int_{t_{i-1}}^{t_i} Tr(V_s)ds\right) \\
&+ Tr\left[\int_{t_{i-1}}^{t_i} \sqrt{V_s}dW_s R + \int_{t_{i-1}}^{t_i} \sqrt{V_s}dZ_s \sqrt{I_n - RR^\top}\right].
\end{aligned}
$$

The increments $\{\varepsilon_{k,i}\}_{1\leq k\leq\beta}$, already used to simulate the Ornstein Uhlenbeck processes (Formulae 3.1), allows to simulate $Tr[\int_{t_{i-1}}^{t_i} \sqrt{V_s}dW_s R]$. For the other terms, a standard Euler scheme is selected giving the following expressions

$$
\int_{t_{i-1}}^{t_i} \sqrt{V_s}dW_s = \sum_{k=1}^{\beta}\int_{t_{i-1}}^{t_i} X_{k,s}dW_{k,s}^\top = \sqrt{\Delta t}\sum_{k=1}^{\beta} X_{k,t_i}\varepsilon_{k,i}^\top, \qquad \int_{t_{i-1}}^{t_i} V_s ds = \bar{V}_{t_{i-1}}\Delta t.
$$

$$
Tr\left[\int_{t_{i-1}}^{t_i} \sqrt{V_s}dZ_s \sqrt{I_n - RR^\top}\right] = \sqrt{\Delta t}\sqrt{Tr(V_{t_{i-1}}(I_n - RR^\top))}Z_i.
$$

where $Z_i \sim \mathcal{N}(0,1)$ are the components of an independent Gaussian vector and being independent with the increments $\varepsilon_{k,i}$

Finally, the Euler scheme for the asset is characterized by

$$
\begin{aligned}
\log(\bar{S}_{t_i}) &= \log(\bar{S}_{t_{i-1}}) + \left[r - \nu Tr(Q^\top R) - \frac{1}{2}Tr(\bar{V}_{t_{i-1}})\right]\Delta t + \sqrt{\Delta t}Tr\left[\sum_{k=1}^{\beta} \bar{X}_{k,t_{i-1}}\varepsilon_{k,i}^\top R\right] \\
&+ \sqrt{\Delta t}\sqrt{Tr(V_{t_{i-1}}) - Tr(V_{t_{i-1}}RR^\top)}Z_i.
\end{aligned}
$$

Therefore, the price of an european option at the maturity T with a payoff f is evaluated by

$$
\begin{aligned}
\pi_0 &= \mathbb{E}_{\mathbb{Q}}[\exp(-rT)f(S_T)] \\
&= \exp[-(r + \nu\, Tr(M))T]\mathbb{E}_{\mathbb{Q}^*}\left\{\left(\frac{\det(V_T)}{\det(V_0)}\right)^{\frac{\nu}{2}}\exp[-\frac{\nu}{2}(K + \nu + n - 1)\int_0^T Tr(V_s^{-1}Q^\top Q)ds]f(S_T)\right\}.
\end{aligned}
$$

The simulation of the integrale appearing in the Radon-Nikodym function can be discretized and approximated as follows

$$
\int_t^{t+\Delta t} Tr(V_s^{-1}Q^\top Q)ds \sim \frac{1}{2}\Delta t Tr[(V_t^{-1} + V_{t+\Delta t}^{-1})Q^\top Q].
$$

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Finally, it is possible to make a Monte Carlo evaluation for an option given that the asset and the Radon-Nikodym derivative of $\mathbb{Q}$ with respect to $\mathbb{Q}^*$ can be simulated using an Euler scheme. One can see that the expectation under the new probability measure depends on the path of $(V_t)_{t\leq T}$ what makes the evaluation costly.

2.3 Evaluation of options by FFT

The Wishart volatility model belongs to the class of affine models for which there exists a closed formulae for the Fourier transform [11]. This transform, noted $\Phi_t(T,\gamma) = \mathbb{E}[\exp(i\gamma Y_T)|\mathcal{F}_t]$, $\gamma \in \mathbb{C}$, is widely used to find options prices by calculating the inverse Fourier transform by FFT.

2.3.1 Expression of the Call price

The premium at date t of a Call option with strike $K = \log(k)$ and maturity T is given by

$$c_t(T,K) = \exp[-r(T-t)]\mathbb{E}[(\exp(Y_T) - \exp(k))_+|\mathcal{F}_t].$$

As it has been done in [4], the calculation of the premium is deduced by the expression of the Fourier transform. Let α define a constant such as $x \to \exp(\alpha x)(\zeta - \exp(x))_+ \in L^1(\mathbb{R})$ and in practice, Carr & Madan consider that $\alpha = 1.1$ is an empirical good value for the Heston model. It appears really important to have a function in $L^1(\mathbb{R})$ in order to use the inverse Fourier transform as explained by S. Levendorskii in [24]. Therefore, a modified price is defined as follows

$$c_t^\alpha(T,k) = \exp(\alpha k)c_t(T,k).$$

By applying a Fubini integration theorem, the Fourier transform of the modified price is given by

$$
\begin{aligned}
\Psi_t^\alpha(T,\nu) &= \int_{-\infty}^{\infty} \exp(i\nu k)c_t^\alpha(T,k)dk \\
&= \exp[-r(T-t)]\mathbb{E}[\int_{-\infty}^{Y_T} \exp[(i\nu+\alpha)k](\exp(Y_T) - \exp(k))dk|\mathcal{F}_t] \\
&= \exp[-r(T-t)]\frac{\Phi_t(T,\nu-(1+\alpha)i)}{(1+\alpha+i\nu)(\alpha+i\nu)}.
\end{aligned}
\tag{2.11}
$$

At last, the premium can be obtained by inversion of the Fourier transform using the fact that the function $\Psi_t^\alpha(T,\nu)$ has an odd imaginary part and an even real part

$$
c_t(T,k) = \frac{\exp(-\alpha k)}{2\pi}\int_{-\infty}^{\infty} \exp(-i\nu k)\Psi_t^\alpha(T,\nu)d\nu
$$

$$= \exp[-r(T-t)]\frac{\exp(-\alpha k)}{\pi}\mathcal{R}e\int_0^\infty \exp(-i\nu k)\frac{\Phi_t(T,\nu-(1+\alpha)i)}{(1+\alpha+i\nu)(\alpha+i\nu)}d\nu \quad (2.12)$$

2.3.2 Numerical upgrade

In this paragraph, we consider a practical point of view and we present a variant of Formulae 2.12 in order to improve numerical results. The idea is to use this formulae for a simple model (like the Black-Scholes model with a constant volatility $\bar\sigma$) and to apply this expression for numerical applications. This improves the effectiveness of numerical integration. It is done by analogy with the variance reduction method for Monte Carlo options pricing through a control variate method using the Black and Scholes price $c_t^{BS}(T,K)$ and its Fourier transform $\Phi_t^{BS}(T,\gamma)$ calculated by :

$$\Phi_t^{BS}(T,\gamma) = \exp[i\gamma Y_t + i\gamma r(T-t) + \frac{i\gamma(i\gamma-1)}{2}\bar\sigma^2(T-t)]. \quad (2.13)$$

The Call price can be calculated as a perturbation of the Call price in the Black-Scholes model with a volatility $\bar\sigma$

$$c_t(T,k) = c_t^{BS}(T,k) + I_t(r,\bar\sigma,\alpha,T)$$

where $I_t(r,\bar\sigma,\alpha,T)$ is given by

$$I_t(r,\bar\sigma,\alpha,T) = \exp[-r(T-t)]\frac{\exp(-\alpha k)}{\pi}\mathcal{R}e[\int_0^\infty \exp(-i\nu k)\frac{(\Phi_t(T,\nu-(1+\alpha)i) - \Phi_t^{BS}(T,\nu-(1+\alpha)i))}{(1+\alpha+i\nu)(\alpha+i\nu)}d\nu]$$

We consider that a good choice for the volatility $\bar\sigma$ is the realized mean volatility in the Wishart volatility model

$$\bar\sigma = \sqrt{\frac{1}{T}\int_0^T \mathbb{E}[Tr(V_t)]dt}.$$

This volatility can be calculated as follows. Let $f(t) = \mathbb{E}(V_t)$, f is the solution of the following differential equation

$$f'(t) = \beta Q^\top Q + Mf(t) + f(t)M^\top, \qquad f(0) = V_0.$$

Then, the solution can be found explicitly

$$f(t) = \exp(tM)V_0\exp(tM^\top) + \beta\int_0^t \exp(sM)Q^\top Q\exp(sM^\top)ds.$$

Assume now that M is a symmetric matrix and take the trace

$$Tr[f(t)] = Tr[V_0\exp(2tM)] + \beta Tr[Q^\top Q(2M)^{-1}(\exp(2tM) - I_n)].$$

Finally, by integrating between 0 and T, the realized mean variance is given by

$$\bar\sigma^2 = \frac{1}{T}Tr[(V_0 + \beta Q^\top Q(2M)^{-1})(2M)^{-1}(\exp(2TM) - I_n)] - \beta Tr[Q^\top Q(2M)^{-1}].$$

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Comment 2. *For a general invertible matrix M, we take the same formulae with $M = \frac{M+M^\top}{2}$ which gives better results in practice.*

3   Smile dynamics in the Wishart volatility model

The hedge of exotic options not only need the trade of the asset but require also the trade of options such as Call and Put options. Therefore, these vanilla options are hedging instruments and the calibration procedure has to take into account of their price dynamics. Then, models considered and used by traders have to be flexible enough to explain market prices. In other words, the model must be able to generate the observable smile effect (strike dependency of the Black-Scholes implied volatility).

Heston model is a single-factor stochastic volatility model that can be calibrated accurately on a market smile surface. However, when the calibration is done on both short and long maturities smile curves, a single-factor model does not provide a flexible modelling of the volatility term structure. Indeed, a single factor model cannot take into account the variability of the skew, also known as correlation risk, and cannot explain independent fluctuations of the smirk in the level and the slope over time. A classic solution is to increase the number of factors in order to achieve this goal like in [5]. For instance, P. Christoffersen and al. [5] introduce a bidimensional extension to solve this problem. The Wishart volatility model is an other multidimensional extension of the Heston model allowing an adequate correlation structure in order to reproduce the skew effect. An analysis of these features is proposed in this section.

The covariance dynamics is represented by a matrix-valued process of size n. The case $n = 2$ is enough for this study allowing to deal with the case of a two-scale volatility (medium maturities). It is also a framework where the computation is not so laborious and explicit formulae are available. In this section, the study of the Wishart volatility model and the effect of some parameters on the smile are considered. Inspired by the singular perturbation method introduced by J.P. Fouque and al. [14, 15, 16] and by standard approximation procedure [21], we present a perturbation algorithm, based on the perturbation of Riccati equations. This method will allow to generate instantly asymptotic smiles with a good level of accuracy. The case of short maturities is a relevant case and is important for a calibration prospect. Moreover, in the two-scale volatility case, the introduction of a slow varying factor gives a much better fit for options with longer maturities [15]

3.1 Specification of the Wishart volatility model

It is important to describe the admissibility of the parametrization for this model. It is important to highlight a canonical model from which all the equivalent models can be obtained by invariant transformation [10]. Considering the case where $M \in S_n(\mathbb{R})$ and $Q \in GL_n(\mathbb{R})$, it is possible to define an equivalent model with M diagonal because the matrix-valued Brownian motion is invariant by any rotation. Indeed, there exists a matrix D diagonal and an orthognal matrix P such as $D = P^\top M P$. Denoting $\tilde{V}_t = P^\top V_t P$, $\tilde{W}_t = P^\top W_t P$ and $\tilde{Q} = P^\top Q P$, an equivalent Wishart process can be deduced by this rotation

$$d\tilde{V}_t = (\beta \tilde{Q}^\top \tilde{Q} + D\tilde{V}_t + \tilde{V}_t D)dt + \sqrt{\tilde{V}_t}d\tilde{W}_t\tilde{Q} + \tilde{Q}^\top d\tilde{W}_t^\top \sqrt{\tilde{V}_t}.$$

Therefore, only eigenvalues of the mean-reversion matrix M matters in the specification of volatility dynamics .

Although, M and Q can not be generally considered both diagonals, this particular case is first analyzed to set up the approximations method and to compare this model to a classic multifactor Heston model as seen later.

In the original article of Bru [3], there is the additional condition that M and $Q^\top Q$ commute, what allows to find a rotation which simultaneously make M and Q diagonal. This condition on the Wishart volatility model is not taken into account in finance but could justify this first choice for a restricted case. The general case of a full matrix Q will be brought up in future researches. In the case of the dimension 2, the parameters of the model can be written as follows

$$M = \begin{pmatrix} -m_1 & 0 \\ 0 & -m_2 \end{pmatrix}, \quad Q = \begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix}, \quad R = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}.$$

Let $\nu_i = \frac{q_i}{\sqrt{m_i}}$ for $i \in \{1, 2\}$, $\sigma(u, w) = \sqrt{u + w}$. The quantities ν_i correspond to a ratio between volatility of volatility and the square root of the mean-reversion parameters. Like for the standard method of singular perturbations introduced by J.P Fouque and al. [14, 15, 16], we keep those quantities as constant values. Indeed, the method of singular perturbation consists in perturbating a process such as its stationary distribution is non-degenerated and particularly has a non-degenerated covariance matrix. This choice allows to preserve the homogeneity between the drift and volatility parameters what allows to have a non degenerated invariant measure.

The covariance matrix of the Wishart process will depend only on $\Sigma(h) = \int_0^h e^{hM}Q^\top Q e^{hM^\top}ds$ which is the covariance matrix of the underlying Ornstein Uhlenbeck processes (Section

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1.2.2). Therefore, in this restricted case, the covariance matrix of the Wishart process depends only on the values ν_i.

Comment 3. *By taking expectation, it is easy to calculate the following expressions*

$$\mathbb{E}(V_T^{11}|\mathcal{F}_t) = \frac{\beta\nu_1^2}{2} + (V_t^{11} - \frac{\beta\nu_1^2}{2})\exp[-2m_1(T-t)].$$

$$\mathbb{E}(V_T^{22}|\mathcal{F}_t) = \frac{\beta\nu_2^2}{2} + (V_t^{22} - \frac{\beta\nu_2^2}{2})\exp[-2m_2(T-t)].$$

$$\mathbb{E}(V_T^{12}|\mathcal{F}_t) = V_t^{12}\exp[-(m_1 + m_2)(T-t)].$$

Consequently, $\tau_i = \frac{1}{2m_i}$ for $i \in \{1, 2\}$ are mean-reverting times of V_T and $\frac{\beta\nu_i^2}{2}$ for $i \in \{1, 2\}$ correspond to the asymptotic mean of the diagonal components of V_T when T goes to infinity.

3.2 Perturbation Method of the Riccati equations

In this part, a method of perturbations based on the affine properties of the model is presented. Indeed, the perturbation of the Riccati equations provides an asymptotic development of the Call price.

This method consists in the pertubation of the parameters of the volatility process like in [14, 15, 16] taking into account the affine properties of the model. In a short maturity case, this method is really similar from an asymptotic approximation in short time like in [13] but is a little different. Indeed, as noted in Section 3.1, during the perturbation process, there is a constraint between the drift and the volatility of the process (in the Wishart case, the coefficients ν_i remain constant). Besides, this method can be used for a multiscale volatility framework as described in the next section.

Finally, this procedure is efficient and allows us to reach higher orders than standard perturbation methods like the singular perturbation method on partial differential equations that appears really complicated after the first order [2]. The affine properties allow an efficient, easy, and general procedure. Remember the homogeneous Riccati system associated to the Wishart volatility model where for simplicity, we use the notation $\theta = i\gamma$

$$A'(\tau) = \frac{\theta(\theta-1)}{2}I_n + A(\tau)M + (M^\top + 2\theta R^\top Q)A(\tau) + 2A(\tau)Q^\top QA(\tau).$$

$$C'(\tau) = r\theta + \beta Tr[Q^\top QA(\tau)].$$

Then, a development of the function A furnishes instantly a development for C.

Consider the case where of the dimension $n = 2$. Thus, there will be two characteristics orders in the perturbation ϵ and δ that will be defined precisely for the short maturities

case and the two-scale volatility case. Let us focus on a development of the solution $A(\tau)$ of the form

$$A(\tau) \;=\; \sum_{i,j} \varepsilon^{\frac{i}{2}} \delta^{\frac{j}{2}} A^{i,j}(\tau).$$

Injecting this development in the perturbed Riccati equation and identifying terms in ε and δ provides the expected approximation. In the following, this method will be applied for two important cases : the short maturity case and the multiscale volatility case.

3.3 Smile dynamics for short maturities

Observe the case of short maturities $(T - t) \ll \tau_1, \tau_2$ meaning that the option maturity is much less than the mean-reverting times of the volatility components. In other words, the volatility V_t does not fluctuate a lot around its starting value. It is a short maturity smile because the maturity of the option is small in relation to the characteristic times of the volatility process. Consequently $\varepsilon = m_1$ and $\delta = m_2$ are small in relation to $\frac{1}{(T-t)}$. Keep in mind that the quantities ν_i remain constant.

The method is detailed for the short maturities case and the procedure is presented for an approximation at order 1 i.e in $(\sqrt{\varepsilon}, \sqrt{\delta})$ and at order 2 i.e in (ε, δ). First, rewrite the Riccati equations

$$
\begin{aligned}
A'(\tau) \;=\; & \frac{\theta(\theta - 1)}{2} I_n + \varepsilon(A(\tau)M_1 + M_1 A(\tau) + 2A(\tau)Q_1^2 A(\tau)) \\
& + 2\theta\sqrt{\varepsilon}R^{\top}Q_1 A(\tau) + 2\theta\sqrt{\delta}R^{\top}Q_2 A(\tau) \\
& + \delta(A(\tau)M_2 + M_2 A(\tau) + 2A(\tau)Q_2^2 A(\tau)),
\end{aligned}
$$

with the following notations

$$
M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad M = -\varepsilon M_1 - \delta M_2.
$$

$$
Q = \sqrt{\varepsilon}\nu_1 M_1 + \sqrt{\delta}\nu_2 M_2, \quad Q^2 = \varepsilon\nu_1^2 M_1 + \delta\nu_2^2 M_2.
$$

$$
V_0 = \begin{pmatrix} u & v \\ v & w \end{pmatrix}, \quad \theta = i\gamma.
$$

3.3.1 Development of the Riccati function A and C

By perturbating the Riccati equation, at order 2, a solution is looked for under the following development

$$A(\tau) = A^{0,0}(\tau) + \sqrt{\varepsilon}A^{1,0}(\tau) + \sqrt{\delta}A^{0,1}(\tau) + \varepsilon A^{2,0}(\tau) + \delta A^{0,2}(\tau) + \sqrt{\varepsilon}\sqrt{\delta}A^{1,1}(\tau) + o\left(\max\left(\varepsilon, \delta\right)\right).$$

Consequently, injecting this formulation into the Riccati equations and doing calculations step by step, gives the expected functions

$$
\begin{aligned}
A^{0,0}(\tau) &= \frac{\theta\,(\theta-1)}{2}\tau I_2. \\
A^{1,0}(\tau) &= \frac{\theta^2\,(\theta-1)}{2}\nu_1\tau^2\left(R^T M_1\right). \\
A^{0,1}(\tau) &= \frac{\theta^2\,(\theta-1)}{2}\nu_2\tau^2\left(R^T M_2\right). \\
A^{2,0}(\tau) &= \theta^3\,(\theta-1)\frac{\tau^3}{3}\nu_1^2\left(R^\top M_1\right)^2 - \frac{\theta\,(\theta-1)}{2}\tau^2 M_1 + \nu_1^2\theta^2\,(\theta-1)^2\frac{\tau^3}{6}M_1. \\
A^{0,2}(\tau) &= \theta^3\,(\theta-1)\frac{\tau^3}{3}\nu_2^2\left(R^\top M_2\right)^2 - \frac{\theta\,(\theta-1)}{2}\tau^2 M_2 + \nu_2^2\theta^2\,(\theta-1)^2\frac{\tau^3}{6}M_2. \\
A^{1,1}(\tau) &= \theta^3\,(\theta-1)\frac{\tau^3}{3}\nu_1\nu_2\left[R^\top M_1 R^\top M_2 + R^\top M_2 R^\top M_1\right].
\end{aligned}
$$

By using the link between functions C and A, the development for C is deduced

$$
C(\tau) = r\tau\theta + \varepsilon C^{2,0}(\tau) + \delta C^{0,2}(\tau) + o\left(\max\left(\varepsilon, \delta\right)\right).
$$

where

$$
C^{2,0}(\tau) = \beta\nu_1^2\frac{\theta\,(\theta-1)}{4}\tau^2, \qquad C^{0,2}(\tau) = \beta\nu_2^2\frac{\theta\,(\theta-1)}{4}\tau^2.
$$

3.3.2 Approximation of the Call price

By considering the previous expressions, it is possible to find a Taylor expansion of the Fourier transform of $Y_t = \log(S_t)$

$$
\begin{aligned}
\Phi_t\,(T,\theta) &= \exp\left[\frac{\theta(\theta-1)}{2}(T-t)\,(u+w) + \theta y + r(T-t)\theta\right] * \\
&\begin{bmatrix}
1 + \sqrt{\varepsilon}Tr\left(A^{1,0}V_0\right) + \sqrt{\delta}Tr\left(A^{0,1}V_0\right) \\
+\varepsilon\left[Tr\left(A^{2,0}V_0\right) + C^{2,0} + \frac{1}{2}\left(Tr\left(A^{1,0}V_0\right)\right)^2\right] \\
+\delta\left[Tr\left(A^{0,2}V_0\right) + C^{0,2} + \frac{1}{2}\left(Tr\left(A^{0,1}V_0\right)\right)^2\right] \\
+\sqrt{\varepsilon}\sqrt{\delta}\left[Tr\left(A^{1,1}V_0\right) + Tr\left(A^{0,1}V_0\right)Tr\left(A^{1,0}V_0\right)\right] \\
+o\left(\max\left(\varepsilon, \delta\right)\right)
\end{bmatrix}
\end{aligned}
$$

with the following formulae

$$
\begin{aligned}
Tr\left(A^{1,0}V_0\right) &= \frac{\theta^2\,(\theta-1)}{2}\nu_1(T-t)^2\left(\rho_{11}u + \rho_{12}v\right). \\
Tr\left(A^{0,1}V_0\right) &= \frac{\theta^2\,(\theta-1)}{2}\nu_2(T-t)^2\left(\rho_{22}w + \rho_{21}v\right). \\
Tr\left(A^{2,0}V_0\right) &= \theta^3\,(\theta-1)\frac{(T-t)^3}{3}\nu_1^2\left(\rho_{11}^2 u + \rho_{11}\rho_{12}v\right) \\
&\quad- \frac{\theta\,(\theta-1)}{2}(T-t)^2 u + \nu_1^2\theta^2\,(\theta-1)^2\frac{(T-t)^3}{6}u.
\end{aligned}
$$

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$$Tr\left(A^{0,2}V_0\right) \;=\; \theta^3\left(\theta-1\right)\frac{(T-t)^3}{3}\nu_2^2\left(\rho_{22}^2 w + \rho_{22}\rho_{21}v\right)$$
$$-\;\frac{\theta\left(\theta-1\right)}{2}(T-t)^2 w + \nu_2^2\theta^2\left(\theta-1\right)^2\frac{(T-t)^3}{6}w.$$
$$Tr\left(A^{1,1}V_0\right) \;=\; \theta^3\left(\theta-1\right)\frac{(T-t)^3}{3}\nu_1\nu_2\left[\rho_{12}\rho_{21}\left(u+w\right)+\left(\rho_{22}\rho_{12}+\rho_{11}\rho_{21}\right)v\right].$$

We recognize the Fourier transform in the Black-Scholes model (Formulae 2.13) with a volatilty $\bar{\sigma}=\sqrt{u+w}$.

This expansion at order $o\left(\max\left(\varepsilon,\delta\right)\right)$ must be analyzed checking that the neglected terms have to be small in comparison to the first terms of the development. The main question of this procedure is how the price approximation can be infered from the Fourier transform development.

In a general way, assume that there is a development for the Fourier transform under the form

$$\Phi_t(T,\gamma) \;=\; \Phi_t^{BS}(\bar{\sigma})(T,\gamma)\sum_{i,j}\varepsilon^{\frac{i}{2}}\delta^{\frac{j}{2}}P_{i,j}(i\gamma).$$

where $P_{i,j}$ are polynomials in $\mathbb{R}_p[X]$. Noticing that $P[\frac{\partial}{\partial y}]\Phi_t^{BS(\bar{\sigma})}(T,\gamma)=P(i\gamma)\Phi_t^{BS(\bar{\sigma})}(T,\gamma)$ with Formulae 2.13, the price of a european option is given by

$$C_t(T,k) \;=\; \sum_{i,j}\varepsilon^{\frac{i}{2}}\delta^{\frac{j}{2}}P_{i,j}\left(\frac{\partial}{\partial y}\right)C_t^{BS(\bar{\sigma})}(T,k).$$

Moreover, the expressions of the successive derivatives of the price with respect to the log spot $y=log(S_0)$ are needed and can be easily calculated.

Consider non standard notations for $a=\log(Ke^{-r(T-t)})+\frac{1}{2}\sigma^2(T-t)$ and $n\geq 2$, $\Lambda_y^{BS,n}=\frac{\partial^n C_{BS(\sigma)}}{\partial y^n}-\frac{\partial^{n-1}C_{BS(\sigma)}}{\partial y^{n-1}}$. One can notice that $\frac{a-y}{\sigma\sqrt{T-t}}=-d_0$.

A useful recursive relation was found allowing the calculation in the Black-Scholes model of Call price succesive derivatives with respect to the log spot y (see Appendice C for explicit calculation of successive derivatives)

$$\forall n\geq 4,\;\frac{\partial^n C_{BS}}{\partial y^n} \;=\; \frac{\partial^{n-1}C_{BS}}{\partial y^{n-1}}+\frac{(a-y)}{\sigma^2(T-t)}\left(\frac{\partial^{n-1}C_{BS}}{\partial y^{n-1}}-\frac{\partial^{n-2}C_{BS}}{\partial y^{n-2}}\right)$$
$$-\;\frac{(n-3)}{\sigma^2(T-t)}\left(\frac{\partial^{n-2}C_{BS}}{\partial y^{n-2}}-\frac{\partial^{n-3}C_{BS}}{\partial y^{n-3}}\right). \tag{3.1}$$

Using classic formulae in the Black-Scholes model, the development for the Call price can be determined

$$P = P^{0,0}+\sqrt{\varepsilon}P^{1,0}+\sqrt{\delta}P^{0,1}+\varepsilon P^{2,0}+\delta P^{0,2}+\sqrt{\varepsilon}\sqrt{\delta}P^{1,1}+o\left(\max\left(\varepsilon,\delta\right)\right).$$

with

$$P^{0,0} = C_{BS(\sigma)}, \qquad \sigma = \sqrt{u+w}.$$

$$P^{1,0} = \frac{\nu_1(T-t)^2}{2}\left(\rho_{11}u + \rho_{12}v\right)\Lambda_y^{BS,3}, \qquad P^{0,1} = \frac{\nu_2(T-t)^2}{2}\left(\rho_{22}w + \rho_{21}v\right)\Lambda_y^{BS,3}.$$

$$P^{2,0} = \frac{(T-t)^2}{2}\left(\frac{\beta\nu_1^2}{2} - u\right)\Lambda_y^{BS,2} + \frac{\nu_1^2(T-t)^3}{3}\left(\rho_{11}^2 u + \rho_{11}\rho_{12}v + \frac{1}{2}u\right)\Lambda_y^{BS,4}$$
$$- \frac{\nu_1^2(T-t)^3}{6}u\Lambda_y^{BS,3} + \frac{\nu_1^2(T-t)^4}{8}\left(\rho_{11}u + \rho_{12}v\right)^2\left(\Lambda_y^{BS,6} - \Lambda_y^{BS,5}\right).$$

$$P^{0,2} = \frac{(T-t)^2}{2}\left(\frac{\beta\nu_2^2}{2} - w\right)\Lambda_y^{BS,2} + \frac{\nu_2^2(T-t)^3}{3}\left(\rho_{22}^2 w + \rho_{22}\rho_{21}v + \frac{1}{2}w\right)\Lambda_y^{BS,4}$$
$$- \frac{\nu_2^2(T-t)^3}{6}w\Lambda_y^{BS,3} + \frac{\nu_2^2(T-t)^4}{8}\left(\rho_{22}w + \rho_{21}v\right)^2\left(\Lambda_y^{BS,6} - \Lambda_y^{BS,5}\right).$$

$$P^{1,1} = \frac{\nu_1\nu_2(T-t)^3}{3}\left(\rho_{12}\rho_{21}\left(u+w\right) + \left(\rho_{11}\rho_{21} + \rho_{22}\rho_{12}\right)v\right)\Lambda_y^{BS,4}$$
$$+ \frac{\nu_1\nu_2(T-t)^4}{4}\left(\rho_{11}u + \rho_{12}v\right)\left(\rho_{22}w + \rho_{21}v\right)\left(\Lambda_y^{BS,6} - \Lambda_y^{BS,5}\right).$$

Finally, an approximation of the price at order $o(\sqrt{\varepsilon}, \sqrt{\delta})$ is highlighted

$$C_t(T,k) = C_t^{BS}(T,k) + [\sqrt{\varepsilon}\frac{1}{4}\beta\nu_1^3\rho_{11} + \sqrt{\delta}\frac{1}{2}\rho_{22}\nu_2 w(T-t)]\left(\frac{\partial^3 C_t^{BS(\bar{\sigma})}}{\partial y^3}(T,k) - \frac{\partial^2 C_t^{BS(\bar{\sigma})}}{\partial y^2}(T,k)\right)$$
$$+ o(max[\sqrt{\varepsilon}, \sqrt{\delta}]).$$

3.3.3 Approximation of the lognormal implied volatility

From a price development, a development of the lognormal implied volatility can be infered. The same kind of development is considered

$$\Sigma = \Sigma^{0,0} + \sqrt{\varepsilon}\Sigma^{1,0} + \sqrt{\delta}\Sigma^{0,1} + \varepsilon\Sigma^{2,0} + \delta\Sigma^{0,2} + \sqrt{\varepsilon}\sqrt{\delta}\Sigma^{1,1} + o\left(\max\left(\varepsilon, \delta\right)\right). \qquad (3.2)$$

By injecting this development in the Black-Scholes price, a Taylor expansion around $\Sigma^{0,0}$ furnishes another expression for the Call price

$$P = C_{BS(\Sigma^{0,0})} + \sqrt{\varepsilon}\Sigma^{1,0}\left(\frac{\partial C_{BS(\sigma)}}{\partial\sigma}|_{\sigma=\Sigma^{0,0}}\right) + \sqrt{\delta}\Sigma^{0,1}\left(\frac{\partial C_{BS(\sigma)}}{\partial\sigma}|_{\sigma=\Sigma^{0,0}}\right)$$
$$+\varepsilon\left(\Sigma^{2,0}\left(\frac{\partial C_{BS(\sigma)}}{\partial\sigma}|_{\sigma=\Sigma^{0,0}}\right) + \frac{1}{2}\left(\Sigma^{1,0}\right)^2\left(\frac{\partial^2 C_{BS(\sigma)}}{\partial\sigma^2}|_{\sigma=\Sigma^{0,0}}\right)\right)$$
$$+\delta\left(\Sigma^{0,2}\left(\frac{\partial C_{BS(\sigma)}}{\partial\sigma}|_{\sigma=\Sigma^{0,0}}\right) + \frac{1}{2}\left(\Sigma^{0,1}\right)^2\left(\frac{\partial^2 C_{BS(\sigma)}}{\partial\sigma^2}|_{\sigma=\Sigma^{0,0}}\right)\right)$$
$$+\sqrt{\varepsilon}\sqrt{\delta}\left(\Sigma^{1,1}\left(\frac{\partial C_{BS(\sigma)}}{\partial\sigma}|_{\sigma=\Sigma^{0,0}}\right) + \Sigma^{1,0}\Sigma^{0,1}\left(\frac{\partial^2 C_{BS(\sigma)}}{\partial\sigma^2}|_{\sigma=\Sigma^{0,0}}\right)\right)$$
$$+o\left(\max\left(\varepsilon, \delta\right)\right)$$

In the following, we denote $Vega^{BS} = \frac{\partial C_{BS(\sigma)}}{\partial\sigma}|_{\sigma=\Sigma^{0,0}}$ and $Vomma^{BS} = \frac{\partial^2 C_{BS(\sigma)}}{\partial\sigma^2}|_{\sigma=\Sigma^{0,0}}$. Finally, by comparing this expression with the development 3.2 obtained previously and

by using classic expression in the Black-Scholes model (see Appendice B), the development of the implied volatility is specified

$$\Sigma^{0,0} = \sqrt{u+w}.$$

$$\Sigma^{1,0} = \frac{P^{1,0}}{Vega^{BS}} = -\frac{\nu_1\sqrt{T-t}}{2(u+w)}(\rho_{11}u + \rho_{12}v)(d_0).$$

$$\Sigma^{0,1} = \frac{P^{0,1}}{Vega^{BS}} = -\frac{\nu_2\sqrt{T-t}}{2(u+w)}(\rho_{22}w + \rho_{21}v)d_0.$$

$$\Sigma^{2,0} = \frac{P^{2,0} - \frac{1}{2}(\Sigma^{1,0})^2 Vomma^{BS}}{Vega^{BS}}.$$

$$\Sigma^{0,2} = \frac{P^{0,2} - \frac{1}{2}(\Sigma^{0,1})^2 Vomma^{BS}}{Vega^{BS}}.$$

$$\Sigma^{1,1} = \frac{P^{1,1} - \Sigma^{1,0}\Sigma^{0,1} Vomma^{BS}}{Vega^{BS}}.$$

Therefore, we can obtain an explicit expression of asymptotic smile at order (ε, δ) but the formulae is too long. For instance, at order $(\sqrt{\varepsilon}, \sqrt{\delta})$, an explicit and concise formulae for the smile can be deduced

$$
\begin{aligned}
\hat{\Sigma}_t(T,K) &= \sqrt{Tr(V_t)} + \frac{\sqrt{\varepsilon}P^{1,0}(t,Y_t,V_t^{11},V_t^{21},V_t^{22}) + \sqrt{\delta}P^{0,1}(t,Y_t,V_t^{11},V_t^{12},V_t^{22})}{\frac{\partial C_t^{BS}}{\partial \sigma}(T,K,\sqrt{Tr(V_t)})} \\
&= \sqrt{Tr(V_t)} \\
&\quad + \frac{q_1(\rho_{11}V_t^{11} + \rho_{12}V_t^{12}) + q_2(\rho_{22}V_t^{22} + \rho_{21}V_t^{12})}{2(Tr(V_t))^{\frac{3}{2}}}[log(Ke^{-r(T-t)}) - Y_t + \frac{Tr(V_t)(T-t)}{2}].
\end{aligned}
$$

(3.3)

Notice that $\sqrt{Tr(V_t)}$ is the stochastic volatility and is also the implied volatility at the forward money ($K = F_t = e^{Y_t + r(T-t)}$) and for very short maturities ($T = t$). Furthermore, an expression at order $(\sqrt{\varepsilon}, \sqrt{\delta})$ of the skew at the forward money is obtained and one can check that this formulae corresponds to the one obtained by Durrleman (Theorem 3)

$$\frac{\partial \Sigma_t}{\partial K}(T,F_t) \sim \frac{q_1(\rho_{11}V_t^{11} + \rho_{12}V_t^{12}) + q_2(\rho_{22}V_t^{22} + \rho_{21}V_t^{12})}{2(Tr(V_t))^{\frac{3}{2}}F_t}.$$

This expression shows that the sign of the skew can be driven by the parameters of the Wishart model and particularly with the correlation parameters. Finally, we have obtained a development of the smile for all strikes (not just at the forward money) and in maturities. This approximation is constrained by the fact that the maturity option T has to be much less than the characteristics times of the volatility process τ_i and it is also clear that this approximation will be more accurate in the neighborhood of the forward money (Section 3.5). In the next section, we will deal with the case of options with a medium maturity.

3.4   Smile dynamics for two-scale volatility

In this section, the case of a two-scale of volatility is brought up what means $\tau_1 \ll (T - t) \ll \tau_2$ or in other words $\frac{1}{2m_1} \ll (T - t) \ll \frac{1}{2m_2}$. Consequently, there is a first component of the volatility with a fast mean reversion and a second with a slow evolution. The introduction of a multiscale volatility allows to obtain a persistent skew for a medium maturity. The perturbation method, already presented before, is applied to prove this observation and to analyze the smile. The method is the same than for the case for short maturities but the calculations are more complicated. The most important steps of the procedure as well as the important results are described but some intermediate calculations are left to the reader. The first Riccati equation can be studied under the form

$$
\begin{aligned}
A'(\tau) &= \frac{\theta(\theta - 1)}{2} I_n + \frac{1}{\varepsilon}(-A(\tau)M_1 - M_1 A(\tau) + 2A(\tau)Q_1^2 A(\tau)) \\
&\quad + 2\theta \frac{1}{\sqrt{\varepsilon}} R^\top Q_1 A(\tau) + 2\theta\sqrt{\delta} R^\top Q_2 A(\tau) \\
&\quad + \delta(-A(\tau)M_2 - M_2 A(\tau) + 2A(\tau)Q_2^2 A(\tau)).
\end{aligned}
$$

with the following notations

$$
M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad M = -\frac{1}{\varepsilon} M_1 - \delta M_2,
$$

$$
Q = \frac{1}{\sqrt{\varepsilon}} \nu_1 M_1 + \sqrt{\delta} \nu_2 M_2, \quad Q^2 = \frac{1}{\varepsilon} \nu_1^2 M_1 + \delta \nu_2^2 M_2,
$$

$$
V_0 = \begin{pmatrix} u & v \\ v & w \end{pmatrix}, \quad \theta = i\gamma.
$$

The development of the function A at order (ε, δ) is given by

$$
A(\tau) = A^{0,0}(\tau) + \sqrt{\varepsilon} A^{1,0}(\tau) + \sqrt{\delta} A^{0,1}(\tau) + \varepsilon A^{2,0}(\tau) + \delta A^{0,2}(\tau) + \sqrt{\varepsilon}\sqrt{\delta} A^{1,1}(\tau) + o\left(\max\left(\varepsilon, \delta\right)\right).
$$

The first steps of the procedure are detailled but the reader has to carry on some calculations. Indeed, contrary to the case of a short maturity, the absence of a closed formulae for the functions $A^{i,j}(\tau)$ requires an ardous identification step by step

Order $\frac{1}{\varepsilon}$:

$$
0 = -A^{0,0}(\tau)M_1 - M_1 A^{0,0}(\tau) + 2A^{0,0}(\tau)Q_1^2 A^{0,0}(\tau).
$$

Hence, $A^{0,0}(\tau)$ the form is deduced

$$
A^{0,0}(\tau) = \begin{pmatrix} 0 & 0 \\ 0 & a_{2,2}^{0,0}(\tau) \end{pmatrix}.
$$

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Order $\frac{1}{\sqrt{\varepsilon}}$:

$$0 = -A^{1,0}(\tau)M_1 - M_1 A^{1,0}(\tau) + 2A^{1,0}(\tau)Q_1^2 A^{0,0}(\tau) + 2A^{0,0}(\tau)Q_1^2 A^{1,0}(\tau)$$
$$+ 2\theta R^T Q_1 A^{0,0}(\tau).$$

Since $A^{0,0}(\tau)Q_1 = Q_1 A^{0,0}(\tau) = 0$, the equation becomes $0 = A^{1,0}(\tau)M_1 + M_1 A^{1,0}(\tau)$. Therefore, $A^{1,0}(\tau)$ is given by

$$A^{1,0}(\tau) = \begin{pmatrix} 0 & 0 \\ 0 & a_{2,2}^{1,0}(\tau) \end{pmatrix}.$$

Order 1 :

$$(A^{0,0})'(\tau) = -A^{2,0}(\tau)M_1 - M_1 A^{2,0}(\tau) + 2A^{2,0}(\tau)Q_1^2 A^{0,0}(\tau) + 2A^{0,0}(\tau)Q_1^2 A^{2,0}(\tau)$$
$$+ 2A^{1,0}(\tau)Q_1^2 A^{1,0}(\tau) + 2\theta R^T Q_1 A^{1,0}(\tau)$$
$$+ \frac{\theta(\theta-1)}{2} I_2.$$

By noticing that $A^{0,0}(\tau)Q_1 = Q_1 A^{0,0}(\tau) = A^{1,0}(\tau)Q_1 = Q_1 A^{1,0}(\tau) = 0$, one can deduce

$$A^{0,0}(\tau) = \begin{pmatrix} 0 & 0 \\ 0 & \frac{\theta(\theta-1)}{2}\tau \end{pmatrix}, \qquad A^{2,0}(\tau) = \begin{pmatrix} \frac{\theta(\theta-1)}{4} & 0 \\ 0 & a_{2,2}^{2,0}(\tau) \end{pmatrix}.$$

Step by step, all the coefficients $A^{i,j}(\tau)$ can be calculated. The constants obtained by integration are considered equal to 0 given that the global function A satisfies $A(0) = 0$. For information, the results of the procedure at order 2 are given so as to carry on the approximation procedure and to give a benchmark to the reader.

$$A^{0,0}(\tau) = \begin{pmatrix} 0 & 0 \\ 0 & \frac{\theta(\theta-1)}{2}\tau \end{pmatrix}, \quad A^{1,0}(\tau) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad A^{0,1}(\tau) = \begin{pmatrix} 0 & 0 \\ 0 & \frac{\theta^2(\theta-1)}{2}\rho_{22}\nu_2\tau^2 \end{pmatrix},$$

$$A^{2,0}(\tau) = \begin{pmatrix} \frac{\theta(\theta-1)}{4} & 0 \\ 0 & 0 \end{pmatrix}, \quad A^{0,2}(\tau) = \begin{pmatrix} 0 & 0 \\ 0 & \frac{\theta^3(\theta-1)}{3}\rho_{22}^2\nu_2^2\tau^3 - \frac{\theta(\theta-1)}{2}\tau^2 + \frac{\theta^2(\theta-1)^2}{6}\nu_2^2\tau^3 \end{pmatrix},$$

$$A^{1,1}(\tau) = \begin{pmatrix} 0 & 0 \\ 0 & \theta^3(\theta-1)\rho_{12}\rho_{21}\nu_1\nu_2\tau^2 \end{pmatrix}, \quad A^{2,1}(\tau) = \begin{pmatrix} 0 & \theta^2(\theta-1)\rho_{21}\nu_2\tau \\ 0 & * \end{pmatrix},$$

$$A^{3,0}(\tau) = \begin{pmatrix} \frac{\theta^2(\theta-1)}{4}\rho_{11}\nu_1 & 0 \\ \frac{\theta^2(\theta-1)}{2}\rho_{12}\nu_1 & * \end{pmatrix}, \quad A^{3,1}(\tau) = \begin{pmatrix} 0 & 2\theta^3\rho_{11}\nu_1\nu_2\rho_{21}(\theta-1)\tau \\ 0 & * \end{pmatrix},$$

$$A^{4,0}(\tau) = \begin{pmatrix} \frac{\theta^2(\theta-1)^2}{16}\nu_1^2 + \frac{\theta^3(\theta-1)}{4}\rho_{11}^2\nu_1^2 & 0 \\ \frac{\theta^3(\theta-1)}{2}\rho_{11}\nu_1^2\rho_{12} & * \end{pmatrix}.$$

Concerning the function C, the third Riccati equation allows to obtain the following development

$$C(\tau) = C^{0,0}(\tau) + \sqrt{\varepsilon}C^{1,0}(\tau) + \sqrt{\delta}C^{0,1}(\tau) + \varepsilon C^{2,0}(\tau) + \delta C^{0,2}(\tau) + \sqrt{\varepsilon}\sqrt{\delta}C^{1,1}(\tau) + o\left(\max\left(\varepsilon, \delta\right)\right).$$

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where

$$C^{0,0}(\tau) = r\tau\theta + \beta \int_0^\tau Tr(Q_1^2 A^{20}(s))ds = r\tau\theta + \frac{\theta(\theta-1)}{4}\beta\nu_1^2\tau.$$

$$C^{1,0}(\tau) = \beta \int_0^\tau Tr(Q_1^2 A^{30}(s))ds = \frac{\theta^2(\theta-1)}{4}\beta\nu_1^3\rho_{11}\tau.$$

$$C^{0,1}(\tau) = \beta \int_0^\tau Tr(Q_1^2 A^{21}(s))ds = 0.$$

$$C^{2,0}(\tau) = \beta \int_0^\tau Tr(Q_1^2 A^{40}(s))ds = \frac{\theta^2(\theta-1)^2}{16}\beta\nu_1^4\tau + \frac{\theta^3(\theta-1)}{4}\beta\nu_1^4\rho_{11}^2\tau.$$

$$C^{0,2}(\tau) = \beta \int_0^\tau Tr(Q_2^2 A^{00}(s))ds = \frac{\theta(\theta-1)}{4}\beta\nu_2^2\tau^2.$$

$$C^{1,1}(\tau) = \beta \int_0^\tau Tr(Q_1^2 A^{31}(s))ds = 0.$$

3.4.1 Call price and smile approximation

As before, the Fourier tranform expression can be calculated

$$\Phi_t(T,\theta) = \exp\left[\frac{\theta(\theta-1)}{2}(T-t)\left(\frac{\beta\nu_1^2}{2}+w\right)+\theta y+r(T-t)\theta\right] *$$
$$\begin{bmatrix} 1 + \sqrt{\varepsilon}\left[Tr\left(A^{1,0}V_0\right)+C^{1,0}\right]+\sqrt{\delta}Tr\left(A^{0,1}V_0\right) \\ +\varepsilon\left[Tr\left(A^{2,0}V_0\right)+C^{2,0}+\frac{1}{2}\left(\left(Tr\left(A^{1,0}V_0\right)+C^{1,0}\right)^2\right)\right] \\ +\delta\left[Tr\left(A^{0,2}V_0\right)+C^{0,2}+\frac{1}{2}\left(Tr\left(A^{0,1}V_0\right)^2\right)\right] \\ +\sqrt{\varepsilon}\sqrt{\delta}\left[Tr\left(A^{1,1}V_0\right)+\left(Tr\left(A^{1,0}V_0\right)+C^{1,0}\right)Tr\left(A^{0,1}V_0\right)\right] \\ +o\left(\max\left(\varepsilon,\delta\right)\right) \end{bmatrix}$$

with

$$Tr\left(A^{1,0}V_0\right) = 0.$$

$$Tr\left(A^{01}V_0\right) = \frac{\theta^2(\theta-1)}{2}\rho_{22}\nu_2 w(T-t)^2.$$

$$Tr\left(A^{11}V_0\right) = \theta^3(\theta-1)\rho_{21}\rho_{12}\nu_2\nu_1 w(T-t)^2.$$

$$Tr\left(A^{2,0}V_0\right) = \frac{\theta(\theta-1)}{4}u.$$

$$Tr\left(A^{0,2}V_0\right) = \frac{\theta^3(\theta-1)}{3}\rho_{22}^2\nu_2^2 w(T-t)^3 - \frac{\theta(\theta-1)}{2}w(T-t)^2 + \frac{\theta^2(\theta-1)^2}{6}\nu_2^2 w(T-t)^3.$$

In regard to the smile for short maturities, the same procedure allows to find the expected approximation. Moreover, this method is easier to implement than the singular method [2] and an expansion at order 2 is given. The premium can be developed as follows

$$P = P^{0,0} + \sqrt{\varepsilon}P^{1,0} + \sqrt{\delta}P^{0,1} + \varepsilon P^{2,0} + \delta P^{0,2} + \sqrt{\varepsilon}\sqrt{\delta}P^{1,1} + o\left(\max\left(\varepsilon,\delta\right)\right).$$

with

$$P^{0,0} = C_{BS(\sigma)}, \qquad \sigma = \sqrt{\frac{\beta\nu_1^2}{2}+w}.$$

$$P^{1,0} = \frac{\beta\rho_{11}\nu_1^3(T-t)}{4}\Lambda_y^{BS,3}, \qquad P^{0,1} = \frac{\rho_{22}\nu_2(T-t)^2w}{2}\Lambda_y^{BS,3}.$$

$$P^{2,0} = \frac{1}{4}u\Lambda_y^{BS,2} + \beta\nu_1^4(T-t)\frac{1}{16}\left(\Lambda_y^{BS,4} - \Lambda_y^{BS,3}\right) + \frac{1}{4}\beta\nu_1^4\rho_{11}^2(T-t)\Lambda_y^{BS,4}$$

$$+\frac{1}{2}\frac{\beta^2\rho_{11}^2\nu_1^6(T-t)^2}{16}\left(\Lambda_y^{BS,6} - \Lambda_y^{BS,5}\right).$$

$$P^{0,2} = \frac{1}{4}\beta\nu_2^2(T-t)^2\Lambda_y^{BS,2} - \frac{1}{2}w(T-t)^2\Lambda_y^{BS,2} + \frac{1}{3}\rho_{22}^2\nu_2^2w(T-t)^3\Lambda_y^{BS,4}$$

$$+\frac{1}{6}\nu_2^2w(T-t)^3\left(\Lambda_y^{BS,4} - \Lambda_y^{BS,3}\right) + \frac{1}{2}\frac{\rho_{22}^2\nu_2^2(T-t)^4w^2}{4}\left(\Lambda_y^{BS,6} - \Lambda_y^{BS,5}\right).$$

$$P^{1,1} = \rho_{21}\rho_{12}\nu_2\nu_1w(T-t)^2\Lambda_y^{BS,4} + \frac{\beta\rho_{11}\nu_1^3(T-t)^3}{4}\frac{\rho_{22}\nu_2w}{2}\left(\Lambda_y^{BS,6} - \Lambda_y^{BS,5}\right).$$

As it was done before for the short maturities case, by injecting the Taylor expansion of the implied volatility for Call options in the Black-Scholes model, and by comparing both approximations of the price, an approximation for the implied volatility at order (ε, δ) is deduced

$$\Sigma^{0,0} = \sqrt{w + \frac{\beta\nu_1^2}{2}}.$$

$$\Sigma^{1,0} = \frac{P^{1,0}}{Vega^{BS}} = -\frac{1}{\sqrt{T-t}(\frac{\beta\nu_1^2}{2} + w)}\frac{\nu_1^3}{4}\beta\rho_{11}d_0.$$

$$\Sigma^{0,1} = \frac{P^{0,1}}{Vega^{BS}} = -\frac{1}{\sqrt{T-t}(\frac{\beta\nu_1^2}{2} + w)}\frac{\nu_2}{2}\rho_{22}wd_0$$

$$\Sigma^{2,0} = \frac{P^{2,0} - \frac{1}{2}\left(\Sigma^{1,0}\right)^2 Vomma^{BS}}{Vega^{BS}}.$$

$$\Sigma^{0,2} = \frac{P^{0,2} - \frac{1}{2}\left(\Sigma^{0,1}\right)^2 Vomma^{BS}}{Vega^{BS}}.$$

$$\Sigma^{1,1} = \frac{P^{1,1} - \Sigma^{1,0}\Sigma^{0,1}Vomma^{BS}}{Vega^{BS}}.$$

The approximation is available at order (ε, δ) but for a better readibility, we will present only the expression of the smile at $(\sqrt{\varepsilon}, \sqrt{\delta})$

$$\hat{\Sigma}_t(T, K) = \sqrt{\frac{\beta\nu_1^2}{2} + V_t^{22}} + \frac{\sqrt{\varepsilon}P^{1,0}(t, Y_t, V_t^{22}) + \sqrt{\delta}P^{0,1}(t, Y_t, V_t^{22})}{\frac{\partial C_t^{BS}}{\partial\sigma}(T, K, \sqrt{\frac{\beta\nu_1^2}{2} + V_t^{22}})}$$

$$= \sqrt{\frac{\beta\nu_1^2}{2} + V_t^{22}}$$

$$+ \frac{1}{(\frac{\beta\nu_1^2}{2} + V_t^{22})^{\frac{3}{2}}}[\frac{q_1^3}{4m_1^2}\beta\rho_{11} + \frac{q_2}{2}\rho_{22}(T-t)V_t^{22}][\frac{\log(Ke^{-r(T-t)}) - Y_t}{(T-t)} + \frac{\frac{\beta\nu_1^2}{2} + V_t^{22}}{2}].$$

For a two-scale volatility, the implied volatility $\Sigma^{0,0} = \sqrt{\frac{\beta\nu_1^2}{2} + V_t^{22}}$ at order 0 is a combination of the long term and the short term through the stationary value of V_s^{11} and the

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initial value of V_s^{22}. Moreover, the skew around the forward money $(K = F_t = e^{Y_t + r(T-t)})$ can also be infered at order $(\sqrt{\varepsilon}, \sqrt{\delta})$. The analysis is focused on the skew in order to give characteristics of the model related to the stochastic correlation. But it is obvious that the same reasoning can be done for convexity and slope.

$$
\begin{aligned}
\frac{\partial \Sigma_t}{\partial K} &\sim \frac{1}{F_t} \frac{1}{(\frac{\beta \nu_1^2}{2} + V_t^{22})^{\frac{3}{2}}} [\frac{q_1^3}{4m_1^2(T-t)} \beta \rho_{11} + \frac{q_2}{2} \rho_{22} V_t^{22}] \\
&\sim \frac{1}{F_t} \frac{1}{(\frac{\beta \nu_1^2}{2} + V_t^{22})^{\frac{3}{2}}} [\nu_1^3 \frac{\sqrt{\varepsilon}}{T-t} \beta \rho_{11} + \frac{\nu_2}{2} \sqrt{\delta} \rho_{22} V_t^{22}].
\end{aligned}
\tag{3.4}
$$

This formulae underlines that the *skew* splits up into two components : the first one, proportional to ρ_{11}, coming from the fast mean-reversion volatility component, and the second one, proportional to ρ_{22}, is persistent and proceeds from the slow variation component of the volatility. Besides, this expression is also obtained as a combination of Heston formulaes for short and long maturities [1].

3.5 Numerical Applications

This part is devoted to numerical applications. Indeed, this section lays the emphasis on the error of the approximation as well as the influence of some parameters like the matrix R and the matrix Q on the smile. It is noticeable that for a range of parameters (a set of parameters in a classic market), this approximation can be used for calibration. But when the parameters overstep some bounds, the approximation does not furnish a calibration tool but carries on providing an idea of the level and the shape of the smile as well as the influence of parameters.

3.5.1 Short maturity smile

Let us handle now the case of a short maturity smile with $(T - t) \ll \tau_1, \tau_2$

Consider the followings values for the parameters of the model allowing standard dynamics for the volatility in short maturities

$$
\beta = 4, \quad r = 0, \quad V_0 = \begin{pmatrix} 0.02 & 0.01 \\ 0.01 & 0.02 \end{pmatrix}, \quad M = \begin{pmatrix} -0.05 & 0 \\ 0 & -0.05 \end{pmatrix}, \quad Q = \begin{pmatrix} 0.02 & 0 \\ 0 & 0.02 \end{pmatrix}.
$$

The matrix of mean-reversion M gives the speed at which the volatility process returns to its mean. The matrix Q corresponds to the volatility of volatility which is the most significant parameter of the stochastic volatility model concerning the evolution of the volatility and the dispersion of the volatility around its expected value. First, the matrix Q was chosen so that the volatility does not reach quickly the asymptotic volatility when this one

is far from the initial one ($\Sigma_0 = \sqrt{V_0^{11} + V_0^{22}} = 20\%$ and $\Sigma_\infty = \sqrt{\frac{\beta}{2}(\nu_1^2 + \nu_2^2)} \sim 17,8\%$).
This case is common in a classic market.

Numerical applications underline the effect of the correlation matrix R on a smile for a maturity $T = 6$ months. Indeed, when R is diagonal, parameters can be found so that the Wishart volatility model corresponds to a multifactor Heston model described for example in [5]. Then, adding non-diagonal components for R allows more flexibility concerning the stochastic correlation and the skew because they will also depend on the new factor V_t^{12}. The study is restricted to the case when M and Q are diagonals because it is the framework of this approximation, nevertheless a general matrix R can be considered in order to glimpse the flexibility of the Wishart volatility model. The impact of the non-diagonal components for M and Q will be studied in future researches.

Characteritic times of the process V are denoted $\tau_1 = \frac{1}{2m_1} = 10y$ and $\tau_2 = \frac{1}{2m_2} = 10y$. Then, this framework deals with a "short maturity" case. At $t = 0$, we have calculated the smile by FFT (Fig. 1) and the approached smiles at order 1 and order 2 (Fig. 2 and Fig. 3).
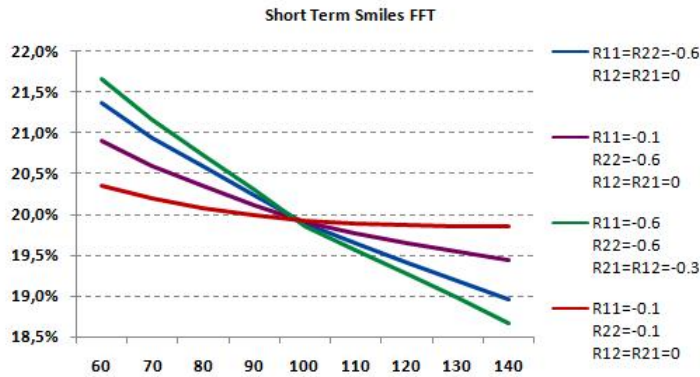
FIGURE 1 – Short term smile by FFT ($T = 6m \ll \tau_1 = \tau_2 = 10y$)

As expected, all components of the correlation matrix R contribute for the generation of the skew. Indeed, the more the components ρ_{11} and ρ_{22} are negative, the more the skew is negative. That is not true for ρ_{12} and ρ_{21} since they are multiplied by V_{12} and V_{21} whose are not necessarly positive.
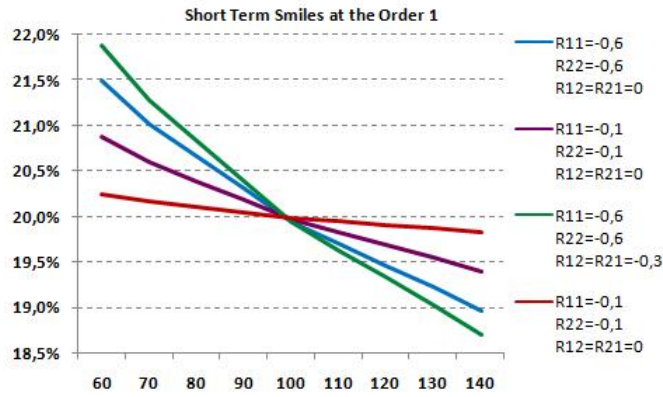
FIGURE 2 – Short term smile at order $(\sqrt{\varepsilon}, \sqrt{\delta})$ $(T = 6m \ll \tau_1 = \tau_2 = 10y)$
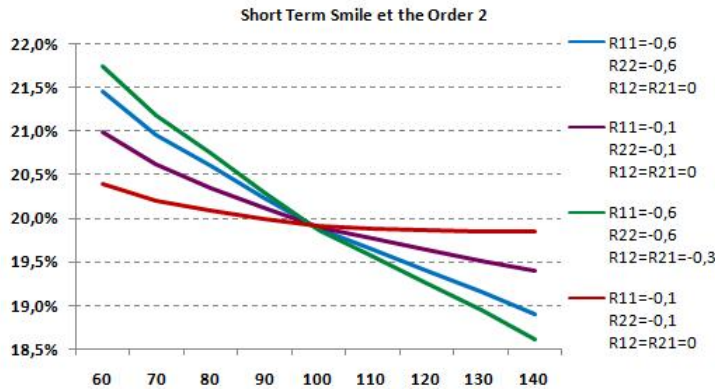
FIGURE 3 – Short term smile at order (ε, δ) $(T = 6m \ll \tau_1 = \tau_2 = 10y)$

In order to quantify the accuracy of this approximation, figures are presented (Fig. 4 and Fig. 5) describing the difference between the approached smiles and the smile obtained by FFT.
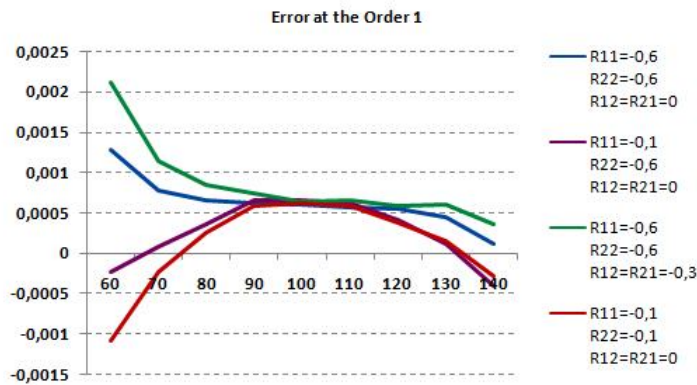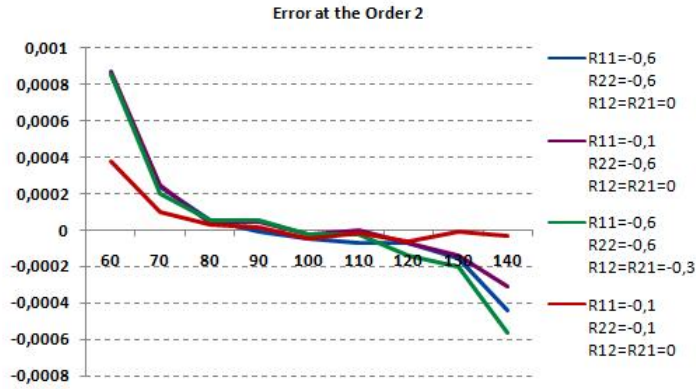

FIGURE 4 – Error for short term smile at order $(\sqrt{\varepsilon}, \sqrt{\delta})$ $(T = 6m \ll \tau_1 = \tau_2 = 10y)$

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Error at the Order 2

FIGURE 5 – Error for short term smile at order (ε, δ) $(T = 6m \ll \tau_1 = \tau_2 = 10y)$

The approximation gives very good results and the approximation at order 2 is accurate enough to be used as a calibration tool : one can see that between the strike 80 and 120, the error is below 5×10^{-5}. This error is totally acceptable by practitioners in a calibration prospect.

An unrealistic case is presented where the volatility of volatility matrix Q is outstanding so that the volatility explodes when maturity tends to infinity. Only the matrix Q is changed to see its influence on the smile and the behaviour of the approximation in a "tense" framework. With these market conditions, the approximation is not good enough in a calibration prospect, but keeps on giving a good shape and level of the smile and consequently a good idea of the impact of the parameters.

$$\beta = 4, \quad r = 0, \quad V_0 = \begin{pmatrix} 0.02 & 0.01 \\ 0.01 & 0.02 \end{pmatrix}, \quad M = \begin{pmatrix} -0.05 & 0 \\ 0 & -0.05 \end{pmatrix}, \quad Q = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix}.$$

In this case, a great asymptotic volatility $(\Sigma_\infty = \sqrt{\frac{\beta}{2}(\nu_1^2 + \nu_2^2)} \sim 89\%)$ is considered and the smile is wealthier as it can be seen in Fig. 6. First, the level at the money (ATM) of the implied volatility at $T = 6$ months is about 24% and is then very different from the initial volatility which is $\Sigma_0 = \sqrt{V_0^{11} + V_0^{22}} = 20\%$. Moreover, on can obtain a real "smirk" with these parameters, in the fourth case for instance (Fig. 6).
The Black-Scholes volatility used in the FFT pricing with Formulae 2.14 is $\bar{\sigma} = 24, 22\%$ which is the realized mean volatility giing an account of the real level of the volatility ATM.
Although the calibration with this approximation is impossible for those parameters, the approximation appears to be stable enough in relation to shock on paremeters and

reproduces a good shape for the smile like the level ATM, the skew and a good behaviour of the convexity.
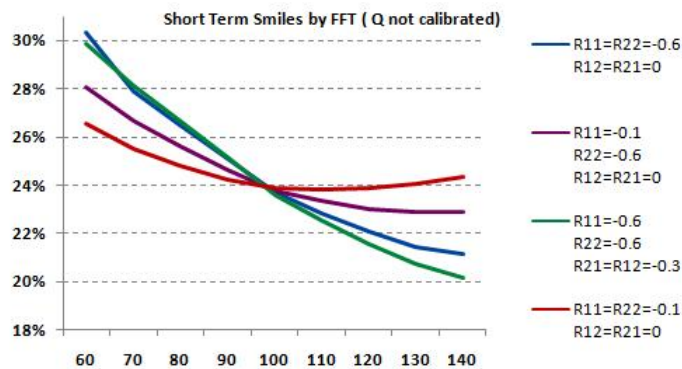

FIGURE 6 – Short term smile by FFT ($T = 6m \ll \tau_1 = \tau_2 = 10y$)

The approximation allows to have an idea of the smile and the influence of some parameters (very useful for a trader) but cannot furnishes a calibration tool. For the 4 sets of correlation coefficients considered, comparing the smile by FFT and the smiles approached by the method at order 1 and 2 allows to estimate the convergence of this method.
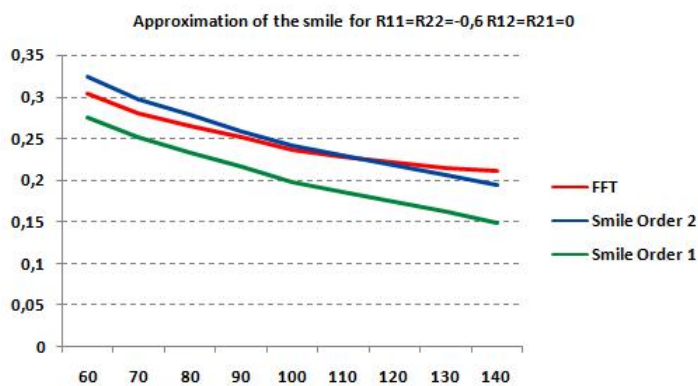
FIGURE 7 – Convergence in the case : $\rho_{11} = \rho_{22} = -0, 6$ $\rho_{12} = \rho_{21} = 0$
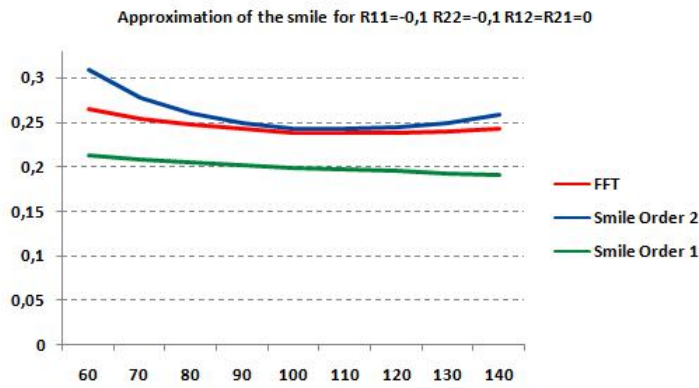
FIGURE 8 – Convergence in the case : $\rho_{11} = \rho_{22} = -0,1$ $\rho_{12} = \rho_{21} = 0$
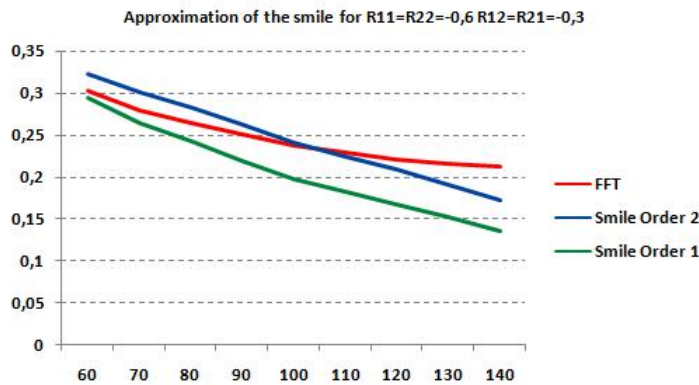
FIGURE 9 – Convergence in the case : $\rho_{11} = \rho_{22} = -0.6$ $\rho_{12} = \rho_{21} = -0,3$
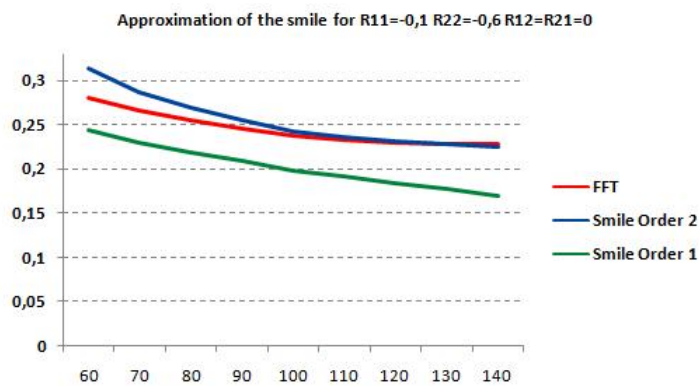
FIGURE 10 – Convergence in the case : $\rho_{11} = -0.1$ $\rho_{22} = -0.6$ $\rho_{12} = \rho_{21} = 0$

The perturbation method becomes more accurate at order 2 : the approximation reaches the good level ATM and the shape of the curves is reproduced more precisely. Indeed, the smirk in the fourth case is faithfully reproduced and the smile level ATM is not far from

the one obtained by FFT (24% Fig. 6).

For information, a complete surface of a short term smile is presented functions of maturity and strike (Fig. 11). The slope appears very high and the smile does not flatten itself when the maturity increases in opposition to the case of a local volatility model.
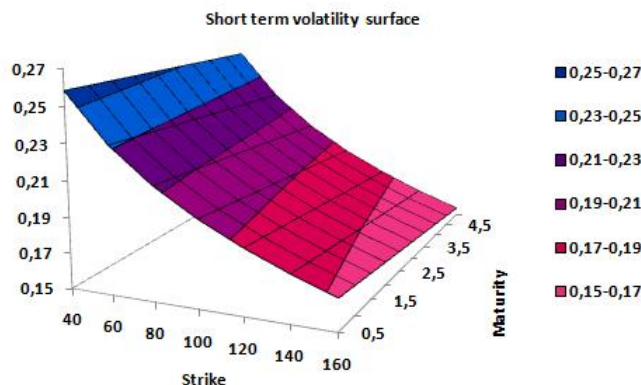

FIGURE 11 – Smile surface for short maturities

3.5.2 Smile with multiscale volatility

At present, those new values for the model parameters are taken as input

$$\beta = 4, \quad r = 0, \quad V_0 = \begin{pmatrix} 0.02 & 0 \\ 0 & 0.02 \end{pmatrix}, \quad M = \begin{pmatrix} -2 & 0 \\ 0 & -0.02 \end{pmatrix}, \quad Q = \begin{pmatrix} 0.05 & 0 \\ 0 & 0.02 \end{pmatrix}.$$

Consider a maturity $T = 3y$ what allows the case of a medium maturity with $\tau_1 = 3m \ll T = 3y \ll \tau_2 = 25y$. The matrix Q is chosen so that the volatility has gone from 20% until $\sqrt{\beta \frac{\nu_1^2}{2} + V_0^{22}} \sim 15,8\%$ in 3 years (Fig. 12) : it is an outstanding change of volatility and the quality of the method has to be tested seeing how the approximation converges toward the real smile with these exceptional market conditions. At order 1, the perturbation method furnishes a good shape for the skew but the level of the smile at the money is far from the good one (15% Fig. 13). However, at order 2, the approximated smile is accurate and the level is really close from the good one (15, 7% Fig. 14).
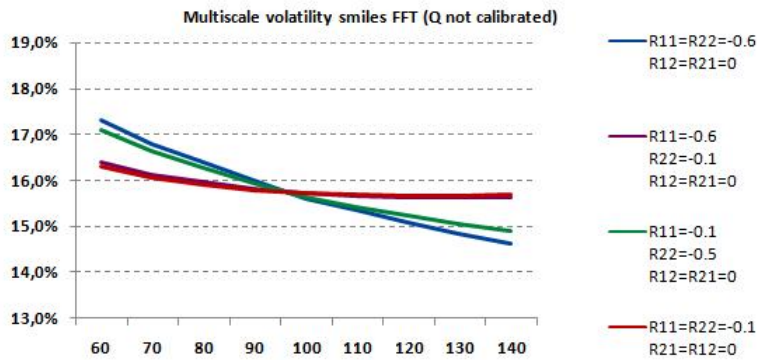
FIGURE 12 – Multiscale volatility smiles by FFT ($\tau_1 = 3m \ll T = 3y \ll \tau_2 = 25y$)
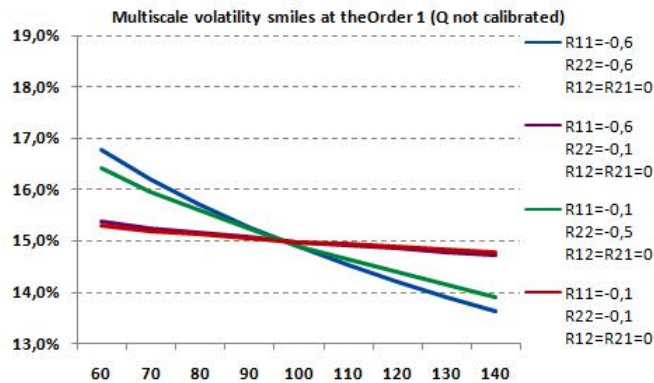

FIGURE 13 – Multiscale volatility smiles at order $(\sqrt{\varepsilon}, \sqrt{\delta})$ ($\tau_1 = 3m \ll T = 3y \ll \tau_2 = 25y$)
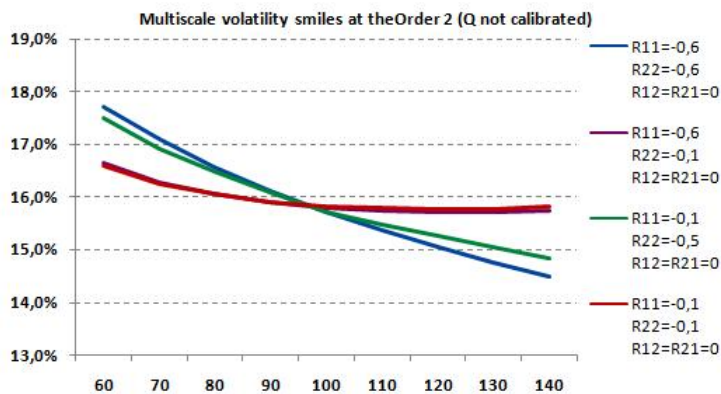

FIGURE 14 – Multiscale volatility smiles at order (ε, δ) ($\tau_1 = 3m \ll T = 3y \ll \tau_2 = 25y$)

It is important to point out that the approximation is maybe not accurate enough for calibration because the set of parameters was taken such as the volatility significantly

changes. Moreover, as in the short maturity case, the procedure gives a good idea of the smile as well as the influence of some parameters (very useful for traders). The smile by FFT is approached by the method at order 1 and at order 2 for the same sets of correlation coefficients (Fig. 15 and Fig. 16).
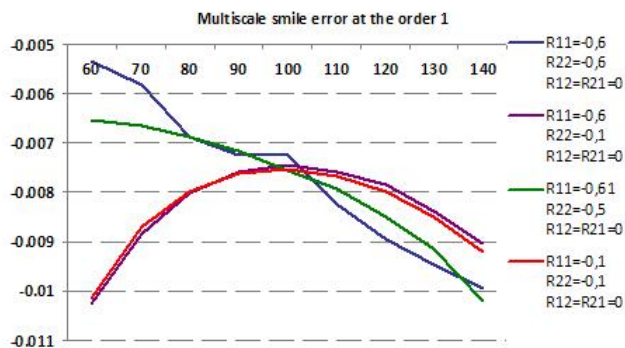

FIGURE 15 – Error for multiscale volatility smile at order $(\sqrt{\varepsilon}, \sqrt{\delta})$ $(\tau_1 = 3m \ll T = 3y \ll \tau_2 = 25y)$
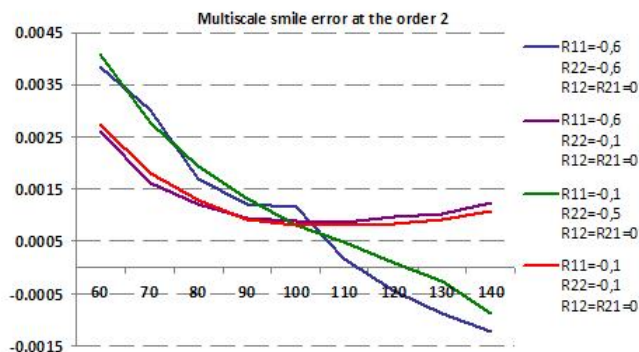

FIGURE 16 – Error for multiscale volatility smile at order (ε, δ) $(\tau_1 = 3m \ll T = 3y \ll \tau_2 = 25y)$

As expected, the skew comes from the component ρ_{22} and the slope of the smile curve is almost insensible to a change of ρ_{11} given that $\tau_1 \ll T$. This is what it was expected after an analysis of the smile through the approximation formulae 3.4. We can also describe the implied volatility surface in the case of a medium maturity option 17.
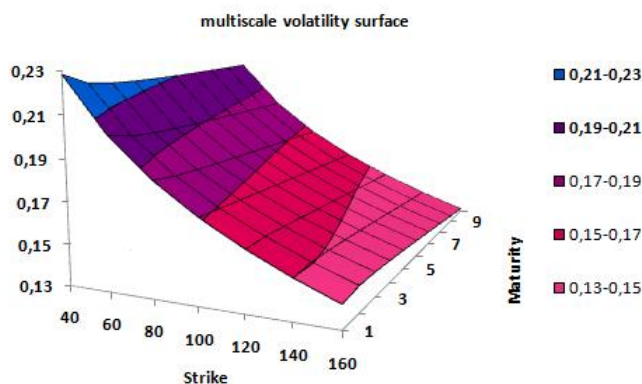
FIGURE 17 – Smile Surface for medium maturities

Conclusion

Having observed the problem of the data fitting for short or long maturities, the Wishart volatility model, which is a multifactor extension of the Heston model, seems to be adapted to solve efficiently this problem. Indeed, preserving advantages of affine models provides semi-closed formulae for the price of some options like Call options and put options, as well as a good understanding of the volatility behaviour through the parameters of the model.

It was natural to find a procedure, based on affine properties, that allows an accurate estimation of the smile. This paper suggests mainly some developments for smile dynamics in the Wishart volatility model. Using an adaptation of the singular perturbations method, closed-form expressions were found concerning asymptotic smiles in the cases of short and medium maturities. Those accurate approximation can be used as a calibration tool in practice with classic conditions and gives also a good idea of the parameters' effect on the smile that could be very useful for traders.

Moreover, for an evaluation by Monte Carlo methods, a procedure of simulation was studied for the Wishart volatility model knowing that a standard Euler scheme is prohibited. Indeed, using a relationship between a Wishart process (with a degree integer) and Ornstein-Uhlenbeck processes, an extension was suggested for a general Wishart process (with a degree real) : in order to handle the simulation in the general case, a helpful change of probability measure was employed using the dynamics of determinant.

Many points are reserved for future works : comprehension of the influence of the non-diagonal components for the matrices M, Q and V_0 that provides the main interest of this model, investigations about the forward smile in order to deal more complex options in the

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Wishart volatility model, and the generalization of this model in the interest rate world where a large number of factors is needed. One expects that this kind of model, where the volatility is a function of a Wishart process, exhibits precisely different source of risk existing in the market and provides enough flexibility for the generation of whealty smiles.

Appendice

Appendice A : About exponentials of matrices

3.5.3   Définitions and properties

1. Let A be a complex matrix. The exponential of A, denoted $\exp(A)$, is the $n \times n$ matrix given by the power series

$$\exp(A) = \sum_{i=0}^{+\infty} \frac{A^n}{n!}.$$

2. Let A, B be two complex matrices. If $AB = BA$, then

$$\exp(A + B) = \exp(A)\exp(B) = \exp(B)\exp(A).$$

3. Let A be a complex matrix and $\{\lambda_1, ..., \lambda_n\}$ its eigenvalues, then $\{\exp(\lambda_1), ..., \exp(\lambda_n)\}$ are the $\exp(A)$ eigenvalues.

4. Let A be a complex matrix, then $\exp(A)$ is invertible and its inverse is given by

$$(\exp(A))^{-1} = \exp(-A).$$

5. Let A be a complex matrix, then

$$\det(\exp(A)) = \exp(Tr(A)).$$

6. Let $F : \mathbb{R} \to GL_n(\mathbb{C})$, then

$$Tr\left[\int_a^b (F(s))^{-1} F(s) ds \right] = \log\left[\frac{\det(F(b))}{\det(F(a))} \right].$$

3.5.4   Exponential of matrices approximation

Many numerical methods are available to approach an exponential of matrix [25]. The Padé approximation method was the one chosen associated with the *Scaling ans Squaring* method. Consider $A \in \mathcal{M}_n(\mathbb{C})$ and $(p, q) \in \mathbb{N}^* \times \mathbb{N}^*$. Then, an approximation of the exponential of matrix

$$\exp(A) \sim R_{pq}(A).$$

where

$$R_{pq}(A) = [D_{pq}(A)]^{-1} N_{pq}(A).$$
$$N_{pq}(A) = \sum_{j=0}^{p} \frac{(p+q-j)!p!}{(p+q)!j!(p-j)!} A^j.$$

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$$D_{pq}(A) \quad = \quad \sum_{j=0}^{q} \frac{(p+q-j)!q!}{(p+q)!j!(q-j)!} A^j.$$

Moreover, the article [25] underlines that if $j \in \mathbb{N}$ such as $\|A\| \leq 2^{j-1}$, then

$$[R_{pq}(2^{-j}A)]^{2^j} = \exp(A + E),$$

where

$$\|E\| \leq 2^{3-p-q} \frac{p!q!}{(p+q)!(p+q+1)!}.$$

This approximation of $\exp(A)$ by $[R_{pq}(2^{-j}A)]^{2^j}$ instead of $R_{pq}(A)$ is the *scaling and squaring* method.

Appendice B : Simulation of Ornstein Uhlembeck processes

Exact simulation of Ornstein-Uhlenbeck processes   Let X be an Ornstein-Uhlenbeck process which follows the dynamics

$$dX_t = MX_t dt + Q^\top dB_t.$$

Then, it is easily shown that

$$
\begin{aligned}
\mathbb{E}(X_{t+h}|X_t) &= \exp(hM)X_t. \\
\mathbb{V}ar(X_{t+h}|X_t) &= \int_0^h \exp(sM)Q^\top Q \exp(sM^\top)ds.
\end{aligned}
$$

The function, defined by $f(t) = \int_0^t \exp(sM)Q^\top Q \exp(sM^\top)ds$, is solution of the differential equation

$$f'(t) \quad = \quad Q^\top Q + Mf(t) + f(t)M^\top.$$

Using the linearization method ([7]), the solution of this equation can be written by

$$
\begin{aligned}
f(t) &= g^{-1}(t)h(t). \\
(g(t)\ h(t)) &= (0\ I_n) \exp\left[t \begin{pmatrix} M^\top & 0 \\ Q^\top Q & -M \end{pmatrix} \right].
\end{aligned}
$$

Al last, $\bar{X}_{t_i}$ can be deduced by $\mathcal{N}(\exp(\Delta t M)\bar{X}_{t_{i-1}}, f(\Delta t))$.

Euler scheme of Ornstein-Uhlenbeck processes   Although there exists an explicit expression of $\bar{X}_{t_i}$ with its transition law, using this method to simulate an Ornstein-Uhlenbeck process appears less effective than an Euler scheme. Indeed, as explained in the next section, the Brownian increments of the Euler ccheme of the Wishart process

can also be used to simulate the asset itself. Therefore, the simultaneaous simulation by an Euler scheme of both the processesappears to be the most efficient method. The discretization of an Ornstein-Uhlenbeck process is given by

$$\bar{X}_{k,t_i} \quad = \quad (I_n + \Delta t M)\bar{X}_{k,t_{i-1}} + \sqrt{\Delta t}Q^\top \varepsilon_{k,i}, \quad 1 \le k \le \beta. \qquad (3.1)$$

Consequently, the simulation of a Wishart process by an Euler scheme is obtained as

$$\bar{X}_{k,t_i} \quad = \quad (I_n + \Delta t M)\bar{X}_{k,t_{i-1}} + \sqrt{\Delta t}Q^\top \varepsilon_{k,i}, \quad 1 \le k \le \beta.$$

$$\bar{V}_{t_i} \quad = \quad \sum_{k=1}^{\beta} \bar{X}_{k,t_i}\bar{X}_{k,t_i}^\top$$

Finally, the simulation of a Wishart process in the case $\beta \in \mathbb{N}$ is easy. The case $\beta \in \mathbb{R}$ is deduced by the change of the probability measure detailed in the previous section. For financial applications, the Wishart volatility model is a stochastic volatility model where the volatility is driven by the trace of a Wishart process. The parameters of the volatility process will be calibrated on market prices. Hence, the simulation of a Wishart process in its general form (β real deduced from calibration) is essential for an evaluation by Monte Carlo methods in the Wishart volatility model and will be developed in the next section.

Appendice C : Standard calculations in the Black-Scholes model

Recall the pricing notations in the Black-Scholes model with the log spot denoted $y = \log(x)$

$$d_1 = \frac{1}{\sigma\sqrt{T-t}}(y + \log(\frac{e^{+r(T-t)}}{K})) + \frac{1}{2}\sigma\sqrt{T-t}.$$

$$d_0 = \frac{1}{\sigma\sqrt{T-t}}(y + \log(\frac{e^{+r(T-t)}}{K})) - \frac{1}{2}\sigma\sqrt{T-t}.$$

The following derivatives can be deduced easily

$$\frac{\partial d_1}{\partial x} = \frac{\partial d_0}{\partial x} = \frac{1}{\sigma x\sqrt{T-t}}.$$

$$\frac{\partial d_1}{\partial y} = \frac{\partial d_0}{\partial y} = \frac{1}{\sigma\sqrt{T-t}}.$$

$$\frac{\partial d_1}{\partial \sigma} = -\frac{1}{\sigma^2\sqrt{T-t}}\log(\frac{xe^{r(T-t)}}{K}) + \frac{1}{2}\sqrt{T-t}.$$

$$\frac{\partial d_0}{\partial \sigma} = -\frac{1}{\sigma^2\sqrt{T-t}}\log(\frac{xe^{r(T-t)}}{K}) - \frac{1}{2}\sqrt{T-t}.$$

Using the previous expression, the Greeks of a Call option can be calculated

$$\Delta^x = \frac{\partial C_{BS}}{\partial x} = N(d_1), \qquad \Delta^y = \frac{\partial C_{BS}}{\partial y} = xN(d_1).$$

$$\Gamma^x = \frac{\partial^2 C_{BS}}{\partial x^2} = \frac{1}{\sigma x \sqrt{T-t}} N'(d_1), \qquad \Gamma^y = \frac{\partial^2 C_{BS}}{\partial y^2} = x^2 \Gamma^x + \Delta^y.$$

Hence, recursive calcultation by considering Formulae 3.1 and the delta and gamma greeks yields

$$\frac{\partial^3 C_{BS}}{\partial y^3} = \Gamma^y + (\Gamma^y - \Delta^y)\frac{(a-y)}{\sigma^2(T-t)}.$$

$$\frac{\partial^4 C_{BS}}{\partial y^4} = \frac{\partial^3 C_{BS}}{\partial y^3} + \left(\frac{\partial^3 C_{BS}}{\partial y^3} - \frac{\partial^2 C_{BS}}{\partial y^2}\right)\frac{(a-y)}{\sigma^2(T-t)} - \frac{1}{\sigma^2(T-t)}(\Gamma^y - \Delta^y).$$

$$\frac{\partial^5 C_{BS}}{\partial y^5} = \frac{\partial^4 C_{BS}}{\partial y^4} + \left(\frac{\partial^4 C_{BS}}{\partial y^4} - \frac{\partial^3 C_{BS}}{\partial y^3}\right)\frac{(a-y)}{\sigma^2(T-t)} - \frac{2}{\sigma^2(T-t)}\left(\frac{\partial^3 C_{BS}}{\partial y^3} - \frac{\partial^2 C_{BS}}{\partial y^2}\right).$$

$$\frac{\partial^6 C_{BS}}{\partial y^6} = \frac{\partial^5 C_{BS}}{\partial y^5} + \frac{(a-y)}{\sigma^2(T-t)}\left(\frac{\partial^5 C_{BS}}{\partial y^5} - \frac{\partial^4 C_{BS}}{\partial y^4}\right) - \frac{3}{\sigma^2(T-t)}\left(\frac{\partial^4 C_{BS}}{\partial y^4} - \frac{\partial^3 C_{BS}}{\partial y^3}\right).$$

The expression of the price successive derivatives with respect to σ is given by

$$Vega^{BS} = \frac{\partial C_{BS}}{\partial \sigma} = \sigma(T-t)(\Gamma^y - \Delta^y).$$

$$Vomma^{BS} = \frac{\partial^2 C_{BS}}{\partial \sigma^2} = (T-t)d_0 d_1(\Gamma^y - \Delta^y).$$

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