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Abstract

This paper studies the impact of a small probability event on strategic behavior in incomplete information games with non-common priors. It is shown that the global impact of a small probability event (i.e., its propensity to affect strategic behavior at all states in the state space) has an upper bound that is an increasing function of a measure of discrepancy from the common prior assumption. In particular, its global impact can be arbitrarily large under non-common priors, but is bounded from above under common priors. These results quantify the different implications common prior and non-common prior models have on the (infinite) hierarchies of beliefs. *Journal of Economic Literature* Classification Numbers: C72, D82.

KEYWORDS: common prior assumption; higher order belief; rationalizability; contagion; belief potential.

Contents

1 Introduction

While controversial, the common prior assumption (hereafter, CPA) is used in most models of incomplete information in game theory and economics. This assumption says that the beliefs of all players are generated from a single prior, updated by Bayes' rule, so that differences in their beliefs are due solely to differences in information that they receive. It is well known that the CPA is crucial for many results in incomplete information games (e.g., Aumann's (1976) result on agreeing to disagree and no trade theorems by Milgrom and Stokey (1982)). The purpose of this paper is to clarify the restrictions that we implicitly impose on strategic behavior in game theoretic models when we accept the CPA. Specifically, we focus on "contagion" effects that a small amount of payoff uncertainty has on strategic behavior through players beliefs about payoffs, their beliefs about others' beliefs, and so on, i.e., hierarchies of beliefs.

It has been known that once we depart from common knowledge of payoffs by introducing a small amount of incomplete information, strategic behavior may change dramatically through higher order beliefs. Rubinstein (1989), Carlsson and van Damme (1993), and Morris, Rob, and Shin (1995), among others, show how a small probability event can have a large impact on strategic behavior (under common prior). To see the logic behind, suppose that player 1 is known to take a certain action at some information set which has a very small ex ante probability. If player 2 puts high conditional probability on that event at his information sets where the first information set is thought possible, this knowledge might imply a unique best response by player 2 at these information sets. This, in turn, implies how player 1 responds to that knowledge at larger information sets, and so on. If this iterative argument results in a unique action profile played everywhere on the state space, then we have a *contagion* of this action profile. Then the question is when it is the case that a certain action profile being chosen at some event (which, again, may have a very small probability) implies that this action profile is chosen everywhere on the state space: in other words, when is an action profile contagious?

To answer this question, Morris, Rob, and Shin (1995) propose to measure the impact of an event by the notion of *belief potential*. First, say that an event E has an *impact* p on a state ω (we refer to this as the local impact of event E) if the statement of the form "player 1 believes with probability at least p that 2 believes with probability at least p that 1 believes ... that the true state is in the original event E" is true at state ω. Then, the belief potential of event E is the largest probability p such that E has impact p on *all* states in the state space.

In Section 2, we first demonstrate with an example that under heterogeneous priors, there exists an information system where small probability events have an arbitrarily large belief potential. In the example, we show that any strict Nash equilibrium can be contagious under heterogeneous prior beliefs.

We then find the measure of discrepancy from the CPA that provides an upper bound on the global impact of small probability events as an increasing function of this measure. The measure of discrepancy is the supremum of the ratios between the players' prior probabilities over the states in the state space. This result implies, first, that for a small probability event to have a maximum global impact, this measure of discrepancy from the CPA has to be large. Second, it implies that the global impact of a small probability event is bounded from above under the CPA. The latter result also quantifies the implications of the CPA on infinite hierarchies of beliefs.

For a small probability event to have a large impact on all states in the state space, it is necessary to drop the CPA. However, we point out that if we are interested in the local impact of this small probability event (i.e., on a given state), then this impact can be arbitrarily large even under CPA. Indeed, we show that for any integer $N > 0$, we can find an information system with a common prior, and a small probability event E such that at some state ω players mutually know up to order N that E did not occur but where still this small probability event has an arbitrarily large impact on state ω. This construction sheds light on a result by Yildiz (2004) that for any strict Nash equilibrium a^* of a complete information type,[1] there exist nearby types (with respect to product topology in the universal type space, see Mertens and Zamir (1985) and Brandenburger and Dekel (1993)) from models *with common prior* such that a^* is the unique rationalizable outcome for these types.[2] This result crucially relies on Lipman (2003, 2005) who considers the implications of the CPA for finite hierarchies of beliefs. He shows that for any state in a partition model where players may have heterogenous priors (but with common support), there is a corresponding state in another partition model with a common prior that is close to the original state with respect to product topology. That is, the CPA does not have any implication on finite order beliefs, if one is interested only in local properties of the beliefs (i.e., properties at a given state). However, Lipman's (2003, 2005) results say nothing about the restrictions imposed on global properties of the whole state space. Indeed, we show that under the CPA, the set of states on which a vanishingly probability event has a large local impact has an arbitrarily small probability with respect to the common prior distribution. This is the sense in which we say that the global impact

[1]A complete information type is a degenerate type in the universal type space where it is common knowledge that a given complete information game is played.

[2]Yildiz' (2004) result is much more general: firstly he consider types that need not be complete information types, secondly he deals not only with strict Nash equilibria but also with rationalizable outcomes that need not be strict Nash equilibria. For our purpose, it is not necessary to consider the most general version of Yildiz' result. See also Weinstein and Yildiz (2007).

of a small probability event cannot be arbitrarily large under the CPA. In relation to Yildiz' (2004) result, this implies that for some complete information type and for some strict Nash equilibria a^* (e.g., the risk-dominated equilibrium in a 2×2 coordination game), the ex ante probability of the set of types (in models from which nearby types are extracted) for which a^* is uniquely rationalizable vanishes, if we require the set of types such that the complete information game is not played to be vanishingly small with respect to the common prior. Hence, our result is strongly related to the so-called "critical path result" found by Kajii and Morris (1997). Indeed, using the critical path result in two player games, one can show that our result that under the CPA, the set of states on which a vanishingly probability event has a large local impact has an arbitrarily small probability with respect to the common prior distribution.

Our example in Section 2 shows that in 2×2 coordination games, where there are two strict Nash equilibria, the risk-dominated equilibrium can be contagious if the players are allowed to have heterogenous priors. On the other hand, Kajii and Morris (1997) have shown that under the CPA, this is not possible. In their terminology, the risk-dominant equilibrium is robust to incomplete information. In a companion paper (Oyama and Tercieux (2005)), we show that in generic games, a Nash equilibrium is robust to incomplete information under heterogeneous priors if and only if it is a unique action profile that survives iterative elimination of strictly dominated actions.

The remainder of the paper is organized as follows. Section 2 provides examples that summarizes the analyses in the subsequent sections. It illustrates why without the CPA, every strict Nash equilibrium can be contagious and how it is related to the discrepancy from the CPA. It also provide a first insight of why the local impact of a small probability event can be arbitrarily large even under CPA and relates this to Yildiz (2004). Section 3 introduces the concept of belief potential and states our results relating the measure of discrepancy from the CPA with the belief potential of small probability events. Section 4 compares the local and the global impacts of small probability events, relating our result to the results by Lipman (2003, 2005) and Yildiz (2004). Section 5 discusses an alternative distance measure from the CPA as well as an extension to the many player case.

2 Example

In this section, we illustrate the analyses in the subsequent sections with a simple example. Consider the following 2×2 coordination game with complete information which we denote by $\mathbf{g}$. There are two players, 1 and 2, each of whom has two actions L and R. Throughout the paper, for $i = 1, 2$ we write $-i$ for player $j \neq i$. The payoffs are given by

	L	R
L	p, p	$0, 0$
R	$0, 0$	$1-p, 1-p$

where $p \in (1/2, 1)$, so that (L, L) is (both Pareto-dominant and) risk-dominant. We will say that (L, L) is a strict $(1-p)$-dominant equilibrium while (R, R) is a strict p-dominant equilibrium (see Definition 3.3). As p becomes close to one, the strict Nash equilibrium (R, R) becomes "weaker".

Now, we ask the following question: For each strict Nash equilibrium $a^* = (L, L), (R, R)$ of $\mathbf{g}$, are there "perturbations" arbitrarily "close" to $\mathbf{g}$ in which a^* is played as a *unique* rationalizable strategy outcome? The question, of course, is not well defined unless what we mean by "perturbations" being "close" to $\mathbf{g}$ is specified.

2.1 Perturbations via Incomplete Information Games

Here, as perturbations of $\mathbf{g}$ we consider incomplete information games with an information partition structure as well as the same sets of players and actions as in $\mathbf{g}$, where we allow the players to have different prior beliefs. The complete information game $\mathbf{g}$ is considered as a degenerate incomplete information game. We regard a perturbed incomplete information game to be close to $\mathbf{g}$ if the event that both players know that their payoffs are given by $\mathbf{g}$ has probability close to one with respect to both players' prior distributions.

To address the question, we consider the following class of perturbations of $\mathbf{g}$. The state space Ω is given by $\{1, 2\} \times \mathbb{Z}_+$. Player $i = 1, 2$ has information partition $\mathcal{Q}_i$ which consists of (i) the event $\{(-i, 0)\}$ and (ii) all the events of the form $\{(i, k-1), (-i, k)\}$ for $k \geq 1$. Observe that this partition structure is of the same type as that in the electronic mail game of Rubinstein (1989).

The players may have different prior beliefs. For $r \in [1, \infty)$ and $\varepsilon \in (0, 1)$, let player i's prior P_i be defined by

$$P_i(i, k) = \frac{r}{r+1} \cdot \varepsilon(1-\varepsilon)^k,$$
$$P_i(-i, k) = \frac{1}{r+1} \cdot \varepsilon(1-\varepsilon)^k.$$

The players have a common prior if and only if $r = 1$. Observe that for all $\omega \in \Omega$, $P_i(\omega)/P_{-i}(\omega) = r$ if $\omega = (i, k)$, while $P_{-i}(\omega)/P_i(\omega) = r$ if $\omega = (-i, k)$. We will use the parameter r to measure the degree of discrepancy from the CPA.

Finally, let $E_i = \{(-i, 0)\}$ and $E = E_1 \cup E_2$. The payoffs of each player i are given by $\mathbf{g}$ at all states in $\Omega \setminus E_i$, while a_i^* is a strictly dominant action for player i on event E_i, where $a^* = (a_1^*, a_2^*)$ will be (L, L) or (R, R). Verify that $P_i(E) = \varepsilon$ for each i. Let us denote this incomplete information game by $\mathcal{U}(r, \varepsilon; a^*)$.

(1) Common prior case ($r = 1$): As demonstrated by Morris, Rob, and Shin (1995),[3] if L is a dominant action for each player i at state $(-i, 0)$, then however small $\varepsilon > 0$ is, the incomplete information game $\mathcal{U}(1, \varepsilon; (L, L))$ has a unique rationalizable strategy profile, where (L, L) is played at all $\omega \in \Omega$: that is, we have a "contagion" of the risk-dominant action L. On the other hand, as established by Kajii and Morris (1997), even if R is a dominant action for each player i at state $(-i, 0)$, the incomplete information game $\mathcal{U}(1, \varepsilon; (R, R))$ has a Bayesian Nash equilibrium in which (L, L) is played with high (ex ante) probability whenever ε is sufficiently small. We may say that under a common prior, the event E, however small its (ex ante) probability is, has an impact large enough to make the risk-dominant action contagious, but not large enough to make the risk-dominated one contagious.

(2) Non-common prior case ($r > 1$): We show that for r sufficiently large, each action is contagious: for each equilibrium a^*, if a_i^* is a dominant action at state $(-i, 0)$ for each player i, then there exists $\bar{r}$ such that for all $r > \bar{r}$ and all $\varepsilon \in (0, 1)$, the incomplete information game $\mathcal{U}(r, \varepsilon; a^*)$ has a unique rationalizable strategy profile, where a^* is played at all $\omega \in \Omega$. To see this, suppose that for each player i, R is a dominant action at $(-i, 0)$. Observe that

$$P_i(\{(i, k-1)\} | \{(i, k-1), (-i, k)\}) = \frac{r}{r + 1 - \varepsilon} \tag{2.1}$$

for all $k \geq 1$. Now, given $p \in (1/2, 1)$, let $\bar{r} = p/(1-p)$ (> 1), and take any $r \geq \bar{r}$ and $\varepsilon \in (0, 1)$. Then, if player $-i$ plays R at $(i, k-1)$ in any rationalizable strategy, then it implies that player i plays R at $(-i, k)$ in any rationalizable strategy, since i assigns a probability $r/(r + 1 - \varepsilon) > p$ to the event $-i$ plays R, which makes R the unique best response. We may hence say that under non-common priors, the event E, however small its (ex ante) probability is, may have an impact large enough that any strict Nash equilibrium is contagious. The key to this result is that by increasing the value of r, we can have the relevant conditional probabilities, $P_i(\{(i, k-1)\} | \{(i, k-1), (-i, k)\})$, be as close to one as possible. The supremum of such conditional probabilities relevant to the contagion argument will be called the *belief potential* of the event E (see Definition 3.1 for the precise definition). In this particular information system with given r and ε, the belief potential of E is $r/(r + 1 - \varepsilon)$, as given by (2.1). But it will turn out that this is the "best case", in which a small probability event has the largest impact. We will show that given values of discrepancy measure, r, and small probability, ε, the value $r/(r + 1 - \varepsilon)$ is the maximum of the belief potential of a small probability event *over information systems* (see

[3]Kajii and Morris (1997) extend their argument to the countable state space case.

Theorem 3.4 for the precise statement). This implies that an event can have a larger impact on higher order beliefs under non-common prior than under common prior.

2.2 Perturbations via Types

The second notion uses as perturbations, states instead of incomplete information games. A state is considered to be close to the complete information game $\mathbf{g}$ if at this state (together with the associated partition model) players know up high level that payoffs are given by $\mathbf{g}$. The corresponding notion in the universal type space is known as the product topology and has been studied by Yildiz (2004), among others. He identifies the complete information game $\mathbf{g}$ with a point $t_{\mathbf{g}}$ in the universal type space, i.e., the hierarchy of degenerate beliefs, and considers as "perturbations" being "close" to $\mathbf{g}$, types in the universal type space that are close to $t_{\mathbf{g}}$ with respect to product topology.[4] His results imply, in particular, that for any strict Nash equilibrium a^* of complete information game $\mathbf{g}$, there exists a sequence of types converging to $t_{\mathbf{g}}$ each of which plays a^* as a unique rationalizable strategy outcome. Moreover, by appealing to Lipman's (2003) result, he shows that those converging types can be taken from models (i.e., belief-closed subspaces) with common prior. We will discuss in Section 4 the relationship between Lipman's (2003, 2005) and Yildiz' (2004) results and ours.

Yildiz' (2004) result can be stated in our framework as follows: for any strict Nash equilibrium a^* of complete information game $\mathbf{g}$, there exists a sequence of perturbed incomplete information games $\mathcal{U}^k$ with common prior and states ω^k such that any rationalizable strategy profile of $\mathcal{U}^k$ plays a^* at ω^k, where in each $\mathcal{U}^k$, a_i^* is a strictly dominant action for player i on an event E_i^k, and at each ω^k, players mutually know up to kth order that the payoffs are given by $\mathbf{g}$. To see this in our example, let $a^* = (R, R)$. Modifying the incomplete information game in the previous subsection with given $p \in (1/2, 1)$, $\mathcal{U}^k$ can be constructed as follows (common for all k). Let the state space be $\bar{\Omega} = \Omega \cup \{\infty\}$, and the information partition for each player i be $\bar{\mathcal{Q}}_i = \mathcal{Q}_i \cup \{\{\infty\}\}$, where Ω and $\mathcal{Q}_i$ are the state space and the information partitions defined in the previous subsection. Define the common prior $\bar{P}$ by

$$\bar{P}(1, k) = \bar{P}(2, k) = \frac{1}{2}\varepsilon \left(\frac{1-\varepsilon}{r} \right)^k$$

for $k \geq 0$ and

$$\bar{P}(\infty) = 1 - \frac{r}{r - (1-\varepsilon)}\varepsilon,$$

[4] A formal definition of the universal type space and of the product topology is given in Subsection 4.2.

where r is such that $r \geq p/(1-p)$. Note that we need to add a state, denoted ∞, in order for $\bar{P}$ to sum up to one. The payoffs of each player i are given by $\mathbf{g}$ at all states in $\bar{\Omega} \setminus E_i$, while a_i^* is a strictly dominant action for player i on event E_i. Then, since the relevant posteriors are given by

$$\bar{P}(\{(i,k-1)\}|\{(i,k-1),(-i,k)\}) = \frac{r}{r+1-\varepsilon}$$

for all $k \geq 1$, the same argument in the previous subsection shows that any rationalizable strategy plays R in every state in $\bar{\Omega} \setminus \{\infty\}$. In addition, it is easy to check that for each k, at $(i, k+1)$ $(i = 1, 2)$, it is mutually known up to order k that payoffs are given by $\mathbf{g}$.

Now, if we require that $\bar{P}(E_1 \cup E_2)$ $(= \varepsilon)$ vanish along the sequence, then $\bar{P}(\bar{\Omega} \setminus \{\infty\})$ must vanish accordingly, and so the ex ante probability of the event that R is played as a unique rationalizable strategy action converges to 0. In fact, as we will argue in Section 4, this is the case not only in this particular construction of incomplete information games, but also in any such construction. This is to be contrasted with the non-common prior case in the previous subsection, where any strict Nash equilibrium can be contagious over the state space. In this sense, if one is interested in strategic behavior on the whole state space, rather than local behavior (i.e., behavior at a particular state as in Yildiz (2004)), then models with common priors may be significantly different from those with non-common priors.

Remark 2.1. In the construction above, we could have assumed that the payoffs of each player i are given by $\mathbf{g}$ at all states in $\Omega \setminus E_i$, while a_i^* is a strictly dominant action for player i on event E_i and at state ∞ some action b_i^* (possibly equal to a_i^*) is a strictly dominant action for each player i. Clearly, we would have obtained a dominance solvable game with a common prior (note, however, that the prior probability of the event "payoffs are given by $\mathbf{g}$" converges to 0 as ε vanishes). As we will show in Section 4, this implies that for any strict Nash equilibrium a^* of complete information game $\mathbf{g}$, there exists a sequence of types in the universal type space converging to $t_{\mathbf{g}}$ (in the product topology) each of which plays a^* as a unique rationalizable strategy outcome. Moreover, those converging types can be taken from models (i.e., belief-closed subspaces) that are *both* dominance-solvable and with a common prior.

3 Belief Potential

3.1 Information Systems and Belief Potential

An *information system* is the structure $(\Omega, (P_i)_{i=1,2}, (\mathcal{Q}_i)_{i=1,2})$, where Ω is a countable set of states, P_i is the prior distribution on Ω for player $i = 1, 2$, and $\mathcal{Q}_i$ is the partition of Ω representing the information of player i. We write $Q_i(\omega)$ for the element of $\mathcal{Q}_i$ containing ω. Given an information system, we write $\mathcal{F}_i$ for the sigma algebra generated by $\mathcal{Q}_i$, We assume that $P_i(Q_i(\omega)) > 0$ for all $i = 1, 2$ and $\omega \in \Omega$. Under this assumption, the conditional probability of ω' given $Q_i(\omega)$, $P_i(\omega'|Q_i(\omega))$, is well-defined by $P_i(\omega'|Q_i(\omega)) = P_i(\omega')/P_i(Q_i(\omega))$. Given an information system, define the following measure of discrepancy from the common prior case, ρ, by

$$\rho\big((P_i)_{i=1,2}\big) = \max_{i \neq j} \sup_{\omega \in \Omega\colon P_j(\omega)>0} \frac{P_i(\omega)}{P_j(\omega)} \tag{3.1}$$

with a convention that $q/0 = \infty$ for $q > 0$, and $0/0 = 1$. Note that $\rho\big((P_i)_{i=1,2}\big) < \infty$ only if $(P_i)_{i=1,2}$ has common support. The information system satisfies the CPA if and only if $\rho((P_i)_{i=1,2}) = 1$.

We use the notion of p-belief as defined by Monderer and Samet (1989). For $p \in (0, 1]$, the p-belief operator for player $i = 1, 2$, $B_i^p \colon 2^\Omega \to 2^\Omega$, is defined by

$$B_i^p(E) = \{\omega \in \Omega \mid P_i(E|Q_i(\omega)) \geq p\}.$$

That is, $B_i^p(E)$ is the set of states where player i believes E with probability at least p (with respect to his own prior P_i). We will also use the knowledge operator for player i, $K_i \colon 2^\Omega \to 2^\Omega$, is defined by

$$K_i(E) = \{\omega \in \Omega \mid Q_i(\omega) \subset E\}.$$

That is, $K_i(E)$ is the set of states where player i knows that event E is true. Let $K_*(E) = \bigcap_{i \in \mathcal{I}} K_i(E)$ be the set of states where it is mutual knowledge that event E is true, i.e., where every player knows that event E is true. At a state ω, an event E is said to be mutual knowledge at order N if $\omega \in \bigcap_{n=1}^{N} [K_*]^n(E)$. Finally, at state ω, an event E is said to be common knowledge if $\omega \in \bigcap_{n=1}^{\infty} [K_*]^n(E)$.

We define the operator $H_i^p \colon 2^\Omega \to 2^\Omega$ by

$$H_i^p(E) = B_i^p(B_{-i}^p(E)) \cup E.$$

We denote $(H_i^p)^0(E) = E$ and for $k \geq 1$, $(H_i^p)^k(E) = H_i^p((H_i^p)^{k-1}(E))$.

Denote $(H_i^p)^\infty(E) = \bigcup_{k=1}^{\infty} (H_i^p)^k(E)$. We follow Morris, Rob, and Shin (1995) to measure the impact of an event by the notion of *belief potential.* The belief potential of an event E is the largest probability p such that a statement of the form "player i believes with probability at least p that player $-i$ believes with probability at least p that i believes ... that the true state is in E" is true at every state in Ω.

Definition 3.1. The *belief potential of event E*, $\sigma(E)$, is

$$\sigma(E) = \max_{i=1,2} \sigma_i(E),$$

where

$$\sigma_i(E) = \sup\{p \in [0, 1] \mid (H_i^p)^\infty(E) = \Omega\}.$$

Similarly, we measure the impact of an event at a given state in the following way. Event E is said to have impact p on a state ω if a statement of the form "player i believes with probability a least p that player $-i$ believes with probability at least p that i believes ... that the true state is in E" is true at ω.

Definition 3.2. Event E is said to have *impact p at state ω* if $\omega \in (H_1^p)^\infty(E) \cap (H_2^p)^\infty(E)$.

The *belief potential of event E at state ω*, $\sigma(\omega|E)$, is

$$\sigma(\omega|E) = \sup\{p \in [0,1] \mid E \text{ has impact } p \text{ at } \omega\}.$$

To illustrate these concepts, consider first the information system and the event $E = \{(1,0),(2,0)\}$ in Subsection 2.1. Note that this information system satisfies $\rho\big((P_i)_{i=1,2}\big) = r$. Observe first that for each $i = 1,2$, $B_i^p(E) = \{(-i,0)\} \cup \{(i,0),(-i,0)\}$ if $p \leq r/(r+1-\varepsilon)$, and $B_i^p(E) = \{(-i,0)\}$ otherwise. Thus,

$$(H_i^p)^K(E) = \{(-i,0)\} \cup \bigcup_{k=1}^{K} \{(i,k-1),(-i,k)\}$$

and therefore $(H_i^p)^\infty(E) = \Omega$ if $p \leq r/(r+1-\varepsilon)$, and $(H_i^p)^\infty(E) = \{(-i,0)\}$ otherwise. This implies that for this information system,

$$\sigma(E) = \frac{r}{r+1-\varepsilon}.$$

In Subsection 3.3, we will show that, given $r \geq 1$ and $\varepsilon > 0$, this is the maximum value of the belief potential of an event with probability ε over the information systems such that $\rho\big((P_i)_{i=1,2}\big) = r$.

Now, consider the information system and the event $E = \{(1,0),(2,0)\}$ in Subsection 2.2. Recall that this information system satisfies the CPA. Note that, for any $p \in [0,1)$, if $r \geq p/(1-p)$, we have $(H_i^p)^\infty(E) = \Omega$ ($\neq \bar{\Omega}$). This implies that, for any $p \in [0,1)$ and $N > 0$ (provided that r is large enough), there exists ω, such that it is mutually known up to order N that payoffs are given by the complete information game and where still E has a large impact on ω, more formally $\omega \in \bigcap_{n=1}^N (K_*)^n(E^c)$ and $\sigma(\omega \mid E) \geq p$. This is the sense in which we will say that a small probability event can have an arbitrarily large local impact. This is true irrespective of whether one assumes the CPA.

3.2 Incomplete Information Games and p-Dominance

To relate the impact of a small probability event to the contagion of Nash equilibria played at that event (as demonstrated in Section 2), we consider both complete and incomplete information games.

A (two-player) complete information game consists of a finite set of action A_i and a payoff function g_i for each $i = 1, 2$. Throughout our analysis, we fix a complete information game, simply denoted by $\mathbf{g} = (g_i)_{i=1,2}$. We restate the definition of strict p-dominant equilibrium as defined by Morris, Rob, and Shin (1995) and Kajii and Morris (1997).

Definition 3.3. Let $p \in [0, 1)$. An action profile $a^* \in A$ is a *strict p-dominant equilibrium* if for each $i = 1, 2$ and all $a_i \neq a_i^*$,

$$g_i(a_i^*, \pi_i) > g_i(a_i, \pi_i)$$

holds for all $\pi_i \in \Delta(A_{-i})$ with $\pi_i(a_{-i}^*) > p$.

An incomplete information game is represented by $\mathcal{U} = (IS, (A_i)_{i=1,2}, (u_i)_{i=1,2})$, where IS is an information system as described above, A_i is the set of actions for player i, and $u_i \colon A \times \Omega \to \mathbb{R}$ is the payoff function for player i. We denote $A = A_1 \times A_2$. We assume that players know their own payoffs, i.e., for each i and every $a \in A$, $u_i(a, \cdot)$ is measurable with respect to $\mathcal{Q}_i$. We also note $\Omega_{\mathbf{g},i} = \{\omega \mid u_i(., \omega) = g_i(.)\}$ and $\Omega_{\mathbf{g}} = \bigcap_{i=1,2} \Omega_{\mathbf{g},i}$. Note that by the latter assumption, $\Omega_{\mathbf{g},i} \in \mathcal{F}_i$. For player $i = 1, 2$ and action $a_i \in A_i$, we write the expected payoff against a conjecture $\nu_i \in \Delta(\Omega \times A_{-i})$ as

$$U_i(a_i, \nu_i) = \sum_{\omega \in \Omega} \sum_{a_{-i} \in A_{-i}} \nu_i(\omega, a_{-i})\, u_i(a_i, a_{-i}, \omega).$$

The set of i's (pure) best responses against $\nu_i \in \Delta(\Omega \times A_{-i})$ is denoted by

$$BR_i(\nu_i) = \arg\max_{a_i \in A_i} U_i(a_i, \nu_i).$$

As the solution concept, we employ interim correlated rationalizability (Battigalli and Siniscalchi (2003) and Dekel, Fudenberg, and Morris (2007)). For each $i = 1, 2$, let $R_i^0[Q_i] = A_i$ for all $Q_i \in \mathcal{Q}_i$. Then, for each $i = 1, 2$, and for $Q_i \in \mathcal{Q}_i$ and for $k = 1, 2, \ldots$, define $R_i^k[Q_i]$ recursively by

$$R_i^k[Q_i] = \left\{ a_i \in A_i \;\middle|\; \begin{array}{l} \exists\, \nu_i \in \Delta(\Omega \times A_{-i}): \\ \nu_i\big(\{(\omega, a_{-i}) \mid a_{-i} \in R_{-i}^{k-1}[Q_{-i}(\omega)]\}\big) = 1; \\ \mathrm{marg}_{\Omega}\, \nu_i = P_i(\cdot | Q_i); \\ a_i \in BR_i(\nu_i) \end{array} \right\}.$$

Let $R_i^\infty[Q_i] = \bigcap_{k=1}^\infty R_i^k[Q_i]$.

Definition 3.4. An action $a_i \in A_i$ is a *rationalizable action* of player i at $\omega \in \Omega$ in $\mathcal{U}$ if $a_i \in R_i^\infty[Q_i(\omega)]$.

The following proposition is a variant of the result by Morris, Rob, and Shin (1995, Theorem 5.1). Roughly, it states that if E has a belief potential equal to σ, then any strict p-dominant equilibrium with $p < \sigma$ can be contagious.

Proposition 3.1. *Consider an incomplete information game $\mathcal{U}$. Suppose that* (1) $\Omega_{\mathbf{g}}^c$ *has belief potential* $\sigma > 0$, (2) (a_1^*, a_2^*) *is a strict* p*-dominant equilibrium of* $\mathbf{g}$ *for some* $p < \sigma$, *and* (3) *for each player* i, a_i^* *is a strictly dominant action at each* $\omega \in \Omega_{\mathbf{g},i}^c$. *Then, for each player* i, a_i^* *is the unique rationalizable action at all* $\omega \in \Omega$.

Proof. See Appendix. ▮

Now we relate the notion of strict p-dominance to the belief potential of an event on a given state. Roughly, it states that if E has a belief potential σ at a given state ω, then any strict p-dominant equilibrium with $p < \sigma$ will be played at ω.

Proposition 3.2. *Consider an incomplete information game $\mathcal{U}$ and a state* ω. *Suppose that* (1) $\Omega_{\mathbf{g}}^c$ *has belief potential* $\sigma > 0$ *at state* ω, (2) (a_1^*, a_2^*) *is a strict* p*-dominant equilibrium of* $\mathbf{g}$ *for some* $p < \sigma$, *and* (3) *for each player* i, a_i^* *is a strictly dominant action at each* $\omega \in \Omega_{\mathbf{g},i}^c$. *Then, for each player* i, a_i^* *is the unique rationalizable action at* ω.

Proof. See Appendix. ▮

3.3 Upper Bound of Belief Potential

Now we want to characterize the upper bound of the belief potential of small probability events over information systems with a given value of the discrepancy from the CPA (i.e., $\rho((P_i)_{i=1,2})$). Given an information system, we denote

$$\mathcal{F}_1 \oplus \mathcal{F}_2 = \{E \subset \Omega \mid E = E_1 \cup E_2 \text{ for some } E_i \in \mathcal{F}_i \text{ for each } i = 1, 2\}.$$

For $p \in (0, 1]$ and $E \in \mathcal{F}_1 \oplus \mathcal{F}_2$, we define

$$H_*^p(E) = B_1^p(E) \cup B_2^p(E).$$

We denote $(H_*^p)^k(E) = H_*^p((H_*^p)^{k-1}(E))$ for $k \geq 1$, where $(H_*^p)^0(E) = E$, and $(H_*^p)^\infty(E) = \bigcup_{k=1}^\infty (H_*^p)^k(E)$. Verify that $(H_1^p)^\infty(E) \cup (H_2^p)^\infty(E) = (H_*^p)^\infty(E)$, so that if $(H_i^p)^\infty(E) = \Omega$, then $(H_*^p)^\infty(E) = (H_i^p)^\infty(E)$. It is thus sufficient to characterize the (ex ante) probability of $(H_*^p)^\infty(E)$. The following result is the "conjugate" of Proposition 5.2 in Oyama and Tercieux (2005), where the upper bound for $P_j([(H_*^p)^\infty(E^c)]^c)$ is obtained for the many-player case. For its proof, we thus report only crucial steps in the Appendix.

Lemma 3.3. *For any* $r \geq 1$, *if* $p > r/(1+r)$, *then in any information system with* $\rho((P_i)_{i=1,2}) = r$, *any event* $E \in \mathcal{F}_1 \oplus \mathcal{F}_2$ *satisfies*

$$P_i((H_*^p)^\infty(E)) \leq \frac{p}{(1+r)p - r} \max\{P_1(E), P_2(E)\}$$

for all $i = 1, 2$.

Proof. See Appendix. ▮

The following is the main result of this section, which shows that the belief potential of small probability events has an upper bound that is an increasing function of the discrepancy from the CPA.

Theorem 3.4. *For any $r \geq 1$ and any information system with $\rho((P_i)_{i=1,2}) = r$, if $E \in \mathcal{F}_1 \oplus \mathcal{F}_2$ and $P_i(E) \leq \varepsilon$ for each $i = 1, 2$, then*

$$\sigma(E) \leq \frac{r}{1 + r - \varepsilon}.$$

Proof. Take any $q > r/(1+r-\varepsilon)$ $(> r/(1+r))$. If $\max\{P_1(E), P_2(E)\} \leq \varepsilon$, then by Lemma 3.3, for each $i = 1, 2$,

$$P_i((H_*^q)^\infty(E)) \leq \frac{q}{(1+r)q - r}\varepsilon < 1,$$

meaning that $(H_*^q)^\infty(E) \neq \Omega$, and hence $(H_i^q)^\infty(E) \neq \Omega$. This implies that $\sigma(E) \leq r/(1 + r - \varepsilon)$, as claimed. ▮

Note that the upper bound given above is tight: it is attained by the event E in the information system considered in Subsection 2.1.

Theorem 3.4 proves that for small probability events to have a large global impact, the measure $\rho((P_i)_{i=1,2})$, must be large. In light of Proposition 3.1, this shows that for any strict Nash equilibrium to be contagious, the discrepancy from the CPA (measured by $\rho((P_i)_{i=1,2})$) must also be (arbitrarily) large. Otherwise stated, whenever this measure is assumed to be bounded, we can find complete information games and strict Nash equilibria that cannot be contagious. In addition, under CPA where this measure is by definition minimal, the global impact of a small probability event is bounded from above.

Remark 3.1. By the previous result, it is clear that any strict Nash equilibrium may be contagious under non-common priors even if $P_i(\Omega_{\mathbf{g}})$ can be made arbitrarily close to 1 for each $i = 1, 2$. Indeed, more than this, it is also true that for all $N > 0$, $P_i(\bigcap_{n=1}^{N}(K_*)^n(\Omega_{\mathbf{g}}))$ for each $i = 1, 2$ can be made arbitrarily close to 1. This latter notion of proximity between incomplete information games and complete information game is studied in a companion paper Oyama and Tercieux (2005). In that paper, we characterize equilibria that cannot be "eliminated" by such perturbations.

4 Common Prior versus Non-Common Prior

Lipman (2003, 2005) shows that given any partition model IS with common support (and tail consistency in the case of infinite state space) and any state ω in the model, for any finite $N > 0$ there is a partition model with a common prior $\bar{IS}$ and a state $\bar{\omega}$ in that model at which all the same facts about the world are true and all the same statements about beliefs and knowledge of order less than N are true. That is, the common prior assumption does not impose any restriction on finite order beliefs. A similar argument allowed us to show in the example of Subsection 2.2 that the small probability E can have an arbitrarily large local impact even if the common prior assumption holds.

On the other hand, the global properties of the state space in $\bar{IS}$ may be very different from the one in IS (as illustrated by the example). If one is interested in global properties of the whole state space, models with non-common priors may be quite far from any model with a common prior. In this section, we formalize this observation with the notions of global and local impact as well as with the universal type space setting.

4.1 Local versus Global Impact of an Event

By the example in Subsection 2.2, we know that the local impact of a small probability event E can be arbitrarily large on a given state irrespective of whether one assumes a common prior. This is true even if at the given state it is mutually known at arbitrarily large order that E did not occur.

This result allows us to show the following. Fix some strict Nash equilibrium (a_1^*, a_2^*) of $\mathbf{g}$. Given any arbitrarily large number N, we can find an information system and a state ω such that, it is mutually known at order N at ω that $\mathbf{g}$ is played, but the strict Nash equilibrium is the unique rationalizable action profile at ω.

To understand this point, let us state the following proposition.

Proposition 4.1. *Let $\mathbf{g}$ be a complete information game and (a_1^*, a_2^*) a strict p-dominant equilibrium for some $p < 1$. For all $N > 0$ and $\varepsilon > 0$, there exists an incomplete information game $\mathcal{U}$ with a common prior and a state $\omega \in \Omega$ such that (1) $P(\Omega_{\mathbf{g}}^c) < \varepsilon$; (2) $\omega \in \bigcap_{n=1}^{N} (K_*)^n(\Omega_{\mathbf{g}})$; (3) for each player i, a_i^* is the unique rationalizable action at ω.*

However, we show that the global impact of a small probability event E cannot be arbitrarily large under the CPA. The main point of this section is that under common prior, the set of states on which a given small probability event has a "large" impact is small with respect to prior probabilities. The following lemma formalizes this point.

Lemma 4.2. *Let $r \geq 1$ and $p > r/(r+1)$. For any $\delta > 0$, there exists $\varepsilon > 0$ such that for any information system IS with $\rho((P_i)_{i \in \mathcal{I}}) \leq r$ and any event $E \in \mathcal{F}_1 \oplus \mathcal{F}_2$ such that $P_i(E) \leq \varepsilon$ for all $i \in \mathcal{I}$, we have*

$$P_i(\{\omega \in \Omega \mid \sigma(\omega|E) \geq p\}) \leq \delta$$

for all $i \in \mathcal{I}$.

Proof. Given $p > r/(r+1)$ and $\delta > 0$, set $\varepsilon = \delta\{(1+r)p - r\}/p$. Then by Lemma 3.3, we have for each $i = 1, 2$,

$$P_i((H_i^p)^\infty(E)) \leq P_i((H_*^p)^\infty(E)) \leq \frac{p}{(1+r)p - r}\varepsilon \leq \delta,$$

as claimed. ▮

As a corollary of the previous lemma, we have the following main result of this subsection.

Proposition 4.3. *For any $p > 1/2$ and any $\delta > 0$, there exists $\varepsilon > 0$ such that for any information system $\overline{IS}$ that satisfies the CPA and any event $\bar{E} \in \bar{\mathcal{F}}_1 \oplus \bar{\mathcal{F}}_2$ such that $\bar{P}(\bar{E}) \leq \varepsilon$, we have*

$$\bar{P}(\{\omega \in \bar{\Omega} \mid \sigma(\omega|\bar{E}) \geq p\}) \leq \delta.$$

In terms of contagion of Nash equilibria, while any strict Nash equilibrium at a small probability event can spread in some partition model with non-common priors, it may not be the case for partition models with a common prior. Indeed, in 2×2 coordination games, the risk-dominated equilibrium cannot spread from a small probability event when we assume the existence of a common prior, as shown by Kajii and Morris (1997).

To summarize, under heterogeneous priors, both the local and the global impact of any small probability event can be arbitrarily large, whereas under common prior, only the local impact can be arbitrarily large.

This distinction will allow us to shed light on a result of Yildiz (2004) which shows that for any type in the universal type space, there exists arbitrarily close types where rationalizability yields a unique action profile, and moreover, such a type can always be taken from a model with a common prior.

We will first see the connection between this point and the local impact of an event. Then, we will see that if one is interested in such a statement on a whole model (and not only on a specific state of the world / type) dropping the common prior assumption is necessary (and sufficient).

4.2 Topology on Types versus Topology on Subspaces

In this subsection, we embed our results in the universal type space setting. This allows us to compare our results with those of Yildiz (2004), who shows that, under standard assumptions, for any type in the universal type space, there exists nearby types where a unique rationalizable action profile is played. This result is obtained irrespective of whether one assumes that players share a common prior. We claim that if one is interested in the behavior of players not only at a given type but on a whole model, then dropping the common prior assumption is crucial to obtain Yildiz' (2004) type of statement.

4.2.1 The Universal Type Space Setting

Let Θ be a compact metric space of payoff-relevant parameters θ, which is identified as the set of possible payoff functions. To make things simple, we assume that $\Theta = \Theta_1 \times \Theta_2$ where $\Theta_i = [0,1]^A$. We write $\Delta(X)$ for the set of probability measures on the Borel field $\mathcal{B}(X)$ of any topological space X. When X is a set of probability measures, it will be endowed with the weak topology.

Define recursively $X_0 = \Theta$, $X_1 = \Delta(X_0)$, $X_2 = \Delta(X_0 \times X_1)$ .... Let us now describe a type in the setting of the universal type space. A type of a player i is an infinite hierarchy of beliefs $t_i = (t_i^1, t_i^2, \ldots)$ where $t_i^1 \in X_1$ is a probability distribution on Θ, representing the (first order) beliefs of player i about Θ, $t_i^2 \in X_2$ is a probability distribution representing the (second order) beliefs of player i, i.e., his beliefs about Θ as well as his beliefs about the other player's beliefs over Θ, and so on. We also assume that it is common knowledge that the beliefs are coherent.[5] Denote the set of all such types by T_i^*, and let $T^* = T_1^* \times T_2^*$.

For each type t_i, let $\kappa_{t_i} \in \Delta(\Theta \times T_{-i}^*)$ be the unique probability distribution that represents the beliefs of t_i about (θ, t_{-i}). Mertens and Zamir (1985) show that the mapping $t_i \mapsto \kappa_{t_i}$ is a homeomorphism. A set $T \subset T^*$ is said to be a *belief-closed subspace* (or *subspace*, in short) if $\kappa_{t_i}(\Theta \times T_{-i}) = 1$ for each $t_i \in T_i$.

Also for a complete information game $\mathbf{g} \in \Theta$, we define the complete information type $t_{\mathbf{g}}$ as the type in the universal type space where $\mathbf{g}$ is common knowledge, formally for each i, $t_{\mathbf{g},i}^1 = \delta_{\mathbf{g}}^1$ where $\delta_{\mathbf{g}}^1$ is the probability distribution in X_1 assigning probability 1 to $\mathbf{g}$, and $t_{\mathbf{g},i}^k = \delta_{\mathbf{g}}^k$ where $\delta_{\mathbf{g}}^k$ is the probability distribution in X_k assigning probability 1 to $(\mathbf{g}, \delta_{\mathbf{g}}^1, \ldots, \delta_{\mathbf{g}}^{k-1})$. We denote $t_{\mathbf{g}} = (t_{\mathbf{g},1}, t_{\mathbf{g},2})$.

We now describe the connection between the partition model setting and the universal type space setting. In particular, we show how a partition model together with a state *induces* a type in the universal type space. Since the parameter space (namely Θ) has been added to the description of the basic uncertainty, we now need to refer to it in the definition of a partition model. Hence a partition model $\mathcal{M}$ now consists of an information system $[\Omega, (\mathcal{Q}_i)_{i\in I}, (P_i)_{i\in I}]$ together with a function $f\colon \Omega \to \Theta$ where $f(\omega)$ is the value of the unknown parameter at state ω.

Any partition model together with any state ω in that model uniquely identifies a particular type in the universal type space denoted $t[\omega]$ by the the so-called unravelling procedure.[6] Given a state ω in a partition model,

[5]A type of player i, t_i is coherent if for every $n \geq 2$, $\mathrm{marg}_{X_{n-2}}\, t_i^n = t_i^{n-1}$.

[6]The converse of this statement (i.e., that any type in the universal type space can be constructed using the unravelling procedure from some partition model) is true as long as we allow for a less restrictive class of partitions models from those we use in this paper. See, e.g., Brandenburger and Dekel (1993).

we can identify each player's first order beliefs at ω. Denote player i's first order beliefs at state ω by $t_i^1[\omega] \in X_1$. For each measurable set $B \subset \Theta$, we define

$$t_i^1[\omega](B) = P_i(f^{-1}(B) \mid Q_i(\omega)).$$

In the same way, player i's second order beliefs at ω, say, $t_i^2[\omega] \in X_2$ is defined by

$$t_i^2[\omega](B) = P_i(\{\omega' \mid (f(\omega'), t_{-i}^1[\omega']) \in B\} \mid Q_i(\omega))$$

for each measurable set $B \subset \Theta \times \Delta(\Theta)$. Continuing recursively, we can define $t_i^n[\omega]$ for every n. Let $t_i[\omega] = (t_i^1[\omega], t_i^2[\omega], ...)$ and $t[\omega] = (t_1[\omega], t_2[\omega])$.

4.2.2 Product Topology

The straightforward topology used in the universal type space, and in particular the one used by Yildiz (2004), is the product topology. Let us review convergence of a sequence of types in the universal type space with respect to product topology. For our purpose, it is sufficient to restrict our attention to convergence toward a complete information type. For a complete information game $\mathbf{g}$, we will write $t_{\mathbf{g}}$ for the complete information type where $\mathbf{g}$ is common knowledge. Since complete information types, considered as singleton sets, can also be seen as complete information subspaces, this will allow us to compare convergence of a sequence of types toward $t_{\mathbf{g}}$ and convergence of sequences of subspaces toward $\{t_{\mathbf{g}}\}$.

Definition 4.1. Let $t_{\mathbf{g}}$ be a complete information type where $\mathbf{g}$ is common knowledge. $t_m \to t_{\mathbf{g}}$ as $m \to \infty$ if for each i and k, $t_{i,m}^k \to t_{i,\mathbf{g}}^k$ as $m \to \infty$.

To relate the statements of the previous subsection to statements in the universal type space as in Yildiz (2004), note first that if $\omega \in \bigcap_{n=1}^N (K_*)^n(f^{-1}(\mathbf{g}))$, then for each i, $t_i^{N'}[\omega] = \delta_{\mathbf{g}}^{N'}$ for all $N' \leq N$. Let us now state the following simple observation.

Observation 4.4. *Consider a sequence $(\mathcal{M}^m, \omega^m)$ of partition models and states in this model. If for all $N > 0$, there exists $\bar{m}$ such that for all $m \geq \bar{m}$ $(\mathcal{M}^m, \omega^m)$ satisfies $\omega^m \in \bigcap_{n=1}^N (K_*)^n(f^{-1}(\mathbf{g}))$, then $t[\omega^m] \to t_{\mathbf{g}}$ as $m \to \infty$.*

We also use a notion of convergence of subspaces toward complete information subspaces. First, we provide a definition that will allow us to extract a (set of) measure(s) from a given subspace.

Definition 4.2. Let $T \subset T^*$ be a subspace. A profile of priors over T, $(P_i)_{i \in \mathcal{I}}$, is said to be *belief consistent* with T if for all i and all type t_i, $\kappa_{t_i} = P_i(\cdot | \{t_i\} \times T_{-i})$.

We say that T satisfies the *common prior assumption* if there is a profile $(P_i)_{i \in \mathcal{I}}$ of priors belief consistent with T such that $P_1 = P_2$.

Definition 4.3. Let $(\{t_{\mathbf{g}}\}, (\delta_{\mathbf{g}})_{i=1,2})$ be a complete information subspace. $(T^m, (P_i^m)_{i=1,2}) \to (\{t_{\mathbf{g}}\}, (\delta_{\mathbf{g}})_{i=1,2})$ as $m \to \infty$ if for all $\delta > 0$,

$$P_i^m(\{t \in T_m \mid t_{i,m}^1 = t_{\mathbf{g},i}^1 \text{ for } i = 1,2\}) \to 1 \quad \text{as } m \to \infty.$$

To discuss the relationship between our results and Yildiz' result on generic uniqueness, we first define the impact of an event in the universal type space which is the corresponding definition of belief potential in the universal type space.

Consider $\hat{\Theta} \in \mathcal{B}(\Theta)$. Define recursively the sequence of sets of distributions:

$$\Pi_1^p(\hat{\Theta}) = \{\pi \in X_1 \mid \pi(\hat{\Theta}) \geq p\}$$

and for all $k \geq 2$,

$$\Pi_k^p(\hat{\Theta}) = \{\pi \in X_k \mid \operatorname{marg}_{X_{k-1}} \pi(\Pi_{k-1}^p(\hat{\Theta})) \geq p\}.$$

Definition 4.4. $\hat{\Theta} \in \mathcal{B}(\Theta)$ has impact p on type t if for all $i \in \{1,2\}$ and K such that

$$\operatorname{marg}_{X_{K-1}} t_i^K(\Pi_{K-1}^p(\hat{\Theta})) \geq p.$$

We say that $\hat{\Theta}$ has impact p on the subspace T if it has impact p on any $t \in T$.

Proposition 4.5. *Fix any $\mathbf{g} \in \Theta$, $p \in [0,1)$ and $\hat{\Theta} \in \mathcal{B}(\Theta)$. There exists a sequence of profiles $\{(T^m, t^m)\}_{m=0}^{\infty}$ such that for each m, T^m is a subspace satisfying the common prior assumption and t^m is a type in T^m such that $t_m \to t_{\mathbf{g}}$ (as $m \to \infty$) and where for each m, $\hat{\Theta}$ has impact p on t_m.*

Proof. Pick $(\theta_1, \theta_2) \in \hat{\Theta}$. Add to the information system in the example in Subsection 2.2 a function f so that $f(\omega) = (\mathbf{g}_i, \theta_{-i})$ when $\omega \in E_i$ and $f(\omega) = \mathbf{g}$ otherwise. Define the subspace $T = T_1 \times T_2$ where $T_i = \bigcup_{\omega \in \bar{\Omega}} t_i[\omega]$. Note that for all $N > 0$, there exists ω such that $\omega \in \bigcap_{n=1}^{N} (K_*)^n (f^{-1}(\mathbf{g}))$. Observation 4.4 completes the proof. ▮

In terms of contagion of strict Nash equilibria, we have the following result.

Proposition 4.6. *Fix any $\mathbf{g} \in \Theta$, and consider (a_1^*, a_2^*) a strict p-dominant equilibrium for some $p < 1$. There exists a sequence of profiles $\{(T^m, t^m)\}_{m=0}^{\infty}$ such that for each m, T^m is a subspace satisfying the common prior assumption and t^m is a type in T^m such that $t_m \to t_{\mathbf{g}}$ (as $m \to \infty$) and where for each m, (a_1^*, a_2^*) is the unique rationalizable action profile at t_m.*

Remark 4.1. Using Remark 2.1, we can have that T^m is a subspace that satisfies *both* the common prior assumption and dominance-solvability.

Note that it is easy to show that T^k can indeed be chosen to be finite. Thus, it is now clear that for any $\hat{\Theta}$ and any $p < 1$, for any open neighborhood (with respect to product topology) of any complete information type $t_{\mathbf{g}}$, there exists a type t coming from a subspace with common prior so that $\hat{\Theta}$ has impact p on t.

However, as we have claimed earlier, in the subspace to which t belongs, the set of types where $\hat{\Theta}$ has impact $p > 1/2$ is assigned probability close to zero by the common prior whenever this prior assigns a small probability to $\hat{\Theta}$.

However, allowing for heterogeneous priors enables us to obtain a result that explicitly refers to subspaces. We want to underline that when the object of interest for a modeler is a subspace, then a statement of the type of proposition 4.6 holds only when we allow for heterogeneous priors.

Proposition 4.7. *Fix any* $\mathbf{g} \in \Theta$, $p \in [0, 1)$ *and* $\hat{\Theta} \in \mathcal{B}(\Theta)$. *There exists a sequence of subspaces* $\{(T^m, (P_i^m)_{i=1,2})\}_{m=0}^{\infty}$ *where* $(T^m, (P_i^m)_{i=1,2}) \to (\{t_{\mathbf{g}}\}, (\delta_{\mathbf{g}})_{i=1,2})$ *as* $m \to \infty$ *and for each* m*:* $\hat{\Theta}$ *has impact* p *on* T^m.

Remark 4.2. The notion of convergence of subspaces provided in definition 4.3 could have been strengthened as follows. Say that $T_m \to \{t_{\mathbf{g}}\}$ as $m \to \infty$ if for all $k > 0$,

$$P_{i,m}(\{t \in T_m \mid t_{i,m}^k = t_{\mathbf{g},i}^k \text{ for } i = 1, 2\}) \to 1 \quad \text{as } m \to \infty$$

where for each m, $(P_{i,m})_{i \in \mathcal{I}}$ is some profile of priors belief consistent with T_m. Proposition 4.7 would stay unchanged under this definition.

5 Discussion

5.1 Alternative Discrepancy Measure

In this subsection, we consider an alternative measure of discrepancy from the CPA. This measure is denoted by d_0 and defined by

$$d_0\big((P_i)_{i=1,2}\big) = \sup_{E \subset \Omega} |P_1(E) - P_2(E)|.$$

Note that the information system satisfies the CPA if and only if $d_0((P_i)_{i=1,2}) = 0$.

The analogue of Lemma 3.3 using the distance d_0 is the following.

Lemma 5.1. *For any* $\xi \geq 0$, *if* $p > 1/2$, *then in any information system with* $d_0((P_i)_{i=1,2}) \leq \xi$, *any event* $E \in \mathcal{F}_1 \oplus \mathcal{F}_2$ *satisfies*

$$P_i((H_*^p)^{\infty}(E)) \leq \frac{p}{2p-1} P_i(E) + \frac{\xi}{2p-1}$$

for all $i = 1, 2$.

Proof. See Appendix. ▮

The following theorem is the analogue of Theorem 3.4.

Proposition 5.2. *For any $\xi \geq 0$ and any information system with $d_0(P_1, P_2) \leq \xi$, if $E \in \mathcal{F}_1 \oplus \mathcal{F}_2$ and $P_i(E) \leq \varepsilon$ for each $i = 1, 2$, then*

$$\sigma(E) \leq \frac{1+\xi}{2-\varepsilon}.$$

Proof. Take any $q > (1+\xi)/(2-\varepsilon)$ ($> 1/2$). If $\max\{P_1(E), P_2(E)\} \leq \varepsilon$, then by Lemma 5.1, for each $i = 1, 2$,

$$P_i((H_*^q)^\infty(E)) \leq \frac{p}{2p-1}\varepsilon\frac{\xi}{2p-1} < 1,$$

meaning that $(H_*^q)^\infty(E) \neq \Omega$, and hence $(H_i^q)^\infty(E) \neq \Omega$. This implies that $\sigma(E) \leq (1+\xi)/(2-\varepsilon)$, as claimed. ▮

One can show that the upper bound given above is asymptotically tight. However, this measure does not allow to prove the analogue of Lemma 4.2. We illustrate this in the following simple example.[7] Let the state space and the information partition for each player i be as in subsection 2.1. Let player i's prior P_i be defined by $P_i(j, 0) = \varepsilon/2$ for $j = 1, 2$; $P_i(i, 1) = \xi$ and $P_i(-i, 1) = \varepsilon^2/2$. Finally, for each $k \geq 2$, let

$$P_i(j,k) = \frac{\alpha(1-\alpha)^{k-2}}{2} \times \big\{1 - \varepsilon(1-\varepsilon) - \xi\big\}$$

for $j = 1, 2$, where $\alpha \in (0, 1)$. Note that this information system satisfies $d_0\big((P_i)_{i=1,2}\big) \leq \xi$. In addition, fixing $p \in [0, 1)$, whenever ε is small enough, we have $P_i(\{\omega \mid \sigma(\omega \mid E) \leq p\}) \geq \xi$ for each $i = 1, 2$.

Note that following Proposition 3.2, this example shows that contagion of any strict Nash equilibrium can occur on a set of states of probability greater than ξ that does not vanish when the prior probability of $\Omega_{\mathbf{g}}^c$ converges to 0.

The point behind this example is that there exists no direct connection between proximity of priors using d_0 and proximity of conditional beliefs. This has to be contrasted with the measure ρ used in this paper. To understand this point, let us consider (without loss of generality) a fixed Ω. For the measure ρ, we have that for any sequence $\{\big(P_i^m\big)_{i=1,2}\}_{m=0}^\infty$: $\rho\big((P_i^m)_{i=1,2}\big) \to 0$ if and only if

$$\max_{i \neq j} \sup_{E,F \subset \Omega} \frac{P_i^m(E|F)}{P_j^m(E|F)} \to 0,$$

[7] This example also shows that a result of the type of the critical path result found by Kajii and Morris (1997) cannot hold even if we only consider information systems where $d_0(P_1, P_2)$ is arbitrarily small (as long as it is strictly positive). This has to be contrasted again with the measure $\rho(P_1, P_2)$.

where $P_i^m(E|F)$ is i's conditional belief on event E given F in the information system $\big(\Omega, (P_i^m)_{i=1,2}, (\mathcal{Q}_i)_{i=1,2}\big)$.

5.2 Many-Player Extension

In this subsection, we briefly discuss an extension of belief potential to the case of many players. We denote by $\mathcal{I} = (1, 2, \ldots, I)$ the finite set of players. As previously, an information system $(\Omega, (P_i)_{i\in\mathcal{I}}, (\mathcal{Q}_i)_{i\in\mathcal{I}})$ consists of a countable state space Ω, the prior distribution P_i and the information partition $\mathcal{Q}_i$ for each player $i \in \mathcal{I}$. Denote by $\mathcal{F}_i$ the sigma algebra generated by $\mathcal{Q}_i$.

Let $\mathbf{E}$ be a profile $(E_1, \ldots, E_I)$ where $E_i \in \mathcal{F}_i$. Define

$$\big(\widehat{\mathbf{B}}^p\big)_i(\mathbf{E}) = B_i^p\left(\textstyle\bigcap_{j\neq i} E_j\right),$$

and $\widehat{\mathbf{B}}^p(\mathbf{E}) = \big(\big(\widehat{\mathbf{B}}^p\big)_i(\mathbf{E})\big)_{i\in\mathcal{I}}$.

Then, define $\big\{\big(\widehat{\mathbf{H}}^p\big)^k(\mathbf{E})\big\}_{k=0}^{\infty}$ recursively by $\big(\widehat{\mathbf{H}}^p\big)_i^0(\mathbf{E}) = E_i$ and for $k \geq 1$,

$$\begin{aligned}\big(\widehat{\mathbf{H}}^p\big)_i^k(\mathbf{E}) &= B_i^p\left(\textstyle\bigcap_{j\neq i}\big(\widehat{\mathbf{H}}^p\big)_j^{k-1}(\mathbf{E})\right) \cup \big(\widehat{\mathbf{H}}^p\big)_i^{k-1}(\mathbf{E})\\ &= \big(\widehat{\mathbf{B}}^p\big)_i\left(\big(\widehat{\mathbf{H}}^p\big)^{k-1}(\mathbf{E})\right) \cup \big(\widehat{\mathbf{H}}^p\big)_i^{k-1}(\mathbf{E}).\end{aligned}$$

Definition 5.1. Let $\mathbf{E} = (E_1, \ldots, E_I)$ where $E_i \in \mathcal{F}_i$. The *belief potential of event profile* $\mathbf{E}$, $\sigma(\mathbf{E})$, is

$$\sigma(\mathbf{E}) = \max_{i\in\mathcal{I}} \min_{j\neq i} \sigma_j(\mathbf{E}),$$

where

$$\sigma_i(\mathbf{E}) = \sup\left\{p \in [0,1] \;\middle|\; \textstyle\bigcup_{k=0}^{\infty}\big(\widehat{\mathbf{H}}^p\big)_i^k(\mathbf{E}) = \Omega\right\}.$$

We want to relate the belief potential to the p-dominance of Nash equilibria. Incomplete information games $\mathcal{U} = (IS, (A_i)_{i\in\mathcal{I}}, (u_i)_{i\in\mathcal{I}})$ are defined analogously to the two player case, where IS is an information system as described above. We denote $A = \prod_{i\in\mathcal{I}} A_i$ and $A_{-i} = \prod_{j\neq i} A_j$.

Definition 5.2. Let $p \in [0, 1)$. An action profile $a^* \in A$ is a *strict p-dominant equilibrium* of $\mathbf{g}$ if for each $i \in \mathcal{I}$ and all $a_i \neq a_i^*$,

$$g_i(a_i^*, \pi_i) > g_i(a_i, \pi_i)$$

holds for all $\pi_i \in \Delta(A_{-i})$ with $\pi_i(a_{-i}^*) > p$.

We have the following.

Proposition 5.3. *Suppose that* (1) $\Omega_{\mathbf{g}}^{c}$ *has belief potential* $\sigma > 0$, (2) a^* *is a strict p-dominant equilibrium at every state for some* $p < \sigma$, *and* (3) *for each player* $i \in \mathcal{I}$, a_i^* *is a strictly dominant action at every* $\omega \in \Omega_{\mathbf{g},i}^{c}$. *Then, for each player* i, a_i^* *is the unique rationalizable action at all* $\omega \in \Omega$.

Appendix

A.1 Proof of Proposition 3.1

Given (a_1^*, a_2^*), denote $\Omega_i = \{\omega \in \Omega \mid R^\infty[Q_i(\omega)] = \{a_i^*\}\}$ for each $i = 1, 2$.

Lemma A.1.1. *Consider an incomplete information game $\mathcal{U}$. Suppose that* (i) (a_1^*, a_2^*) *is a strict p-dominant equilibrium of* $\mathbf{g}$*, and* (ii) *for each player i, a_i^* is a strictly dominant action at each $\omega \in \Omega^c_{\mathbf{g},i}$. Then, for any $q > p$, $(H_i^q)^\infty(\Omega^c_{\mathbf{g}}) \subset \Omega_i$ for each i.*

Proof. Fix $q > p$ and $i = 1, 2$. We first show by induction that $(H_i^q)^k(\Omega^c_{\mathbf{g}}) \subset \Omega_i \cup \Omega^c_{\mathbf{g},-i}$ for all k. This is true for $k = 0$ since $\Omega^c_{\mathbf{g},i} \subset \Omega_i$ by assumption (ii). Assume now that it is true for $k-1$, that is, $(H_i^q)^{k-1}(\Omega^c_{\mathbf{g}}) \subset \Omega_i \cup \Omega^c_{\mathbf{g},-i}$. Then we have

$$B^q_{-i}((H_i^q)^{k-1}(\Omega^c_{\mathbf{g}})) \subset B^q_{-i}(\Omega_i \cup \Omega^c_{\mathbf{g},-i}) = B^q_{-i}(\Omega_i) \cup \Omega^c_{\mathbf{g},-i},$$

where the equality follows from $\Omega^c_{\mathbf{g},-i} \in \mathcal{F}_{-i}$. Since $B^q_{-i}(\Omega_i) \subset \Omega_{-i}$ and $\Omega^c_{\mathbf{g},-i} \subset \Omega_{-i}$ by assumptions (i) and (ii), respectively, it follows that $B^q_{-i}((H_i^q)^{k-1}(\Omega^c_{\mathbf{g}})) \subset \Omega_{-i}$. Again by (i) and (ii) as well as the induction hypothesis, we have

$$\begin{aligned}(H_i^q)^k(\Omega^c_{\mathbf{g}}) &= B_i^q(B^q_{-i}((H_i^q)^{k-1}(\Omega^c_{\mathbf{g}}))) \cup (H_i^q)^{k-1}(\Omega^c_{\mathbf{g}}) \\ &\subset B_i^q(\Omega_{-i}) \cup (H_i^q)^{k-1}(\Omega^c_{\mathbf{g}}) \subset \Omega_i \cup (\Omega_i \cup \Omega^c_{\mathbf{g},-i}) = \Omega_i \cup \Omega^c_{\mathbf{g},-i},\end{aligned}$$

as desired.

Now note that since $(H_i^q)^\infty(\Omega^c_{\mathbf{g}}) \setminus \Omega_i \in \mathcal{F}_i$, we have $B_i^q((H_i^q)^\infty(\Omega^c_{\mathbf{g}}) \setminus \Omega_i) = (H_i^q)^\infty(\Omega^c_{\mathbf{g}}) \setminus \Omega_i$. Thus, by (i), together with $(H_i^q)^\infty(\Omega^c_{\mathbf{g}}) \setminus \Omega_i \subset \Omega^c_{\mathbf{g},-i}$, we get $(H_i^q)^\infty(\Omega^c_{\mathbf{g}}) \setminus \Omega_i \subset \Omega_i$. Hence, $(H_i^q)^\infty(\Omega^c_{\mathbf{g}}) \subset \Omega_i$. ∎

Proof of Proposition 3.1. Let $\sigma(\Omega^c_{\mathbf{g}}) = \sigma_i(\Omega^c_{\mathbf{g}})$. By Lemma A.1.1, assumptions (2) and (3) imply that $(H_i^\sigma)^\infty(\Omega^c_{\mathbf{g}}) \subset \Omega_i$. But (1) implies that $(H_i^\sigma)^\infty(\Omega^c_{\mathbf{g}}) = \Omega$, so that $\Omega_i = \Omega$. Also, by (2) it must be that $\Omega_{-i} = \Omega$. ∎

Proof of Proposition 3.2. By Lemma A.1.1, assumptions (2) and (3) imply that $(H_i^\sigma)^\infty(\Omega^c_{\mathbf{g}}) \subset \Omega_i$ for each $i = 1, 2$. But (1) implies that $\omega \in (H_1^\sigma)^\infty(\Omega^c_{\mathbf{g}}) \cap (H_2^\sigma)^\infty(\Omega^c_{\mathbf{g}}) \subset \Omega_1 \cap \Omega_2$. ∎

A.2 Proof of Lemma 3.3

We first note the following, which is essentially equivalent to Lemma A in Kajii and Morris (1997).

Lemma A.2.1. *Let $p > 0$. For any event E and player i, if $F_i \in \mathcal{F}_i$ and $F_i \subset B_i^p(E)$, then $P_i(F_i \setminus E) \leq ((1-p)/p)P_i(F_i \cap E)$.*

Fix $r \geq 1$, and consider any information system with $\rho((P_i)_{i=1,2}) = r$ and any event $E = E_1 \cup E_2$, each $E_i \in \mathcal{F}_i$. In the following, we want to obtain an upper bound for $P_j((H_*^p)^K(E))$.

Let $E_i^0 = E_i$ and $E^0 = E_1^0 \cup E_2^0$. Given $K \geq 1$ and $p \in (0,1]$, define $\{E_1^k, E_2^k, E^k\}_{k=1}^{K+1}$ recursively by

$$E_i^k = B_i^p(E^{k-1}), \quad E^k = E_1^k \cup E_2^k.$$

Then, $(H_*^p)^K(E) = E^K$. Let $D_i^0 = E_i^0$ and $D_i^k = E_i^k \setminus E_i^{k-1}$ for $k = 1, \dots, K+1$. Observe that $\{D_i^k\}_{k=0}^{K+1}$ is a partition of Ω, which is coarser than $\mathcal{Q}_i$.

For $i, j = 1, 2$, let $x_i(j, 0) = 0$, and

$$x_i(j,k) = \sum_{\ell=1}^{k} P_i(D_j^\ell \setminus E^{\ell-1}) \tag{A.1}$$

and

$$x_i(k) = x_i(1,k) + x_i(2,k)$$

for $k = 1, \dots, K$. Let also

$$z_i(j,k) = \sum_{\ell=1}^{k} P_i(D_j^\ell \cap E^{\ell-1}) \tag{A.2}$$

for $k = 1, \dots, K$. Note that

$$P_i((H_*^p)^K(E)) \leq P_i(E) + x_i(1,K) + x_i(2,K) \tag{A.3}$$

and that

$$z_i(j,k) \leq x_i(-j, k-1) + P_i(E_{-j}^0 \setminus E_j^0). \tag{A.4}$$

By using Lemma A.2.1, we have the following.

Lemma A.2.2. *For all $k = 1, \dots, K$ and $i = 1, 2$,*

$$x_i(i,k) \leq \frac{1-p}{p} z_i(i,k).$$

Now, $\rho((P_i)_{i=1,2}) = r$ implies that $x_i(j,k) \leq r x_{-i}(j,k)$ and $z_i(j,k) \leq r z_{-i}(j,k)$. Thus by Lemma A.2.2, we have the following.

Lemma A.2.3. *For all $k = 1, \dots, K$ and $i = 1, 2$,*

$$x_i(k) \leq \frac{r(1-p)}{p} x_{-i}(k-1) + \frac{r(1-p)}{p} P_{-i}(E^0).$$

Proof. By Lemma A.2.2 and (A.4),

$$\begin{aligned} x_i(k) &= x_i(i,k) + x_i(-i,k) \\ &\le x_i(i,k) + r x_{-i}(-i,k) \\ &\le \frac{1-p}{p} z_i(i,k) + \frac{r(1-p)}{p} z_{-i}(-i,k) \\ &\le \frac{r(1-p)}{p}\big(z_{-i}(i,k) + z_{-i}(-i,k)\big) \\ &\le \frac{r(1-p)}{p} x_{-i}(k-1) + \frac{r(1-p)}{p} P_{-i}(E^0), \end{aligned}$$

as claimed. ▮

By recursively using Lemma A.2.3, we obtain the upper bound of $P_i((H_*^p)^K(E))$.

Lemma A.2.4. *In any information system with $\rho((P_i)_{i=1,2}) = r$, any event $E \in \mathcal{F}_1 \oplus \mathcal{F}_2$ satisfies*

$$P_i((H_*^p)^K(E)) \le \max\{P_1(E), P_2(E)\} \sum_{k=0}^{K} \left\{\frac{r(1-p)}{p}\right\}^k \tag{A.5}$$

for all $i = 1, 2$.

We are now in a position to prove Lemma 3.3. It remains to consider the limit of the right hand side of (A.5) as $K \to \infty$. This is where the assumption that $p > r/(1+r)$ is used.

Proof of Lemma 3.3. If $p > r/(1+r)$, or $r(1-p)/p < 1$, then the right hand side of (A.5), $\sum_{k=0}^{K}\{r(1-p)/p\}^k$, converges to

$$\frac{1}{1 - \frac{r(1-p)}{p}} = \frac{p}{(1+r)p - r}$$

as $K \to \infty$. Hence, by Lemma A.2.4 we have the desired inequality. ▮

A.3 Proof of Lemma 5.1

Fix $\xi \ge 0$, and consider any information system with $d_0((P_i)_{i=1,2}) = \xi$ and any event $E = E_1 \cup E_2$, each $E_i \in \mathcal{F}_i$. We use the same labeling of events as in Subsection A.2. Let $E_i^0 = E_i$ and $E^0 = E_1^0 \cup E_2^0$. Define $\{E_1^k, E_2^k, E^k\}_{k=1}^{\infty}$ recursively by $E_i^k = B_i^p(E^{k-1})$ and $E^k = E_1^k \cup E_2^k$. Let $D_i^k = E_i^k \setminus E_i^{k-1}$. Note again that for all $k \ge 0$, $(H_*^p)^k(E) = E^k$.

Observe that

$$\begin{aligned} z_i(i,\infty) + z_i(-i,\infty) &\le P_i((H_*^p)^\infty(E)) \\ &\le P_i(E) + x_i(i,\infty) + x_i(-i,\infty), \end{aligned} \tag{A.6}$$

where $x_i(\cdot)$ and $z_i(\cdot)$ are as in (A.1) and (A.2), respectively. The condition $d_0((P_i)_{i=1,2}) = \xi$ implies that $x_i(j,\infty) \le x_{-i}(j,\infty) + \xi$ and $z_i(j,\infty) \le z_{-i}(j,\infty) + \xi$.

Proof of Lemma 5.1. By Lemma A.2.2 and (A.6),

$$\begin{aligned} P_i((H_*^p)^\infty(E)) &\le P_i(E) + x_i(i,\infty) + x_i(-i,\infty) \\ &\le P_i(E) + x_i(i,\infty) + \big(x_{-i}(-i,\infty) + \xi\big) \\ &\le P_i(E) + \frac{1-p}{p} z_i(i,\infty) + \frac{1-p}{p} z_{-i}(-i,\infty) + \xi \\ &\le P_i(E) + \frac{1-p}{p}\big(z_i(i,\infty) + z_i(-i,\infty)\big) + \frac{1}{p}\xi \\ &\le P_i(E) + \frac{1-p}{p} P_i((H_*^p)^\infty(E)) + \frac{1}{p}\xi. \end{aligned}$$

If $p > 1/2$, or $(2p-1)/p > 0$, then this implies

$$P_i((H_*^p)^\infty(E)) \le \frac{p}{2p-1} P_i(E) + \frac{\xi}{2p-1},$$

as claimed. ∎

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