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Conditional Value-at-Risk Constraint and Loss Aversion Utility Functions

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Abstract

We provide an economic interpretation of the practice consisting in incorporating risk measures as constraints in a classic expected return maximization problem. For what we call the infimum of expectations class of risk measures, we show that if the decision maker (DM) maximizes the expectation of a random return under constraint that the risk measure is bounded above, he then behaves as a "generalized expected utility maximizer" in the following sense. The DM exhibits ambiguity with respect to a family of utility functions defined on a larger set of decisions than the original one; he adopts pessimism and

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performs first a minimization of expected utility over this family, then performs a maximization over a new decisions set. This economic behaviour is called "Maxmin under risk" and studied by Maccheroni (2002). This economic interpretation allows us to exhibit a loss aversion factor when the risk measure is the Conditional Value-at-Risk.

Keywords. Risk measures, Utility functions, Nonexpected utility theory, Maxmin, Conditional Value-at-Risk, Loss aversion.

1 Motivation

Taking risk into account in decision problems in a mathematical formal way is more and more widespread. For instance, liberalization of energy markets displays new issues for electrical companies which now have to master both traditional problems (such as optimization of electrical generation) and emerging problems (such as integration of spot markets and risk management, see e.g. Eichhorn and Römisch (2006)). The historical issue which consisted in managing the electrical generation at lowest cost evolved: liberalization of energy markets and introduction of spot markets lead to consider a problem of revenue maximization under Earning-at-Risk constraint, because financial risks are now added to the traditional risks.

Let us now be slightly more formal. Consider a decision maker (DM) whose return $J(\mathtt{a}, \xi)$ depends on a decision variable $\mathtt{a}$ (for instance, proportions of assets) and a random variable ξ (random return of assets, for example). The question of how to take risk into account in addition has been

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studied since long. Let us briefly describe two classical approaches to deal with risk in a decision problem. On the one hand, the DM may maximize the expectation of $J(\mathsf{a},\xi)$ under explicit risk constraints, such as variance (as in Markowitz (1952)) or Conditional Value-at-Risk; we shall coin this practice as belonging to the engineers or practitioners world. On the other hand, the DM may maximize the expectation of $U\big(J(\mathsf{a},\xi)\big)$ where U is a utility function wich captures more or less risk aversion (in the so called expected utility theory, or more general functionals else); this is the world of economists. In this paper, we shall focus on the links between these two approaches.

The paper is organized as follows. In Section 2, we specify a wide class of risk measures which will prove useful to provide an economic interpretation of profit maximization under risk constraint. Section 3 gives our main result and points out a specific nonexpected utility theory which is compatible with the original problem. Our approach uses duality theory and Lagrange multipliers. However it does not focus on the optimal multiplier, and this is how we obtain a *family* of utility functions and an economic interpretation (though belonging to nonexpected utility theories), and not a single utility function. This differs from the result in Dentcheva and Rusczynki (2006) where the authors prove, in a way, that utility functions play the role of Lagrange multipliers for second order stochastic dominance constraints. With this, Dentcheva and Rusczynki (2006) prove the equivalence between portfolio maximization under second order stochastic dominance constraints and expected utility maximization, for *one single* utility function. However, such utility function is not given *a priori* and may not be interpreted economically

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before the decision problem. A concluding economic discussion is given in Section 4. Proofs are gathered in Appendix A.

2 The infimum of expectations class of risk measures

Let be given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{P}$ a probability measure on a σ-field $\mathcal{F}$ of events on Ω. We are thus in a *risk* decision context. The expectation of a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ will be denoted by $\mathbb{E}$. We introduce a class of risk measures which covers many of the usual risk measures and which will be prove adequate for optimization problems. We will also briefly examine the connections with coherent risk measures.

2.1 Definition and examples

Let a function $\rho : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be given. Let $L_\rho(\Omega, \mathcal{F}, \mathbb{P})$ be a set of random variables X defined on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\rho\big(X, \eta\big)$ is integrable for all η and such that the following risk measure

$$\mathcal{R}_\rho(X) := \inf_{\eta \in \mathbb{R}} \mathbb{E}\big[\rho\big(X, \eta\big)\big] \ , \tag{1}$$

is finite ($\mathcal{R}_\rho(X) > -\infty$).

In the sequel, we shall require the following properties for the function $\rho : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and for $L_\rho(\Omega, \mathcal{F}, \mathbb{P})$.

H1. $\eta \mapsto \rho(x, \eta)$ is a convex function,

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H2. for all $X \in L_\rho(\Omega, \mathcal{F}, \mathbb{P})$, $\eta \mapsto \mathbb{E}\big[\rho\big(X, \eta\big)\big]$ is continuous[1] and has limit $+\infty$ when $\eta \to +\infty$.

The random variable X represents a loss. Hence, risk constraints will be of the form $\mathcal{R}_\rho(X) \leq \gamma$. For the so called safety measures, the safety constraint is rather $\mathcal{S}_\rho(X) \geq \gamma$. We pass from one to the other by $\mathcal{S}_\rho(X) = -\mathcal{R}_\rho(-X) = \sup_{\eta \in \mathbb{R}} \mathbb{E}\big[-\rho\big(-X, \eta\big)\big]$.

Several well-known risk measures belong to the infimum of expectations class of risk measures.

Variance

In Markowitz (1952), Markowitz uses variance as a risk measure. A well known formula for the variance is

$$\mathsf{var}\left[X\right] = \inf_{\eta \in \mathbb{R}} \mathbb{E}\big[\big(X - \eta\big)^2\big] \,.$$

The function

$$\rho_{\mathsf{var}}(x, \eta) := (x - \eta)^2$$

is convex with respect to η (H1.). Taking $L_\rho(\Omega, \mathcal{F}, \mathbb{P}) = L^2(\Omega, \mathcal{F}, \mathbb{P})$, assumption H2. is satisfied.

Conditional Value-at-Risk

[1]We could also formulate assumptions directly on the function ρ, ensuring Lebesgue dominated convergence validity, but this is not our main concern here.

Rockafellar and Uryasev (2000) give the following formula for the Conditional Value-at-Risk risk measure CVaR at confidence level[2] $0 < p < 1$:

$$\mathsf{CVaR}_p(X) = \inf_{\eta \in \mathbb{R}} \left(\eta + \frac{1}{1-p} \mathbb{E}\big[\max\{0, X - \eta\} \big] \right) .$$

The function

$$\rho_{\mathsf{CVaR}}(x, \eta) := \eta + \frac{1}{1-p} \max\{0, x - \eta\}$$

is convex with respect to η. Taking $L_\rho(\Omega, \mathcal{F}, \mathbb{P}) = L^1(\Omega, \mathcal{F}, \mathbb{P})$, assumption H2. is satisfied.

Weighted mean deviation from a quantile

Let us introduce ψ_X^{-1} the left-continuous inverse of the cumulative distribution function ψ_X of the random variable X and $\psi_X^{-2}(p) = \int_0^p \psi_X^{-1}(\alpha) \mathrm{d}\alpha$. The weighted mean deviation from a quantile WMd is

$$\mathsf{WMd}_p(X) = \mathbb{E}[X]\, p - \psi_X^{-2}(p) \,.$$

In Ogryczak and Ruszczynski (1999), one finds the expression

$$\mathsf{WMd}_p(X) = \inf_{\eta \in \mathbb{R}} \mathbb{E}\Big[\max \big\{ p(X - \eta), (1-p)(\eta - X) \big\} \Big] .$$

The function

$$\rho_{\mathsf{WMd}}(x, \eta) := \max \big\{ p(x - \eta), (1-p)(\eta - x) \big\} ,$$

is continuous, convex with respect to η.

[2] In practice, p is rather close to 1 ($p = 0.95$, $p = 0.99$). The Value-at-Risk $\mathsf{VaR}_p(X)$ is such that $\mathbb{P}(X \le \mathsf{VaR}_p(X)) = p$. Then $\mathsf{CVaR}_p(X)$ is interpreted as the expected value of X knowing that X exceeds $\mathsf{VaR}_p(X)$.

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Optimized Certainty Equivalent

The Optimized Certainty Equivalent was introduced in Ben-Tal and Teboulle (1986) (see also Ben-Tal and Teboulle (2007)). This concept is based on the economic notion of certainty equivalent. Let $U : \mathbb{R} \to [-\infty, +\infty[$ be a proper closed concave and nondecreasing utility function[3]. The Optimized Certainty Equivalent $\mathcal{S}_U$ of the random return X defined on $(\Omega, \mathcal{F}, \mathbb{P})$ is

$$\mathcal{S}_U(X) = \sup_{\nu \in \mathbb{R}} \big(\nu + \mathbb{E}\big[U(X-\nu)\big]\big) .$$

The associated risk measure is

$$\mathcal{R}_\rho(X) := -\mathcal{S}_U(-X) = \inf_{\eta \in \mathbb{R}} \big(\eta - \mathbb{E}\big[U(\eta - X)\big]\big)$$

where

$$\rho_U(x, \eta) := \eta - U(\eta - x)$$

satisfies assumptions H1. and H2. whenever U satisfies appropriate continuity and growth assumptions.

A Summary Table

We sum up the above cases. We denote $a_+ := \max\{a, 0\}$. Notice that all the functions $x \mapsto \rho(x, \eta)$ in Table 1 are convex.

[3]With additional assumptions: U has effective domain $\text{dom}U = \{x \in \mathbb{R} \mid U(x) > -\infty\} \neq \emptyset$, supposed to satisfy $U(0) = 0$ and $1 \in \partial U(0)$, where ∂U denotes the subdifferential map of U.

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Risk measure $\mathcal{R}_\rho$	$\rho(x,\eta)$
Variance	$(x-\eta)^2$
Conditional Value-at-Risk	$\eta + \frac{1}{1-p}(x-\eta)_+$
Weighted Mean Deviation	$\max\{p(x-\eta), (1-p)(\eta-x)\}$
Optimized Certainty Equivalent	$\eta - U(\eta - x)$

Table 1: Risk measures given by an infimum of expectations

2.2 Infimum of expectations and coherent risk measures

Coherent risk measures were introduced in Artzner et al. (1999). When the risk measure $\mathcal{R}_\rho$ is given by (1), we shall provide sufficient conditions on ρ ensuring monotonicity, translation invariance, positive homogeneity, convexity, subadditivity.

Proposition 1 *Under the assumption that the set $L_\rho(\Omega, \mathcal{F}, \mathbb{P})$ in §2.1 is a vector space containing the constant random variables, we have the following properties.*

1. *If $x \mapsto \rho(x,\eta)$ is increasing, then $\mathcal{R}_\rho$ is monotonous.*
2. *If $\rho(x+m,\eta) = \rho(x,\eta'_m) - m$ where $\eta \mapsto \eta'_m$ is one-to-one, then $\mathcal{R}_\rho$ satisfies translation invariance.*

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3. *If* $\rho(\theta x, \eta) = \theta\rho(x, \eta'_\theta)$ *where* $\eta \mapsto \eta'_\theta$ *is one-to-one for* $\theta > 0$, *then* $\mathcal{R}_\rho$ *satisfies positive homogeneity.*

4. *If* $(x, \eta) \mapsto \rho(x, \eta)$ *is jointly convex then* $\mathcal{R}_\rho$ *is convex.*

5. *If* $(x, \eta) \mapsto \rho(x, \eta)$ *is jointly subadditive, then* $\mathcal{R}_\rho$ *is subadditive.*

Proof: We shall only prove item 4 since item 5 is proved in the same way, and that all other assertions are straightforward.

Assume that $(x, \eta) \mapsto \rho(x, \eta)$ is jointly convex. Let X_1 and X_2 be two random variables, $(\eta_1, \eta_2) \in \mathbb{R}^2$ and $\theta \in [0, 1]$. We have

$$\rho(\theta X_1 + (1-\theta)X_2, \theta\eta_1 + (1-\theta)\eta_2) \leq \theta\rho(X_1, \eta_1) + (1-\theta)\rho(X_2, \eta_2).$$

By using expectation operator and positivity of θ, one obtains

$$\inf_\eta \mathbb{E}\big[\rho(\theta X_1 + (1-\theta)X_2, \eta)\big] \leq \inf_{(\eta_1,\eta_2)\in\mathbb{R}\times\mathbb{R}} \Big\{\theta\mathbb{E}\big[\rho(X_1, \eta_1)\big] + (1-\theta)\mathbb{E}\big[\rho(X_2, \eta_2)\big]\Big\}.$$

It yields

$$\mathcal{R}_\rho(\theta X_1 + (1-\theta)X_2) \leq \theta\mathcal{R}_\rho(X_1) + (1-\theta)\mathcal{R}_\rho(X_2)\ .$$

□

For the Conditional Value-at-Risk, $\rho_{\mathsf{CVaR}}(x + m, \eta) = \rho_{\mathsf{CVaR}}(x, \eta'_m) - m$ where $\eta'_m = \eta + m$. For the Weighted Mean Deviation, $\rho_{\mathsf{WMd}}(\theta x, \eta) = \theta\rho(x, \eta')$ where $\eta'_\theta = \eta/\theta$.

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3 Profit maximization under risk constraints: a reformulation with utility functions

We now state our main result, which is an equivalence between a profit maximization under risk constraints problem and a maxmin problem involving an infinite number of utility functions. We relate this (non) expected utility economic interpretation to the "maxmin" representation proposed in Maccheroni (2002). For the specific case of CVaR risk constraint, we exhibit a link with "loss aversion utility functions" *à la* Kahneman and Tversky (see Kahneman and Tversky (1992)).

3.1 A maxmin reformulation

To formulate a maximization problem under risk constraint, let us introduce

- a set $\mathbb{A} \subset \mathbb{R}^n$ of actions or decisions,
- a random variable ξ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with values in $\mathbb{R}^p$,
- a mapping $J : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$, such that for any action $\mathsf{a} \in \mathbb{A}$, the random variable $J(\mathsf{a}, \xi)$ represents the prospect (profit, benefit, etc.) of the decision maker,
- a risk measure $\mathcal{R}_\rho$ defined in (1), together with the level constraint $\gamma \in \mathbb{R}$.

Our main result is the following (the proof is given in Appendix A).

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Theorem 2 *Assume that* ρ *and* $L_\rho(\Omega, \mathcal{F}, \mathbb{P})$ *satisfy assumptions H1. and H2. in §2.1, that* $J(\mathsf{a}, \xi) \in L_\rho(\Omega, \mathcal{F}, \mathbb{P})$ *for all* $\mathsf{a} \in \mathbb{A}$*, and that the infimum in* (1) *is achieved for any loss* $X = -J(\mathsf{a}, \xi)$ *when* a *varies in* $\mathbb{A}$*.*

The maximization under risk constraint[4] *problem*

$$\begin{cases} \sup_{\mathsf{a} \in \mathbb{A}} \mathbb{E}\big[J(\mathsf{a}, \xi)\big] \\ \mathcal{R}_\rho\big(-J(\mathsf{a}, \xi)\big) = \inf_{\eta \in \mathbb{R}} \mathbb{E}\big[\rho\big(-J(\mathsf{a}, \xi), \eta\big)\big] \leq \gamma, \end{cases} \tag{2}$$

is equivalent to the following maxmin problem

$$\sup_{(\mathsf{a}, \eta) \in \mathbb{A} \times \mathbb{R}} \inf_{U \in \mathcal{U}} \mathbb{E}\left[U\big(J(\mathsf{a}, \xi), \eta\big)\right] . \tag{3}$$

The set $\mathcal{U}$ *of functions* $\mathbb{R}^2 \to \mathbb{R}$ *over which the infimum is taken is*

$$\mathcal{U} := \left\{ U^{(\lambda)} : \mathbb{R}^2 \to \mathbb{R}\,,\ \lambda \geq 0 \mid U^{(\lambda)}(x, \eta) = x + \lambda\big(-\rho(-x, \eta) + \gamma\big) \right\} .$$

Table 2 sums up parameterized functions corresponding to usual risk measures.

Notice that the formulation (3) of Problem (2) leads us to introduce a new decision variable $\eta \in \mathbb{R}$. This decision variable comes from our choice of risk measure given by (1).

We shall now see that the formulation (3) has connections with the so called "maxmin" representation of Maccheroni (2002), that we briefly sketch.

Let us consider a continuous and convex weak order $\succcurlyeq$ over the set ΔZ of all lotteries (simple probability distributions) defined on an outcome space

[4] The risk constraint is not on the prospect $J(\mathsf{a}, \xi)$, but on the loss $-J(\mathsf{a}, \xi)$.

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Risk measure $\mathcal{R}_\rho$	$U^{(\lambda)}(x,\eta)$, $\lambda \geq 0$
$\rho(-x,\eta)$	$x - \lambda\rho(-x,\eta) + \lambda\gamma$
Variance	$x - \lambda(x+\eta)^2 + \lambda\gamma$
Conditional Value-at-Risk	$x - \frac{\lambda}{1-p}(x+\eta)_- - \lambda\eta + \lambda\gamma$
Weighted Mean Deviation	$x - \lambda \max\big\{-p(x+\eta), (1-p)(x+\eta)\big\} + \lambda\gamma$
Optimized Certainty Equivalent	$x + \lambda U(x+\eta) + \lambda\eta + \lambda\gamma$

Table 2: Usual risk measures and their corresponding family functions

Z. The main result of Maccheroni (2002) is the following: if there exists a best outcome[5] and if decisions are made independently of it, then there exists a closed and convex set $\mathcal{U}$ of utility functions defined on Z such that, for any lotteries r and q, we have

$$r \succcurlyeq q \Leftrightarrow \min_{U\in\mathcal{U}} \int U(z)\mathrm{d}r(z) \geq \min_{U\in\mathcal{U}} \int U(z)\mathrm{d}q(z)\ .$$

The interpretation given by Maccheroni for this decision rule is the following one: a conservative investor has an unclear evaluation of the different outcomes when facing lotteries. He then acts as if he were considering many expected utility evaluations and taking the worst one[6].

[5]A best outcome for a preference relation $\succcurlyeq$ is an element $z^* \in Z$ such that $z^* \succcurlyeq r$ for all $r \in \Delta Z$.

[6]This reformulation is, in a sense, dual to the well known form $\inf_{\mathbb{P}\in\mathcal{P}} \int U(w)\,\mathrm{d}\mathbb{P}(w)$, where $\mathcal{P}$ is a convex set of probability measures (Gilboa and Schmeidler (1989)).

3.2 Conditional Value-at-Risk and loss aversion

Suppose that the risk constraint $\mathcal{R}_\rho$ is the Conditional Value-at-Risk. The utility functions associated to this risk measure are

$$U^{(\lambda)}(x,\eta) = x - \frac{\lambda}{1-p}(-x-\eta)_+ - \lambda\eta + \lambda\gamma\,.$$

We consider only the x argument (profit, benefit, etc.). We interpret the $-\eta$ argument as an *anchorage* parameter: for $x \geq -\eta$, $x \mapsto U^{(\lambda)}(x,\eta) = x - \lambda\eta + \lambda\gamma$ has slope 1, while it has slope

$$\theta := 1 + \frac{\lambda}{1-p}$$

for x lower than $-\eta$, as in Figure 1. We interpret the parameter θ as a *loss aversion* parameter introduced by Kahneman and Tversky (see Kahneman and Tversky (1992)). Indeed, this utility function $x \mapsto U^{(\lambda)}(x,\eta)$ expresses the property that one monetary unit than the anchorage $-\eta$ gives one unit of utility, while one unit less gives $-\theta$.

3.3 An illustration

Suppose that a DM splits its investment between a risk free asset ξ^0 (deterministic[7] and a risky asset ξ^1 (random following a Normal law $\mathcal{N}(M,\Sigma)$[8], in proportion $1-\mathsf{a} \in [0,1]$), giving the value of the portfolio

$$J\big(\mathsf{a},\xi\big) = \mathsf{a}\xi^0 + (1-\mathsf{a})\xi^1 = \mu(\mathsf{a}) + \sigma(\mathsf{a})N, \quad N \sim \mathcal{N}(0,1)$$

with

$$\mu(\mathsf{a}) = \mathsf{a}\xi^0 + (1-\mathsf{a})M \quad \text{and} \quad \sigma(\mathsf{a}) = (1-\mathsf{a})\Sigma\,.$$

[7]Numerical value of 1 030 €.

[8]Mean M=1 144 € and standard deviation Σ=249 €, French stock market index.

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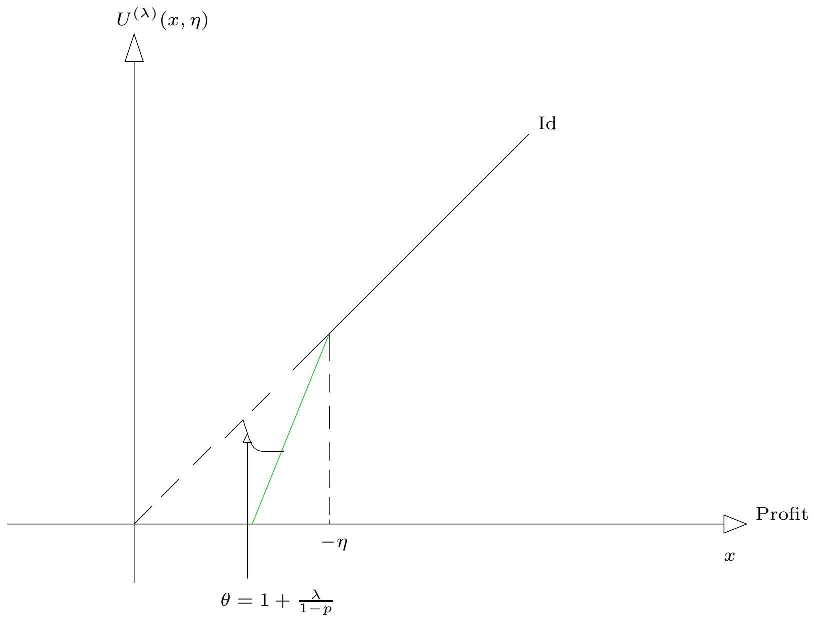

Figure 1: Utility function attached to CVaR

The Conditional Value-at-Risk constraint on $-J\big(\mathsf{a},\xi\big)$ is

$$\sigma(\mathsf{a})\mathsf{CVaR}_p\big(-N\big) - \mu(\mathsf{a}) \leq \gamma\,.$$

The portfolio maximization problem subject to Conditional Value-at-Risk risk constraint is:

$$\sup_{\mathsf{a}\in[0,1]} \mu(\mathsf{a}) \tag{4a}$$

$$\sigma(\mathsf{a})\mathsf{CVaR}_p\big(-N\big) - \mu(\mathsf{a}) \leq \gamma\,. \tag{4b}$$

By duality we find solutions of problem (4):

$$\mathsf{a}^\sharp = \frac{M + \Sigma\mathsf{CVaR}_p(-N) - \gamma}{M - \xi^0 + \Sigma\mathsf{CVaR}_p(-N)} \quad \text{and} \quad \lambda^\sharp = \frac{1}{1 + \frac{\Sigma\mathsf{CVaR}_p(-N)}{M-\xi^0}}\,.$$

p=0.95				p=0.99			
γ	$\mathsf{a}^\sharp$	$\eta^\sharp$	θ	γ	$\mathsf{a}^\sharp$	$\eta^\sharp$	θ
-630 €	0	-735	6.6	-496 €	0	-565.1	22
-772.5 €	0.36	-839.5	6.6	-772.5 €	0.47	-785.9	22
-978.5 €	0.87	-991.9	6.6	-978.5 €	0.84	-955.6	22
-1030 €	1	-1030	6.6	-1030 €	1	-1030	22

Table 3: Loss aversion parameter with confidence levels p=0.95 and p=0.99

Hence the optimal value of η is $\eta^\sharp = \sigma(\mathsf{a}^\sharp)\mathsf{VaR}_{1-p}(N) - \mu(\mathsf{a}^\sharp)$.

For two confidence levels $p = 0.95$ and $p = 0.99$, we exhibit the loss aversion parameter in Table 3. This latter takes high values, well above the empirical findings (median value of 2.25 in Kahneman and Tversky (1992)).

4 Economic discussion

Our main result establishes a connection between some risk measures and parameterized families of multi-attribute "utility functions". We hope to be able to "read" some properties of the risk measure from these latter. For instance, focusing only on the x argument in utility functions of Table 2, we notice that the variance risk measure is associated to quadratic utility functions. Now, the latter are well known for their poor economic qualities (see Gollier (2001)), such as exhibiting risk aversion increasing with wealth,

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for instance. In the CVaR risk constraint case, the corresponding utility function is interpreted as loss aversion utility function *à la* Kahneman and Tversky (see Kahneman and Tversky (1992)).

The role of the variable η can also be discussed. It is known that the optimal η is the expectation of the optimal profit in the variance case, and the Value-at-Risk in the Conditional Value-at-Risk. Our formalism amounts to attributing a cost (disutility) to such a variable η when it is let loose, not necessarily fixed at its optimal value. In the Optimized Certainty Equivalent case the optimal η gives the optimal allocation between η consumption and $(J - \eta)$ investment.

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A Appendix: proof of the main result

We shall show that (2) is equivalent to

$$\sup_{\mathsf{a}\in\mathbb{A}} \sup_{\eta\in\mathbb{R}} \inf_{\lambda\in\mathbb{R}_+} \mathbb{E}\Big[J(\mathsf{a},\xi) - \lambda\Big(\rho\big(-J(\mathsf{a},\xi),\eta\big) - \gamma\Big)\Big] \,.$$

We suppose that all assumptions of Theorem 2 are satisfied.

Equivalent Lagrangian formulation

As is well known, the Lagrangian associated to maximization problem (2) is

$$L(\mathsf{a},\lambda) := \mathbb{E}\big[J(\mathsf{a},\xi)\big] - \lambda\Big(\mathcal{R}_\rho\big(-J(\mathsf{a},\xi)\big) - \gamma\Big) \,,$$

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where $\lambda \in \mathbb{R}_+$ is a Lagrange multiplier, and we have (2) $\iff \sup_{\mathsf{a}\in\mathbb{A}} \inf_{\lambda\geq 0} L(\mathsf{a},\lambda)$.

We have:

$$\begin{aligned}
L(\mathsf{a},\lambda) &= \mathbb{E}\big[J(\mathsf{a},\xi)\big] - \lambda\Big(\inf_{\eta\in\mathbb{R}} \mathbb{E}\Big[\rho\big(-J(\mathsf{a},\xi),\eta\big)\Big] - \gamma\Big) \text{ by (1)} \\
&= \mathbb{E}\big[J(\mathsf{a},\xi)\big] - \lambda\Big(-\sup_{\eta\in\mathbb{R}} \mathbb{E}\Big[-\rho\big(-J(\mathsf{a},\xi),\eta\big)\Big] - \gamma\Big) \\
&= \mathbb{E}\big[J(\mathsf{a},\xi)\big] + \lambda \sup_{\eta\in\mathbb{R}} \mathbb{E}\Big[-\rho\big(-J(\mathsf{a},\xi),\eta\big)\Big] + \lambda\gamma \\
&= \sup_{\eta\in\mathbb{R}} \Big(\mathbb{E}\big[J(\mathsf{a},\xi)\big] + \lambda\mathbb{E}\Big[-\rho\big(-J(\mathsf{a},\xi),\eta\big)\Big] + \lambda\gamma\Big), \\
&\quad \text{because } \lambda \geq 0 \text{ and } \mathbb{E}\big[J(\mathsf{a},\xi)\big] \text{ does not depend upon } \lambda \\
&= \sup_{\eta\in\mathbb{R}} \mathbb{E}\Big[J(\mathsf{a},\xi) - \lambda\rho\big(-J(\mathsf{a},\xi),\eta\big) + \lambda\gamma\Big].
\end{aligned}$$

Since (2) $\iff \sup_{\mathsf{a}\in\mathbb{A}} \inf_{\lambda\geq 0} L(\mathsf{a},\lambda)$, it follows that

$$(2) \Leftrightarrow \sup_{\mathsf{a}\in\mathbb{A}} \inf_{\lambda\in\mathbb{R}_+} \sup_{\eta\in\mathbb{R}} \mathbb{E}\Big[J(\mathsf{a},\xi) - \lambda\rho\big(-J(\mathsf{a},\xi),\eta\big) + \lambda\gamma\Big]. \tag{5}$$

We now show that we can exchange $\inf_{\lambda\in\mathbb{R}_+}$ and $\sup_{\eta\in\mathbb{R}}$.

Exchanging $\inf_{\lambda\in\mathbb{R}_+}$ and $\sup_{\eta\in\mathbb{R}}$

Let $\mathsf{a} \in \mathbb{A}$ be fixed. Define $\Psi_{\mathsf{a}} : \mathbb{R}^2 \to \mathbb{R}$ by

$$\Psi_{\mathsf{a}}(\lambda,\eta) := \mathbb{E}\Big[J(\mathsf{a},\xi) - \lambda\rho\big(-J(\mathsf{a},\xi),\eta\big) + \lambda\gamma\Big]. \tag{6}$$

We can exchange $\inf_{\lambda\in\mathbb{R}_+}$ and $\sup_{\eta\in\mathbb{R}}$ in (5) by the two following Lemmas.

Lemma 3 *Let* $\mathsf{a} \in \mathbb{A}$ *be fixed. If* $\gamma < \mathcal{R}_\rho\big(-J(\mathsf{a},\xi)\big)$, *then* $\inf_{\lambda\in\mathbb{R}_+} \sup_{\eta\in\mathbb{R}} \Psi_{\mathsf{a}}(\lambda,\eta) = \sup_{\eta\in\mathbb{R}} \inf_{\lambda\in\mathbb{R}_+} \Psi_{\mathsf{a}}(\lambda,\eta) = -\infty$.

Proof.

By (1) and $\lambda \geq 0$, we have $\sup_{\eta\in\mathbb{R}} \Psi_{\mathtt{a}}(\lambda,\eta) = \mathbb{E}\big[J(\mathtt{a},\xi)\big] - \lambda\big(\mathcal{R}_\rho\big(-J(\mathtt{a},\xi)\big) - \gamma\big)$.
Thus $\inf_{\lambda\in\mathbb{R}_+} \sup_{\eta\in\mathbb{R}} \Psi_{\mathtt{a}}(\lambda,\eta) = -\infty$, since $\gamma < \mathcal{R}_\rho\big(-J(\mathtt{a},\xi)\big)$.

We always have $\sup_{\eta\in\mathbb{R}} \inf_{\lambda\in\mathbb{R}_+} \Psi_{\mathtt{a}}(\lambda,\eta) \leq \inf_{\lambda\in\mathbb{R}_+} \sup_{\eta\in\mathbb{R}} \Psi_{\mathtt{a}}(\lambda,\eta)$.

It holds that $\sup_{\eta\in\mathbb{R}} \inf_{\lambda\in\mathbb{R}_+} \Psi_{\mathtt{a}}(\lambda,\eta) = -\infty$.

□

The proof of the following Lemma is based on the Theorem hereafter (see (Barbu and Precupanu, 1986, Chap. 2. Corollary 3.8)).

Theorem 4 *Assume that $\Phi : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ is a convex-concave* [9] *and l.s.c.-u.s.c.* [10] *mapping. Assume that Y and Z are two closed convex subsets of $\mathbb{R}^p$ and $\mathbb{R}^q$ respectively, and that there exists $(y^*, z^*) \in Y \times Z$ such that*

$$\begin{cases} \Phi(y^*, z) & \to -\infty, \quad \textit{when } \|z\| \to +\infty \textit{ and } z \in Z \\ \Phi(y, z^*) & \to +\infty, \quad \textit{when } \|y\| \to +\infty \textit{ and } y \in Y\,. \end{cases}$$

Then Φ admits a saddle point $(\bar{y}, \bar{z}) \in Y \times Z$:

$$\Phi(\bar{y}, z) \leq \Phi(\bar{y}, \bar{z}) \leq \Phi(y, \bar{z}) \quad \forall (y,z) \in Y \times Z\,.$$

Lemma 5 *If $\gamma \geq \mathcal{R}_\rho\big(-J(\mathtt{a},\xi)\big)$ then $\Psi_{\mathtt{a}}$ defined by (6) admits a saddle point in $\mathbb{R}_+ \times \mathbb{R}$ and thus $\sup_{\eta\in\mathbb{R}} \inf_{\lambda\in\mathbb{R}_+} \Psi_{\mathtt{a}}(\lambda,\eta) = \inf_{\lambda\in\mathbb{R}_+} \sup_{\eta\in\mathbb{R}} \Psi_{\mathtt{a}}(\lambda,\eta)$.*

Proof.

Let $\eta^\star$ be such that $\mathcal{R}_\rho\big(-J(\mathtt{a},\xi)\big) = \mathbb{E}\big[\rho\big(-J(\mathtt{a},\xi),\eta^\star\big)\big]$. We have indeed supposed that the infimum in (1) is achieved for any $X = -J(\mathtt{a},\xi)$ when $\mathtt{a}$ varies in $\mathbb{A}$. We distinguish two cases.

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[9]convex with respect to its first argument, and concave with respect to its second argument.

[10]lower semicontinuous with respect to its first argument and upper semicontinuous with respect to its second argument.

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a. If $\gamma = \mathcal{R}_\rho\big(-J(\mathsf{a},\xi)\big)$, any $(\lambda,\eta^\star)$ is a saddle point because (6) gives $\Psi_{\mathsf{a}}(\lambda,\eta^\star) = \mathbb{E}\big[J(\mathsf{a},\xi)\big]$.

b. Assume now that $\gamma > \mathcal{R}_\rho\big(-J(\mathsf{a},\xi)\big)$, and let us check the conditions of existence of a saddle point in Theorem 4.

The function $\Psi_{\mathsf{a}}(\lambda,\eta) = \mathbb{E}\Big[J(\mathsf{a},\xi)\Big] - \lambda\mathbb{E}\Big[\rho\big(-J(\mathsf{a},\xi),\eta\big) - \gamma\Big]$ is

- linear with respect to λ and thus convex in λ;
- concave with respect to η (the function : $\eta \mapsto -\rho\big(-J(\mathsf{a},\xi),\eta\big)$ is concave, $\lambda \geq 0$ and the expectation operator preserves concavity).

Now, by assumption ρ and $L_\rho(\Omega,\mathcal{F},\mathbb{P})$ satisfy assumption H2 with $J(\mathsf{a},\xi) \in L_\rho(\Omega,\mathcal{F},\mathbb{P})$, we have

- $\eta \mapsto \mathbb{E}\Big[\rho\big(-J(\mathsf{a},\xi),\eta\big) - \gamma\Big]$ is continuous;
- $\lim_{\eta\to+\infty} \mathbb{E}\Big[\rho\big(-J(\mathsf{a},\xi),\eta\big) - \gamma\Big] = +\infty$.

Thus, the function Ψ_{a} is convex-concave, l.s.c.-u.s.c. and satisfies

$$\Psi_{\mathsf{a}}(\lambda,\eta) \to -\infty, \text{ when } \eta \to +\infty \text{ for any } \lambda > 0\,.$$

Since $\gamma > \mathcal{R}_\rho\big(-J(\mathsf{a},\xi)\big) = \mathbb{E}\big[\rho\big(-J(\mathsf{a},\xi),\eta^\star\big)\big]$, we have

$$\Psi_{\mathsf{a}}(\lambda,\eta^\star) = \mathbb{E}\big[J(\mathsf{a},\xi)\big] + \lambda\Big(\gamma - \mathcal{R}_\rho\big(-J(\mathsf{a},\xi)\big)\Big) \to +\infty, \text{ when } \lambda \to +\infty\ .$$

Hence, the function Ψ_{a} admits a saddle point in $\mathbb{R}_+ \times \mathbb{R}$.

□

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