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# Rational Impatience?

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**Abstract**

This paper introduces a life-cycle model where impatience, instead of being driven by an exogenous discount function, results from the combination of risk aversion and mortality risks. Opting for such a formulation provides novel views on the impact of longevity extension on welfare, saving behavior and capital accumulation. In particular, we show that longevity extension may have much larger impacts on capital accumulation and equilibrium rate of interest than is usually thought. Moreover, we show that the adherence to the additive life cycle model introduced by Yaari (1965) may lead to significantly overestimating the welfare gains due to mortality risk reduction.

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JEL-classification: J10, D91.

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# 1 Introduction

It has long been recognized that uncertainty, in particular lifetime uncertainty, contributes to human impatience. However, economic theory has developed around the consensus that uncertainty could not be the main cause of human impatience. Early economists, such as Jevons, Marshall, Böhm-Bawerk, Pigou or Fisher emphasized the "irrationality" of human impatience driven by "the incompleteness of the imaginations" (Böhm-Bawerk, 1891), "a lack of self control" (Marshall, 1890, Fisher, 1930) or "a faulty telescopic faculty" (Pigou, 1920)[1]. Moreover, this "defect in will" (Böhm-Bawerk, 1891) was claimed to be stronger among "savages", "uneducated" or "uncivilized" people[2]. Modern economic theory differs by its terminology, but not that much by its underlying conceptualization of human impatience. Time preference is still viewed as an intrinsic element of preferences, which is not to be discussed *(de gustibus non est disputandum)*. Human beings, possibly as a result of biological evolution[3], are impatient by nature. And some of them (e.g. men and non-whites) tend to be, for unexplained reasons, more impatient than others (women and whites)[4].

The present paper challenges this view by introducing a life cycle model, where impatience results from uncertainty and risk aversion. The suggested model, which remains in the standard expected utility framework, involves abandoning the assumption that agents have pure time preferences (i.e.: a non constant subjective discount function) but allowing for temporal risk aversion. It is worth emphasizing that what is suggested is not a negation of human impatience, but an alternative theory for human impatience. In short, to the view that human impatience is unrelated to agents' risk aversion and mainly

[1]Peart (2000) provides an excellent discussion on how early economists used to view time preferences.

[2]For Jevons (1871): "The untutored savage, like the child, is wholly occupied with the pleasures and the troubles of the moment; the morrow is dimly felt; the limit of his horizon is but a few days off."

According to Böhm-Bawerk (1891): "We systematically underestimate future wants, and the goods which are to satisfy them. Of the fact itself there can be no doubt; but, of course, in particular nations, at various stages of life, in different individuals, the phenomenon makes its appearance in very varying degree. We find it most frankly expressed in children and savages."

In the same line, Fisher (1930) argues that: "In the case of primitive races, children, and other uninstructed groups in society, the future is seldom considered in its true proportions."

[3]See for example Rogers (1994) or Robson and Samuelson (2008) for discussions about the evolutionary fundations of time preferences.

[4]See for example the empirical results of Lawrance (1991) and Warner and Pleeter (2001) that are discussed in Section 6.

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driven by an exogenous parameter of preferences, is opposed the view whereby human impatience results from risk aversion and mortality risk.

This alternative way of modelling life cycle preferences has several attractive features. Although impatience is “endogenized”, all forms of impatience can be obtained. Even if mortality rates are small, lifetime uncertainty may generate large rates of time discounting. In fact, it will be formally shown that, as long as we consider life-cycle behaviors under a given (non-degenerate) mortality pattern, this novel formulation can reproduce (up to infinitesimally small differences) all the predictions of Yaari’s (1965) standard life cycle model, which assumes additively separable utilities and exogenous time preferences. Since, the validity of Yaari’s model has never been tested by empirical studies using heterogeneity in mortality across agents, it follows that the alternative approach I suggest has, up to now, at least as much empirical support as Yaari’s model.

Nonetheless, this alternative approach affords new insights as to the impact of uncertain lifetime on intertemporal choice. Individuals with identical preferences, but with different mortality, may exhibit very different degrees of impatience. The discussion will explain why, contrary to conventional wisdom, mortality decline may generate very significant changes in human impatience. This is of crucial importance for discussing the impact of longevity extension on capital accumulation and equilibrium interest rate. In particular, we find that longevity extension may have had much larger impacts on capital accumulation and equilibrium rate of interest than is usually thought.

This novel approach also suggests a revision of the literature on endogenous mortality. Temporal risk aversion enhances risk aversion with respect to life duration and, therefore, increases the willingness to pay for reducing mortality at young ages, relatively to the willingness to pay for reducing mortality at old ages. The value of life is thus found to decrease more rapidly (or to increase less rapidly) with age. This turns out to be significant for the evaluation of welfare gains associated with longevity extension. Simulations based on 1970 and 2000 U.S. demographic data show that using this novel model, instead of the standard additive life cycle model, would lead to dividing by approximately two the estimates of the value of longevity gains.

Abandoning Yaari’s model for a model that assumes temporal risk aversion suggests

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therefore a significant reformulation of the economics of ageing. More generally this paper emphasizes that, because of the longitudinal nature of human life, temporal risk aversion is a key element of individual preferences. The suggested model permits to account for temporal risk aversion, in a simplest way, without abandoning expected utility nor assuming time inconsistencies.

The structure of the paper is as follows. In Section 2 we return to the standard model of intertemporal choice under uncertain lifetime, due to Yaari (1965), and introduce the alternative "time neutral" model. In Section 3 and 4, we discuss the fundamental properties (time preferences and temporal risk aversion) that distinguish both models. In Section 5, we consider life cycle behavior under a given mortality pattern. Section 6 provides theoretical results on the impact of mortality changes. It will be complemented by Section 7 that provides illustrations based on historical demographic data. Section 8 explores issues related to the value of life. Section 9 discusses the main conclusions that can be drawn from the present paper. The appendix contains proofs of the results and Section A that looks at technical difficulties that appear when working with the non-additive model and suggests ways to deal with them.

## 2 Two Models of Individual Preferences

In this paper, we will view a "life" as being a pair $(c, T)$, where $c$ is an infinitely long consumption profile, and $T$ a (finite) length of life. The set of possible lives will, therefore, be:

$$X = C^{\infty}(\mathbb{R}^+, \mathbb{R}^+) \times \mathbb{R}^+$$

This representation might seem odd at first sight, since consumption has not been constrained to zero after death. Instead, consumption after death can theoretically take any non-negative value. However, as the models that will be considered assume that people do not care for consumption after death, my results will be formally equivalent to what we would obtain if consumption was constrained to zero after death. The paper is about preferences that make it possible to rank lotteries whose outcomes are in $X$.

## 2.1 Additive model (Yaari's model)

The most common approach to deal lifetime uncertainty involves stating that individuals maximize an expected utility of the form:

$$\int_0^{+\infty} s(t)\alpha(t)u(c(t))dt \tag{1}$$

where $c(t)$ is the consumption at age $t$, $s(t)$ the probability of being alive at age $t$ and $\alpha(t)$ the subjective discount function. This model, which was introduced by Yaari (1965), has become the model of reference for discussing the economic impact of mortality changes, as in Blanchard (1985), Boucekkine, de la Croix and Licandro (2002) or Sheshinski (2007). It is also used for providing policy recommendations on major social issues, such as the optimal level of health spending or pollution regulation[5].

For our purpose it proves useful to come back on the path followed by Yaari to derive this representation. At the origin of Yaari's model is the fundamental assumption that preferences over lotteries involving lives of different lengths can be modeled within the standard expected utility framework using an additively separable Bernoulli utility function[6]. More precisely, we will say throughout the paper that:

**Definition 1** *Preferences are additive (or of Yaari's type) if they are represented within the expected utility framework with a Bernoulli utility function of the form:*

$$U^{add}(c,T) = \int_0^{T} \alpha(t)u(c(t))dt \tag{2}$$

*with $\alpha > 0$ and $u' > 0$.*

In order to clarify the link between the above definition and the formulation in terms of expected utility shown in (1), consider the case of lotteries that are characterized by a

[5] For example in the recent contributions of Murphy and Topel (2006) and Hall and Jones (2007) as well as in EPA (1997).

[6] We use the terms "Bernoulli utility function" to avoid any possible confusion between "utility function" and "expected utility function". Bernoulli utility functions are defined over $X$. A Bernoulli utility function composed with the expectation operator, gives an expected utility function, defined over the set of lotteries with outcomes in $X$.



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given consumption profile, $c$, and a lottery on life duration. The distribution of the age of death that is associated with the lottery on life duration is denoted $d(T)$.

The expected utility associated with such a lottery is

$$EU^{add}(c) = \int_0^{+\infty} d(T)U^{add}(c,T)dT$$

Noting $s(t) = \int_t^{\infty} d(T)dt$ the the survival function, and integrating by parts the above equation gives:

$$EU^{add}(c) = \int_0^{+\infty} s(t)\frac{\partial U^{add}(c,T)}{\partial T}|_{T=t}dt = \int_0^{+\infty} s(t)\alpha(t)u(c(t))dt$$

which corresponds to (1). The popular formulation of Yaari's model given in (1) is therefore nothing other than a straight application of specification (2) to the case where uncertainty bears on life duration.

## 2.2 Time neutral model

As with any economic model, there would be no difficulty in arguing that the additive model is, by its structure, too restrictive to provide a faithful representation of human rationality. A "safe criticism" would then involve pointing to some particular assumption and relaxing it to obtain a more general and less structured representation. This drift towards more generality and less structure would, however, magnify identification problems. Moreover, it would fail to question the necessity to assume the existence of pure time preferences.

The present paper follows a different kind of argument, challenging the additive model with a model having the same level of complexity, when measured in terms of degrees of freedom. More precisely we will consider time neutral preferences, defined as follows:

**Definition 2** *Preferences are "time neutral" if they are represented within the expected utility framework with a Bernoulli utility function of the form:*

$$U^{tn}(c,T) = \phi\left(\int_0^T u(c(t))dt\right) \tag{3}$$

*with* $\phi' > 0$ *and* $u' > 0$.

The function $\phi$ that enters the time neutral utility function has no impact on ordinal preferences, but determines individuals' attitude towards risk. In particular as is known from Kihlstrom and Mirman (1974), increasing the concavity of the function $\phi$ involves increasing individuals' risk aversion. The class of time neutral utility functions is therefore a natural candidate to analyze the role of risk aversion within the expected utility framework and appeared as such in Kihlstrom and Mirman (1974). Epstein and Zin (1989) and Epstein (1992) argued, however, that working with such utility functions is problematic because in a dynamic time consistent approach, they would generate "unreasonable" forms of history dependence, with the distant past mattering as much as the closer past[7]. The critique relies therefore on the presupposition that consumption in the remote past should be given a smaller weight than consumption in the closer past; in other words, that people have time preferences. It is precisely this presupposition that the time neutral model challenges. In a fully consistent manner, the time neutral model assumes that -in absence of uncertainty- individuals would exhibit no time discounting for the assessment of the future or for the assessment of the past. This might seem a surprising assumption, but a key result of the paper (Proposition 5) shows that this apparently heterodox model is able to reproduce all forms of time discounting when mortality risk is taken into account and may look very similar to the additive model which assumes history independence. Denigrating the time neutral model, on the ground that it rules out pure time preferences, or that it suggests undiscounted forms of history dependence, would be no different than authoritatively stating that the existence of time preferences may not be debated.

The additive and time neutral models have many features in common. Indeed, both formulations can be seen as diverging extensions of the simplest case where preferences are represented by the Bernoulli utility function :

$$U_0(c,T) = \int_0^T u(c(t))dt$$

[7] The questions of time consistency and history dependence are discussed in greater details in Appendix A.



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with $u' > 0$. Preferences represented by $U_0$ are both additive and time neutral. As we will see in the following two sections, the additive preferences extend the above formulation by introducing time preferences, while the time neutral preferences introduce temporal risk aversion.

# 3 Time Preferences

The concept of pure time preference is an ordinal concept representing impatience in a context without uncertainty. It can be summarized by the rate of time preference, which in the continuous time framework is usually defined as follows:

**Definition 3** *For any length of life $T$, any time $t < T$ and any consumption path $c$, the rate of time preference is defined by:*

$$\rho(c,t,T) = -\frac{d}{dt}\left(\log(\frac{\partial U(c,T)}{\partial c(t)})\right)|_{\frac{dc(t)}{dt}=0}$$

The notation $\rho(c,t,T)$ is used to stress that, in general, the rate of time preference can depend on $c$, $t$ and $T$. However, with the preferences we are considering, the rate of time preference at time $t$ depends only on $t$. Indeed:

**Proposition 1** *In the additive model, the rate of time preference is given by:*

$$\rho^{add}(c,t,T) = \frac{-\alpha'(t)}{\alpha(t)} \tag{4}$$

*In the time neutral model, it is given by:*

$$\rho^{tn}(c,t,T) = 0 \tag{5}$$

**Proof.** From (2) we derive $\frac{\partial U^{add}(c,T)}{\partial c(t)} = \alpha(t)u'(c(t))$, which implies (4). From (3) we derive $\frac{\partial U^{tn}(c,T)}{\partial c(t)} = u'(c(t))\phi'\left(\int_0^T u(c(\tau))d\tau\right)$, which implies (5). ■

This proposition emphasizes a fundamental difference between the two models. In the additive case, people can have pure time preferences while the time neutral model excludes this possibility.



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Nevertheless, as will be explained in Section 5, agents with time neutral preferences may exhibit any kind of (positive) impatience when confronted with lifetime uncertainty. The additive and time neutral models therefore suggest two very different theories for human impatience. In the standard approach, supported by the additive model, impatience is inherent to human nature and would exist even in the absence of uncertainty. On the other hand, the time neutral model, takes for granted that risk aversion and mortality are inherent to human nature. It then suggests that human impatience may exclusively result from a rational response to uncertainty and, in particular, to the risk of death.

The interest of each interpretation might be debated on philosophical grounds and, in particular, in relation with Heidegger's *Being and Time* (Heidegger, 1927). Instead, I will focus on pragmatic matters and show why opting for one or the other interpretation may well be crucial for concrete social issues, and especially in order to grasp the impact of mortality changes.

# 4 Temporal Risk Aversion

Temporal risk aversion is an adaptation of the general notion of "multivariate risk aversion" of Richard (1975) to the case of intertemporal choice under uncertainty. It is used in Ahn (1989) and van der Ploeg (1993). To obtain an intuitive notion of what temporal risk aversion is, consider the simple case of an individual who lives over two periods. An individual is temporally risk averse if for any $c_1 < C_1$ and $c_2 < C_2$ he prefers the lottery that gives $(c_1, C_2)$ or $(C_1, c_2)$ with equal probability to the lottery that gives $(c_1, c_2)$ or $(C_1, C_2)$ with equal probability. To quote Richard (1975), a temporally risk averse consumer prefers getting some of the "best" and some the "worst", to taking a chance on all of the "best" or all of the "worst". Richard (1975) shows that temporal risk aversion is related to the cross derivative of the Bernoulli utility function. In continuous time, temporal risk aversion can be defined as follows:

**Definition 4** *An individual exhibits:*

- *temporal risk aversion if* $\frac{\partial^2 U(c,T)}{\partial c(t_1)\partial c(t_2)} < 0$ *for all* $t_1,\ t_2 < T$ *with* $t_1 \neq t_2$.

- *temporal risk neutrality if* $\frac{\partial^2 U(c,T)}{\partial c(t_1)\partial c(t_2)} = 0$ *for all* $t_1,\ t_2 < T$ *with* $t_1 \neq t_2$.



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- *temporal risk proneness if* $\frac{\partial^2 U(c,T)}{\partial c(t_1)\partial c(t_2)} > 0$ *for all* $t_1, t_2 < T$ *with* $t_1 \neq t_2$.

It is then fairly simple to note that:

**Proposition 2** *Agents with additive preferences exhibit temporal risk neutrality. Agents with time neutral preferences exhibit temporal risk aversion if* $\phi$ *is concave, temporal risk neutrality if* $\phi$ *is linear, and temporal risk proneness if* $\phi$ *is convex.*

**Proof.** In the additive case $\frac{\partial^{add} U(c,T)}{\partial c(t_1)} = \alpha(t_1)u(c(t_1))$ and $\frac{\partial^2 U^{add}(c,T)}{\partial c(t_1)\partial c(t_2)} = 0$. In the time neutral case, $\frac{\partial^{tn} U(c,T)}{\partial c(t_1)} = u'(c(t_1))\phi'\left(\int_0^T u(c(t)dt\right)$ which implies that $\frac{\partial^2 U^{tn}(c,T)}{\partial c(t_1)\partial c(t_2)} = u'(c(t_1))u'(c(t_2))\phi''\left(\int_0^T u(c(t)dt\right)$. ■

This is the second fundamental difference between the two models. The additive model rules out temporal risk aversion while the time neutral model allows for it.

Temporal risk aversion matters when considering attitude towards risks that have durable consequences, since risks of this kind affect individuals over several periods of time. This is for example the case for risks related to wealth investment, since current wealth affects individuals' consumption in future periods. This explains why temporal risk aversion plays a central role in Ahn (1989), van der Ploeg (1993) or Bommier and Rochet (2006) who study optimal saving and portfolio choices in models where the horizon is infinite or known with certainty.

A risk that indisputably has longlasting consequences is that of mortality. Indeed, the risk of dying at time $t$ is nothing other than the risk of being put in the "death state" for all times subsequent to $t$. Thus, we expect temporal risk aversion to deeply affect rational attitudes towards the risk of death.

Given the obvious durability of death, it is intriguing that the economic literature that deals with human mortality focuses on the additive specification which assumes temporal risk neutrality. Several papers did discuss the role of "risk aversion" in the context of uncertain lifetime but, in reality, they only considered the additive specification and discussed the role of the curvature of the instantaneous utility function $u$, which has no impact on temporal risk aversion. We know, however, from the fundamental contribution of Kihlstrom and Mirman (1974) that, strictly speaking, increasing individuals' risk aversion does not involve changing the curvature of $u$, but taking a concave transformation



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of the intertemporal utility function. This is what is done with the time neutral model, where temporal risk aversion arises naturally.

To end this section, let us remark that the curvature of the function $\phi$, which generates temporal risk aversion in the time neutral model, can be related to individuals' risk aversion with respect to life duration. Imagine the (fictive) case of individuals who have to choose between lotteries involving a single constant consumption path, but different life durations. Consumption being the same in all outcomes, these individuals only have to rank lotteries on a single dimensional variable: life duration. Their choices are then governed by their risk aversion with respect to life duration which can be measured by a standard Arrow-Pratt coefficient:

$$-\frac{\frac{\partial^2 U(c,t)}{\partial T^2}}{\frac{\partial U(c,t)}{\partial T}}$$

It is a matter of simple calculation to show that, with the time neutral model, this coefficient equals $u(c)\frac{-\phi''(Tu(c))}{\phi'(Tu(c))}$. Considering such simple lotteries may therefore help to understand the economic meaning of assumptions that might be made about $\phi$. For example, assuming $\phi$ is concave (temporal risk aversion) would involve assuming positive risk aversion with respect to length of life. Assuming $-\frac{\phi''}{\phi'}$ is decreasing would involve assuming decreasing risk aversion with respect to length of life.

# 5 Life-Cycle Behavior Under an Exogenous Mortality Pattern

In this section, I consider the case where individuals face an exogenous mortality pattern. Throughout the section, mortality will be described either by the distribution of the age at death $d(t)$, by the survival function $s(t) = 1 - \int_0^t d(\tau)d\tau$ or by the hazard rate of death $\mu(t) = -\frac{s'(t)}{s(t)} = \frac{d(t)}{s(t)}$. Even though, in this section, I do not compare what is obtained with different mortality patterns (this is the purpose of Section 6), I will introduce an index $\mu$ whenever I want to stress that an object depends on the mortality pattern.

Rational individuals with a Bernoulli utility function $U(c, T)$ who face this exogenous mortality pattern have preferences on consumption profiles given by the following expected



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utility:

$$E_\mu U(c) \equiv \int_0^{+\infty} d(T)U(c,T)dT \tag{6}$$

A crucial point is that although the time neutral representation assumes that people have no pure time preferences, temporal risk aversion, together with uncertainty on the length of life, generate non-trivial time discounting. The intuition, stressed in Bommier (2006), is that if people cannot avoid the risk of dying young, they should prefer consuming early in life in order to avoid the very low level of lifetime utility which would result from simultaneously having a short life and low levels of instantaneous consumption. This intuition can be formalized by looking at the rate of discount at time $t$.

**Definition 5** *For any consumption profile c, the rate of discount at time t is defined by:*

$$RD_\mu(c,t) = -\frac{d}{dt}(\log(\frac{\partial E_\mu U}{\partial c(t)})|_{\frac{dc(t)}{dt}=0}$$

This extends Definition 3 to the case where the length of life is not known with certainty, but is described by an exogenous distribution. The rate of discount depends on the mortality pattern considered. Indeed:

**Proposition 3** *In the case of the additive utility function, the rate of discount is given by:*

$$RD_\mu^{add}(c,t) = \mu(t) - \frac{\alpha'(t)}{\alpha(t)} \tag{7}$$

*For the time neutral utility function, the rate of discount is given by*

$$RD_\mu^{tn}(c,t) = \mu(t) - \mu(t)\frac{\int_t^{+\infty} s(t_1)u(c(t_1))\phi''(\int_0^{t_1} u(c(\tau))d\tau)dt_1}{\int_t^{+\infty} d(t_1)\phi'(\int_0^{t_1} u(c(\tau))d\tau)dt_1} \tag{8}$$

**Proof.** See Appendix B. ■

In the additive case, the rate of discount is the sum of the mortality rate and the rate of time preference, as is well known. An implication of this standard result is that, although mortality is a risk, the impact of mortality on impatience is found to be independent of individuals' risk aversion. This is because of the assumption of temporal risk neutrality. In the time neutral case, even though individuals have no pure time preferences, in the



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typical case where $u$ is positive, $\phi$ strictly concave and mortality greater than zero, the rate of discount is greater than the hazard rate of death. Temporal risk aversion together with lifetime uncertainty does generate impatience. The intuition for this result will be discussed at length in the following section, after Proposition 6, once comparative statics related to risk aversion (Proposition 4) and mortality rates (Proposition 6) will be provided.

Considering time neutral preferences makes it possible to highlight a relation between risk aversion and impatience. The more concave the function $\phi$, the greater risk aversion. But, increasing the concavity of $\phi$ also has simple consequences on the rate of time discounting.

**Proposition 4** *For a given instantaneous utility function $u$, a given consumption profile $c$ such that $u(c(t)) \geq 0$ for all $t$, and a given mortality pattern, the greater the concavity*[8] *of the function $\phi$ the greater $RD_{\mu}^{tn}(c,t)$.*

**Proof.** See Appendix C. ■

Hence, the greater risk aversion, the greater human impatience[9]. The intuition for this result will also be given in the following section, after Proposition 6.

Once we found that impatience may result from temporal risk aversion and mortality, it is natural to wonder about the impatience patterns that can be generated. Bommier (2006) considers realistic mortality rates and exponential or hyperbolic functions $\phi$. This leads to discount functions that are approximately exponential or hyperbolic. In fact, by adjusting the functions $\phi$ and $u$, any decreasing discount function can be generated. Indeed, taking matters further, we will see in the following proposition that for any given mortality pattern, any additive preferences with non-negative rates of time preference can be obtained as the limit of time neutral preferences.

[8] A real function $f$ is said to be at least as concave as a function $g$ if

$$-\frac{f''(x)}{f'(x)} \geq -\frac{g''(x)}{g'(x)} \text{ for all } x$$

[9] This relation may be extended to a broader class of models. In particular, the proof of Proposition 4 would also work with specifications that would include both temporal risk aversion and time preferences.

**Proposition 5** *Assume that individuals face an exogenous mortality pattern and that the hazard rate of death is always positive. For any additive preferences that generate positive rates of discount*[10]*, there exists a sequence of time neutral preferences such that the corresponding expected utility functions converge (weakly and up to positive affine transformations*[11]*) towards the expected utility function obtained from the additive representation.*

**Proof.** See appendix D. ■

An implication of Proposition 5 is that, when modeling life-cycle behavior under a given mortality pattern, all the predictions of the additive models with non-negative rates of time preference can be reproduced, up to infinitesimally small differences, by time neutral models. Thus, there is no chance to infer from micro data on individual behavior that the additive formulation with non-negative rates of time preference is better than the time neutral one, unless heterogeneity in mortality across agents is considered. This point is particularly important since, to my knowledge, the validity of the additive assumption has never been challenged by empirical studies that consider heterogeneity in mortality. In other words, Proposition 5 tells us that, up to now, there is no piece of empirical evidence that can afford more credit to the additive model than to the time neutral one.

Interestingly enough, Proposition 5 is not symmetrical. Indeed, from equation (20), in appendix, we see that the expected utility function that represents the preferences over consumption profiles in the time neutral is, in general, not additive. Thus, it cannot be obtained as the limit of a sequence of additive expected utility functions. Although the additive and time neutral models have the same degree of complexity, the time neutral models provide a wider class of preferences with positive rates of discount than the additive models, when a given non-degenerate mortality pattern is considered. This is because preferences over consumption profiles under an exogenous mortality pattern do not depend on the rate of substitution between consumption and the length of life in the additive model[12], while they do depend on it in the time neutral model.

[10] In the additive case, the age-specific rates of discount are given by (7). As mortality rates are assumed to be positive, the rates of discount are positive whenever the rates of time preference (equation (4)) are non-negative.

[11] What is meant by "weak convergence up to positive affine transformations" is formalized in the proof (equations (23) and (24)).

[12] Preferences over consumption profiles provided by the expected utility function shown in equation



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# 6 The Consequences of Mortality Changes

In the previous section, we saw that there may be some similarity between the predictions of the time neutral and the additive models on life-cycle behavior under an exogenous mortality pattern. More precisely, I showed that for any given mortality pattern, with positive hazard rates of death, and any additive preferences, with non-negative rates of time preference, I could define a sequence of time neutral Bernoulli utility functions such that the corresponding expected utility functions converge towards the expected utility function obtained with the additive formulation. I could not, however, find a sequence of time neutral utility functions that satisfy this property for all mortality patterns. In other words, although additive and time neutral preferences may give similar predictions when a given mortality pattern is considered, they will predict, in general, contrasted effects of mortality changes.

In particular, a fundamental difference between the two models is that the rate of discount (Definition 5) will react quite differently to mortality. To stress this point, we can examine the Volterra derivative $\frac{\partial RD_{\mu}(c,t_1)}{\partial\mu(t_2)}$, which gives the effects of a change in mortality around age $t_2$ on the rate of discount at age $t_1$:

**Proposition 6** *In the additive case:*

$$\frac{\partial RD_{\mu}^{add}(c,t_1)}{\partial\mu(t_2)} = \delta(t_2 - t_1) \text{ where } \delta \text{ is the Dirac delta function}$$

*In the time neutral case:*

$$\begin{aligned}\frac{\partial RD_{\mu}^{tn}(c,t_1)}{\partial\mu(t_2)} &= \frac{1}{\mu(t_1)} RD_{\mu}^{tn}(c,t_1)\delta(t_2-t_1) \\ &+\mu(t_1)\frac{\phi'(\int_0^{t_1} u(c(\tau))d\tau)\int_{t_2}^{+\infty}\frac{s(\tau)}{s(t_1)}u(c(\tau))\phi''(\int_0^{\tau} u(c(\tau_1))d\tau_1)d\tau}{\left(\phi'(\int_0^{t_1} u(c(\tau))d\tau)+\int_{t_1}^{+\infty}\frac{s(\tau)}{s(t_1)}u(c(\tau))\phi''(\int_0^{\tau} u(c(\tau_1))d\tau_1)d\tau\right)^2}1_{(t_2>t_1)}\end{aligned} \quad (9)$$

*where $\delta$ is the Dirac delta function and $1_{(t_2>t_1)}$ a dummy that equals one if $t_2 > t_1$, and zero otherwise.*

**Proof.** See Appendix E. ■

(19) do not change if a constant is added to $u$.



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In the additive case, the result is very simple: an increase in the hazard rate of death at age $t_2$ of $\delta\mu$ causes an increase in the rate of discount at age $t_2$ of $\delta\mu$, and has no impact on the rate of discount at other ages. This is because the rate of discount is simply the sum of the hazard rate of death and an exogenous parameter.

In the time neutral case, the result is very different. In fact, there are two fundamental differences. First, an increase in the hazard rate of death at age $t_2$ affects positively and in *the same proportion* the rate of discount at age $t_2$ (first term in equation (9)). In other words, the elasticity of the rate of discount at age $t_2$ with respect to the hazard rate of death at age $t_2$ equals 1. Second, a change in the hazard rate of death at time $t_2$ will affect the rate of discount at all ages smaller than $t_2$. More precisely, if the hazard rate of death increases by $\delta\mu$ between ages $t_2$ and $t_2 + dt$ then, for all ages $t_1 < t_2$, the rate of discount will change from $RD^{tn}(t_1)$ to:

$$RD^{tn}(t_1) + \frac{\int_{t_2}^{+\infty} \frac{s(\tau)}{s(t_1)} u(c(\tau))\phi''(\int_0^{\tau} u(c(\tau_1))d\tau_1)d\tau}{\phi'(\int_0^{t_1} u(c(\tau))d\tau) + \int_{t_1}^{+\infty} \frac{s(\tau)}{s(t_1)} u(c(\tau))\phi''(\int_0^{\tau} u(c(\tau_1))d\tau_1)d\tau} RD^{tn}(t_1) \times dt\delta\mu$$

If $u$ is positive and $\phi$ strictly concave (that is, if individuals are willing to live longer and are temporally risk averse), the adjustment is negative. Thus, in that case, the time neutral model predicts that an increase in the mortality rate at age $t_2$ will have a positive impact on the rate of discount at age $t_2$ and a negative impact on the rate of discount at all ages before $t_2$.

Proposition 6 together with Propositions 3 and 4 clarify how mortality and risk aversion contribute to impatience. An intuitive interpretation of these results can be given. Mortality actually generates two kinds of risk: a risk on consumption (consumption is contingent on survival) and a risk on lifetime utility (lifetime utility is typically low in the case of an early death and high in the case of a late death). In both the additive and time neutral models, the risk on consumption affects the discount rates in the simplest way: mortality rate at age $t$ contributes additively to the rate of discount at age $t$ (this explains the first terms of equations (7) and (8) found in Proposition 3). The risk on lifetime utility has no effect in the additive model because of the underlying assumption of temporal risk neutrality. In the time neutral model, when $\phi$ is strictly concave, individuals exhibit

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temporal risk aversion. This incites them to re-allocate consumption towards young ages in order to reduce the risk on lifetime utility. Indeed, by consuming early in the life cycle, individuals avoid the low levels of lifetime utility that would result from having a short life with low levels of consumption. In other words, they see the intertemporal allocation of consumption as a way to (partially) insure themselves against the risk of death. But the need for insurance at a given age results from three parameters: (1) risk aversion, (2) the probability of incurring damage (death, in the present case) at that age and (3) the magnitude of the damage (the expected quantity of future pleasures in case of survival). The greater risk aversion the greater the need for insurance. That explains Proposition 4. Mortality affects both the second and third parameters, but in opposite directions. It increases the probability of damage, but diminishes the magnitude of the damage. More precisely, mortality at age $t$ increases the probability of incurring damage at age $t$ and decreases the magnitude of the damage in case of death before age $t$. The first point explains why the second term of (8) (and hence the rate of discount at age $t$) increases with mortality at age $t$. The second point clarifies why an increase in the mortality rate at age $t$ also causes a decrease in the rate of discount at all ages under $t$. Note also that the damage caused by a death at age $t$ increases with the consumption that was planned after age $t$ in case of survival. This explains why the rate of time discounting at age $t$ increases with consumption at ages greater than $t$ (as can easily been seen from equation 22 in the appendix).

In practice, we would like to know what happens when there is a global mortality decline that is characterized by a decrease in mortality rates at all ages. According to the additive model, the result is unambiguous: such a global mortality decline implies a decline in the rate of discount at all ages. This is no longer true in the time neutral model. In this latter model, in the typical case where $u$ is positive and $\phi$ is strictly concave, such a global mortality decline may have a positive or a negative impact on the age-specific rates of discount. Indeed, the rate of discount at an age $t$ was shown to depend positively on the mortality rate at age $t$ and negatively on the mortality rates at ages greater than $t$. There are, therefore, two opposing effects, which can aggregate into a positive or a negative effect. The computations based on historical mortality rates that

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will be provided in Subsection 7.1 show examples of both positive and negative aggregate effects. Thus, we know that it is impossible to provide a general result on the impact of a global mortality decline on the rates of discount for the time neutral model. Some interesting results can, however, be obtained if additional assumptions are made on how age specific mortality rates are affected by a global mortality decline:

**Proposition 7** *Consider two mortality patterns described by hazard rates of death $\mu_1(t)$ and $\mu_2(t)$, with:*

$$\frac{\mu_2(t)}{\mu_1(t)} \leq \frac{\mu_2(t')}{\mu_1(t')} \leq 1 \text{ for all } t \leq t' \tag{10}$$

*Then, for all consumption paths such that $u(c(t)) > 0$ for all $t$, we have:*

$$RD^{tn}_{\mu_2}(c,t) \leq RD^{tn}_{\mu_1}(c,t) \text{ for all } t.$$

*Moreover, if, in addition, $\phi$ is strictly concave, $\frac{-\phi''}{\phi'}$ non-increasing and mortality non-decreasing with age, for all constant consumption paths such that $u(c) > 0$, we have:*

$$RD^{tn}_{\mu_1}(c,t) - RD^{tn}_{\mu_2}(c,t) \geq RD^{add}_{\mu_1}(c,t) - RD^{add}_{\mu_2}(c,t) = \mu_1(t) - \mu_2(t) \quad \text{for all } t$$

**Proof.** See appendix F. ■

According to the first point of Proposition 7, if we consider a “high mortality” context ($\mu_1$) and a “low mortality” context ($\mu_2$), such that mortality is higher at all ages in the “high mortality” context and the relative difference in mortality rates, $|\log(\frac{\mu_1}{\mu_2})|$, decreases with age, we know that the time neutral model will predict higher rates of discount in the “high mortality” context.

Moreover, the second point of Proposition 7 indicates that if $\frac{-\phi''}{\phi'}$ is positive and non-increasing[13] and if mortality increases with age[14], the difference in the rates of discount will exceed the differences in the mortality rates. That means that the rates of discount are, in that case, more sensitive to mortality in the time neutral model than in the additive model.

[13]This is equivalent to stating that individuals provided with a constant consumption profile exhibit a positive and non-increasing risk aversion with respect to life duration.

[14]Demographic studies show that this is generally the case after age 25.

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Interestingly enough, the results of Proposition 7 can be compared with the findings of empirical studies on heterogeneity in discount rates. Indeed, differential mortality has been quite well documented by demographic studies. It is well known that in the USA, being a woman, or being rich, educated or white are factors that are negatively correlated with mortality[15]. Moreover, it is also often found that whatever the socioeconomic status considered (e.g. gender, education, etc.), the differential mortality, measured by the absolute value of the difference in the log of mortality rates tends to decrease with age after ages 30 or 40. Thus, from Proposition 7, according to the time neutral model, we expect to find that in the USA, female, rich, educated or white individuals have lower values of $RD_\mu - \mu$ (the difference between the rate of discount and the mortality rate). Conversely, the additive model predicts that $RD_\mu - \mu$ should be the same across the population.

Two well-known empirical studies concur with the predictions of the time neutral model. Lawrance (1991), who used data from the PSID, found that the rate of discount is negatively correlated with education, wealth and being white[16]. Moreover the differences in the rates of discount she observed are much larger than the differences in mortality rates[17]. Warner and Pleeter (2001), who analyzed how US military servicemen chose between lump-sum payments and pensions, found that men, less educated people, blacks and those with low incomes had higher rates of discount. They also found a heterogeneity in the rates of discounts that largely exceeds the differences in mortality rates. These findings are consistent with the time neutral model, while they cannot be explained by the additive model, without introducing further assumptions on the relation between mortality and the discount function[18].

[15] See for example the data provided by the Berkeley Mortality Database for comparison by gender or by race, and the results of Brown, Liebman and Pollet (2002) for data on differential mortality by gender, race and education.

[16] Lawrance used household data and did not explore the role of gender.

[17] Remember that in a country such as the USA, the mortality rate is only about 0.2 % at age 40 and does not reach 1% before age 60. Differences in age-specific mortality rates across socio-economic groups are typically a fraction of a percent and much smaller than the differences in the rates of discount found by Lawrance (which are of a few percent).

[18] It could be argued, for example, that the discount function $\alpha(\cdot)$ is related to morbidity, and decreases more rapidly for individuals who have higher mortality rates.

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# 7 Illustrations Using Historical Mortality Rates

The recent history of developed countries is characterized by a huge decline in mortality rates. In order to show how significant the difference between the additive models and the time neutral models can be when considering historical mortality decline, we conduct below three exercises. The first aims at illustrating that even if impatience is driven by mortality in the time neutral model, it may well happen, even in realistic cases, that mortality risk reductions lead to increase human impatience. The second deals with the effect of mortality decline on consumption smoothing. The third discusses the macroeconomic consequences of mortality decline.

## 7.1 Impatience at age 30

Imagine that in 1937, the year in which Samuelson's paper on the Discounted Utility Model was published (Samuelson, 1937), we observed that individuals of age 30 exhibited a rate of discount of 4%. Let us explore the three following possibilities:

- Case A (Additive preferences): this rate of discount is due to the fact that individuals had additive preferences and expected to die according to the average age-specific mortality rates observed in the USA in 1937.
- Case B (Time neutral preferences with a constant absolute risk aversion with respect to length of life): this rate of discount is due to the fact that individuals had time neutral preferences with a function $\phi$ of the form $\phi_1(x) = \frac{1-e^{-kx}}{k}$, and that they expected to have a constant quality of life and to die according to the mortality rates of 1937.
- Case C (Time neutral preferences with a constant relative risk aversion with respect to length of life): this rate of discount is due to the fact that individuals had time neutral preferences, with a function $\phi$ of the form $\phi_2(x) = \frac{x^{1-\kappa}-1}{1-\kappa}$, and expected to have a constant quality of life and to die according to the mortality rates of 1937.

Now, let us ask the following question: in each case, what would have been these individuals' rates of discount if they had expected to face the mortality rates observed in subsequent years? In solving this problem, we find what the effect of mortality decline on the rate of discount at age 30 would have been if individuals' preferences had remained the same.

In practice, I used the historical cross-sectional mortality rates provided by the Berkeley Mortality Database. As shown in Figure 1, the mortality rate at age 30 decreased rapidly between 1937 and 1960. Between 1960 and 2000, the mortality rate at age 30 had a non-monotonic evolution, but its global trend indicates a slow decline. Life expectancy at age 30 increased during the whole period (Figure 2).

For our exercise, I calibrated the rate of time preference (for case A), the function $\phi_1$ (for case B) and the function $\phi_2$ (for case C), so that the rate of discount of a 30 year-old individual was of 0.04 per year with the mortality of 1937. Then, for each year from 1938 to 2000, I computed the rate of discount that followed from the mortality observed in those years.

The results are shown in Figure 3. We know from Proposition 6 that in the case of additive preferences, the rate of discount is just the sum of the mortality rate and the rate of time preference. Thus, the solid line that gives the rate of discount in the additive case exactly follows the evolution of the mortality rate shown in Figure 1. However, as the mortality rate is very small compared to the rate of time preference (note that the scales of Figures 1 and 3 differ by a factor of 10), the rate of discount is found to decrease only very slightly. It equals 0.03754 in 1960 and 0.03739 in 2000.

The two dashed lines, which represent the time neutral preferences, show radically different patterns. In Case B, the mortality decline that occurred between 1937 and 1960 leads to a drop of 0.01743 in the rate of discount. That is 7.1 times greater than what we would have predicted using the additive model! This is due to the major decline in the mortality rate at age 30. After 1960, the rate of discount goes up and down, but the average trend shows a slight increase. Thus, during this period, the change in the rate of discount shows a global trend that does not follow the evolution of the mortality rate. In fact, during the period 1960-2000, the mortality rate at age 30 declined only slightly while life expectancy increased considerably. I explained after Proposition 6 that in the time neutral model, the rate of discount at age 30 is linked to mortality through two different



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channels. It is positively related to the mortality rate at age 30, and negatively related to the mortality rate at older ages. We see from our results that during the period from 1937 to 1960, it is the first factor that dominates, while after 1960, if we look at the global trend, it is the second one that predominates.

The results in Case C are comparable to those in Case B, although they further diverge from the results of the additive model. The interpretation is similar to Case B.

Overall, we found that the time neutral model can lead to radically different predictions of the impact of mortality decline. A drop of 1.743 % or of 1.928 % in the rate of discount at age 30 between 1937 and 1960, as we respectively found in Cases B and C, is likely to generate a substantial impact on savings, human capital investment, and henceforth, on economic growth. The additive model would have predicted a drop in the rate of discount of only 0.25 %.

## 7.2 Life cycle consumption smoothing

To deal with more concrete issues, let us look at consumption smoothing behaviors. Consider the case of an individual who earns 20000 dollars a year between ages 20 and 60 and nothing afterwards. Assume that there are perfect annuity markets and only one risk free asset whose rate of return equals 3% per year. How would such an individual smooth consumption and save along the life cycle? Let us consider three specifications for individuals' preferences:

$$\begin{aligned}
&\text{1 - Additive model} &:\ & U^{add} = \int_0^T e^{-\rho t}\left[u_0 + \frac{c(t)^{1-\gamma}-1}{1-\gamma}\right]dt \\
&\text{2 - Time neutral model (CARA)} &:\ & U^{tn}_{cara} = 1-\exp\left(-k\int_0^T\left[\widehat{u}_0 + \frac{c(t)^{1-\gamma}-1}{1-\gamma}\right]dt\right) \\
&\text{3 - Time neutral model (CRRA)} &:\ & U^{tn}_{crra} = \frac{1}{1-\kappa}\left(\int_0^T\left[\widetilde{u}_0 + \frac{c(t)^{1-\gamma}-1}{1-\gamma}\right]dt\right)^{1-\kappa}
\end{aligned}$$

For each specification, we can compute the optimal life cycle behavior for two different mortality patterns[19]. The first one is given by the mortality rates that were observed

[19] In all three specifications, the intertemporal elasticity of substitution, $\frac{1}{\gamma}$, was set at 0.9. The constants $\rho$ ,$k$, $\kappa$, $u_0, \widehat{u}_0$ and $\widetilde{u}_0$ were chosen so that, with 1950 mortality, 40 year-old individuals have a rate of discount of 0.03 per year and a Value of a Statistical Life of 5 million dollars. Consumption before age



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in 1950 in the USA. The second one corresponds to the year 2000 mortality rates. The predicted age-specific consumption and wealth profiles are shown in Figure 4.

When preferences are additive, the optimal consumption profile has the same shape, whether we consider 1950 or 2000 mortality rates. The "2000 consumption" is obtained from the 1950 one by a simple scaling down. It is in fact well-known that, with perfect annuity markets and a constant intertemporal elasticity of substitution, the rate of consumption growth is independent of mortality rates. Consumption is lower with 2000 mortality rates, because longevity extension generates a dilution effect.

The time neutral specifications suggest very different pictures. Firstly, the 2000 consumption and 1950 consumption no longer have the same shape. In both the CARA and the CRRA cases, 2000 consumption lies below 1950 consumption at young ages, and above at old ages. This reflects the fact that mortality decline has a two-fold effect. Firstly, there is a dilution effect, as in the additive case. Secondly, and here is the novelty, there is a significant impatience effect.

To see how significant the divergence in predictions is, we can consider individuals' wealth at retirement. The additive specification suggests that wealth at retirement increases by 14% when passing from 1950 to 2000 mortality rates. Rational individuals increase their savings because the retirement period becomes longer. However, the time neutral specifications suggest much larger increases (26% for the CARA case and 28% for the CRRA case). Even in such a rough example, where retirement age does not adapt to mortality decline, accounting for the change in impatience appears to be as important as accounting for the extension of the retirement period.

## 7.3 Aggregate wealth and equilibrium rate of interest

We have explored the impact of mortality change on impact individual behavior. The additive and time neutral models were found to provide contrasted predictions. Naturally we expect that this will also translate into quite different macro-economic predictions for the consequences of mortality decline.

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20 is assumed to be exogenous and equal to 16000 dollars per year. The optimal consumption profiles were numerically computed with the method detailed in Appendix A.

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In order to illustrate this point we consider below a model inspired from the one developed in Blanchard (1985) and discuss the aggregate impact of mortality decline in this setting. For any time $T$ there are $e^{nT}dT$ agents that are born between time $T$ and $T + dT$. Mortality rates may vary with age but, for a given age, not with calendar time (for simplicity we will focus on the comparison of steady states). To an age-specific hazard risk of death $\mu(.)$ corresponds a survival function $s(t) = \exp(-\int_0^t \mu(\tau)d\tau)$. At any time $T$ there are therefore $e^{n(T-t)}s(t)dt$ agents of age between $t$ and $t + dt$ in the population. Agents have an exogenous age-specific productivity profile $\omega(t)$. The labor income of $t$ year-old individual is $y\omega(t)$ where $y$ is the market price for one unit of productivity. There are perfect intertemporal and insurance markets and no uncertainty beyond lifetime uncertainty.

In order to show how crucial are the assumptions on individual preferences we proceed with the following exercise. Imagine that society is observed in steady state $A$ which is characterized by a survival function $s_A(.)$, a rate of interest $r_A$, a wage per unit of productivity $y_A$ and an age specific consumption profile $c_A(.)$. Assume that this steady state may be rationalized by two different macro-economic models that only differ by the specifications of individual preferences (the assumptions regarding the production sector being identical in both models and specified later). More precisely, we assume that one model relies on the additive specification of individual preferences, while the other uses the time neutral one[20]. Consider then a society where mortality rates are exogenously set to different levels $\mu_B(t) < \mu_A(t)$, corresponding to a survival function $s_B(.)$ and ask how the steady state $B$ associated with that mortality pattern would compare to $A$, according to both models. Doing so, we show to what extent the assumptions made on individual preferences influence predictions about the macro-economic impact of mortality decline.

### 7.3.1 Aggregate wealth in an open economy

As in Blanchard (1985), we consider in turn an open and a closed economy. Markets are assumed to be perfect. In a small open economy the rate of interest and labor income

[20] In both cases we will assume a constant intertemporal elasticity of substitution $\frac{1}{\gamma}$. From Proposition 5, we can deduce that, when intertemporal markest are perfect, rationalization by both an additive and a time neutral model is possible as soon as $\frac{\frac{d}{dt}c_A(t)}{c_A(t)} < \frac{r_A}{\gamma}$.

are exogenous. We denote $r = r_A = r_B$ and $y = y_A = y_B$. We will discuss how the ratio of aggregate wealth over aggregate income depends on mortality. Using the results of Bommier and Lee (2003), the ratio of aggregate wealth over aggregate income in a steady state $i \quad (i = A, B)$ is:

$$\frac{W_i}{Y_i} = \frac{1}{r-n}\left(\frac{\int_0^\infty s_i(t)c_i(t)e^{-nt}dt}{y\int_0^\infty s_i(t)\omega(t)e^{-nt}dt} - 1\right) \tag{11}$$

With perfect intertemporal markets, the individual budget constraint is:

$$\int_0^\infty s_i(t)c_i(t)e^{-rt}dt = y\int_0^\infty s_i(t)\omega(t)e^{-rt}dt$$

Plugging this latter equality into (11) we obtain:

$$\frac{W_B}{Y_B} - \frac{W_A}{Y_A} = \frac{1}{r-n}(I+J)$$

where

$$I = \frac{\frac{\int_0^\infty s_B(t)c_A(t)e^{-nt}dt}{\int_0^\infty s_B(t)c_A(t)e^{-rt}dt}}{\frac{\int_0^\infty s_B(t)\omega(t)e^{-nt}dt}{\int_0^\infty s_B(t)\omega(t)e^{-rt}dt}} - \frac{\frac{\int_0^\infty s_A(t)c_A(t)e^{-nt}dt}{\int_0^\infty s_A(t)c_A(t)e^{-rt}dt}}{\frac{\int_0^\infty s_A(t)\omega(t)e^{-nt}dt}{\int_0^\infty s_A(t)\omega(t)e^{-rt}dt}}$$

$$J = \frac{\int_0^\infty s_B(t)\omega(t)e^{-rt}dt}{\int_0^\infty s_B(t)\omega(t)e^{-nt}dt}\left(\frac{\int_0^\infty s_B(t)c_B(t)e^{-nt}dt}{\int_0^\infty s_B(t)c_B(t)e^{-rt}dt} - \frac{\int_0^\infty s_B(t)c_A(t)e^{-nt}dt}{\int_0^\infty s_B(t)c_A(t)e^{-rt}dt}\right) \tag{12}$$

The change in the ratio of aggregate savings over aggregate income is broken down into the sum of two terms. Term $I$ may be qualified as a structural effect, that measures the aggregate impact of a change in mortality if agents had no reaction other than rescaling their consumption in order to match their budget constraint. Term $J$ may be called the behavioral impact that accounts for the fact that agents' saving strategies may vary with mortality.

Since term $I$ depends on $c_A$ but not on $c_B$ it will be the same whether the additive or time neutral rationalization are chosen. The choice of a model of individual rationality is however determinant for evaluating $J$. Rescaling the consumption profiles $c_A$ or $c_B$ by positive factors in equation (12) does not change $J$. In other words, $J$ depends on the shape of the consumption profile in states $A$ and $B$, but is independent of the levels of



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these consumption profiles. An immediate consequence is that $J = 0$ when individual preferences are assumed to be additive.

When preferences are time neutral, the shape of the optimal consumption profile depends of the mortality pattern (as illustrated in Figures 4b and 4c). Thus $J$ does not necessarily equal zero. The sign of $J$, however, cannot be determined without making further assumptions on the mortality patterns $\mu_A$ and $\mu_B$. In particular the fact that mortality rates are lower in state $B$ than in state $A$ does not imply that $J$ is positive: this reflects the fact that, even if impatience is generated by mortality, mortality decline has an ambiguous impact on human impatience (see Proposition 6 and the related discussion). Still, if mortality decline generates a drop in the rate of discount that is greater than the drop in mortality rates (which was shown to be the case under a set of sufficient conditions in Proposition 7) we get that $\frac{-\frac{d}{dt}c_B(t)}{c_B(t)} < \frac{-\frac{d}{dt}c_A(t)}{c_A(t)}$ which implies that $\frac{1}{(r-n)}J > 0$. In such a case, the additive formulation would lead to under-estimate the impact of mortality decline on wealth accumulation, as compared to the time neutral one.

As an illustration we consider the case where $\mu_A$ corresponds to the 1950 mortality rates and $\mu_B$ to the 2000 mortality rates. We assume that the steady state $A$ correspond to a rate of interest of 3% per year, a zero population growth rate and the income and consumption profiles that are plotted in Figure 4b. Rationalization by a time neutral model is obtained by construction, since Figure 4b was precisely drawn using time neutral preferences. Rationalization by an additive model is straightforward and relies on a non exponential discount function[21].

According to the additive model, the ratio of aggregate wealth[22] over aggregate income would have increased by 28.6 % when passing from 1950 to 2000 mortality rates. The time neutral model predicts an increase of 47.6%, the behavioral term, $J$, being almost as large as the structural term, $I$.

[21] Several arguments have been suggested in the literature for using non exponential discount functions. Murphy and Topel (2006) argue for example that non exponential discount functions may result from life-cycle changes in health.

[22] We only consider savings of people of age 20 or above.



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### 7.3.2 The equilibrium rate of interest

We now consider the case of a closed economy. We assume a zero population growth rate ($n = 0$). The production function is given by a Cobb-Douglas function of the form $F(K, L) = AK^{\beta}L^{1-\beta} =$ where $K$ and $L$ represent aggregate capital and labor in the economy, and $\beta$ a constant that will be chosen to fit a target rate of interest at the calibration stage. Capital and labor are remunerated at their marginal productivity, the rate of interest and wages being therefore endogenously determined.

Assume that a reference steady state is observed where mortality rates correspond to those of the 2000 US life table, the rate of interest equals 3% per year and the consumption and income profiles are those given in Figure 4b. Then, ask what steady state would be obtained if preference and production parameters were hold constant, but if age-specific mortality rates were set at the levels that were observed in past periods or at the levels they will plausibly reach in the future (we use historical and projected US-lifetable from 1900 to 2080). Doing so we identify the "comparative steady state" impact of mortality changes, holding everything else constant[23]. Again we compare the case where the consumption profile $c_A$ plotted in Figure 4b is rationalized through the time neutral model (with the parameters used to draw Figure 4b), to the case where it is rationalized by an additive model with the same intertemporal elasticity of substitution and a non exponential discount function.

Whether one chooses the time neutral or the additive model to rationalize the reference steady-state, we find that the historical and projected mortality decline leads to a decrease in the rate of interest (see Figure 5). The magnitude of the decline is, however, quite different. In the additive model the effect is driven by the fact that people live longer and have to save for a longer period of retirement. In the time neutral model, in addition to this effect, there is an impatience effect that plays a major role. Mortality decline makes people appear less impatient on average and aggregate savings supply increases. The equilibrium rate of interest decreases accordingly. As a consequence, the time neutral

[23] Of course, in reality, economies are not in steady-states and many dynamic aspects may turn out to play a key role. This comparative steady state exercice aims at providing an order of magnitude of the long term impact of mortality decline, but not at explaining what might occur in a realistic dynamic setting.

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model predicts a decrease in the interest rate of 2.33 percentage points between 1900 and 2000, while the additive one only predicts a decrease of 0.79 percentage point. Predictions for the future are also quite different since the time neutral model predicts a decline in the rate of interest between 2000 and 2080 that is about twice as much as the one predicted by the additive one (0.53 percentage points versus 0.25).

Historical data on interest rates show very large fluctuations. That makes it difficult to identify the long term change in interest rates. A linear regression on the series from Siegel (1992) indicate an average decline of the real risk free rate of about 3.5 percentage points per century during the period 1800-1990. The 1889-2000 series of Mehra and Prescott (2003) indicates a decline of about 1.6 percentage points per century. These numbers have, however, to be taken with extreme precaution since the confidence interval are very large. It would therefore be unfair to use them as a definitive argument against the additive model. Still, they do not seem to be at odds with the predictions of the time neutral model.

Whether we consider the case of an open or a closed economy, the contrast between the predictions of the additive and time neutral models is quite substantial. What basically comes out of the several simulations is that using the additive model (and therefore ignoring the endogeneity of human impatience) when assessing the impact of mortality decline may lead to ignore about half of the story.

# 8 The value of longevity gains

An important aspect of mortality changes is that they are not exogenous. Mortality largely depends on (public and private) expenditure. A natural question is whether enough resources are (and have been) allocated to the reduction of mortality risks. A number of studies based on the additive model aim to measure the welfare gains associated with mortality risk reduction (Becker, Philipson, Soares, 2005, Murphy and Topel, 2006, Hall and Jones, 2007, EPA, 1997). Some of these studies reached very strong conclusions on the optimal amount of public health expenditure or on the interest of continuing environmental policies such as those delineated by the Clean Air Act in the USA. An

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intriguing question is to what extent the conclusions of theses studies would be altered if the time neutral model were used instead of the additive specification.

For matters related to endogenous mortality, the sharpest contrast between the additive and time neutral models occurs when looking at how the value of a statistical life (VSL) varies with age. The intuition is that introducing temporal risk aversion involves introducing risk aversion with respect to life duration (see Bommier 2006). Risk aversion increases individuals' willingness to avoid particularly unfavorable consequences (such as an early death) relatively to the willingness to avoid less dramatic consequences (such as a death at old age). Consequently, and this is confirmed by the formal result given in Section 8.1, the intuition is that the VSL should decline faster with the time neutral model than with the additive model.

Theoretical considerations about the relation between age and VSL are of crucial importance since empirical estimates of the VSL are typically derived from choices of relatively young individuals (workers, in most cases), while policy guidance requires to evaluate the impact of mortality reduction that mainly occur at much greater ages. For example, Hall and Jones (2007) calibrate their model using empirical values of the VSL for 35-39 year-old individuals, and then use that calibrated model to predict the VSL at ages up to 100. A similar strategy was adopted in Murphy and Topel (2006). There is therefore a great deal of extrapolation underlying the conclusions of theses studies.

The question we address below is: to what extent this required extrapolation would change when opting for the time neutral specification instead of the additive one. Our discussion will gather a formal result as well as numerical illustrations.

## 8.1 Formal result

In the following, the VSL at age $t$ is defined as the opposite of the marginal rate of substitution between mortality risk at age $t$ and consumption at time $t$.[24]

$$VSL(c,t) = \frac{-\frac{\partial E_\mu U}{\partial \mu(t)}}{\frac{\partial E_\mu U}{\partial c(t)}}$$

Using a model with endogenous labor supply, Bommier and Villeneuve (2006) explain how VSL may be directly revealed from empirical studies on wage risk trade-offs. Notation $VSL^{add}(c,t)$ and $VSL^{tn}(c,t)$ refer then to the $VSL$ that is obtained when assuming that preferences are additive or time neutral, respectively.

**Proposition 8** *Consider time neutral and additive preferences that assume the same intertemporal elasticity of substitution and that are such that (for a given rate of interest, $r$, a survival function, $s(.)$, and an initial wealth $W_0$):*

*(i) They provide the same optimal consumption profile $c(t)$ to the problem*

$$\begin{aligned} &\max E_\mu U \\ s.t.\ &\int_0^\infty s(t)e^{-rt}c(t) \leq W_0 \end{aligned} \tag{13}$$

*(ii) They provide the same value of life at a given age $t_0$ :*

$$VSL^{tn}(c,t_0) = VSL^{add}(c,t_0)$$

*Assume moreover than the ratio of the rate of discount over the rate of mortality, $\frac{RD}{\mu}$, is greater than 1 and decreases with age. Assume also that $c$, which solves (13), is non increasing.*

[24] Note that whenever agents maximize their lifetime utility under budget constraints that are linear in saving (such as $\int p(t)(c(t) - y(t))dt \leq W_0$ where $p(.)$ is an exogenous function), the VSL also equals the opposite of the marginal rate of substitution between mortality risk at age $t$ and wealth age $t$.



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*Then, we have:*

$$VSL^{tn}(c,t) \quad < \quad VSL^{add}(c,t) \;\; \textit{for all } t > t_0$$
$$VSL^{tn}(c,t) \quad > \quad VSL^{add}(c,t) \; \textit{for all } t < t_0$$

**Proof.** See Appendix G. ■

As already mentioned, models used to discuss the social value of longevity are usually calibrated to fit a given consumption profile and a single value of life (corresponding either to the value of life at a given age, or the average value of life for a given age range). The fit may be obtained both with the additive and time neutral specifications. Proposition 8 shows, however, that opting for one or the other specification is not without consequences. Relying on the additive model leads to assume greater values of VSL at old ages than when using the time neutral model. This may significantly alter policy recommendations. In practice the “additivists” would ascribe greater significance to the reduction of mortality risk at old ages than the “time-neutralists”; the latter would be more attentive to mortality at younger ages. Moreover, as it is the extrapolation towards old ages that is of greater relevance for policy recommendation (because most mortality reductions occur at old ages while value of life empirical estimates are derived from younger people), the additivists would typically allocate a greater amount of public resources to mortality risk reduction than the time-neutralists. The following Section seeks to provide some insights into the magnitude of this divergence, when using realistic mortality data.

## 8.2 Numerical illustration

In this illustration, we assume that we observe the consumption profile shown in figure 4b (while the rate of interest is assumed to be 3% and mortality rates equal to those observed in year 2000 in the USA), and a VSL that equals 5 million dollars at age 40. Again, such an observation may be consistent both with the additive model (using a non exponential discount function) and with the time neutral model. Both interpretations lead to different extrapolations when evaluating the VSL at ages other than 40. The result of these extrapolations is shown in Figure 6. We find, in agreement with Proposition 8, that



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the time neutral model predicts that the VSL declines more rapidly with age than the additive model. When looking at the VSL around age 80, where most deaths occur, we find that the additive model predicts a VSL that is 3.4 times larger than the time neutral one.

We may now wonder whether this divergence in the age adjustment of the VSL is likely to generate contrasted estimates of the value of longevity gains. To answer that question, we compute how much adults living in the USA in 2000 would have been ready to pay to maintain mortality rates at their 2000 levels instead of having them set back to 1970 levels. This measure of gains from increased longevity was suggested by Murphy and Topel (2006), the results being reported in Figure 6 of their paper. Unlike Murphy and Topel, who only provided results based on the additive model, we provide in Figure 7 of the present paper two kinds of estimates: one relying on the additive model (and therefore similar to the one reported by Murphy and Topel) and another using the time neutral model. Compared to the additive model, the time neutral model provides lower estimates for the value of longevity gains. This is because both models were calibrated to provide the same value of life at age 40, while most of longevity gains occurred after that age. If we aggregate the gains for the whole 2000 US population, the time neutral model provides an estimates that is 1.98 smaller than the additive model. In this case, switching from the additive model to the time neutral one would approximately lead to divide the estimates of the value of longevity gains by a factor of two.

# 9 Discussion

More than fifty years ago, George Stigler, in a discussion bearing on the way precursors modelled preferences over several commodities wrote:

"The faithful adherence for so long to the additive utility function strikes one as showing at least a lack of enterprise. I think it showed also a lack of imagination: no economic problem has only one avenue of approach" (Stigler, 1950, p394).

One might argue that the same statement currently applies to the theory of choice under uncertain lifetime.

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Mortality risks were first considered by Yaari in a simple model that assumed additively separable preferences. Yaari's choice, which he did not discuss, involved making an assumption of temporal risk neutrality. A major consequence of this choice is that the rate of time discounting equals the sum of the rate of time preference and the mortality rate. Since mortality rates are typically much lower than observed rates of discount, Yaari's model eventually provides a theory where time discounting owes very little to mortality. The ongoing adherence to Yaari's approach ended up generating the belief that mortality could not significantly contribute to human impatience, unless particularly high mortality rates (either due to diseases or advanced age) were considered. Moreover, it has popularized the idea in economic theory that risk aversion and impatience are orthogonal aspects of individual preferences.

The present paper shows that a different path, which is no more complex than the one followed by Yaari, might have been pursued. It involves assuming that individuals have no time preferences but exhibit temporal risk aversion, giving the "time neutral model". With this model, impatience has no ordinal origins, but results from the combination of mortality risks and temporal risk aversion.

In order to show the plausibility of the time neutral model, I formally showed that it can reproduce all the predictions of Yaari's model, as long as heterogeneity in mortality across agents is ignored. To my knowledge, Yaari's model has never been challenged by studies that used heterogeneity in mortality. Thus, today, there is no empirical evidence indicating that Yaari's model is better than the time neutral model. In particular, the fact that Yaari's formulation proved useful to study consumption patterns, saving behaviors, labor supply, etc. cannot be considered as an argument supporting Yaari's model. The time neutral model would do at least as well.

The time neutral model provides new insights on the relation between risk aversion, mortality and impatience. In Yaari's model, impatience, measured by the rate of time discounting, is almost exogenous (mortality having a minor role). In the time neutral model, impatience is exclusively driven by mortality. Thus, unsurprisingly, we find that both models have very different predictions about the impact of changes in mortality. Illustrations using historical data mortality were provided, showing that the time neu-

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tral model may shed new light on the interrelation between longevity extension, capital accumulation and the equilibrium rate of interest.

For matters related to the value of life, introducing temporal risk aversion, as with the time neutral model, generates risk aversion with respect to length of life. As a consequence, agents are more willing to avoid the risk of an early death. The value of a statistical life is then found to decline more rapidly with age with the time neutral model than with the additive one. This is of particular importance for applied issues, since debating the welfare impact of longevity extension generally requires a great deal of extrapolation for assessing the benefits of reducing mortality at old ages. Numerical illustrations, based on realistic demographic data, show that switching from the additive model to the time neutral one would lead to significantly revise the estimation of the value of longevity gains.

Ultimately, one would like to discriminate between both models from empirical data. The paper naturally suggests two lines of research. One possibility is to use empirical data on the relation between age and the value of life. Unfortunately, as discussed in Bommier and Villeneuve (2006), this approach cannot be conclusive today given the lack of consensus in the empirical research that addresses this question. The other possibility is to explore the interrelation between mortality and human impatience. Getting the right intuitions about the role of mortality in the time neutral model requires, however, a rigorous look at the formal expression of the rate of time discounting (Propositions 6 and 7). Even if impatience is driven by mortality, it is not always the case that greater mortality implies greater impatience. In fact, in the time neutral model, mortality in the short term increases impatience while mortality in the long term decreases it. Sufficient conditions were provided for the first effect to dominate the second one, but one should bear in mind that these conditions are not always fulfilled. As a consequence, in order to discriminate between Yaari's model and the time neutral model, it is necessary to have, on the one hand, very good knowledge of differential mortality (so that heterogeneity in short term mortality and heterogeneity in long term mortality can be compared) and, on the other hand, excellent data on intertemporal choice that make it possible to measure individuals' rates of discount.

The ideal data set does not yet exist. A number of surveys report data on health,

health shocks, etc. but it is generally impossible to accurately translate this information into short-term and long-term mortality rates. To my mind, the best available option is to confront the well documented heterogeneity in mortality rates across gender, ethnic, education and income groups with the heterogeneity in discount rates. I explained that this confrontation actually supports the time neutral model over Yaari's model. This certainly does not provide sufficient evidence to abandon the notion of time preference. However, there are even less arguments in favor of ignoring temporal risk aversion. Accounting for temporal risk aversion is in fact crucial to understanding the origins of human impatience and the transformation that societies are going through along with the rapid change in mortality rates.

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# APPENDIX

## A The time neutral model in practice

One attractive feature of Yaari's model is its mathematical tractability. With Yaari's model, preferences over consumption profiles, conditional on an exogenous mortality pattern, are represented by the utility function:

$$E_{\mu}^{add}(c) = \int_0^{+\infty} s(t)\alpha(t)u(c(t))dt$$

Dealing with such a utility function is, technically speaking, particularly convenient for several reasons. First, preferences over consumption after age $t$ are independent of consumption before age $t$. Therefore, in a dynamic setting, individuals need not remember the past to have time consistent behaviors. Moreover the additive structure of the expected utility function often leads to relatively simple optimization problems. A number of life cycle problems (e.g. consumption smoothing, portfolio choices) can be studied with standard techniques, such as dynamic programming, and, for particular functions $u$ , yield to simple solutions.

The object of this section is to discuss how the landscape is transformed when working with the time neutral model. It will be split into three parts. A first subsection identifies the technical difficulties that emerge when dealing with the time neutral model. As we will see, there are no fundamental obstacles for using standard techniques, such as dynamic programming. The main difference, however, is that explicit solutions cannot readily be found. Nevertheless, it is possible to work with the time neutral model without developing cumbersome numerical computations. First, as will be explained in Subsection A.2, a linear approximation makes it possible to recover all the simplicity of the additive model, while maintaining key aspects of the time neutral model. Secondly, Subsection A.3 provides a very simple method for numerically computing exact solutions when there are complete markets.

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## A.1 History dependence and dynamic programming

Agents with time neutral preferences who face an exogenous mortality pattern have preferences over (stochastic) consumption profiles represented by the utility function:

$$E_\mu U^{tn}(c) = \int_0^{+\infty} d(T)\phi\left(\int_0^T u(c(\tau))d\tau\right) dT \tag{14}$$

or, equivalently (through an integration by parts):

$$E_\mu U^{tn}(c) = \int_0^{+\infty} s(\tau)u(c(\tau))\phi'\left(\int_0^\tau u(c(\tau'))d\tau'\right) d\tau \tag{15}$$

A noteworthy difference with the additive formulation is that preferences over consumption after time $t$ generally depend on consumption prior to $t$. From (15), given a consumption profile $\widetilde{c}$ between times 0 and $t$, preferences over consumption profiles after time $t$ are represented by the utility function:

$$\int_t^{+\infty} s(\tau)u(c(\tau))\phi'\left(H_t + \int_t^\tau u(c(\tau'))d\tau'\right) d\tau \tag{16}$$

where

$$H_t = \int_0^t u(\widetilde{c}(\tau))d\tau$$

is the "stock of felicity" that has been accumulated up to time $t$. In a dynamic setting, under the assumption of time consistency, the utility function (16) represents the preferences of an agent of age $t$ with past consumption $\widetilde{c}$. Preferences may then exhibit history dependence, since past consumption affects $H_t$ which enters into the agents' utility functions.

At this point, however, it is useful to distinguish the case where $\phi'$ is exponential from the general case. When $\phi'(x) = e^{-kx}$, past consumption only matters in (16) through a positive multiplicative factor, $e^{-kH_t}$, and therefore has no impact on individual preferences. Thus, precisely as with the additive model, individuals do not need to remember the past to be time consistent.

When $\phi'$ is not exponential, preferences over consumption after age $t$ depend on con-

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sumption before $t$. Thus, in order to have time consistent behaviors, individuals have to bear in mind some of their history[25]. History dependence however takes a very simple form. Preferences over consumption after age $t$ depend on the past only through $H_t$, the stock of felicity that has been accumulated at age $t$. This largely resembles habit formation problems, where preferences at age $t$ depend on the past only through the stock of habits that has been accumulated at time $t$. Dynamic programming can then be implemented in a standard way, even though the technical problems that one has to face are indisputably more complex. The dynamics involve two scalar state variables (wealth and the stock of past felicity) instead of one (wealth) with the usual additive case. For numerical applications going from one to two state variables only represents a slight increase in complexity. Gomes and Michaelides (2003) and Polkovnichenko (2007) produced papers on portfolio choice with habit formation that successfully deal with similar (and significantly greater) technical difficulties. Their approach could be replicated. Alternatively, one may opt for the simpler approaches developed in sections A.2 and A.3.

Since the economic literature has mostly focused on models that exhibit history independence, one may wonder whether we should not restrict our attention to the case where $\phi'$ is exponential. One can interpret the time neutral model as the case where agents have an additive lifetime felicity, $\int_0^T u(c(t))dt$, and are risk averse with respect to lifetime felicity (risk aversion with respect to lifetime felicity being related to the curvature of the function $\phi$). By restricting the analysis to the case where $\phi'$ is exponential, one would impose constant absolute risk aversion with respect to lifetime felicity. This is less restrictive than imposing risk neutrality (as is done when using the additively separable specification) but it is difficult to find any compelling reason for ruling out decreasing or increasing absolute risk aversion with respect to lifetime felicity. The assumption of decreasing absolute risk aversion, which has become extremely popular for modelling preferences over monetary lotteries, may also prove to be very natural for modelling preferences over lotteries over lifetime felicity. With such an assumption, an individual's risk aversion has to depend on the stock of felicity that he/she has accumulated, and therefore on his/her age and past consumption.

[25] Another route, pursued in Bommier (2006), involves assuming the history independence of preferences and allowing for time-inconsistencies.

## A.2 A linear approximation of the time neutral

From Proposition 5, we know that all the predictions of Yaari's model can be reproduced, up to infinitesimally small differences. Thus it must be the case that for some specifications the time neutral model has the same tractability as the additive specification. The strategy involves assuming that consumption remains in a range $[c_{\min}, c_{\max}]$, such that the difference in welfare between having a high or a low level of consumption is much smaller than the difference of welfare between being alive with a low level of consumption and being dead[26]:

$$\frac{u(c_{\max}) - u(c_{\min})}{u(c_{\min}) - 0} << 1$$

For any $c^*$ in $[c_{\min}, c_{\max}]$ one can write

$$u(c) = u(c^*)[1 + \varepsilon v(c)]$$

with $\varepsilon = \frac{u(c_{\max})-u(c_{\min})}{u(c^*)} << 1$ and $v(c) = \frac{u(c)-u(c^*)}{u(c_{\max})-u(c_{\min})}$. The idea is then to approximate the utility function (15) by a first order approximation in $\varepsilon$. Following the lines of the proof of Proposition 5, one can compute:

$$E_\mu U^{tn} \simeq A + \varepsilon \int_0^{+\infty} s(t)\alpha_\mu(t)v(c(t))dt$$

where $A$ is a constant and $\alpha_\mu$ is a discount function given by

$$\alpha_\mu(t) = \frac{1}{s(t)} \int_t^{+\infty} d(\tau)\phi'(\tau u(c^*))d\tau$$

Thus, individuals approximately behave as if they were maximizing the expectation of:

$$\int_0^{+\infty} s(t)\alpha_\mu(t)v(c(t))dt$$

We are then back to an additive specification and recover all the tractability of Yaari's formulation. The fundamental difference with Yaaris's formulation is that the discount

[26]This actually involves assuming that the value of life is very large. In fact, the linear approximation developped below corresponds to the limit case where the value of life is infinite.



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function is now related to mortality. This is of course of crucial importance for studying the role of mortality changes.

Such an approximation preserves one of the main features of the time neutral model (the strong relation between mortality and impatience) and proves to be relatively efficient for studying the impact of mortality on consumption smoothing[27]. However, by "forcing additivity", we necessarily lose some features of the time neutral model, as with its ability to separate risk aversion and intertemporal elasticity of substitution. This linear approximation will then be less advisable to study life cycle portfolio choices, since it would lead to the same shortcomings as the standard additive case[28].

## A.3 Numerical solutions

In this section we explain how optimal life cycle behavior can be readily and quickly numerically computed when financial markets are complete. We give accounts of the method without addressing the technical questions as to the conditions that would ensure this method's efficiency.

Following the martingale approach, when financial markets are complete, life-cycle optimization is equivalent to finding the consumption process $c$ that solves:

$$\max_c E\left[E_\mu U^{tn}(c)\right] \quad \text{subject to } W = E\left[\int_0^{+\infty} p(t)c(t)dt\right] \tag{17}$$

where $p$ is a contingent price process. From (15), it is clear that, when functions $\phi$ and $u$ are concave, $E_\mu U^{tn}$ is concave. Resolution of the maximization problem (17) can therefore be achieved using standard numerical methods of convex optimization, as described in Boyd and Vandenberghe (2004). However, given the particular structure of the objective function it generally proves easier to solve the optimization problem using the utility

[27] For example, if we use this additve approximation to study the example developed in Section 7.2, we find that switching from 1950 to 2000 mortality should induce an increase of wealth at retirement by 28%, in the CARA case, and 30% in the CRRA case. These predictions are close to those obtained from an exact resolution of the time neutral (26% and 28%) and sharply contrast with those of the additive model (14%).

[28] As explained in Bommier and Rochet (2006), when preferences are not additively separable, as in the time neutral model, the optimal degree of risk taking varies along the life cycle. This is an intersting feature that would be lost with the linear approximation.



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gradient approach. Basically, one has to compute the gradient of the utility function and to invert it. In the present case, the utility gradient admits a simple expression. Under regularity conditions on the functions $\phi$ and $u$, for any "small perturbation process" $\delta c$:

$$E\left[E_\mu U^{tn}(c+\delta c)-E_\mu U^{tn}(c)\right]\simeq E\left[\int_0^{+\infty} d(T)\left(\int_0^T u'(c(t))\delta c(t)dt\right)\phi'\left(\int_0^T u(c(\tau))d\tau\right)dT\right]$$

Switching the order of integration:

$$E\left[E_\mu U^{tn}(c+\delta c)-E_\mu U^{tn}(c)\right]\simeq E\left[\int_0^{+\infty}\delta c(t)\pi(t)d\tau\right]$$

with:

$$\pi(t)=u'(c(t))\int_t^{+\infty} d(T)\phi'(\int_0^T u(c(\tau))d\tau)dT$$

The first order conditions of the optimization problem (17) are thus:

$$u'(c(t))\int_t^{+\infty} d(T)\phi'(\int_0^T u(c(\tau))d\tau)dT=\lambda p(t)\text{ for all }t \tag{18}$$

The core of the problem involves inverting this equation, that is to obtain $c(.)$ from $\lambda p(.)$. Denoting $z(t)=\log(u'(c(t)))$ the problem is to find a fixed point of the mapping:

$$\Omega:\begin{cases} z\in C(\mathbb{R}^+,\mathbb{R})\rightarrow\Omega\left[z\right]\in C(\mathbb{R}^+,\mathbb{R})\\ \Omega\left[z\right](t)=\log(\lambda p(t))-\log\left(\int_t^{+\infty} d(T)\phi'\left(\int_0^T g(z(\tau))d\tau\right)dT\right)\end{cases}$$

where $g=u\circ\left[u'\right]^{-1}\circ\exp$. Here $\left[u'\right]^{-1}$ denotes the reciprocal of $u'$ and $\circ$ the composition operator[29].

It occurs that when relative risk aversion with respect to length of life is small enough, or when the value of life large enough, the mapping $\Omega$ is a contraction[30]. Its fixed point can be found by a simple iteration process, looking at the limit of a sequence such that $z_{n+1}=\Omega(z_n)$, the limit being independent of $z_0$.

The strategy to solve the optimization problem (17) is then as follows. Step 1: for any $\lambda$, find the consumption process $c_\lambda$ that solves the first order conditions (18) by computing

[29] In the standard isoelastic case, $u(c)=\frac{c^{1-\gamma}-1}{1-\gamma}+u_0$, we have $g(z)=\frac{e^{-\frac{1-\gamma}{\gamma}z}-1}{1-\gamma}+u_0$.
[30] A formal proof is available upon request.

the fixed point of $\Omega$ by iteration. Step 2: compute $W_\lambda = E\left[\int_0^{+\infty} p(t)c_\lambda(t)dt\right]$ and look for the value of $\lambda$ such that $W_\lambda - W$ equals zero[31].

Moreover, when $u$ is isoelastic (that is when $u(c) = \frac{c^{1-\gamma}-1}{1-\gamma} + u_0$), the function $[u']^{-1}$ is homogenous, which makes it possible to merge steps 1 and 2 into a single fixed point search. Resolution of (17) involves finding the fixed point that solves $z = \widehat{\Omega}[z]$ where:

$$\widehat{\Omega}[z](t) = \gamma \log\left(E\left[\int_0^{+\infty} p(t)\exp(-\frac{1}{\gamma}\widetilde{\Omega}[z](t))dt\right]\right) - \gamma\log(W) + \widetilde{\Omega}[z](t)$$

with:

$$\widetilde{\Omega}\,[z]\,(t) = \log(p(t)) - \log\left(\int_t^{+\infty} d(T)\phi'\left(\int_0^T \frac{e^{-\frac{1-\gamma}{\gamma}z(\tau)} - c_0^{1-\gamma}}{1-\gamma}d\tau\right)dT\right)$$

With the parameters that allow standard estimates of the rate of discount and the value of a statistical life to be matched, the mapping $\widehat{\Omega}$ was found to be a contraction, and its fixed point could easily be found by looking at the limit of a sequence such that $z_{n+1} = \widehat{\Omega}(z_n)$, whatever initial value $z_0$ was chosen. Intuitively, working along this line involves (i) starting with an arbitrary initial consumption profile, (ii) computing the age-specific rates of time discounting that would be obtained with such a consumption profile, (iii) looking for the optimal consumption profile, assuming exogenous age-specific rates of time discounting equal to those found in step (ii). Steps (ii) and (iii) are then iterated until a satisfactory convergence is reached. The method proved to be extremely efficient and the consumption profiles shown in Figure 4 were thus computed in less than a second.

# B Proof of Proposition 3

By integrating by parts (6), we find that:

$$E_\mu U(c) = \int_0^{+\infty} s(t)\frac{\partial U(c,T)}{\partial T}|_{T=t}dt$$

[31]Remark that $c_\lambda$ solves (17) when $W$ is replaced by $W_\lambda$. This means that $\lambda$ is the marginal utility of wealth when wealth equals $W_\lambda$. Thus $\lambda$ and $W_\lambda$ are negatively related when $E_\mu U^{tn}$ is concave (which is the case when $\phi$ and $u$ are concave). Solving $W_\lambda - W = 0$ then involves finding the zero of a decreasing function.



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where $s(t)$ is the survival function.

In the additive case, $\frac{\partial U^{add}(c,T)}{\partial T}|_{T=t} = \alpha(t)u(c(t))$ and

$$E_\mu U^{add}(c) = \int_0^{+\infty} s(t)\alpha(t)u(c(t))dt \tag{19}$$

which implies that $\frac{\partial E_\mu U^{add}(c)}{\partial c(t)} = s(t)\alpha(t)u'(c(t))$ and $RD^{add}_\mu(c,t) = \frac{-s'(t)}{s(t)} - \frac{\alpha'(t)}{\alpha(t)}$.

In the time neutral case, $\frac{\partial U^{tn}(c,T)}{\partial T}|_{T=t} = u(c(t))\phi'(\int_0^t u(c(t))dt)$, and we find:

$$E_\mu U^{tn}(c) = \int_0^{+\infty} s(t)u(c(t))\phi'\left(\int_0^t u(c(\tau))d\tau\right)dt \tag{20}$$

so that:

$$\frac{\partial E_\mu U^{tn}(c)}{\partial c(t)} = u'(c(t))\left[s(t)\phi'(\int_0^t u(c(\tau))d\tau) + \int_t^{+\infty} s(t_1)u(c(t_1))\phi''\left(\int_0^{t_1} u(c(\tau))d\tau\right)dt_1\right]$$

and

$$RD^{tn}_\mu(c,t) = \frac{-s'(t)\phi'(\int_0^t u(c(\tau))d\tau)}{s(t)\phi'(\int_0^t u(c(\tau))d\tau) + \int_t^{+\infty} s(t_1)u(c(t_1))\phi''(\int_0^{t_1} u(c(\tau))d\tau)dt_1} \tag{21}$$

or also:

$$RD^{tn}_\mu(c,t) = \mu(t) - \frac{\mu(t)\int_t^{+\infty} s(t_1)u(c(t_1))\phi''(\int_0^{t_1} u(c(\tau))d\tau)dt_1}{s(t)\phi'(\int_0^t u(c(\tau))d\tau) + \int_t^{+\infty} s(t_1)u(c(t_1))\phi''(\int_0^{t_1} u(c(\tau))d\tau)dt_1}$$

which, after integration by parts of the denominator of the fraction, gives (8).

# C Proof of proposition 4

From (8), through integration by parts of the numerator we obtain:

$$RD^{tn}_\mu(c,t) = \mu(t) - \mu(t)\frac{-s(t)\phi'(\int_0^t u(c(\tau))d\tau) + \int_t^{+\infty} d(t_1)u(c(t_1))\phi'(\int_0^{t_1} u(c(\tau))d\tau)dt_1}{\int_t^{+\infty} d(t_1)\phi'(\int_0^{t_1} u(c(\tau))d\tau)dt_1}$$



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and therefore:

$$RD_\mu^{tn}(c,t) = \frac{-s'(t)}{\int_t^{+\infty} d(t_1)\frac{\phi'(\int_0^{t_1} u(c(\tau))d\tau)}{\phi'(\int_0^{t} u(c(\tau))d\tau)}dt_1}$$

Now for all $x$ and $x_0$:

$$\frac{\phi'(x)}{\phi'(x_0)} = \exp\left(-\int_{x_0}^{x}\frac{-\phi''(z)}{\phi'(z)}dz\right)$$

Thus:

$$RD_\mu^{tn}(c,t) = \frac{-s'(t)}{\int_t^{+\infty} d(t_1)\exp\left(-\int_{g(t)}^{g(t_1)}\frac{-\phi''(z)}{\phi'(z)}dz\right)dt_1} \quad (22)$$

where $g(t) = \int_0^t u(c(\tau))d\tau$. The function $g(t)$ is an increasing function since it is assumed that $u(c(\tau)) \geq 0$ for all $\tau$. Thus $g(t_1) \geq g(t)$ for all $t_1 \geq t$. It is then clear from (22) that the greater the concavity of $\phi$ the greater the rate of discount $RD_\mu^{tn}(c,t)$.

# D Proof of Proposition 5

As in equation (2), denote by $\alpha$ and $u$ a pair of discount and instant utility functions that characterize the additive preferences. The corresponding expected utility function, $E_\mu U^{add}$, defined by (6), can be rewritten as in (19). The positivity of the rates of discount implies that $\mu(t) - \frac{\alpha'}{\alpha}(t) > 0$ for all $t$.

For any $\varepsilon > 0$ define $U_\varepsilon^{tn}$ by:

$$U_\varepsilon^{tn}(c,T) = \phi_\mu\left(\int_0^T u_\varepsilon(c(t))dt\right)$$

with

$$u_\varepsilon(c(t)) = 1 + \varepsilon u(c(t)) \text{ and } \phi_\mu(x) = \int_0^x (\alpha(t) - \frac{\alpha'(t)}{\mu(t)})dt$$

Because $\alpha > 0$ and $\mu(t) - \frac{\alpha'}{\alpha}(t) > 0$, the function $\phi_\mu$ has a positive derivative. Also $u'_\varepsilon = \varepsilon u' > 0$. Thus, the utility functions $U_\varepsilon^{tn}$ represent time neutral preferences.

From (20), we know that the corresponding expected utility function can be written



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as:

$$E_\mu U_\varepsilon^{tn}(c) = \int_0^{+\infty} s(t) u_\varepsilon(c(t)) \phi'_\mu \left( \int_0^t u_\varepsilon(c(\tau)) d\tau \right) dt$$

I show below that, for any consumption paths $c_0$, $c_1, c$, such that $c_1(t) > c_0(t)$ for all $t$,we have:

$$E_\mu U_\varepsilon^{tn}(c_1) - E_\mu U_\varepsilon^{tn}(c_0) > 0 \tag{23}$$

and

$$\lim_{\varepsilon \to 0} \left( \frac{E_\mu U_\varepsilon^{tn}(c) - E_\mu U_\varepsilon^{tn}(c_0)}{E_\mu U_\varepsilon^{tn}(c_1) - E_\mu U_\varepsilon^{tn}(c_0)} \right) = \frac{E_\mu U^{add}(c) - E_\mu U^{add}(c_0)}{E_\mu U^{add}(c_1) - E_\mu U^{add}(c_0)} \tag{24}$$

This is what is meant by "converges weakly up to positive affine transformations". Clearly, these conditions guarantee that at the limit $\varepsilon \to 0$ the expected utility function $E_\mu U_\varepsilon^{tn}$ will represent the same preferences over consumption profiles as $E_\mu U^{add}$.

Inequality (23) is a direct consequence of the fact that the utility functions $U_\varepsilon^{tn}$ are increasing in consumption that occurs before death. Equality (24) is shown thereafter using a Taylor expansion in $\varepsilon$. We have:

$$E_\mu U_\varepsilon^{tn}(c) = \int_0^{+\infty} s(t) u_\varepsilon(c(t)) \phi'_\mu \left( \int_0^t u_\varepsilon(c(\tau)) d\tau \right) dt$$

Replacing $u_\varepsilon(c, t)$ by $1 + \varepsilon u(c(t))$ and keeping only the zero and first order terms in $\varepsilon$ we find:

$$\begin{aligned} E_\mu U_\varepsilon^{tn}(c) \;=\; & \textstyle\int_0^{+\infty} s(t) \phi_\mu(t) dt \\ & + \varepsilon \textstyle\int_0^{+\infty} s(t) u(c(t)) \phi'_\mu(t) dt \\ & + \varepsilon \textstyle\int_0^{+\infty} s(t) \phi''_\mu(t) \left( \int_0^t u(c(\tau)) d\tau \right) dt \\ & + o(\varepsilon) \end{aligned} \tag{25}$$

The first term is a constant, independent of $c$ and $\varepsilon$, that I denote by $A$. Switching the order of integration in the third term, we find that:

$$\int_0^{+\infty} s(t) \phi''_\mu(t) \left( \int_0^t u(c(\tau)) d\tau \right) dt = \int_0^{+\infty} u(c(t)) \left( \int_t^{+\infty} s(\tau) \phi''_\mu(\tau) d\tau \right) dt \tag{26}$$

Proceeding with an integration by parts and using $\phi'_\mu(t) = \alpha(t) - \frac{\alpha'(t)}{\mu(t)}$, we compute:

$$\begin{aligned}
\int_t^{+\infty} s(\tau)\phi''_\mu(\tau)d\tau &= \left[s(t)\left(\alpha(t) - \frac{\alpha'(t)}{\mu(t)}\right)\right]_t^{+\infty} - \int_t^{+\infty} s'(\tau)\left(\alpha(t) - \frac{\alpha'(t)}{\mu(t)}\right)d\tau \\
&= -s(t)\left(\alpha(t) - \frac{\alpha'(t)}{\mu(t)}\right) - \int_t^{+\infty}\left[s'(\tau)\alpha(\tau) + \alpha'(\tau)s(\tau)\right]d\tau \\
&= s(t)\frac{\alpha'(t)}{\mu(t)}
\end{aligned} \tag{27}$$

Using (26) and (27), and replacing $\phi'_\mu(t)$ by $\alpha(t) - \frac{\alpha'(t)}{\mu(t)}$ in the second term of (25), we finally obtain:

$$E_\mu U_\varepsilon^{tn}(c) = A + \varepsilon \int_0^{+\infty} s(t)\alpha(t)u(c(t))dt + o(\varepsilon) \tag{28}$$

The first order term is thus precisely $E_\mu U^{add}(c)$ and (24) follows directly from (28).

# E Proof of Proposition 6

The result for the additive case is immediate from equation (7). For the time neutral case, the result is also obvious from equation (8) when $t_2 \leq t_1$. The only difficult case is when $t_2 > t_1$. In this instance, equation (21) can be rewritten as:

$$RD_\mu^{tn}(c,t) = \mu(t_1)\frac{\phi'(\int_0^{t_1} u(c(\tau))d\tau)}{\phi'(\int_0^{t_1} u(c(\tau))d\tau) + \int_{t_1}^{+\infty}\frac{s(\tau)}{s(t_1)}u(c(\tau))\phi''(\int_0^{t_1} u(c(\tau_1))d\tau_1)d\tau} \tag{29}$$

Note that $\frac{s(\tau)}{s(t_1)} = \exp(-\int_{t_1}^{\tau}\mu(t)dt)$. So we have:

$$\begin{aligned}
\frac{\partial\left(\frac{s(\tau)}{s(t_1)}\right)}{\partial\mu(t_2)} &= -\frac{s(\tau)}{s(t_1)} \text{ for } t_1 < t_2 < \tau \\
\frac{\partial\left(\frac{s(\tau)}{s(t_1)}\right)}{\partial\mu(t_2)} &= 0 \text{ for } t_1 < \tau < t_2
\end{aligned}$$

This implies that for $t_2 > t_1$ :

$$\frac{\partial}{\partial\mu(t_2)}\left(\int_{t_1}^{+\infty}\frac{s(\tau)}{s(t)}u(c(\tau))\phi''(\int_0^{t_1} u(c(\tau_1))d\tau_1)d\tau\right) = \\ -\int_{t_2}^{+\infty}\frac{s(\tau)}{s(t_1)}u(c(\tau))\phi''(\int_0^{\tau} u(c(\tau_1))d\tau_1)d\tau$$

which explains why we obtain (9) by taking the derivative of (29) with respect to $\mu(t_2)$.



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# F Proof of Proposition 7

For the first point, rewrite (21) for $i = 1, 2$:

$$RD^{tn}_{\mu_i}(c,t) = \mu_i(t)\frac{\phi'(\int_0^t u(c(\tau))d\tau)}{\int_t^{+\infty} \mu_i(\tau)\exp\left(-\int_t^\tau \mu_i(\tau_1)d\tau_1\right)\phi'(\int_0^\tau u(c(\tau_1))d\tau_1)d\tau}$$

Inequality (10) implies that $\mu_2(t) \leq \mu_1(t)$. Thus $\exp\left(-\int_t^\tau \mu_2(\tau_1)d\tau_1\right) \geq \exp\left(-\int_t^\tau \mu_1(\tau_1)d\tau_1\right)$ for all $\tau \geq t$. Moreover we also know from inequality (10) that $\mu_2(\tau) \geq \mu_1(\tau)\frac{\mu_2(t)}{\mu_1(t)}$for all $\tau \geq t$. It follows that $RD^{tn}_{\mu_2}(c,t) \leq RD^{tn}_{\mu_1}(c,t)$.

For the second point, use (8) to write that for $i = 1, 2$

$$RD^{tn}_{\mu_i}(c,t) = \mu_i(t) - \mu_i(t)\frac{\int_t^{+\infty} \exp\left(-\int_t^\tau \mu_i(\tau_1)d\tau_1\right)u(c(\tau))\phi^{''}(I_\tau)d\tau}{\int_t^{+\infty} \mu_i(\tau)\exp\left(-\int_t^\tau \mu_i(\tau_1)d\tau_1\right)\phi^{'}(I_\tau)d\tau}$$

where $I_\tau = \int_0^\tau u(c(\tau_1))d\tau_1$. Using $\mu_2(\tau) \geq \mu_1(\tau)\frac{\mu_2(t)}{\mu_1(t)}$ and $\phi^{''} < 0$ we obtain

$$RD^{tn}_{\mu_1}(c,t) - RD^{tn}_{\mu_2}(c,t) \geq \mu_1(t) - \mu_2(t) + \mu_1(t)\Delta$$

with

$$\Delta = \frac{\int_t^{+\infty} k(\tau)g(\tau)d\tau}{\int_t^{+\infty} g(\tau)d\tau} - \frac{\int_t^{+\infty} k(\tau)h(\tau)g(\tau)d\tau}{\int_t^{+\infty} h(\tau)g(\tau)d\tau}$$

where $k(\tau) = -\frac{\phi''(I_\tau)}{\phi'(I_\tau)}\frac{u(c)}{\mu_1(\tau)}$, $h(\tau) = \exp\left(-\int_t^\tau(\mu_2(\tau_1) - \mu_1(\tau_1))d\tau_1\right)$ and $g(\tau) = \mu_1(\tau)\exp\left(-\int_t^\tau \mu_1(\tau_1)d\tau_1\right)u(c(\tau))\phi^{'}(I_\tau)$. The functions $k$, $g$ and $h$ are non-negative. Note also that, by assumption, $h$ is non-decreasing while $k$ is non-increasing. Thus, $\Delta$ is non-negative[32] and $RD^{tn}_{\mu_1}(c,t) - RD^{tn}_{\mu_2}(c,t) \geq \mu_1(t) - \mu_2(t)$. The fact that $RD^{add}_{\mu_1}(c,t) - RD^{add}_{\mu_2}(c,t) = \mu_1(t) - \mu_2(t)$ is a direct consequence of (7). The proof of Proposition 7 is then complete.

[32] To prove that $\Delta \geq 0$ one can show that the function

$$f(x) = \left(\int_t^x k(\tau)g(\tau)d\tau\right)\left(\int_t^x h(\tau)g(\tau)d\tau\right) - \left(\int_t^x k(\tau)h(\tau)g(\tau)d\tau\right)\left(\int_t^x g(\tau)d\tau\right)$$

is non-decreasing (and therefore non-negative) for $x \geq t$.

# G Proof of Proposition 8

Write

$$VSL(t) = -\frac{\frac{\partial E_\mu U}{\partial \mu(t)}}{\frac{\partial E_\mu U}{\partial c(t)}}$$

Now, since:

$$s(t) = \exp\left(-\int_0^t \mu(\tau)d\tau\right)$$

we have

$$\frac{\partial E_\mu U}{\partial \mu(t)} = -\int_t^{+\infty} \frac{\partial E_\mu U}{\partial s(\tau)} d\tau$$

and

$$VSL(t) = \int_t^{+\infty} \frac{\frac{\partial E_\mu U}{\partial s(\tau)}}{\frac{\partial E_\mu U}{\partial c(\tau)}} \frac{\frac{\partial E_\mu U}{\partial c(\tau)}}{\frac{\partial E_\mu U}{\partial c(t)}} d\tau = \frac{1}{s(t)e^{-rt}} \int_t^{+\infty} s(\tau)e^{-r\tau} \frac{\frac{\partial E_\mu U}{\partial s(\tau)}}{\frac{\partial E_\mu U}{\partial c(\tau)}} d\tau$$

Now compute

$$\frac{\frac{\partial E_\mu U^{add}}{\partial s(\tau)}}{\frac{\partial E_\mu U^{add}}{\partial c(\tau)}} = \frac{1}{s(\tau)} \frac{u^{add}(c(\tau))}{u'(c(\tau))}$$

so that

$$VSL^{add}(t) = \frac{1}{s(t)e^{-rt}} \int_t^{+\infty} e^{-r\tau} \frac{u^{add}(c(\tau))}{u'(c(\tau))} d\tau$$

For time neutral specification:

$$\frac{\frac{\partial E_\mu U^{tn}}{\partial s(\tau)}}{\frac{\partial E_\mu U^{tn}}{\partial c(\tau)}} = \frac{u^{tn}(c(\tau))\phi'\left(\int_0^\tau u(c(\tau_1))d\tau_1\right)}{u'(c(\tau)) \int_t^{+\infty} d(\tau)\phi'\left(\int_0^\tau u(c(\tau_1))d\tau_1\right) d\tau} = \frac{1}{s(\tau)} \frac{RD(\tau)}{\mu(\tau)} \frac{u^{tn}(c(\tau))}{u'(c(\tau))}$$

And

$$VSL^{tn}(t) = \frac{1}{s(t)e^{-rt}} \int e^{-r\tau} \frac{RD(\tau)}{\mu(\tau)} \frac{u^{tn}(c(\tau))}{u'(c(\tau))} d$$

That implies

$$\frac{VSL^{tn}(t)}{VSL^{add}(t)} = \frac{\int_t^{+\infty} \frac{RD(\tau)}{\mu(\tau)} \frac{u^{tn}(c(\tau))}{u^{add}(c(\tau))} h(\tau)d\tau}{\int_t^{+\infty} h(\tau)d\tau}$$

with $h(\tau) = e^{-r\tau}\frac{u^{add}(c(\tau))}{u'(c(\tau))} > 0$. Now, because it is assumed the same intertemporal elasticity of substitution in both the time neutral model and the additive models, it must be



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the case that $u^{tn}(c(\tau)) = u^{add}(c(\tau)) + \delta$ where $\delta$ is a constant. We obtain

$$\frac{VSL^{tn}(t)}{VSL^{add}(t)} = \frac{\int_t^{+\infty} \frac{RD(\tau)}{\mu(\tau)}(1 + \frac{\delta}{u^{add}(c(\tau))})h(\tau)d\tau}{\int_t^{+\infty} h(\tau)d\tau}$$

Now recall that $RD(\tau) > \mu(\tau)$. Since $VSL^{tn}(t_0) = VSL^{add}(t_0)$, it must be the case that $\delta < 0$. Since $c$ is non-increasing, $(1 + \frac{\delta}{u^{add}(c(\tau))})$ is non increasing. It is also the case of $\frac{RD(\tau)}{\mu(\tau)}$, by assumption. That implies that $\frac{VSL^{tn}(t)}{VSL^{add}(t)}$ is a non increasing function, which completes the proof of Proposition 8.

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# Figure 1: Mortality Rate at Age 30 (Historical Data from the USA)

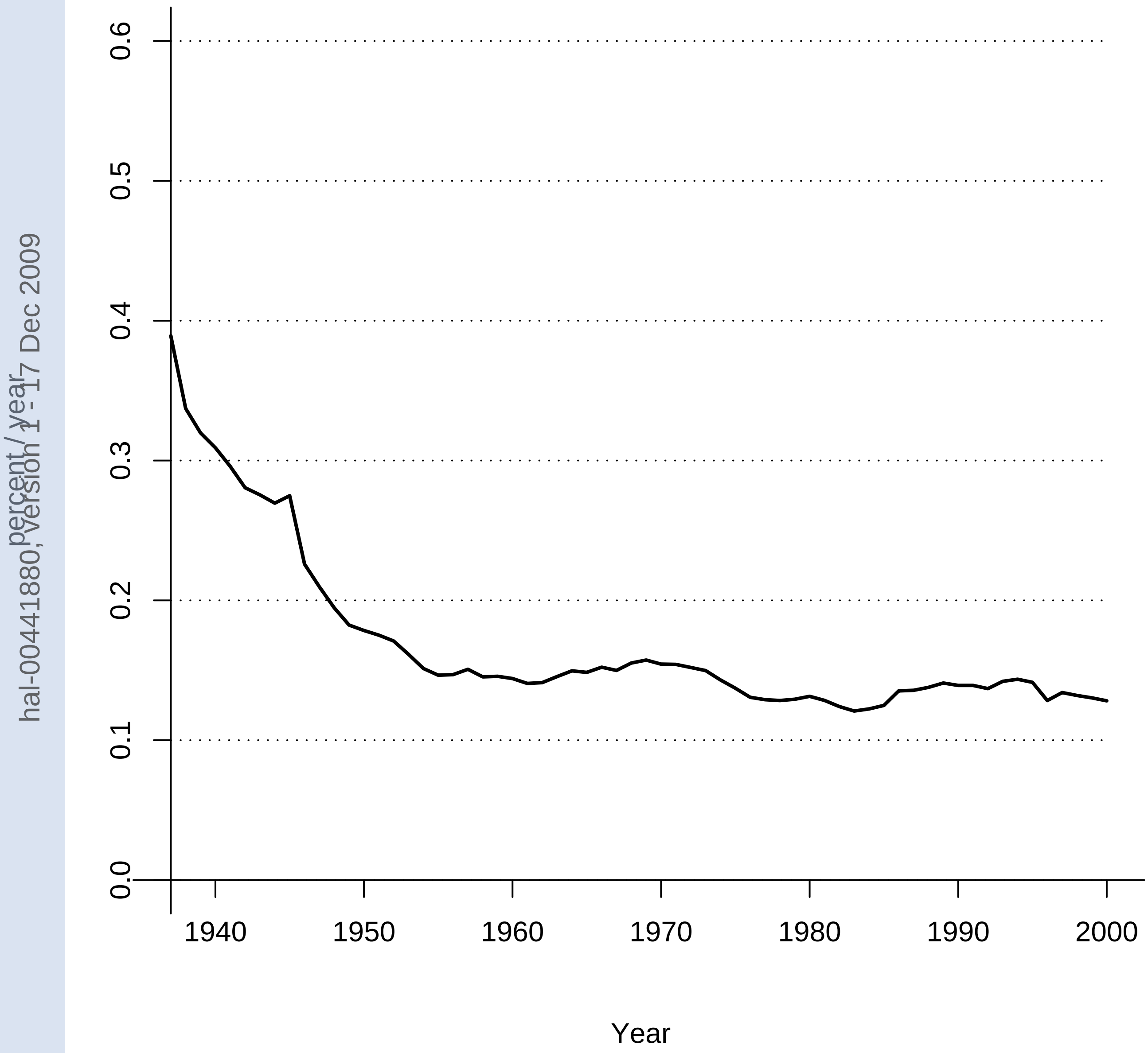


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**Figure 2: Life Expectancy at Age 30 (Historical Data from the USA)**

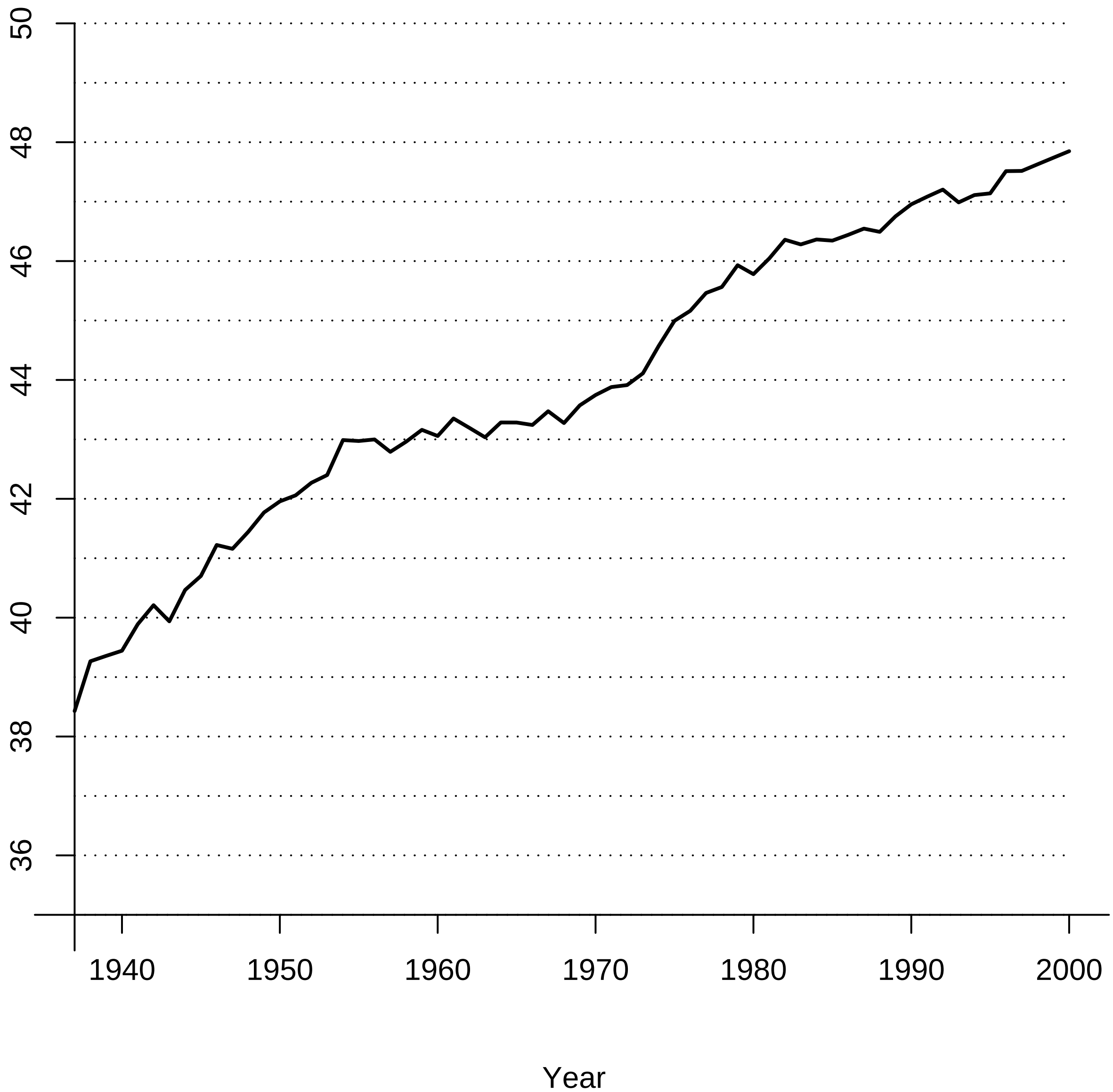


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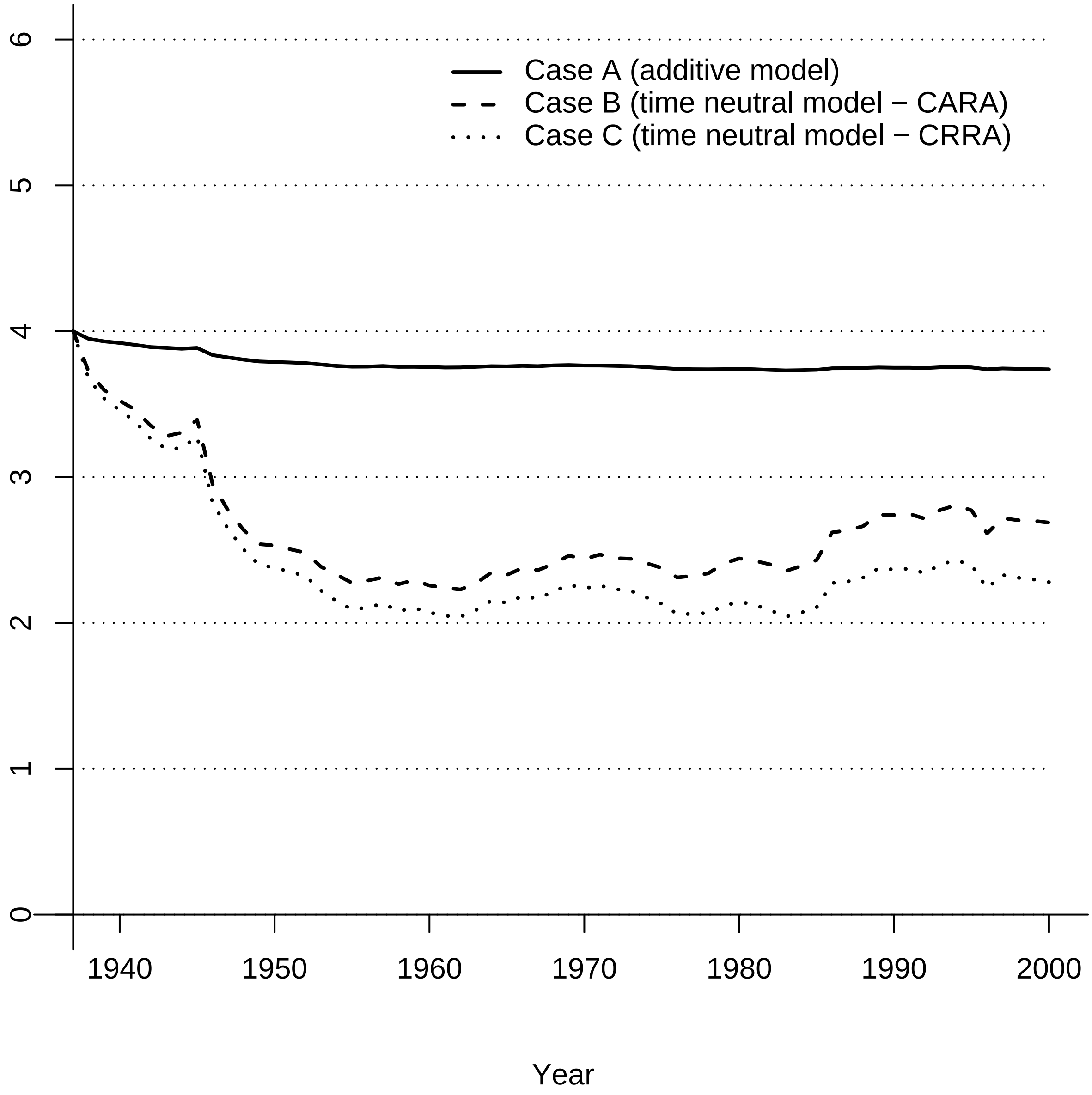
**Figure 3: Rate of Discount at Age 30 According to Historical Mortality rates**

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# Figure 4

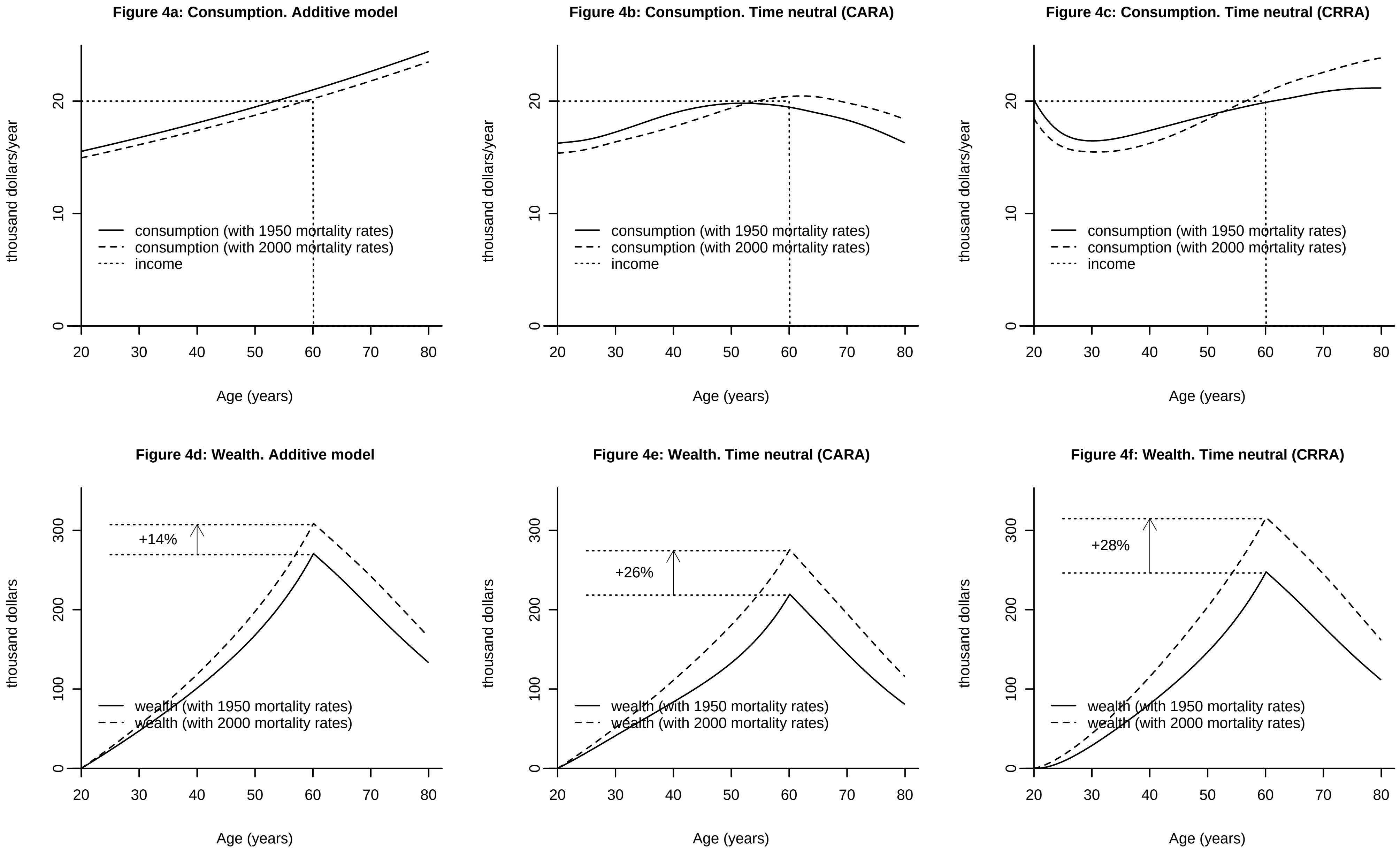

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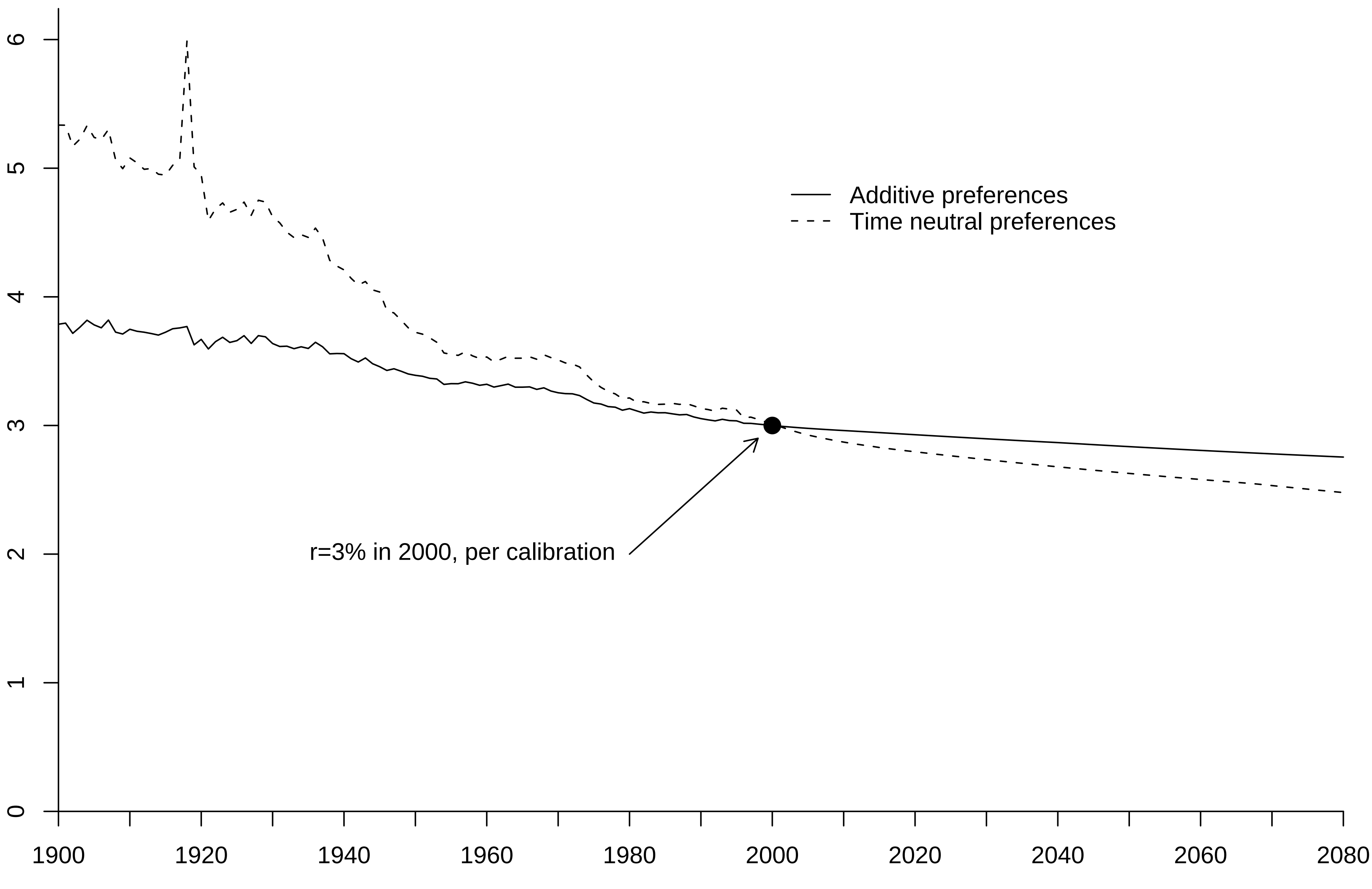
**Figure 5: Rate of interest in steady−state general equilibria**




Historical and projected US mortality.

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**Figure 6: Value of Statistical Lives**

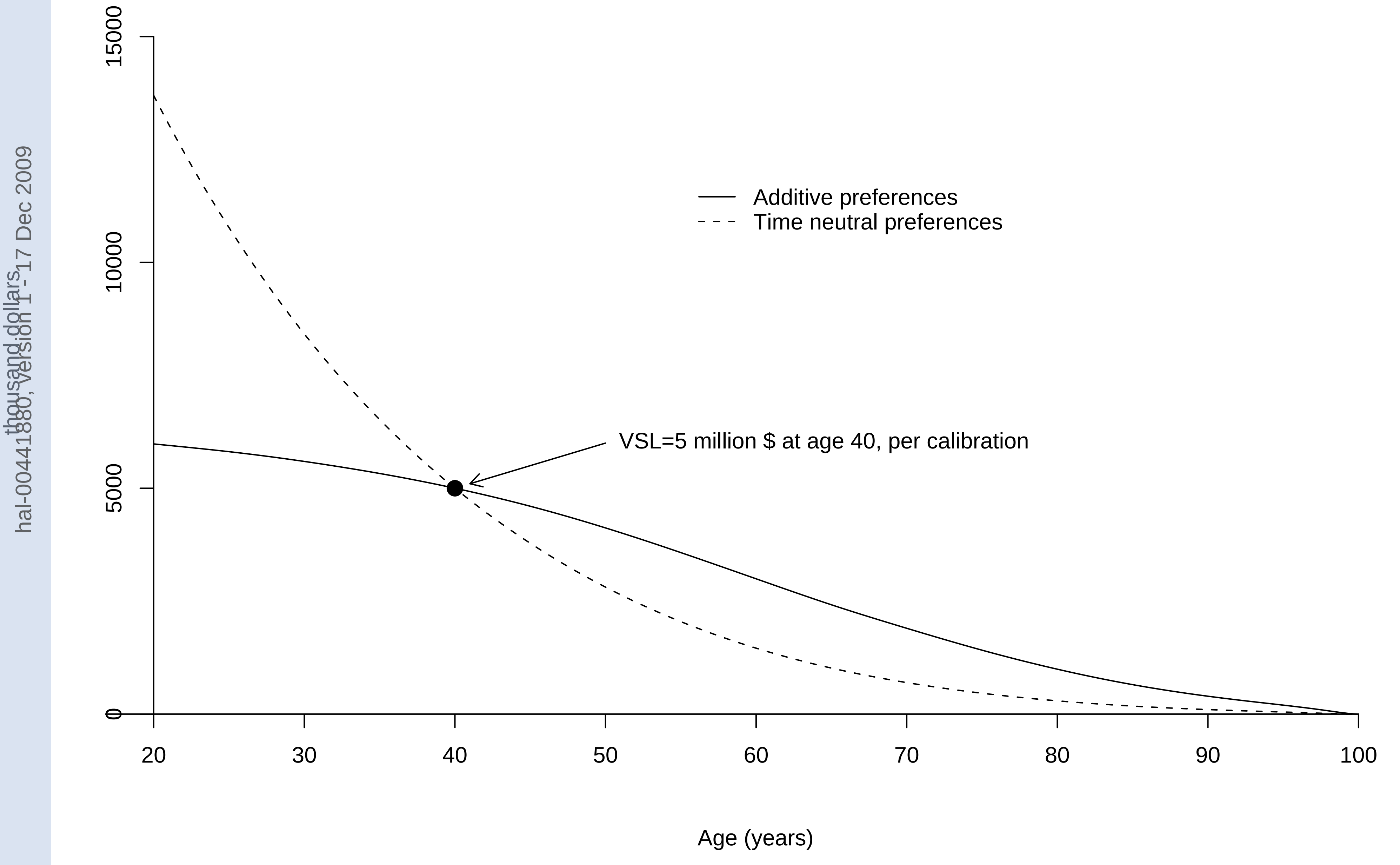
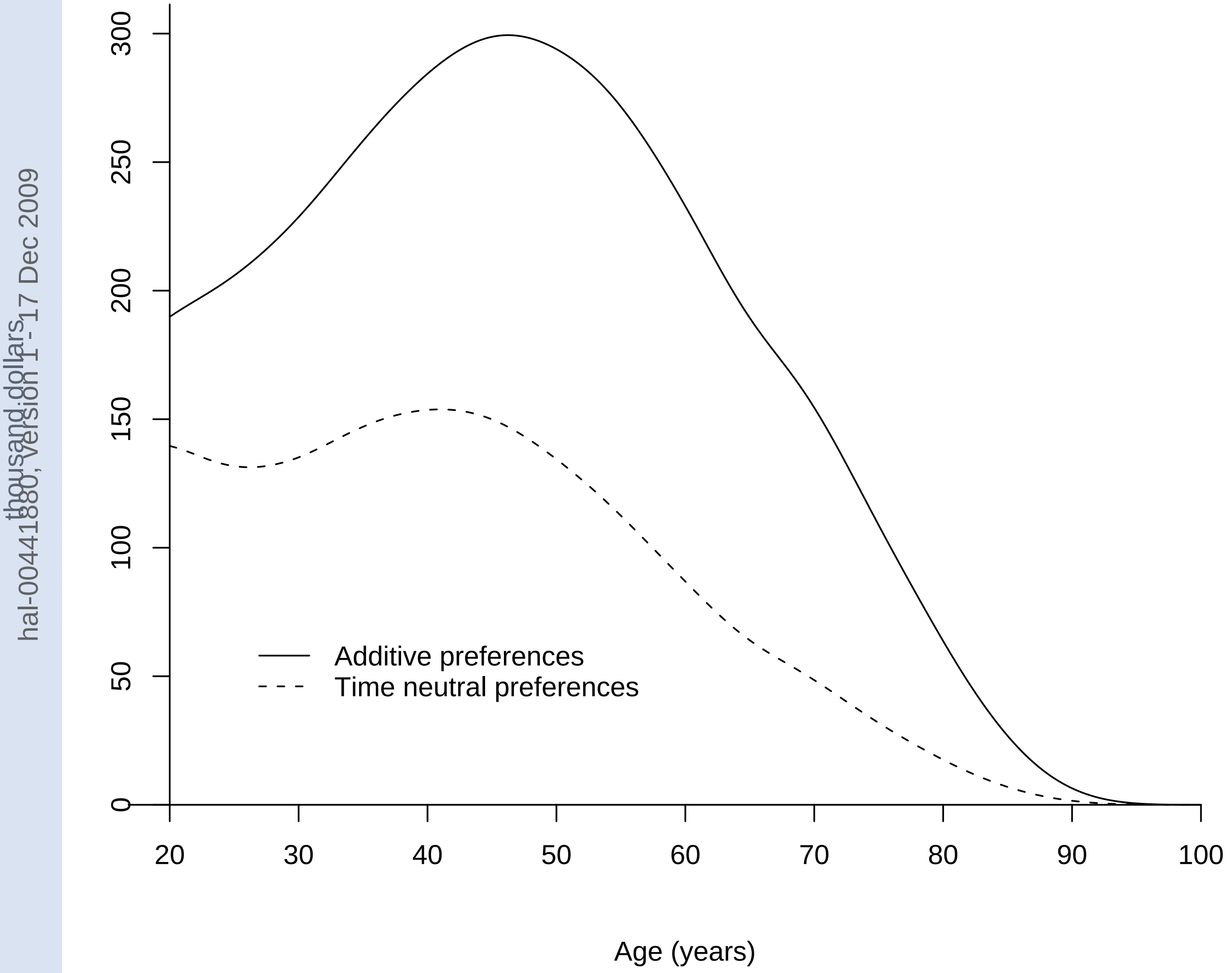


Figure 7: Gains from increased longevity, 1970−2000

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