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## COALITIONAL EQUILIBRIA OF STRATEGIC GAMES

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# Coalitional Equilibria of Strategic Games \*

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## Abstract

Let  $N$  be a set of players,  $\mathcal{C} \subset 2^N$  a set of permissible coalitions and  $G$  an  $N$ -player strategic game. A profile is a coalitional-equilibrium if no coalition in  $\mathcal{C}$  has a unilateral deviation that profits all its members. Nash-equilibria [10] correspond to  $\mathcal{C} = \{\{i\}, i \in N\}$ , Aumann-equilibria [2] (usually called strong-equilibria) to  $\mathcal{C} = 2^N$ . A new fixed point theorem allows to obtain a condition for the existence of coalitional equilibria that covers Glicksberg [7] for the existence of Nash-equilibria and is related to Ichiishi's [9] condition for the existence of Aumann-equilibria.

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## Introduction

Nash [10] and Aumann [2] equilibria (also known as strong equilibria) are the two main solution concepts in non-cooperative game theory. The first asks the stability of the strategy profile against all single player unilateral deviations while the second asks the stability against all coalitions unilateral deviations. The technics that are used to show existence are of different types. To establish the existence of Nash equilibria, Glicksberg [7] needs quasi-concavity of the payoff functions and uses a standard fixed point theorem (i.e. Brouwer/Kakutani). For the existence of Aumann equilibria, Ichiishi [9] assumes the game to be balanced and uses the KKMS-lemma (for Knaster-Kuratowski-Mazurkiewicz-Shapley, lemma 22.4 in [4]) established by Shapley.

The paper has two main contributions. First, it provides a new fixed point theorem. Second, it defines the concept of coalitional equilibrium and provides a sufficient condition for its existence. Coalitional equilibria lies between Nash and Aumann. An exogenous coalitional structure defines which coalitions are admissible to jointly deviate. A strategy profile is a coalitional-equilibrium if it is stable against all admissible coalitions unilateral

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deviations. When only single-player coalitions are permissible it is Nash-equilibrium and when all coalitions are admissible it is Aumann-equilibrium. The motivation is straightforward. In many applications (voting, council of Europe, or market competition) some coalitions are not natural and so cannot be expected to coordinate (extreme-leftists and -rightists, western and eastern countries, or firms of different areas).

Ray and Vohra [12] consider a solution concept that uses coalitional equilibria but only for partitions. First, for each partition of the players, they associate the set of coalitional equilibria. Second, given a coalitional-equilibrium for a given partition, one can define internal stability with respect to stable coalitional equilibria of all finer partitions. The set of stable equilibria for a given partition is constructed recursively. Starting from the finest partition structure (that contains only singleton), one can construct stable coalitional equilibria for larger partition structures, and so on. Clearly, a similar construction could be done if the partition constraint is relaxed. This naturally leads to what may be called stable coalitional equilibria.

In the next section, a new fixed point theorem is established. It is closely related to Sonnenschein's [13] theorem. It provides conditions for a non-transitive preference to have a maximal element. Its proof is based on KKM-lemma (lemma 17.43 in [1]). It is then applied to establish a new version of the well known Gale and Mas-Colell [5] theorem. In the last section, the fixed point theorem is applied to obtain a quasi-concavity condition for the existence of coalitional equilibria as well as for the existence of Berge equilibria [3] (defined below). This induces a unified existence result for Nash, Aumann and Berge equilibria. The theorem may be useful in other contexts such as competitive equilibria as in Sonnenschein [13] or social coalitional equilibria as in Ichiishi [8]. The latter is an equilibrium concept in a general game model that mixes cooperation and conflict and links Aumann equilibria to the Core.

## A new fixed point theorem

Let  $S$  denote a compact convex subset of a Hausdorff and locally convex topological vector space (TVS). The interior of  $X \subset S$  relatively to  $S$  is denoted  $intX$ , its convex envelope  $coX$  and the closed convex hull  $\overline{co}X$ . Recall Ky Fan's lemma, which is nothing but a generalization of the well known KKM-lemma to correspondences (see [1] theorem 17.46).

**Lemma 1 (Ky Fan)** *If the correspondence  $F$  from  $S$  to  $S$  has closed values and if for any finite family  $\{x_1, \dots, x_k\}$ ,  $co\{x_1, \dots, x_k\} \subset \cup_{i=1, \dots, k} F(x_i)$  then  $\cap_{x \in S} F(x) \neq \emptyset$ .*

Let  $A$  be a correspondence on  $S$  (i.e. from  $S$  to  $S$ ), best viewed as a (not necessarily) transitive preference where  $A(x) \subset S$  is the set of points in  $S$  better than  $x$ . The set of maximal elements of a  $A$  is  $E = \{x \in S \text{ such that } A(x) = \emptyset\}$ .

A famous implication of the Ky Fan lemma is the following important result of Sonnenschein [13]. For its proof and the numerous applications, see [1] and [4].

**Theorem 2 (Sonnenschein)** *Let  $A$  be a correspondence on  $S$ . **If** (i) for all  $x \in S$ ,  $x \notin coA(x)$  and (ii) for any  $y \in A^{-1}(x)$  there exists  $x' \in S$  (possibly  $x' = x$ ) such that  $y \in intA^{-1}(x')$ , **then** the set of maximal elements of  $A$  is compact and non-empty.*

The second assumption of Sonnenschein is very strong (in applications, it asks the lower-sections  $A^{-1}(x)$  to be open for every  $x$ ). The fixed point theorem that follows reinforces slightly assumption (i) and relaxes sufficiently (ii) to allow  $A$  to be only lower-semi-continuous (i.e.  $A^{-1}(W)$  is asked to be open for any  $W$  that is open).

**Theorem 3** *Let  $A$  be a correspondence on  $S$ . **If** (i') for all  $x \in S$ ,  $x \notin \overline{co}A(x)$  and (ii') for any  $y \in A^{-1}(x)$  and any convex open neighborhood  $W$  of zero, there exists  $x' \in S$  (possibly  $x' = x$ ) such that  $y \in \text{int}A^{-1}(x' + W)$ , **then** the set  $E$  of maximal elements of  $A$  is compact and non-empty.*

We first establish a useful lemma (to be compared with lemma 17.47 in [1]).

**Lemma 4** *If there is a convex open neighborhood  $W$  of zero such that for each  $x \in S$ ,  $x \notin coA(x) - W$ , then  $\bigcap_{x \in S} F(x) \neq \emptyset$  where  $F(x)$  is the closure of the complement of  $A^{-1}(x + W)$ :  $F(x) = \overline{S/A^{-1}(x + W)}$ .*

**Proof.** From Ky Fan's lemma, it is sufficient to show that for any finite family  $\{x_1, \dots, x_k\}$  and any  $y \in co\{x_1, \dots, x_k\}$ ,  $y \in \bigcup_{i=1, \dots, k} F(x_i)$ . Suppose not. This implies that for any  $i$ ,  $y \notin F(x_i)$  implying that  $x_i \in A(y) - W$ . Since  $W$  is convex, conclude that  $y \in coA(y) - W$ , a contradiction. ■

We now prove the fixed point theorem, inspired by the proof of Sonnenschein's theorem 17.48 in [1].

**Proof.** If for any convex open neighborhood  $W$  of zero, there is an  $x$  in  $S$  such that  $x + W \cap coA(x) \neq \emptyset$  then, by compactness, Hausdorff and local convexity assumptions, there will exist  $x \in S$  such that  $x \in \overline{co}A(x)$ , contradicting (i'). Thus, there is an open and convex neighborhood  $W$  of zero such that, for all  $x \in S$ ,  $x + W \cap coA(x) = \emptyset$ . Note that  $E = \bigcap_{x \in S} S/A^{-1}(x + W)$ . By (ii'),  $E = \bigcap_{x' \in S} S/\text{int}A^{-1}(x' + W)$ , so that it is compact (as the intersection of a family of compact sets). Thus  $\overline{S/A^{-1}(x + W)} \subset S/\text{int}A^{-1}(x + W)$ . Hence,  $\bigcap_{x \in S} \overline{S/A^{-1}(x + W)} \subset \bigcap_{x \in S} S/\text{int}A^{-1}(x + W)$ . By (i') and the last lemma,  $\bigcap_{x \in S} S/A^{-1}(x) \neq \emptyset$ . ■

This provides a new short proof of Brouwer-Schauder-Tychonoff's theorem [1]. Actually, if  $f$  is continuous from  $S$  to  $S$  then  $A(x) = \{f(x)\}$  satisfies (ii') and is never empty, thus there exists  $x$  such that  $x \in \overline{co}A(x) = \{f(x)\}$ .

In [1], it is shown that lemma 1 and theorem 2 hold without the local convexity assumption. Schauder conjectured in 1935 that every convex compact set  $S$  of a Hausdorff TVS has the fixed point property (i.e. any continuous function from  $S$  to  $S$  has a fixed point). His conjecture has been proved only recently by Robert Cauty [6]. Using Cauty's result, one could drop local convexity in theorem 3 as well (but in that case, the proof will depend on a highly non trivial result).

To illustrate the power of the theorem, let us use it to establish a new version of the very helpful Gale and Mas-Colell [5] theorem. Assume  $S = \prod_{i \in N} S_i$  where each  $S_i$  is a compact-convex subset of a locally convex TVS (typically, the strategy set of player  $i$  where  $N$  is the set of players, not necessarily finite.). By Tychonoff [1],  $S$  is

convex, compact, Hausdorff and locally convex for the product topology. As usual, let  $S_{-i} = \prod_{j \neq i} S_j$  be the set of profiles of players other than  $i$ .

Let  $\{A_i\}_{i \in N}$  denotes a family of correspondences, where for each  $i$  in  $N$ ,  $A_i$  is a correspondence from  $S$  to  $S_i$ . The set of maximal elements of  $\{A_i\}_{i \in N}$  is  $\{x \text{ such that } A_i(x) = \emptyset \text{ for all } i \text{ in } N\}$ .

**Theorem 5** *If for each  $i \in N$  and  $x \in S$  (1)  $x_i \notin \overline{co}A_i(x)$  and (2)  $A_i$  is lower-semi-continuous **then** the set of maximal elements of  $\{A_i\}_{i \in N}$  is nonempty and compact.*

**Proof.** Define the correspondence  $A$  on  $S$  as follows:  $A(x) := \bigcup_{i \in N} A_i(x) \times \{x_{-i}\}$ . Clearly  $A$  is lower-semi-continuous. Also, if for some  $x \in S$ ,  $x \in \overline{co}A(x)$  then there is  $i \in N$  such that  $x_i \in \overline{co}A_i(x)$ . This contradicts (1). Thus,  $A$  satisfies (i') and (ii'). Consequently, there is  $x \in S$  such that for all  $i \in N$ ,  $A_i(x)$  is empty. ■

Gale and Mas-Colell [5] proved the result under the assumptions (1')  $x_i \notin coA_i(x)$  for every  $x$  and (2')  $A_i$  has an open graph. The later assumption is very strong. Assumption (2) is much weaker, but (1) is slightly stronger than (1').

## Application to coalitional equilibria

Let  $N$  be a set of players, not necessarily finite. Let  $G = (N, \{S_i\}_{i \in N}, \{g_i\}_{i \in N})$  be a strategic game. Assume that for each  $i$  in  $N$ ,  $S_i$ , the strategy set of player  $i$ , is a compact-convex subset of a Hausdorff and locally convex TVS and the payoff function of player  $i$ ,  $g_i : S \rightarrow \mathbf{R}$ , is continuous. This defines a compact-continuous strategic game. Let  $\mathcal{C} \subset 2^N$  be the set of permissible coalitions. As usual,  $S = \prod_{i \in N} S_i$  is the set of strategy profiles,  $S_C = \prod_{j \in C} S_j$  and  $N/C$  denotes the set of players outside  $C$ .

**Definition 6**  *$s$  is a coalitional-equilibrium of  $G$  if no admissible coalition in  $\mathcal{C}$  has a unilateral deviation that profits all its members; That is, there is no  $C$  in  $\mathcal{C}$  and no  $t_C \in S_C$  such that for any  $i \in C$ ,  $g_i(t_C, s_{N/C}) > g_i(s)$ .*

A particularly interesting class is  $\mathcal{C} = \bigcup_{k=1}^K 2^{P_k}$  where  $\{P_1, \dots, P_K\}$  is some partition of  $N$ . Restricting the model of Ichiishi [8], one could obtain a balanced-condition for the existence of this type of coalitional equilibria. The two conditions are related, as it is proven below. The interpretation is simple. Members in different  $P_k$ 's cannot talk to each others while members inside  $P_k$  could try to coordinate but cannot commit to never deviate.

**Definition 7**  *$G$  is  $\mathcal{C}$ -quasi-concave if for all  $s \in S$ ,  $\epsilon > 0$  and any family of permissible coalitions  $(C_k)_{k \in K}$  with corresponding strategies  $t_{C_k} \in S_{C_k}$ , **if** for all  $k$  and  $i \in C_k$   $g_i(t_{C_k}, s_{N/C_k}) \geq g_i(s) + \epsilon$ , **then**  $s \notin \overline{co}\{(t_{C_k}, s_{N/C_k}), k \in K\}$ .*

In finite dimensional strategy spaces and finitely many players, Caratheodory's theorem implies that  $\overline{co}$  above could be replaced by  $co$  and that only finitely many deviating coalitions are to be considered.

**Theorem 8** *If a compact-convex-continuous strategic game is  $\mathcal{C}$ -quasi-concave, the set of coalitional-equilibria is compact and non-empty.*

**Proof.** Let  $A_C(s) = \{(t_C, s_{N/C}) \text{ such that for all } i \in C, g_i(t_C, s_{N/C}) > g_i(s) + \epsilon\}$  and let  $A = \bigcup_C A_C$ . From the continuity of the game,  $A$  is lower-semi-continuous. Suppose  $s \in \overline{\text{co}}A(s)$  for some  $s$ . Thus there exists a family of permissible coalitions  $(C_k)_{k \in K}$  and strategies  $(t_{C_k} \in S_{C_k})_{k \in K}$  such that  $s \in \overline{\text{co}}\{(t_{C_k}, s_{N/C_k}), k \in K\}$  and  $g_i(t_{C_k}, s_{N/C_k}) \geq g_i(s) + \epsilon$  for all  $k$  and  $i \in C_k$ : a contradiction. Hence,  $A$  has a maximal element  $s_\epsilon$  whose accumulation points are coalitional-equilibria. ■

Other definitions of  $A$  would lead to other concepts. For example, if  $A_C(s) = \{t_C \in S_C : \text{there is } i \in C \text{ such that } g_i(t_C, s_{-C}) > g_i(s) + \epsilon\}$ , the underlying concept requires that an admissible coalition has a deviation if at least one of its members profits (even if all the other players inside the coalition lose: this is too demanding!). Hence, at equilibrium, players outside an admissible coalition forces all the players inside the coalition to play according to the equilibrium. The concept was defined by Claude Berge [3] but only for admissible coalitions of the form  $N/\{i\}$ ,  $i \in N$ . A coalitional Berge-equilibrium exists if  $G$  is compact-convex-continuous and *for all  $s \in S$ ,  $\epsilon > 0$  and any family of permissible coalitions  $(C_k)_{k \in K}$  with corresponding strategies  $t_{C_k} \in S_{C_k}$ , **if for all  $k$  there is  $i \in C_k$  such that  $g_i(t_{C_k}, s_{N/C_k}) \geq g_i(s) + \epsilon$ , **then  $s \notin \overline{\text{co}}\{(t_{C_k}, s_{N/C_k}), k \in K\}$ .*****

When only single player coalitions are permissible, the theorem is an improvement of Nash-Glicksberg [7] theorem (since it considers infinitely many players<sup>1</sup>). When all coalitions are admissible, the theorem provides a quasi-concavity condition for the existence of strong-equilibria. In the later case, quasi-concavity could be related to the balanced-condition in Ichiishi [9] and Border [4] as shown below. Recall that a finite family of nonempty subsets  $\mathcal{B}$  of  $N$  is *balanced* if for each  $B \in \mathcal{B}$ , there are nonnegative real numbers  $\lambda_B$  (balancing weights) such that for each  $i$  in  $N$ ,  $\sum_{B:i \in B} \lambda_B = 1$ .

**Definition 9**  *$G$  is balanced if for all  $\alpha \in [0, 1]^N$  and any finite balanced family of coalitions  $\{C_k\}$  with weights  $\{\lambda_k\}$  and corresponding strategies  $\{t^{C_k}\} \in S$ , **if  $g_i(t^{C_k}) > \alpha_i$  for all  $k$  and  $i \in C_k$  **then  $g_i(s) > \alpha_i$  for all players  $i \in N$ , where  $s_i := \sum_{k:i \in C_k} \lambda_k t_i^{C_k}$ .*****

A slightly stronger condition is:

**Definition 10**  *$G$  is strictly-balanced if for all  $\alpha \in [0, 1]^N$  and any finite balanced family of coalitions  $\{C_k\}$  with weights  $\{\lambda_k\}$  and corresponding strategies  $\{t^{C_k}\} \in S$ , **if  $g_i(t^{C_k}) \geq \alpha_i$  for all  $k$  and  $i \in C_k$  and  $g_i(t^{C_k}) > \alpha_i$  for some  $k$  and  $i \in C_k$  **then  $g_i(s) > \alpha_i$  for some player  $i \in N$ , where  $s_i := \sum_{k:i \in C_k} \lambda_k t_i^{C_k}$ .*****

**Lemma 11** *With finitely many players, finite dimensional strategy spaces and all coalitions permissible, strictly-balanced implies  $\mathcal{C}$ -quasi-concave.*

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<sup>1</sup>Observe that Nash-Glicksberg's theorem could be proved directly from our new version of Gale & Mas-Colell theorem. Reny [11] proved Nash-Glicksberg without the Hausdorff and local convexity assumptions. We believe that a similar improvement is possible for theorem 8.

**Proof.** Take  $(C_k)_{k \in K}$  to be a finite family of coalitions, let  $t_{C_k} \in S_{C_k}$  and  $s = \sum_k \alpha_k(t_{C_k}, s_{N/C_k})$ . The family  $\{\{C_k\} \cup \{N/C_k\}\}_{k \in K}$  with weights  $\{(\alpha_k, \alpha_k)\}_{k \in K}$  is balanced. Define  $t^{C_k} = (t_{C_k}, s_{N/C_k})$  and  $t^{N/C_k} = s$ . Then,  $s_i = \sum_{k:i \in C_k} \alpha_k t_i^{C_k} + \sum_{k:i \in N/C_k} \alpha_k t_i^{N/C_k}$ . Suppose that for each  $k$  and  $i \in C_k$ ,  $g_i(t^{C_k}) > g_i(s)$ . Since for all  $i \in N/C_k$ ,  $g_i(t^{N/C_k}) = g_i(s)$ , taking  $\alpha = g(s)$  leads to a contradiction with strictly-balanced. ■

So, quasi-concavity almost extends Ichiishi-Border's balanced-condition and is easier to interpret.

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