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# Algorithm for Financial Derivatives Evaluation in Generalized Double-Heston Model

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**Abstract**

This paper shows how can be estimated the value of an option if we assume the double-Heston model on a message-based architecture. For path trace simulation we will discretize continous model with an Euler division of time.

**Keywords**: Monte Carlo; algorithms; computational financial engineering; derivatives evaluation; Black–Scholes–Merton model; Heston model; double-Heston model; generalized double-Heston model.

## 1. BSM model, Heston model, Double-Heston model

From physical models, the following situation has reached acceptance: a financial asset interest rate follows a normal law, where the mean is the drift rate and the deviation is the volatility. This leads to a model that is currently accepted in finance:, the model of *geometric Brownian motion*. This model (known as *Black–Scholes–Merton model* in finance and financial engineering, see [1]) is a stochastic differential equation (1):

$$dS(t) = m\, S(t)\, dt + s\, S(t)\, dB(t), \qquad (1)$$

where:

a) $(S(t),\ t\geq 0)$ is a stochastic process for the value of stock;
b) $m$ is a static parameter for the drift rate of return;
c) $s^2$ is a static parameter for the volatility of stock ($s\geq 0$);
d) $(B(t),\ t\geq 0)$ is a standard Wiener process.

Another model is assumed by Heston (see [2]) and it consists from two stochastic differential equations. *The Heston* model corrects some inconsistency of the Black–Scholes–Merton model, for example:

a) in reality, volatility is not a static parameter; it can be used as static value only on short periods (this value will obtain on calibration process, usual with a statistical estimator);
b) on long periods, it is possible that interest rate series did not verify a normal law.

The Heston model is described by the following coupled stochastic differential equations (2), (3):

$$dS(t) = A(S(t), v(t), t)\, dt + B(S(t), v(t), t)\, dB_1(t) \qquad (2)$$

$$dv(t) = C(S(t), v(t), t)\, dt + D(S(t), v(t), t)\, dB_2(t) \qquad (3)$$

where:

a) $(S(t),\ t\geq 0)$ is a stochastic process for value of stock;
b) $(v(t),\ t\geq 0)$ is a stochastic process for volatility of value of stock;

c) $A(S, v, t)$, $B(S, v, t)$, $C(S, v, t)$, $D(S, v, t)$ are three parametric algebraic functions;
d) $(B_1(t), t \geq 0)$ and $(B_2(t), t \geq 0)$ are two $r$-correlated standard Wiener processes, i.e. (4):

$$dB_1(t)\, dB_2(t) = r\, dt \qquad (4)$$

For Wiener processes, more details can be found in [3]. For the basic Heston model we have (5):

$$\left.\begin{array}{l} \text{a) } A = S(t)\, m \\ \text{b) } B = S(t)\, v(t) \\ \text{c) } C = K(\theta - v(t)) \\ \text{d) } D = \xi\, v(t) \end{array}\right\} \qquad (5)$$

where:
a) $m$ is a drift of rate;
b) $\theta$ is long run average price volatility; as t tends to infinity, the expected value of $v(t)$ tends to $\theta$;
c) $K$ is the rate at which $v(t)$ reverts to $\theta$;
d) $\xi$ is the volatility of the volatility; as the name suggests, this determines the variance of $v(t)$.

Note that for $C = D = 0$ we obtain a static volatility model (Black–Scholes–Merton) (6)

$$dv(t) = 0.$$

The Double-Heston model (see [4]) is described by the following coupled stochastic differential equations (7), (8), (9):

$$dS(t) = M(S(t), v_1(t), v_2(t), t)\, dt + S_1(S(t), v_1(t), t)\, dB_1(t) + S_2(S(t), v_2(t), t)\, dB_2(t) \qquad (7)$$
$$dv_1(t) = C_1(S(t), v_1(t), t)\, dt + D_1(S(t), v_1(t), t)\, dB_3(t) \qquad (8)$$
$$dv_2(t) = C_2(S(t), v_2(t), t)\, dt + D_2(S(t), v_2(t), t)\, dB_4(t) \qquad (9)$$

where:

a) $(S(t), t \geq 0)$ is a stochastic process for value of stock;
b) $(v_1(t), t \geq 0)$ is a stochastic process for *half*-volatility of value of stock;
c) $(v_2(t), t \geq 0)$ is a stochastic process for *half*-volatility of value of stock;
d) $M(S, v_1, v_2, t)$, $S_1(S, v_1, t)$, $S_2(S, v_2, t)$, $C_1(S, v_1, t)$, $D_1(S, v_1, t)$, $C_2(S, v_2, t)$, $D_2(S, v_2, t)$ are three/four parametric algebraic functions;
e) $(B_1(t), t \geq 0)$ and $(B_3(t), t \geq 0)$ are two $r_1$-correlated standard Wiener processes, i.e. (10):

$$dB_1(t)\, dB_3(t) = r_1\, dt \qquad (10)$$

f) $(B_2(t), t \geq 0)$ and $(B_4(t), t \geq 0)$ are two $r_2$-correlated standard Wiener processes, i.e. (11):

$$dB_2(t)\, dB_4(t) = r_2\, dt \qquad (11)$$

g) $(B_1(t), t \geq 0)$ and $(B_2(t), t \geq 0)$ are two independent standard Wiener processes.

For the basic Double-Heston model we have (12):

$$\left.\begin{array}{l} \text{a) } M = S(t)\, m \\ \text{b) } S_1 = S(t)\, v_1(t) \\ \text{c) } S_2 = S(t)\, v_2(t) \\ \text{d) } C_1 = K_1\, (\theta_1 - v_1(t)) \\ \text{e) } C_2 = K_2\, (\theta_2 - v_2(t)) \\ \text{f) } D_1 = \xi_1\, v_1(t) \\ \text{g) } D_2 = \xi_2\, v_2(t) \end{array}\right\} \quad (12)$$

where:
a) $m$ is a drift of rate;
b) $\theta_1$ is long run average price volatility; as $t$ tends to infinity, the expected value of $v_1(t)$ tends to $\theta_1$;
c) $\theta_2$ is long run average price volatility; as $t$ tends to infinity, the expected value of $v_2(t)$ tends to $\theta_2$;
d) $K_1$ is the rate at which $v_1(t)$ reverts to $\theta_1$;
e) $K_2$ is the rate at which $v_2(t)$ reverts to $\theta_2$;
f) $\xi_1$ is the volatility of the volatility; as the name suggests, this determines the variance of $v_1(t)$;
g) $\xi_2$ is the volatility of the volatility; as the name suggests, this determines the variance of $v_2(t)$.

Any financial derivative based on support with price $S(t)$ at time $t$, with quotation at time $t$ and a value $S$ of support as $V(S, v_1, v_2, t)$, where (12):

$$V : R_+ \times [0,T] \times [0,T] \times [0,T] \to R_+ \quad (12)$$

and at maturity time $T$ will generate an generate an *payoff* (13):

$$payoff: R_+ \to R_+$$

For example, *European options CALL and PUT* has payoff functions (14):

$$\left.\begin{array}{l} payoff(x) = \max\{0, x - E\} \\ payoff(x) = \max\{0, E - x\} \end{array}\right\} \quad (14)$$

where $E$ is *excercise price of option*.

**2. Path trace simulation for option's pricing in generalized Double-Heston model**
First, we discretize continuous dimension of time. Let us denote (15):

$$t[k] = t[0] + k\Delta,\ 0 \le k \le N \quad (15)$$

where:
a) $\Delta = (T - t[0])/N$
b) $T$ is the maturity time of option;
c) $N$ is a number of time units (like days, hours, minutes, etc); note that sometimes is used transaction days - in this case, discretization hasn't a constant step.

Because for a standard Wiener process *(B(t), t≥0)* we can obtain a standard normal random variable series *(X[B(t)], t≥0)* with (16):

$$dB(t) = X(dt)^{1/2} \qquad (16)$$

we can build a simulation step as (17):

$$\left.\begin{array}{l}
M \leftarrow M(S[k], v_1[k], v_2[k], t[k]) \\
S_1 \leftarrow S_1(S[k], v_1[k], t[k]) \\
S_2 \leftarrow S_2(S[k], v_2[k], t[k]) \\
C_1 \leftarrow C_1(S[k], v_1[k], t[k]) \\
D_1 \leftarrow D_1(S[k], v_1[k], t[k]) \\
C_2 \leftarrow C_2(S[k], v_2[k], t[k]) \\
D_2 \leftarrow D_2(S[k], v_2[k], t[k]) \\
S[k+1] \leftarrow S[k] + M\Delta + S_1 X_1 \sqrt{\Delta} + S_2 X_2 \sqrt{\Delta} \\
v_1[k+1] \leftarrow v_1[k] + C_1 \Delta + D_1 \; X_3 \sqrt{\Delta} \\
v_2[k+1] \leftarrow v_2[k] + C_2 \Delta + D_2 \; X_4 \sqrt{\Delta}
\end{array}\right\} \quad (17)$$

where $X_1$ and $X_3$ are $r_1$-correlated, $X_2$ and $X_4$ are $r_2$-correlated. A simple method to generate two *r* correlated normal values is (18):

$$\left.\begin{array}{l}
X \leftarrow NormRand() \\
Z \leftarrow NormRand() \\
Y \leftarrow r\,X + \sqrt{1-r^2}\;Z
\end{array}\right\} \quad (18)$$

where *NormRand* is a function that produces independent real random numbers between 0 and 1, with normal distribution.

A complete simulation for interval $[t_0, T]$ in *N* steps with evaluation of payoff is function S*imulation*, described below:

*FUNCTION Simulation()*

$S \leftarrow S_0$

$v_1 \leftarrow v_{10}$

$v_2 \leftarrow v_{20}$

$t \leftarrow t_0$

$\Delta \leftarrow (T - t_0) / N$

*FOR* $k \leftarrow 1, N$

$t \leftarrow t + \Delta$

$X_1 \leftarrow NormRand()$

$X_2 \leftarrow NormRand()$

$Y_3 \leftarrow NormRand()$

$Y_4 \leftarrow NormRand()$

$X_3 \leftarrow r_1 X_1 + \sqrt{1-r_1^2}\; Y_3$

$X_4 \leftarrow r_2 X_2 + \sqrt{1-r_2^2}\; Y_4$

$M \leftarrow M(S, v_1, v_2, t)$

$S_1 \leftarrow S_1(S, v_1, t)$

$S_2 \leftarrow S_2(S, v_2, t)$
$C_1 \leftarrow C_1(S, v_1, t)$
$D_1 \leftarrow D_1(S, v_1, t)$
$C2 \leftarrow C_2(S, v_2, t)$
$D_2 \leftarrow D_2(S, v_2, t)$
$SS \leftarrow S + M\Delta + S_1 X_1 \sqrt{\Delta} + S_2 X_2 \sqrt{\Delta}$
$vv_1 \leftarrow v_1 + C_1 \Delta + D_1 X_3 \sqrt{\Delta}$
$vv_2 \leftarrow v_2 + C_2 \Delta + D_2 X_4 \sqrt{\Delta}$
$S \leftarrow SS$
$v_1 \leftarrow vv_1$
$v_2 \leftarrow vv_2$
*END FOR*
*RETURN payoff(S)*
*END FUNCTION*

**3. Monte Carlo method for Option's Pricing in Double-Heston Model**

Because for a level of acceptance α, where 0< α <1, a trust interval for $E[S(T)]$ is $[s - a, s + a]$, with (19):

$$s = [Simulation() + Simulation() + ... + Simulation()] / N \qquad (19)$$

and (20):

$$a = F(\alpha/2)\sigma/M^{1/2} \qquad (20)$$

where:
a) $N$ is number of simulations;
b) $F$ is the inverse function for CDF (cumulative distribution function) of standard normal distribution; it means that (21) or (22):

$$Prob(s - a < E[payoff(S(T))] < s + a) = 1 - \alpha \qquad (21)$$
$$Prob(E[payoff(S(T))] = s + O(M^{1/2})) = 1 - \alpha. \qquad (22)$$

where big–O notation is a Buchmann–Landau symbol (see [5]). Algorithm for evaluation of $E[payoff(S(T))]$ is described below, in *Serial_Simulation* function:

*FUNCTION Serial_Simulation()*
*LOCAL x*
$x \leftarrow 0$
*FOR* $i \leftarrow 1, M$
$x \leftarrow x + Simulation()$
*ENDFOR*
*RETURN x/M*
*END FUNCTION*

**4. Further works**

Like in [6] we will to parallelize Monte Carlo algorithm for generalized Double-Heston model. Also, we want to build a Merton-Garman like PDE for option pricing like in [1] for generalized Double-Heston model, and build some parallelization of PDE numerical solving, like in [4].

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